

# ON SPECIALIZATIONS OF MINIMAL $p$ -DIVISIBLE GROUPS

By

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**Abstract.** In this paper, for any pair  $(\zeta, \xi)$  of Newton polygons with  $\zeta \prec \xi$ , we construct a concrete specialization from the minimal  $p$ -divisible group of  $\xi$  to the minimal  $p$ -divisible group of  $\zeta$  by a beautiful induction. This in particular gives the affirmative answer to the unpolarized analogue of a question by Oort on the boundaries of central streams, and gives another proof of the dimension formula of the central leaves in the unpolarized case.

## 1. Introduction

Let  $p$  be a rational prime. In this paper, by a  $p$ -divisible group we mean a Barsotti-Tate group in algebraic and arithmetic geometry, i.e., an inductive limit of finite algebraic group schemes having some properties. The precise definition of  $p$ -divisible groups will be reviewed at the beginning of Section 2. We study  $p$ -divisible groups in characteristic  $p$ . By the Dieudonné-Manin classification (cf. [7]), the isogeny classes of  $p$ -divisible groups over an algebraically closed field in characteristic  $p$  are classified by Newton polygons, see Definition 2.4 for the definition of Newton polygons.

Let  $\xi$  be a Newton polygon. Among  $p$ -divisible groups having Newton polygon  $\xi$ , there is a special  $p$ -divisible group which is called minimal. We denote it by  $H(\xi)$ . The main reference for minimal  $p$ -divisible groups is Oort [12]. As proposed in the latter part of [11, Question 6.10],  $H(\zeta)$  was expected to appear as a specialization of  $H(\xi)$  for  $\zeta \prec \xi$ , see Definition 2.4 for the notation about Newton polygons. Our main theorem (Theorem 5.1) implies that the expectation is true:

**COROLLARY 1.1.** *If  $\zeta \prec \xi$ , then  $H(\zeta)$  appears as a special fiber of a  $p$ -divisible group having  $H(\xi)$  as geometric generic fiber.*

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The converse of this corollary is also true, which is a consequence of Grothendieck-Katz [5], Theorem 2.3.1.

Now several proofs of this corollary have been known (cf. [14], Proposition 1.8 and [3], Corollary 5.1), but the authors could not find any known result which implies Theorem 5.1. Also the proof of Theorem 5.1 has an advantage of giving a very concrete construction of such specializations. For example it would be interesting to study relations between the construction and that in the proof in [3, Corollary 5.1]. The method of our proof is based on the idea of [1], where the second author proved the similar result in the polarized case with application to the theory of stratifications of the moduli space of principally polarized abelian varieties. Remark that, combining Corollary 1.1 and [2], Theorem 1.1, one can prove the unpolarized analogue of Oort's conjecture [11, 6.9], see [3], Corollary 5.2. From Theorem 5.1, we can also give a new proof of the dimension formula of central leaves in the unpolarized case, see Corollary 5.2.

The essential case for the proof of the main theorem is that  $\xi$  consists of two segments and  $\zeta \prec \xi$  is saturated. The case that the slopes of  $\xi$ , say  $\lambda_2 < \lambda_1$ , satisfy  $\lambda_2 < \lambda_1 \leq 1/2$  or  $1/2 \leq \lambda_2 < \lambda_1$  has been proved in [1], 8.4. This paper confirms that the above theorem holds in the remaining case  $\lambda_2 < 1/2 < \lambda_1$ .

This paper is organized as follows. In Section 2, we recall the definition of  $p$ -divisible groups and truncated Dieudonné modules of level one (abbreviated as  $DM_1$ ), which are Dieudonné modules of  $p$ -kernels of  $p$ -divisible groups. We also recall the definitions of  $(DM_1)$ -simple  $DM_1$ 's, their direct sums and minimal  $DM_1$ 's. In Section 3, we recall some facts on specializations of  $DM_1$ 's. In Section 4, we treat some combinatorics on Newton polygons. In Section 5, we state our main results. In Section 6, we give a proof of the main theorem (Theorem 5.1) whose beautiful induction would hopefully influence some future works.

## 2. Background

Let  $p$  be a rational prime, and  $h$  a non-negative integer. Let  $S$  be a scheme. A  $p$ -divisible group (Barsotti-Tate group) of height  $h$  over  $S$  is an inductive system  $X = (G_\nu, i_\nu)$  ( $\nu = 0, 1, 2, \dots$ ), where  $G_\nu$  is a finite locally free commutative group scheme over  $S$  of order  $p^{\nu h}$  and, for each  $\nu$  the sequence of commutative group schemes

$$0 \longrightarrow G_\nu \xrightarrow{i_\nu} G_{\nu+1} \xrightarrow{p^\nu} G_{\nu+1}$$

is exact, that is to say,  $G_\nu$  is identified via  $i_\nu$  with  $\text{Ker}(p^\nu : G_{\nu+1} \rightarrow G_{\nu+1})$ .

For a  $p$ -divisible group  $X = (G_\nu, i_\nu)$  over  $S$  and for a morphism  $T \rightarrow S$  of schemes, we have a  $p$ -divisible group  $X_T$  over  $T$  defined by  $(G_\nu \times_S T, i_\nu \times \text{id})$ . In particular for a closed point  $s = \text{Spec}(k) \rightarrow S$  the  $p$ -divisible group  $X_s$  over  $k$  is

called the *fiber* of  $X$  over  $s$ . A  $p$ -divisible group obtained as the fiber of  $X$  over a closed point is said to be a *special fiber* of  $X$ .

Let  $S$  be an  $\mathbb{F}_p$ -scheme. For any  $S$ -scheme  $T$ , let  $\text{Frob}$  be the absolute Frobenius on  $T$  and  $\text{Fr} : T \rightarrow T^{(p)}$  be the relative Frobenius.

A *truncated Bartotti-Tate group of level one* ( $\text{BT}_1$ ) over  $S$  is a finite locally free commutative group scheme over  $S$  such that  $\text{Im}(\text{Ver} : G^{(p)} \rightarrow G) = \text{Ker}(\text{Fr} : G \rightarrow G^{(p)})$  and  $\text{Ker}(\text{Ver} : G^{(p)} \rightarrow G) = \text{Im}(\text{Fr} : G \rightarrow G^{(p)})$ , where  $G^{(p)} = G \times_{S, \text{Frob}} S$  and  $\text{Ver}$  is the Verschiebung on  $G$ . The  $p$ -kernel  $X[p] := \text{Ker}(p : X \rightarrow X)$  of a  $p$ -divisible group  $X$  is a  $\text{BT}_1$ , and any  $\text{BT}_1$  over an algebraically closed field is the  $p$ -kernel of a  $p$ -divisible group.

Let  $k$  be a perfect field of characteristic  $p$ . Let  $W(k)$  denote the ring of Witt vectors with coefficients in  $k$ . Let  $\sigma$  be the Frobenius map on  $k$ , and use the same symbol  $\sigma$  for the Frobenius map on  $W(k)$ . A *Dieudonné module* over  $k$  is a finite  $W(k)$ -module  $M$  equipped with  $\sigma$ -linear homomorphism  $F : M \rightarrow M$  and  $\sigma^{-1}$ -linear homomorphism  $V : M \rightarrow M$  such that  $F \circ V$  and  $V \circ F$  are equal to the multiplication by  $p$ . In this paper we use the covariant Dieudonné theory. It says that there is an equivalence from the category of  $p$ -divisible group (resp.  $p$ -torsion finite commutative group schemes) over  $k$  to that of Dieudonné modules which is free (resp. is of finite length) as  $W(k)$ -modules. In this paper, Dieudonné modules corresponding to  $\text{BT}_1$ 's via the Dieudonné functor is called  $\text{DM}_1$ 's. The precise definition of them is as follows.

**DEFINITION 2.1.** A truncated Dieudonné module of level one (abbreviated as  $\text{DM}_1$ ) over  $k$  of height  $h$  is a triple  $(N, F, V)$  consisting of a  $k$ -vector space  $N$  of dimension  $h$ , a  $\sigma$ -linear homomorphism  $F : N \rightarrow N$  and a  $\sigma^{-1}$ -linear homomorphism  $V : N \rightarrow N$  satisfying  $\text{Ker } F = \text{Im } V$  and  $\text{Im } F = \text{Ker } V$ .

When  $k$  is an algebraically closed field, the following theorem is known (cf. Kraft [6], Oort [10] and Moonen-Wedhorn [9]).

**THEOREM 2.2.**

*There exists a bijection:*

$$\{0, 1\}^h \longleftrightarrow \{\text{DM}_1 \text{ over } k \text{ of height } h\} / \cong .$$

Before we recall the bijection in this theorem, we give a remark.

**REMARK 2.3.** Giving a  $\text{DM}_1$  over  $k$  is equivalent to giving an  $F$ -zip over  $k$  with support contained in  $\{0, 1\}$  with the terminology in [9], where Moonen-Wedhorn used some subsets of the Weyl group of  $GL_h$  (in this case) as classifying data of them over  $k$ . Using the Weyl group as classifying data is quite natural, but



It follows from  $\text{Ker } F = \text{Im } V = \{e_1, e_3\}$  and  $\text{Im } F = \text{Ker } V = \{e_1, e_2, e_3\}$  that  $(N, F, V)$  is a  $\text{DM}_1$ .

**DEFINITION 2.4.** A *Newton polygon*  $\xi$  is a finite multiple set of pairs  $(m_i, n_i)$  ( $i = 1, \dots, t$ ) of coprime non-negative integers. Conventionally we write  $\xi = \sum_{i=1}^t (m_i, n_i)$ . We call  $\sum_{i=1}^t n_i$  the *dimension* of  $\xi$  and  $\sum_{i=1}^t (m_i + n_i)$  the *height* of  $\xi$ . Put  $\lambda_i = n_i / (m_i + n_i)$ . We arrange  $(m_i, n_i)$  so that  $\lambda_i \geq \lambda_j$  for  $i < j$ . We regard  $\xi$  as the downward-convex line graph starting at  $(0, 0)$  and ending at  $(h, d)$  with breaking points  $\sum_{i>j} (m_i + n_i, n_i)$  for  $j = 0, \dots, t$ . For two Newton polygons  $\xi$  and  $\zeta$  with same ending point, we say  $\zeta \prec \xi$  if any point of  $\zeta$  is above or on  $\xi$ . We say that  $\zeta \prec \xi$  is *saturated* if there is no Newton polygon  $\eta$  such that  $\zeta \not\prec \eta \not\prec \xi$ .

To each Newton polygon, we associate a  $\text{DM}_1$ , which is called minimal:

**DEFINITION 2.5.** Let  $m$  and  $n$  be non-negative integers with  $\text{gcd}(m, n) = 1$ . To  $(m, n)$ , we associate a  $\text{DM}_1$  which corresponds to the sequence

$$A_{m,n} = \underbrace{1 \ 1 \ \cdots \ 1}_m \underbrace{0 \ \cdots \ 0}_n$$

by Theorem 2.2, and we write it as  $N_{m,n}$ . Such  $\text{DM}_1$ 's are called *simple* (more precisely should be called  $\text{DM}_1$ -*simple*). A *minimal*  $\text{DM}_1$  is the direct sum

$$N_\xi := \bigoplus N_{m_i, n_i}$$

for some Newton polygon  $\xi = \sum (m_i, n_i)$ . We may frequently identify a  $\text{DM}_1$   $N$  and  $A \in \{0, 1\}^h$  if  $N$  is the  $\text{DM}_1$  corresponding to  $A$ , and we write  $A \in \{0, 1\}^h$  as  $A_\xi$  if  $A$  corresponds to  $N_\xi$ .

This is the notion of Dieudonné modules of  $p$ -kernels of minimal  $p$ -divisible groups. Let us recall the definition of minimal  $p$ -divisible groups. For each coprime pair  $(m, n)$  of non-negative integers, let  $H_{m,n}$  be the  $p$ -divisible group  $H_{m,n}$  over  $\mathbb{F}_p$  whose Dieudonné module  $\mathbb{D}(H_{m,n})$  is given by

$$\mathbb{D}(H_{m,n}) = \bigoplus_{i=1}^{m+n} \mathbb{Z}_p e_i$$

with  $F, V$ -operations defined by  $F e_i = e_{i-m}$  and  $V e_i = e_{i-n}$ , where  $e_i$  for non-positive  $i$  is inductively defined by  $e_i = p e_{i+m+n}$ . We set

$$H(\xi) = \bigoplus H_{m_i, n_i}$$

for each Newton polygon  $\xi = \sum(m_i, n_i)$ . A *minimal*  $p$ -divisible group is a  $p$ -divisible group which is isomorphic over an algebraically closed field to  $H(\xi)$  for some Newton polygon  $\xi$ . Let  $k$  be an algebraically closed field. The Dieudonné module of the  $p$ -kernel of  $H(\xi)_k$  is isomorphic to  $N_\xi$ . The main theorem of [12] says that for any  $p$ -divisible group  $X$  over  $k$ , if  $\mathbb{D}(X[p]) \simeq N_\xi$ , then  $X$  is isomorphic to  $H(\xi)_k$ .

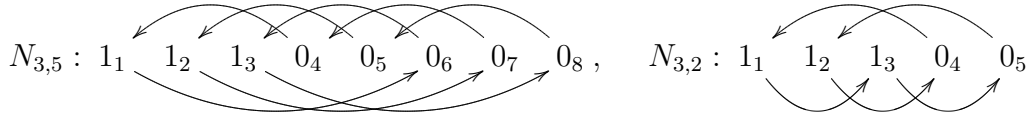
Let  $N_1$  and  $N_2$  be two  $\text{DM}_1$ 's. Let  $A$  and  $B$  be the elements of  $\{0, 1\}^{h_1}$  and  $\{0, 1\}^{h_2}$  corresponding to  $N_1$  and  $N_2$  respectively. Recall how to get the element of  $\{0, 1\}^{h_1+h_2}$  corresponding to the direct sum  $N_1 \oplus N_2$ . For this, we define a real number  $b_A(i)$  with  $0 \leq b_A(i) \leq 1$  for each  $i = 1, \dots, h_1$ . To define  $b_A(i)$ , we consider the  $(F, V^{-1})$ -diagram of  $A$ . Running through the arrows in the reverse direction from  $\delta_i^A$  we set  $b_l = 0$  if the  $l$ -th arrow is  $F$  and  $b_l = 1$  if the  $l$ -th arrow is  $V^{-1}$ . We define  $b_A(i)$  to be the binary expansion

$$b_A(i) = 0.b_1b_2 \dots .$$

In the similar way, we define  $b_B(j)$  for  $j = 1, \dots, h_2$ . The sequence corresponding to  $N_1 \oplus N_2$ , written as  $A \oplus B$ , is obtained by arranging  $\delta_i^A$  ( $i = 1, \dots, h_1$ ) and  $\delta_j^B$  ( $j = 1, \dots, h_2$ ) in ascending order of their binary expansions, namely  $A \oplus B = \dots \delta_i^A \dots \delta_j^B \dots$  if  $b_A(i) < b_B(j)$  and  $\dots \delta_j^B \dots \delta_i^A \dots$  if  $b_B(j) < b_A(i)$ , and so on.

**EXAMPLE 2.6.**

Here, let us see  $N_{3,5} \oplus N_{3,2}$  as an example of direct sum of  $\text{DM}_1$ 's. We write  $N_{3,5}$  and  $N_{3,2}$  in the following using sequences.



Let  $A \in \{0, 1\}^8$  and  $B \in \{0, 1\}^5$  be the sequences of  $N_{3,5}$  and  $N_{3,2}$  respectively.

Consider  $b_A(8)$  for example: we trace vectors in the reverse direction from  $0_8$ :

$$0_8 \xleftarrow{V^{-1}} 1_3 \xleftarrow{F} 0_6 \xleftarrow{V^{-1}} 1_1 \xleftarrow{F} 0_4 \xleftarrow{F} 0_7 \xleftarrow{V^{-1}} 1_2 \xleftarrow{F} 0_5 \xleftarrow{F} 0_8 \xleftarrow{V^{-1}} \dots .$$

Hence we get  $b_A(8) = 0.10100100 \dots$ . Similarly we have

$$\begin{aligned} b_A(8) &= 0.10100100 \dots, & b_B(1) &= 0.01011 \dots, \\ b_A(7) &= 0.10010100 \dots, & b_B(2) &= 0.01101 \dots, \\ b_A(6) &= 0.10010010 \dots, & b_B(3) &= 0.10101 \dots, \\ b_A(5) &= 0.01010010 \dots, & b_B(4) &= 0.10110 \dots. \end{aligned}$$

By the above, we in particular get:

$$b_A(5) < b_B(1) < b_B(2) < b_A(6) < b_A(7) < b_A(8) < b_B(3) < b_B(4).$$

Then the sequence  $A_\xi$  corresponding to  $N_\xi = N_{3,5} \oplus N_{3,2}$  is

$$A_\xi = 1_1^A \ 1_2^A \ 1_3^A \ 0_4^A \ 0_5^A \ 1_1^B \ 1_2^B \ 0_6^A \ 0_7^A \ 0_8^A \ 1_3^B \ 0_4^B \ 0_5^B,$$

where, to avoid confusion, we write each elements  $1_i$  (resp.  $0_i$ ) of  $A$  as  $1_i^A$  (resp.  $0_i^A$ ), and we write each elements  $1_j$  (resp.  $0_j$ ) of  $B$  as  $1_j^B$  (resp.  $0_j^B$ ).

More generally we have

**LEMMA 2.7.** *Let  $\xi = (m_1, n_1) + (m_2, n_2)$  with  $\lambda_2 < 1/2 < \lambda_1$  where  $\lambda_i = n_i/(m_i + n_i)$ . The sequence associated with  $N_\xi = N_{m_1, n_1} \oplus N_{m_2, n_2}$  is given by*

$$\underbrace{1_1^A \cdots 1_{m_1}^A}_{m_1} \underbrace{0_{m_1+1}^A \cdots 0_{n_1}^A}_{n_1 - m_1} \underbrace{1_1^B \cdots 1_{n_2}^B}_{n_2} \underbrace{0_{n_1+1}^A \cdots 0_{m_1+n_1}^A}_{m_1} \underbrace{1_{n_2+1}^B \cdots 1_{m_2}^B}_{m_2 - n_2} \underbrace{0_{m_2+1}^B \cdots 0_{m_2+n_2}^B}_{n_2},$$

where  $A$  and  $B$  are the sequences associated with  $N_{m_1, n_1}$  and  $N_{m_2, n_2}$  respectively.

*Proof.* See [1], Proposition 4.20. □

### 3. Specializations

We introduce the notion of families of  $DM_1$ 's (but such a family will be also called a  $DM_1$  simply), and review some basic facts on specializations of  $DM_1$ 's.

Let  $R$  be a commutative ring of characteristic  $p > 0$ . Let  $\sigma : R \rightarrow R$  be the Frobenius endomorphism defined by  $\sigma(a) = a^p$ .

#### DEFINITION 3.1.

A  $DM_1$  over  $R$  of height  $h$  is a quintuple  $\mathcal{N} = (\mathcal{N}, C, D, F, V^{-1})$  where

- (1)  $\mathcal{N}$  is a free  $R$ -module of rank  $h$ ,
- (2)  $C$  and  $D$  are submodules of  $\mathcal{N}$  which are locally direct summands of  $\mathcal{N}$ ,
- (3)  $F : (\mathcal{N}/C) \otimes_{R, \sigma} R \rightarrow D$  and  $V^{-1} : C \otimes_{R, \sigma} R \rightarrow \mathcal{N}/D$  are  $R$ -linear isomorphisms.

#### REMARK 3.2.

- (1) In Moonen and Wedhorn [9],  $DM_1$ 's over  $R$  here are called  $F$ -zips over  $R$  with support contained in  $\{0, 1\}$ .
- (2) When  $R$  is a perfect field  $k$ , to a  $DM_1 (N, V, F)$  over  $k$  with the notation in Definition 2.1 we associate a quintuple  $(N, VN, FN, F, V^{-1})$ , which

naturally becomes a  $\text{DM}_1$  with the notation in Definition 3.1. By this association, we identify them.

Let  $k$  be an algebraically closed field of characteristic  $p$ , and let  $R = k[[t]]$  be the ring of formal power series over  $k$ . For  $\mathcal{N}$  an arbitrary  $\text{DM}_1$  over  $R$ , we can consider  $\mathcal{N}_k := \mathcal{N} \otimes_R k$ , which is a  $\text{DM}_1$  over  $k$ . Hence we have the canonical map called *specialization*

$$\{\text{DM}_1 \text{ over } R\} \longrightarrow \{\text{DM}_1 \text{ over } k\}.$$

sending  $\mathcal{N}$  to  $\mathcal{N}_k$ .

Let  $K$  be the fractional field of  $R$ . We also consider  $\mathcal{N}_{\overline{K}} := \mathcal{N} \otimes_R \overline{K}$ , which is a  $\text{DM}_1$  over  $\overline{K}$  and is called the geometric generic fiber of  $\mathcal{N}$ .

**DEFINITION 3.3.** Let  $A$  and  $B$  be elements of  $\{0, 1\}^h$ . We say  $B \preceq A$  if there exists an  $\text{DM}_1$  over  $R$  such that  $\mathcal{N}_k$  is associated with  $B$  and  $\mathcal{N}_{\overline{K}}$  is associated with  $A$ .

Let  $A, B$  be the sequences of  $N_1, N_2$  respectively. We denote by  $A \oplus B$  the sequence corresponding to  $N_1 \oplus N_2$ . It is obvious that if  $B \prec A$ , then  $B \oplus P \prec A \oplus P$  holds for  $A, B \in \{0, 1\}^h$  and  $P \in \{0, 1\}^{h'}$ .

**DEFINITION 3.4.** Let  $A$  and  $B$  be elements of  $\{0, 1\}^h$ . We say  $B < A$  if there exist elements  $A^{(1)}, \dots, A^{(\ell-1)}$  of  $\{0, 1\}^h$  with  $A^{(0)} := B$  and  $A^{(\ell)} := A$  such that one can write  $A^{(i)} = P^{(i)} \oplus Q^{(i)}$ , and  $A^{(i-1)} = P^{(i)} \oplus Q^{(i-1)}$  for  $i = 1, \dots, \ell$  for some  $P^{(i)}$ , where  $Q^{(i-1)}$  is constructed by exchanging an adjacent subsequence “0 1” in  $Q^{(i)}$  for “1 0”.

The following is known, see the proof of [10, Proposition 11.1].

**PROPOSITION 3.5.**

*If  $B \leq A$ , then  $B \preceq A$  holds.*

We want to know relations between  $\mathcal{N} \otimes_R \overline{K}$  and  $\mathcal{N} \otimes_R k$ . The aim of this paper is to show the existence of  $\mathcal{N}$  satisfying  $\mathcal{N} \otimes_R \overline{K} \cong N_{m_1, n_1} \oplus N_{m_2, n_2} = N_\xi$  and  $\mathcal{N} \otimes_R k \cong N_{m'_1, n'_1} \oplus N_{m'_2, n'_2} \oplus \dots \oplus N_{m'_i, n'_i} = N_\zeta$  for any  $\zeta \prec \xi$ . Before dealing with the general case, let us see as an example the case of  $\zeta = (2, 3) + 4(1, 1) \prec \xi = (3, 5) + (3, 2)$ .

**EXAMPLE 3.6.**

We consider  $N_\xi = N_{3,5} \oplus N_{3,2}$ . As seen in Example 2.6, the sequence corresponding to  $N_\xi$  is

$$A_\xi = 1 \ 1 \ 1 \ 0 \ \underline{0 \ 1} \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0.$$



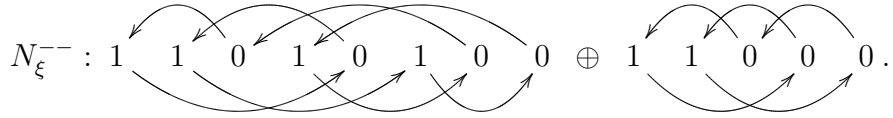
Let  $A_\xi^-$  be the sequence obtained by exchanging 0 and 1 of the above underline part. Then another subsequence 0 1 appears, see the double-underline part below:

$$A_\xi^- = 1\ 1\ 1\ \underline{0\ 1}\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0.$$

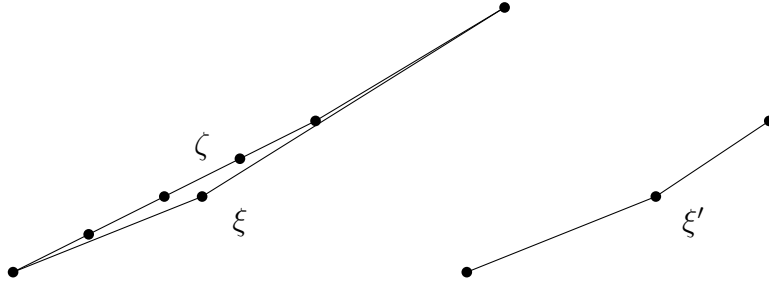
Let  $A_{\xi}^{--}$  be the sequence obtained by exchanging 0 and 1 of the double-underline part:

$$A_{\xi}^{--} = 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0.$$

The  $(F, V^{-1})$ -diagram of  $N_\xi^{--}$  associated with  $A_{\xi}^{--}$  is



The right direct summand  $1\ 1\ 0\ 0\ 0$  is the sequence of  $N_{2,3}$ , and the left direct summand  $A' := 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0$  can be specialized to  $B := 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0$  by exchanging repeatedly 0 1 for 1 0. Note that  $B$  is the sequence of  $N_{1,1}^{\oplus 4}$ . Hence we get a specialization from  $N_\xi$  to  $N_\zeta = N_{2,3} \oplus N_{1,1} \oplus N_{1,1} \oplus N_{1,1} \oplus N_{1,1}$ . The figure of  $\xi$  and  $\zeta$  is as follows.



Put  $\xi' = (1, 2) + (3, 2)$ . Then

$$A_{\xi'} = 1\ 0\ 0 \oplus 1\ 1\ 1\ 0\ 0 = 1\ \underline{0\ 1}\ 1\ 0\ 1\ 0\ 0.$$

We define  $A_{\xi'}^-$  in the similar way as when we defined  $A_\xi^-$  from  $A_\xi$ , i.e.,

$$A_{\xi'}^- = 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0.$$

We observe that  $A_{\xi'}^-$  is equal to  $A'$  above. The key step of our proof is to show that such phenomenon always occurs. As the height of  $\xi'$  is less than that of  $\xi$ , this allows us to prove the main theorem by induction.

#### 4. Combinatorics of Newton polygons

We show some combinatorial facts on Newton polygons, which will be used later on.

For Newton polygons  $\zeta \prec \xi$  of height  $h$  we set

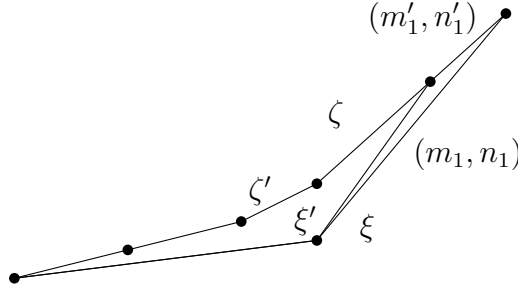
$$c(\zeta, \xi) = 2 \sum_{i=1}^h (\zeta(i) - \xi(i)), \quad (4)$$

where we regard Newton polygons (line graphs in the  $xy$ -plane) as functions on  $\{x \in \mathbb{R} \mid 0 \leq x \leq h\}$ . There is a relationship between  $c(\zeta, \xi)$  and the number of segments of  $\zeta$  in a special case:

**PROPOSITION 4.1.** *Let  $\xi$  be a Newton polygon consisting of two segments. Let  $\zeta \prec \xi$  be a saturated pair of Newton polygons. Then  $c(\zeta, \xi)$  is equal to the number of segments of  $\zeta$ .*

*Proof.* First note that  $c(\zeta, \xi)$  is equal to the area of the part which is surrounded by  $\zeta$  and  $\xi$ .

We prove the proposition by induction on the number  $c$  of the segments of  $\zeta$ . The case of  $c = 1$  is obvious from Lemma 4.2 below. Assume  $c \geq 2$ . Write  $\xi = (m_1, n_1) + (m_2, n_2)$  and  $\zeta = (m'_1, n'_1) + \cdots + (m'_c, n'_c)$ . We set  $\zeta' = (m'_2, n'_2) + \cdots + (m'_c, n'_c)$  and  $\xi' = (m_1 - m'_1, n_1 - n'_1) + (m_2, n_2)$  if  $m_1 \geq m'_1$  and  $n_1 \geq n'_1$  or  $\zeta' = (m'_1, n'_1) + \cdots + (m'_{c-1}, n'_{c-1})$  and  $\xi' = (m_1, n_1) + (m_2 - m'_c, n_2 - n'_c)$  otherwise. We write a proof only in the former case, as the same argument works for the latter case. The figure in the former case is as follows.



We have  $\zeta' \prec \xi'$  and this is saturated. By the hypothesis of induction, the number of segments of  $\zeta'$  is equal to  $c(\zeta', \xi')$ , and therefore the number of segments of  $\zeta$  is equal to  $c(\zeta', \xi') + 1$ . Considering the areas we have the following.

$$c(\zeta, \xi) = c(\zeta', \xi') + n_1 m'_1 - m_1 n'_1$$

We have to show that  $n_1 m'_1 - m_1 n'_1 = 1$  under the assumption that there is no lattice point in the region surrounded by  $(m'_1, n'_1)$ ,  $\xi$  and  $\xi'$ . This follows from the lemma below.  $\square$

For vectors  $\vec{a} = (a_1, a_2)$  and  $\vec{b} = (b_1, b_2)$ , we set  $\langle \vec{a}, \vec{b} \rangle = a_1 b_2 - a_2 b_1$ .

**LEMMA 4.2.** *Let  $\vec{a} = (a_1, a_2)$ ,  $\vec{b} = (b_1, b_2)$  with  $a_1, a_2, b_1, b_2 \in \mathbb{Z}$  and  $\langle \vec{a}, \vec{b} \rangle > 0$ . Assume that there exists no lattice point in the interior of the triangle that is the convex hull of  $(0, 0)$ ,  $\vec{a}$  and  $\vec{b}$ . Then  $\langle \vec{a}, \vec{b} \rangle = 1$ .*

*Proof.* Assume  $v := \langle \vec{a}, \vec{b} \rangle \geq 2$ , and lead a contradiction. Choose  $\vec{x} = (x, y)$  with  $\langle \vec{a}, \vec{x} \rangle = 1$  and  $x, y \in \mathbb{Z}$ . Choose  $r \in \mathbb{Z}$ ,  $s, t \in \mathbb{Q}$  such that  $\vec{x} + r\vec{a} = s\vec{a} + t\vec{b}$  with  $0 \leq s \leq 1$ . Taking  $\langle -, \vec{a} \rangle$  on the both sides, we have  $\langle \vec{a}, \vec{x} \rangle + r\langle \vec{a}, \vec{a} \rangle = s\langle \vec{a}, \vec{a} \rangle + t\langle \vec{a}, \vec{b} \rangle$ , whence  $1 = tv$ . We have  $\vec{x} + r\vec{a} = s\vec{a} + (1/v)\vec{b}$ . As  $s \in (1/v)\mathbb{Z}$  with  $0 \leq s < 1$ , we have  $s + (1/v) \leq 1$ . This means that  $\vec{x} + r\vec{a}$  is a lattice point in the interior of the triangle that is the convex hull of  $(0, 0)$ ,  $\vec{a}$  and  $\vec{b}$ . This is a contradiction.  $\square$

## 5. Main results

Our main theorem is the following.

### THEOREM 5.1.

*Let  $\zeta$  and  $\xi$  be Newton polygons of height  $h$  with  $\zeta \prec \xi$ . Set  $M_0 := N_\zeta$  and  $M_c := N_\xi$  with  $c = c(\zeta, \xi)$ . Then there exist DM<sub>1</sub>'s  $M_1, \dots, M_{c-1}$  such that  $A^{(0)} < A^{(1)} \dots < A^{(c-1)} < A^{(c)}$ , where  $A^{(i)} \in \{0, 1\}^h$  are the sequences corresponding to  $M_i$  ( $0 \leq i \leq c$ ).*

The proof will be given in the next section. Let us see that this theorem implies Corollary 1.1 in Introduction.

*Proof of Corollary 1.1.* It suffices to show the case that  $\zeta$  contains no étale segment  $(1, 0)$ . Indeed write  $\zeta = f(1, 0) + \zeta'$  and  $\xi = f(1, 0) + \xi'$  for  $f \in \mathbb{Z}_{\geq 0}$ , where  $\zeta'$  has no étale segment  $(1, 0)$ . If there exists a specialization from  $H(\xi')$  to  $H(\zeta')$ , then considering the direct sum of it and  $H(f(1, 0))$  we have a specialization from  $H(\xi)$  to  $H(\zeta)$ .

Assume that  $\zeta$  has no étale segment  $(0, 1)$ . Let  $k$  be an algebraically closed field. Theorem 5.1 in particular says that there exists a DM<sub>1</sub>  $\mathcal{N}$  over  $R = k[[t]]$  such that  $\mathcal{N}_k$  is isomorphic to  $N_\zeta$  and  $\mathcal{N}_K$  is isomorphic to  $N_\xi$  over an algebraically closed field. There exists a lifting of DM<sub>1</sub> to a display  $\mathcal{M}$  over  $R$  (cf. [2], Lemma 4.1). Thanks to the theory of display by Zink [16, Theorem 103 on p. 221], we get the  $p$ -divisible group  $\mathcal{X}$  associated with  $\mathcal{M}$ , which is a specialization from  $H(\xi)$  to  $H(\zeta)$ . Indeed by [12] the special fiber  $\mathcal{X}_k$  is isomorphic to  $H(\zeta)$  and the generic fiber  $\mathcal{X}_K$  is isomorphic to  $H(\xi)$  over an algebraically closed field.  $\square$

Here is another corollary. Consider a  $p$ -divisible group  $X$  over  $k$ . Let  $\text{Def}(X) =$

$\mathrm{Spf}(\Gamma)$  be the local deformation space in characteristic  $p$  (cf. [13, 1.9]). Set  $D(X) = \mathrm{Spec}(\Gamma)$  and let  $\mathcal{X} \rightarrow D(X)$  be the  $p$ -divisible group over  $D(X)$  obtained via the category equivalence obtained in [4, 2.4.4]. For a  $p$ -divisible group  $\mathcal{Y} \rightarrow S$  over a scheme  $S$  and for a  $p$ -divisible group  $Y$  over  $k$ , we consider

$$\mathcal{C}_Y(S) = \{s \in S \mid \mathcal{Y}_s \text{ is isomorphic to } Y \text{ over an algebraically closed field}\}.$$

To the dimension formula of  $\mathcal{C}_X(D(X))$ , Oort gave three proofs in [13]. This paper gives the fourth proof.

**COROLLARY 5.2.** *Let  $X$  be a  $p$ -divisible group of Newton polygon  $\xi$  of height  $h$  and of dimension  $d$ , we have*

$$\dim \mathcal{C}_X(D(X)) = c(\xi),$$

where we set  $c(\xi) := c(\sigma, \xi)$  with  $\sigma = (h - d, d)$ .

*Proof.* Let  $\mathcal{S}_{Y[p]}(D(Y))$  be the Ekedahl-Oort stratum on  $D(Y)$ , i.e., the locally closed subset consisting of points whose  $p$ -kernel types are the same as that of  $Y[p]$  (cf. [13], 1.6). We consider it as a locally closed subscheme of  $D(Y)$  by giving it the induced reduced structure. By Oort [13], 7.19 and 7.9, we have

$$\dim \mathcal{C}_X(D(X)) = \dim \mathcal{C}_{H(\xi)}(D(H(\xi))) = \dim \mathcal{S}_{H(\xi)[p]}(D(H(\xi))).$$

Combining Wedhorn [15, 6.10] and Moonen [8, 2.1.2],  $\dim \mathcal{S}_{Y[p]}(D(Y))$  this is equal to the length  $\ell(A)$  of the type  $A$  of  $Y[p]$ , where  $\ell(A) = \#\{(i, j) \mid i < j, \delta_i = 0, \delta_j = 1\}$  for  $A = \delta_1 \cdots \delta_h$ . Let  $\chi = (h - d)(1, 0) + d(0, 1)$ . By Theorem 5.1, we have

$$\ell(A_\sigma) + c(\sigma, \xi) \leq \ell(A_\xi) \leq \ell(A_\chi) - c(\xi, \chi).$$

The corollary follows from the obvious cases  $\ell(A_\sigma) = 0 = c(\sigma)$  and  $\ell(A_\chi) = (h - d)d = c(\chi)$ , using the formula  $c(\zeta, \xi) = c(\xi) - c(\zeta)$ .  $\square$

**REMARK 5.3.** In the proof of Corollary 5.2, we used the dimension formula of Ekedahl-Oort strata due to Wedhorn [15, 6.10] and Moonen [8, 2.1.2], but, as type  $A$  in the proof is combinatorially complicated (no explicit general form of  $A$  is known), determining  $\ell(A)$  is a non-trivial problem.

## 6. Proof

In this section, we prove Theorem 5.1. The essential case is that  $\xi$  consists of two segments, i.e.,  $\xi = (m_1, n_1) + (m_2, n_2)$  with  $\mathrm{gcd}(m_1, n_1) = 1$  and

$\gcd(m_2, n_2) = 1$ . Here we treat the case of  $\lambda_2 < 1/2 < \lambda_1$ , where  $\lambda_i = n_i/(m_i+n_i)$  for  $i = 1, 2$ , since the other case  $\lambda_2 < \lambda_1 \leq 1/2$  or  $1/2 \leq \lambda_2 < \lambda_1$  has already been studied in [1], 8.4.

Let  $\xi = (m_1, n_1) + (m_2, n_2)$  with  $\lambda_2 < 1/2 < \lambda_1$ , until *Proof of Theorem 5.1*. Let  $N_\xi$  be the minimal  $\text{DM}_1$  of  $\xi$ , and we denote the sequence associated with  $N_\xi$  as  $A_\xi$ . In Lemma 2.7, we have seen that the sequence  $A_\xi$  is described as

$$\underbrace{1_1^A \cdots 1_{m_1}^A}_{m_1} \underbrace{0_{m_1+1}^A \cdots 0_{n_1}^A}_{n_1-m_1} \underbrace{1_1^B \cdots 1_{n_2}^B}_{n_2} \underbrace{0_{n_1+1}^A \cdots 0_{m_1+n_1}^A}_{m_1} \underbrace{1_{n_2+1}^B \cdots 1_{m_2}^B}_{m_2-n_2} \underbrace{0_{m_2+1}^B \cdots 0_{m_2+n_2}^B}_{n_2}.$$

For a sequence  $S$  of 0 and 1, let  $S^-$  denote the sequence obtained by exchanging “first” adjacent subsequence “0 1” for “1 0” in  $S$ . To simplify, we write  $(A_\xi^-)^- = A_\xi^{--}$ . Note  $A_\xi^{--} < A_\xi^- < A_\xi$ . Let  $N_\xi^-$  and  $N_\xi^{--}$  be the  $\text{DM}_1$ 's associated with  $A_\xi^-$  and  $A_\xi^{--}$  respectively. We shall prove Theorem 5.1 by induction, using the following three propositions.

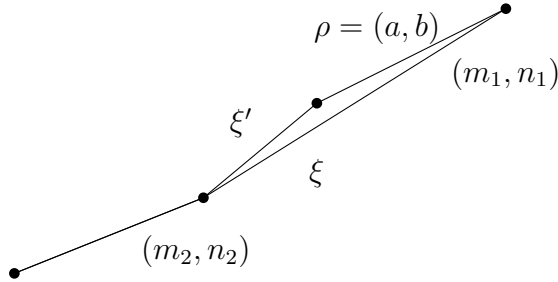
**PROPOSITION 6.1.** *If one of the following conditions*

- (1)  $n_1 - m_1 > 1$  and  $n_2 > 0$ ,
- (2)  $n_1 - m_1 = 1$  with  $m_1 > 0$  and  $n_2 = 1$ ,
- (3)  $n_2 = 0$  and  $m_1 > 1$

*holds, then we have*

$$N_\xi^{--} = N_{\xi'}^- \oplus N_\rho,$$

*with  $\rho = (a, b)$  and  $\xi' = (m_1 - a, n_1 - b) + (m_2, n_2)$ , where  $\rho$  is uniquely determined by  $\xi$  so that the area of the region surrounded by  $\xi$ ,  $\xi'$  and  $\rho$  in the picture below is one.*



*Proof.* First, let us see the case (1). Since  $\gcd(m_1, n_1) = 1$ , we have  $m_1 > 0$ . The sequences  $A = A_{m_1, n_1}$  and  $B = A_{m_2, n_2}$  associated with  $N_{m_1, n_1}$  and  $N_{m_2, n_2}$

are

$$A = \underbrace{1_1^A \cdots 1_{m_1}^A}_{m_1} \underbrace{0_{m_1+1}^A \cdots 0_{n_1}^A}_{n_1-m_1} \underbrace{0_{n_1+1}^A \cdots 0_{m_1+n_1}^A}_{m_1},$$

$$B = \underbrace{1_1^B \cdots 1_{n_2}^B}_{n_2} \underbrace{1_{n_2+1}^B \cdots 1_{m_2}^B}_{m_2-n_2} \underbrace{0_{m_2+1}^B \cdots 0_{m_2+n_2}^B}_{n_2}$$

respectively. By Lemma 2.7, the sequence  $A \oplus B$  of  $N_\xi = N_{m_1, n_1} \oplus N_{m_2, n_2}$  is

$$N_\xi : 1_1^A \cdots 1_{m_1}^A \ 0_{m_1+1}^A \cdots 0_{n_1}^A \ 1_1^B \cdots 1_{n_2}^B \ 0_{n_1+1}^A \cdots 0_{m_1+n_1}^A \ 1_{n_2+1}^B \cdots 1_{m_2}^B \ 0_{m_2+1}^B \cdots 0_{m_2+n_2}^B .$$

Here, we write only arrows of  $F$  and  $V^{-1}$  which are necessary to check the structure of  $N_\xi$ . Of course the  $(F, V^{-1})$ -diagram of  $A \oplus B$  consists of the cycles of  $A$  and  $B$ :

$$N_{m_1, n_1} : \begin{array}{ccc} 0_{m_1+1}^A & \longleftarrow \cdots \longleftarrow & 0_{n_1}^A \\ \downarrow & & \uparrow \\ 1_1^A & \longrightarrow \cdots \longrightarrow & 0_{h_1}^A \end{array} \quad N_{m_2, n_2} : \begin{array}{ccc} 0_{m_2+1}^B & \longleftarrow \cdots \longleftarrow & 1_{n_2}^B \\ \downarrow & & \uparrow \\ 1_1^B & \longrightarrow \cdots \longrightarrow & 0_{h_2}^B \end{array}$$

where  $h_1 = m_1 + n_1$  and  $h_2 = m_2 + n_2$ . The DM<sub>1</sub>  $N_\xi^-$  is obtained by exchanging  $0_{n_1}^A$  and  $1_1^B$  in  $A_\xi$ . The sequence  $A_\xi^-$  corresponding to  $N_\xi^-$  is described as

$$1_1^A \cdots 1_{m_1}^A \ 0_{m_1+1}^A \cdots 0_{n_1-1}^A \ \underline{1_1^B} \ 0_{n_1}^A \ 1_2^B \cdots 1_{n_2}^B \ 0_{n_1+1}^A \cdots 0_{h_1}^A \ 1_{n_2+1}^B \cdots 1_{m_2}^B \ 0_{m_2+1}^B \cdots 0_{h_2}^B .$$

We claim that  $N_\xi^-$  consists of one cycle. Indeed, by the exchange of  $0_{n_1}^A$  and  $1_1^B$ , the destination of the vector from  $0_{m_2+1}^B$  is changed to  $0_{n_1}^A$ , and the destination of the vector from  $0_{m_1+n_1}^A$  is changed to  $1_1^B$ . The claim follows from the diagram of  $N_\xi^-$ :

$$N_\xi^- : \begin{array}{ccccccc} 0_{m_1+1}^A & \longleftarrow \cdots \longleftarrow & 0_{n_1}^A & \longleftarrow & 0_{m_2+1}^B & \longleftarrow \cdots \longleftarrow & 1_{n_2}^B \\ \downarrow & & \uparrow \times & & \downarrow \times & & \uparrow \\ 1_1^A & \longrightarrow \cdots \longrightarrow & 0_{h_1}^A & \longrightarrow & 1_1^B & \longrightarrow \cdots \longrightarrow & 0_{h_2}^B \end{array} \quad (\star)$$

We get the DM<sub>1</sub>  $N_\xi^{--}$  by exchanging  $0_{n_1-1}^A$  and  $1_1^B$  in  $A_\xi^-$ . The sequence  $A_\xi^{--}$  corresponding to  $N_\xi^{--}$  is described as

$$1_1^A \cdots 1_{m_1}^A \ 0_{m_1+1}^A \cdots 0_{n_1-2}^A \ 1_1^B \ 0_{n_1-1}^A \ 0_{n_1}^A \ 1_2^B \cdots 1_{n_2}^B \ 0_{n_1+1}^A \cdots 0_{h_1}^A \ 1_{n_2+1}^B \cdots 1_{m_2}^B \ 0_{m_2+1}^B \cdots 0_{h_2}^B .$$

We shall use these arrows to check the structure of cycles of  $N_\xi^{--}$ . To consider the cycles of  $N_\xi^{--}$ , we rewrite the  $(F, V^{-1})$ -cycle of  $N_\xi^-$  as follows:

$$N_\xi^- : \begin{array}{ccccccc} \bullet & \longleftarrow & \cdots & \longleftarrow & 0_{n_1-1}^A & \longleftarrow & 0_{h_1-1}^A & \longleftarrow & \cdots & \longleftarrow & 0_{n_1}^A \\ \downarrow & & & & & & & & & & \uparrow \\ \bullet & \longrightarrow & \cdots & \longrightarrow & 0_{h_1}^A & \longrightarrow & 1_1^B & \longrightarrow & \cdots & \longrightarrow & 0_{m_2+1}^B \end{array}$$

We claim that  $N_\xi^{--}$  consists of two cycles. By the exchange of  $0_{n_1-1}^A$  and  $1_1^B$ , the destination of the vector from  $0_{h_1}^A$  is changed to  $0_{n_1-1}^A$ , and the destination of the vector from  $0_{h_1-1}^A$  is changed to  $1_1^B$ . The claim follows from the diagram:

$$N_\xi^{--} : \begin{array}{ccccccc} \bullet & \longleftarrow & \cdots & \longleftarrow & 0_{n_1-1}^A & \longleftarrow & \cdots & \longleftarrow & 0_{h_1-1}^A & \longleftarrow & \cdots & \longleftarrow & 0_{n_1}^A \\ \downarrow & & & & \uparrow & & & & \downarrow & & & & \uparrow \\ \bullet & \longrightarrow & \cdots & \longrightarrow & 0_{h_1}^A & \longrightarrow & \cdots & \longrightarrow & 1_1^B & \longrightarrow & \cdots & \longrightarrow & 0_{m_2+1}^B \end{array}$$

Let  $C_1$  be the cycle containing  $0_{h_1}^A$ , and let  $C_2$  be the cycle containing  $1_1^B$ . It suffices to show the following properties:

- (a)  $C_1$  is the  $(F, V^{-1})$ -diagram of  $N_\rho$ ,
- (b)  $C_2$  is the  $(F, V^{-1})$ -diagram of  $N_{\xi'}$ .

As  $C_1$  is a part of the cycle of simple  $DM_1$   $N_{m_1, n_1}$ , the sequence corresponding to  $C_1$  is written as follows:

$$\underbrace{1^A \cdots 1^A}_a \underbrace{0^A \cdots 0_{m_1+n_1}^A}_b$$

Since this cycle coincides with the cycle obtained from  $A$  by applying [1], Lemma 5.6 to the adjacent  $0_{n_1-1}^A$   $0_{n_1}^A$ , we have  $an_1 - bm_1 = 1$ . Hence the property (a) holds.

Next we consider the cycle  $C_2$ . We can write  $C_2$  as follows.

$$\begin{array}{cccccccccccccccccccc} 1^A & \cdots & 1^A & 0^A & \cdots & 0^A & 1_1^B & 0_{n_1}^A & 1^B & \cdots & 1^B & 0^A & \cdots & 0^A & 0_{h_1-1}^A & 1_{n_2+1}^B & \cdots & 1^B & 0_{m_2+1}^B & 0^B & \cdots & 0_{h_2}^B \end{array}$$

Let  $L$  be the sequence associated to  $C_2$ , and let  $L'$  be the sequence constructed by exchanging  $1_1^B$  and  $0_{n_1}^A$  in  $L$ . Then we have

$$L' = \underbrace{1^A \cdots 1^A}_{m_1-a} \underbrace{0^A \cdots 0_{n_1}^A}_{(n_1-b)-(m_1-a)} \underbrace{1_1^B \cdots 1^B}_{n_2} \underbrace{0^A \cdots 0^A}_{m_1-a} \underbrace{1^B \cdots 1^B}_{m_2-n_2} \underbrace{0^B \cdots 0^B}_{n_2}.$$

By Lemma 2.7, we see that this sequence is decomposed into two simple  $DM_1$ 's as follows:

$$\underbrace{1^A \dots 1^A}_{m_1-a} \underbrace{0^A \dots 0^A}_{n_1-b} \oplus \underbrace{1^B \dots 1^B}_{m_2} \underbrace{0^B \dots 0^B}_{n_2} = A_{m_1-a, n_1-b} \oplus A_{m_2, n_2}.$$

It is clear that  $L'$  corresponds to  $N_{\xi'}$  with  $\xi' = (m_1 - a, n_1 - b) + (m_2, n_2)$ , whence the property (b) holds.

Let us see the case (2). In this case, for  $A_{\xi}^-$  given by exchanging the subsequence  $0_{n_1}^A 1_1^B$  for  $1_1^B 0_{n_1}^A$ , we construct  $A_{\xi}^{-}$  by the exchange of  $0_{h_1}^A$  and  $1_2^B$  with  $h_1 = m_1 + n_1$ . The sequence  $A_{\xi}^{-}$  consists of two cycles. The cycle containing  $0_{h_1}^A$  is associated with  $N_{1,1}$ . In fact, the sequence of this cycle is  $1_1^B 0_{h_1}^A$ . Moreover, for the other cycle, one can check that this cycle is associated with  $N_{\xi'}$  with  $\xi' = (m_1 - 1, n_1 - 1) + (m_2, n_2)$  by exchanging  $1_{m_1}^A$  and  $0_{n_1}^A$ .

For the case (3), since  $\gcd(m_2, n_2) = 1$ , we have  $m_2 = 1$ . In this case, we have only one subsequence “0 1” in  $A_{\xi}$ , which is  $0_{h_1}^A 1_1^B$ . We construct  $A_{\xi}^-$  by exchanging this subsequence for  $1_1^B 0_{h_1}^A$ . We exchange  $0_{h_1-1}^A$  and  $1_1^B$  to construct  $A_{\xi}^{-}$ . Then  $A_{\xi}^{-}$  consists of two cycles. The cycle containing  $0_{h_1-1}^A$  is associated with  $N_{a,b}$  as this cycle consists of some terms of  $A$ . Since this cycle coincides with the cycle obtained from  $A$  by applying [1], Lemma 5.6 to the adjacent  $0_{h_1-1}^A 0_{h_1}^A$ , we see  $an_1 - bm_1 = 1$ . One can check that the other cycle is associated with  $N_{\xi'}$  with  $\xi' = (m_1 - a, n_1 - b) + (m_2, n_2)$  by exchanging  $1_1^B$  and  $0_{h_1}^A$ .  $\square$

**EXAMPLE 6.2.** We have dealt with the case of  $\xi = (3, 5) + (3, 2)$  in Example 3.6, where we observed that

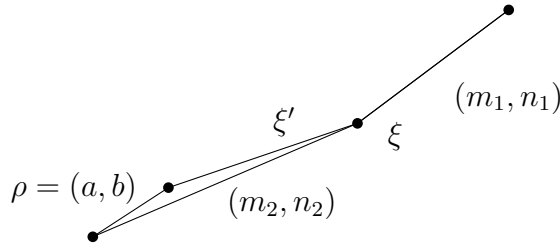
$$N_{(3,5)+(3,2)}^{-} = N_{(1,2)+(3,2)}^{-} \oplus N_{2,3}$$

holds.

**PROPOSITION 6.3.** *If  $n_1 - m_1 = 1$  and  $n_2 > 1$ , then we have*

$$N_{\xi}^{-} = N_{\xi'} \oplus N_{\rho},$$

with  $\rho = (a, b)$  and  $\xi' = (m_1, n_1) + (m_2 - a, n_2 - b)$ , where  $\rho$  is uniquely determined by  $\xi$  so that the area of the region surrounded by  $\xi$ ,  $\xi'$  and  $\rho$  in the picture below is one.







1 1 1 0 1 0 0 1 0 0. One verifies that  $A_{\xi'}^-$  coincides with the left direct summand of (a).

The final proposition treats the remaining case:

**PROPOSITION 6.5.** *We have*

- (1) *If  $m_1 = n_2 = 0$ , then  $N_{\xi}^- = N_{1,1}$ ;*
- (2) *If  $m_1 = 0$  and  $n_2 = 1$ , then  $N_{\xi}^{--} = N_{1,1} \oplus N_{m_2-1,1}$ ;*
- (3) *If  $m_1 = 1$  and  $n_2 = 0$ , then  $N_{\xi}^{--} = N_{1,n_1-1} \oplus N_{1,1}$ .*

*Proof.* In the case (1),  $N_{\xi}$  is expressed by

$$N_{\xi} : \begin{array}{cc} & \curvearrowright \\ 0 & 1 \\ & \curvearrowleft \end{array}.$$

We construct  $N_{\xi}^-$  by exchanging 0 and 1 in  $N_{\xi}$ , then we get

$$N_{\xi}^- : \begin{array}{cc} & \curvearrowleft \\ 1 & 0 \\ & \curvearrowright \end{array}.$$

Hence  $N_{\xi}^- = N_{1,1}$ .

In the case (2), we exchange the subsequence  $0_1^A 1_1^B$  to construct  $A_{\xi}^-$ . Moreover, we construct  $A_{\xi}^{--}$  by exchanging the subsequence  $0_1^A 1_2^B$  for  $1_2^B 0_1^A$ . Then  $A_{\xi}^{--}$  consists of two cycles. The cycle containing  $0_1^A$  is associated with  $N_{1,1}$  since the sequence of this cycle is  $1_1^B 0_1^A$ . Moreover, the other cycle is associated with  $N_{m_2-1,1}$ .

In the case (3), The sequence  $A_{\xi}^-$  is constructed by exchanging  $0_{h_1}^A 1_1^B$  for  $1_1^B 0_{h_1}^A$  with  $h_1 = m_1 + n_1$ . We exchange the subsequence  $0_{h_1-1}^A 1_1^B$  for  $1_1^B 0_{h_1-1}^A$ , and we construct  $A_{\xi}^{--}$ . Then  $A_{\xi}^{--}$  consists of two cycles. The cycle containing  $1_1^B$  is associated with  $N_{1,1}$ . In fact, the sequence of this cycle is  $1_1^B 0_{h_1}^A$ . The other cycle is associated with  $N_{1,n_1-1}$ . This completes the proof.  $\square$

Finally, we see that Propositions 6.1, 6.3 and 6.5 imply Theorem 5.1.

*Proof of Theorem 5.1.* It suffices to show the essential case that  $\zeta \prec \xi$  is saturated and  $\xi$  consists of two segments. Put  $\xi = (m_1, n_1) + (m_2, n_2)$  and assume  $\zeta \prec \xi$  is saturated. Set  $\lambda_1 = n_1/(m_1 + n_1)$  and  $\lambda_2 = n_2/(m_2 + n_2)$ . The case that the slopes of  $\xi$ , say  $\lambda_2 < \lambda_1$ , satisfy  $\lambda_2 < \lambda_1 \leq 1/2$  and  $1/2 \leq \lambda_2 < \lambda_1$  has been proved in [1], 8.4. As it suffices to prove the remaining case, we assume  $\lambda_2 < 1/2 < \lambda_1$ . We claim that there exist  $A^{(1)}, \dots, A^{(c-2)}$  with  $c = c(\zeta, \xi)$  such that  $A_{\zeta} =: A^{(0)} < \dots < A^{(c-2)} < A^{(c-1)} := A_{\xi}^-$ . This claim implies the theorem by  $A_{\xi}^- < A_{\xi}$ . Recall that for a sequence  $S$  we denote by  $S^-$  the sequence obtained by exchanging the first adjacent subsequence “1 0” in  $S$  for “0 1”, and  $S^{--}$  means

$(S^-)^-$ . We use induction on  $h = m_1 + n_1 + m_2 + n_2$ . The claim is clear when  $h$  is smallest, i.e.,  $h = 2$  with  $\zeta = (1, 1) \prec \xi = (0, 1) + (1, 0)$ , see the case of  $m_1 = n_2 = 0$  in Proposition 6.5. Assume  $h > 2$ . Note that  $\xi$  with  $h > 2$  belongs to either of the following six cases.

- (i)  $n_1 - m_1 > 1$  and  $n_2 > 0$ ,
- (ii)  $n_1 - m_1 = 1$  with  $m_1 > 0$  and  $n_2 = 1$ ,
- (iii)  $n_2 = 0$  and  $m_1 > 1$ ,
- (iv)  $n_1 - m_1 = 1$  and  $n_2 > 1$ ,
- (v)  $m_1 = 0$  and  $n_2 = 1$ ,
- (vi)  $m_1 = 1$  and  $n_2 = 0$ .

In the case of (i), (ii) and (iii), we get  $N_{\xi}^{--} = N_{\xi'}^- \oplus N_{\rho}$  by Proposition 6.1, where  $\zeta = \zeta' + \rho$  and  $\zeta' \prec \xi'$  is saturated. In the case of (iv), we get  $N_{\xi}^{--} = N_{\xi'}^- \oplus N_{\rho}$  by Proposition 6.3, where  $\zeta = \zeta' + \rho$  and  $\zeta' \prec \xi'$  is saturated. By the hypothesis of induction, there exist  $B^{(1)}, \dots, B^{(c-3)}$  such that  $A_{\zeta'} =: B^{(0)} < \dots < B^{(c-3)} < B^{(c-2)} := A_{\xi'}^-$ . Here we note that  $c(\zeta', \xi') = c(\zeta, \xi) - 1$  holds by Proposition 4.1. Hence putting  $A^{(i)} = B^{(i)} \oplus A_{\rho}$  ( $i = 0, \dots, c-2$ ), we have

$$A_{\zeta} = A_{\zeta'} \oplus A_{\rho} = A^{(0)} < \dots < A^{(c-2)} = A_{\xi'}^- \oplus A_{\rho} = A_{\xi}^{--} < A_{\xi}^-.$$

In the case of (v) and (vi), we get  $N_{\xi}^{--} = N_{\rho} \oplus N_{\rho'}$  by Proposition 6.5, where  $\zeta = \rho + \rho'$  and  $\zeta \prec \xi$  is saturated, whence we have  $A_{\zeta} = A_{\xi}^{--} < A_{\xi}^-$ . Thus we have proved the claim for all the cases.  $\square$

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