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# LOGARITHMIC ABELIAN VARIETIES, PART V: PROJECTIVE MODELS

By

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*Dedicated to Professor Luc Illusie*

**Abstract.** This is Part V of our series of papers on log abelian varieties. In this part, we study polarization and projective models of log abelian varieties.

## Contents



## Introduction

This is Part V of our series of papers on log abelian varieties. For general ideas of our theory of log abelian varieties, the following references would be helpful: Introduction of Part I [8], Introduction and Section 1 of Part II [9], and a survey [14].

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In Part I [8], we studied a complex analytic theory. In Part II [9], we started to study an algebraic theory. In Part III [10], we illustrated our theory in the case of log elliptic curves. In Part IV [11], we introduced models for a log abelian variety and studied proper models. In this Part V, we study projective models and polarizations on log abelian varieties. In the remaining part of this series, one of the main subjects should be moduli spaces of log abelian varieties.

The main theorem of this paper is the existence of projective models of a polarized log abelian variety (Theorem 1.11). The polarization is defined as a biextension in 1.3. We use the method of theta functions as in the analytic case (Part I [8], 5.4). To this end, we generalize various theorems of line bundles on abelian varieties (cf. [12] the theorem of Appel–Humbert, the theorem of the cube) to log abelian varieties.

The organization of this paper is as follows. In the first two sections, we summarize our results. In Section 1, we state the results of the existence of projective models. In later sections, these are proved based on a systematic study of $\mathbb{G}_m$-torsors, $\mathbb{G}_{m,\log}$-torsors and $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsors on log abelian varieties, which are summarized in Section 2. Sections 3–8 give the proofs of the results described in Sections 1–2 in the case of constant degeneration. Based on them, we prove a large part of the results described in Section 1 on projective models in Section 9. After some preparations in Sections 10–11, we prove the general cases of the results described in Sections 1–2 (Section 12).

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## 1. Main results, I (projective models)

In this section, we state our main results concerning the existence of a projective model of a polarized log abelian variety (Theorem 1.11). The proofs of the propositions in this section will be given in later sections.

**1.1.** First we briefly review an outline of the definition of weak log abelian variety. See [11] Section 1 for the details. We use the same notation there.

Let $S$ be an fs log scheme. A *weak log abelian variety* over $S$ is a sheaf $A$ of abelian groups on the big étale site of $S$ satisfying the following three conditions.

(1) Each geometric fiber corresponds to an admissible and nondegenerate log

1-motif.

(2) Étale locally on $S$, there is a subsheaf of $A$ represented by a semiabelian variety $G$ such that the quotient $A/G$ satisfies the condition explained below.

(3) The diagonal morphism $A \to A \times A$ is represented by finite morphisms. The condition on $A/G$ in (2) is as follows: There are finitely generated free $\mathbb{Z}$-modules $X$ and $Y$, an admissible pairing

$$\langle\ ,\ \rangle\colon X \times Y \to \mathbb{G}_{m,\log}/\mathbb{G}_m$$

on $S$, and an isomorphism $A/G \cong \mathcal{Q}/\overline{Y}$, where $\overline{Y}$ is the image of $Y$ in

$$\mathcal{Q} := \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}.$$

See 1.3 of [11] for the definition of the last group, i.e., the $(Y)$-part of $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$.

Let $\overline{X}$ be the image of $X$ in $\mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)$. Then the induced pairing

$$\langle\ ,\ \rangle\colon \overline{X} \times \overline{Y} \to \mathbb{G}_{m,\log}/\mathbb{G}_m$$

on $S$ glues globally on $S$. The $G$ also glues globally and called the *semiabelian part* of $A$ (cf. [11] 1.7).

If we replace "admissible and nondegenerate" in the above condition (1) by "polarizable" ([9] Definition 2.8), we obtain the definition of log abelian variety ([9] Section 4).

**1.2.** We define a polarization on a weak log abelian variety. First, we discuss the dual weak log abelian variety in a special case.

Assume that a weak log abelian variety $A$ is with constant degeneration ([11] 1.7). Let $M$ be a log 1-motif corresponding to $A$ via [9] Theorem 8.1. Then, the dual $A^*$ of $A$ is defined to be the weak log abelian variety corresponding to the dual log 1-motif $M^*$ of $M$. (We will define the dual weak log abelian variety in the general case in a forthcoming part of this series of papers.)

**1.3.** We define a polarization on a weak log abelian variety $A$ as a biextension. For biextensions, see [7] Exp. VII and VIII, and [4].

First, assume that $A$ is with constant degeneration. Then, we already have the notion of polarization as the one on the log 1-motif $M$ corresponding to $A$. See [9] Definition 2.8 for the definition of a polarization on a log 1-motif. This notion is interpreted in terms of biextension. In fact, if $p$ is a polarization on $M$, it induces a homomorphism $A$ to $A^*$. But, since $A^*$ is canonically embedded into $\mathcal{E}xt(A, \mathbb{G}_{m,\log})$ ([9] Theorem 7.4 (3), cf. [9] Remark 7.5 (2)), it gives a homomorphism $A \to \mathcal{E}xt(A, \mathbb{G}_{m,\log})$. Hence, by Proposition 2.3 below (whose case of constant degeneration is proved in 6.3), it corresponds to a biextension of $(A, A)$ by $\mathbb{G}_{m,\log}$. This biextension is symmetric.

We return to the general case. Let $A$ be a weak log abelian variety. Then, a *polarization on* $A$ is a symmetric biextension of the pair $(A, A)$ by $\mathbb{G}_{m,\log}$ whose pullback to $(\mathrm{fs}/\overline{s})$ for any $s \in S$ is induced by a polarization on the log 1-motif corresponding to $A \times_S \overline{s}$ as above.

By definition, a polarizable weak log abelian variety is a log abelian variety.

**1.4.** Our main result says that a polarization yields a special model. To describe constructions of special models, it is convenient to introduce the following condition on $A$.

**1.4.1.** Étale locally on $S$, there is a homomorphism $\psi\colon \overline{Y} \to \overline{X}$ satisfying the following three conditions:

(1) $\psi$ is compatible with $\langle\ ,\ \rangle$ in the sense that $\langle \psi(y), z\rangle = \langle \psi(z), y\rangle$ for any $y, z \in \overline{Y}$.

(2) $\langle \psi(y), y\rangle \in M_S/\mathcal{O}_S^\times \subset M_S^{\mathrm{gp}}/\mathcal{O}_S^\times$ for any $y \in \overline{Y}$.

(3) $\psi$ induces an isomorphism $\overline{Y} \otimes \mathbb{Q} \xrightarrow{\simeq} \overline{X} \otimes \mathbb{Q}$.

**1.4.2 Remark.** This condition is satisfied if $A$ is a log abelian variety. But the converse is not valid. Further, a weak log abelian variety does not necessarily satisfy the condition 1.4.1. See 4.14–4.15 for the details.

**1.5.** Let $F$ be an abelian sheaf on the étale site $(\mathrm{fs}/S)_{\text{ét}}$ of fs log schemes over $S$. Let $p$ be a biextension of $(A, A)$ by $F$. Let $L_p$ be the $F$-torsor on $A$ defined as follows. For any fs log scheme $U$ over $S$ and any morphism $a\colon U \to A$, the $F$-torsor $L_p(a)$ on $U$ is $p(a, a)$ (= the value of the biextension $p$ at $(a, a)\colon U \to A \times A$).

This construction $p \mapsto L_p$ induces a natural homomorphism, called the *pullback to the diagonal*,

$$\mathrm{Biext}(A, A; F) \to H^1(A, F),$$

which coincides with the homomorphism induced by the diagonal morphism $A \to A \times A$.

**1.6 Proposition.** *Let $A$ be a weak log abelian variety over an fs log scheme $S$. Assume that either $A$ is with constant degeneration or $A$ satisfies the condition* 1.4.1. *Let* $\mathrm{Hom}_{\langle\,,\,\rangle}(\overline{Y}, \overline{X})$ *be the group of homomorphisms $\psi\colon \overline{Y} \to \overline{X}$ which are compatible with $\langle\ ,\ \rangle$ (*1.4.1 (1)*).*

(1) *There are natural isomorphisms*

$$\mathrm{Hom}_{\langle\,,\,\rangle}(\overline{Y}, \overline{X}) \cong \mathrm{Biext}_{\mathrm{sym}}(A/G, A/G; \mathbb{G}_{m,\log}/\mathbb{G}_m) \cong \mathrm{Biext}_{\mathrm{sym}}(A, A; \mathbb{G}_{m,\log}/\mathbb{G}_m).$$

*Here* $\mathrm{Biext}_{\mathrm{sym}}$ *means the group of symmetric biextensions.*

(2) *The pullback to the diagonal*

$$\mathrm{Biext}(A, A; \mathbb{G}_{m,\log}/\mathbb{G}_m) \to H^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) \quad (\text{cf. } 1.5)$$

*sends the image of* $\psi \in \mathrm{Hom}_{\langle\ ,\ \rangle}(\overline{Y}, \overline{X})$ *by the isomorphism in* (1) *to the pullback of the class of the* $\mathbb{G}_{m,\log}/\mathbb{G}_m$*-torsor on* $A/G = \mathcal{Q}/\overline{Y}$*, obtained as the quotient of* $\mathcal{Q} \times \mathbb{G}_{m,\log}/\mathbb{G}_m$ *by the action* $(h, \lambda) \mapsto (yh, h(\psi(y)^{-2})\langle \psi(y), y\rangle^{-1}\lambda)$ *(*$y \in \overline{Y}$*) of* $\overline{Y}$*. Here* $yh$ *denotes* $x \mapsto h(x)\langle x, y\rangle$ *for* $x \in \overline{X}$*.*

This is proved in 12.12.

**1.6.1 Remark.** There is a choice of the sign in this proposition. See Remark 5.2.1 for the details.

**1.7.** We review briefly the model of $A$ associated to a fan. See Section 2 of [11] for details.

We assume that an admissible pairing $X \times Y \to \mathbb{G}_{m,\log}/\mathbb{G}_m$ as in 1.1 exists globally.

Let $\widetilde{A}$ be the fiber product of

$$A \to \mathcal{Q}/\overline{Y} \leftarrow \mathcal{Q}.$$

Then, we have an exact sequence

$$0 \to G \to \widetilde{A} \to \mathcal{Q} \to 0.$$

We define certain subsheaves of $\widetilde{A}$ and $A$. Étale locally on the base $S$, we can take an fs monoid $\mathcal{S}$, a homomorphism $\mathcal{S} \to M_S/\mathcal{O}_S^\times$, and an $\mathcal{S}$-admissible pairing

$$\langle\ ,\ \rangle \colon X \times Y \to \mathcal{S}^{\mathrm{gp}}$$

which lifts the above $\mathbb{G}_{m,\log}/\mathbb{G}_m$-valued pairing. We assume that they exist and we fix them.

Let $C$ be the subcone of $\mathrm{Hom}(\mathcal{S}, \mathbb{N}) \times \mathrm{Hom}(X, \mathbb{Z})$ defined as

$$C := \{(N, l) \mid l(X_{\mathrm{Ker}(N)}) = 0\}$$

(cf. [8] 3.4.2). By a *finitely generated subcone of* $C$, we mean a finitely generated submonoid $\sigma$ of the additive monoid $C$ such that for any $a \in C$ satisfying $na \in \sigma$ for some $n \geq 1$, we have $a \in \sigma$.

If $\sigma$ is a finitely generated subcone of $C$, we write

$$\widetilde{A}^{(\sigma)}$$

for the pullback of $\overline{V}(\sigma)$ by $\widetilde{A} \to \mathcal{Q}$ and call it the $\sigma$*-part* of $\widetilde{A}$, where $\overline{V}(\sigma)$ is the $\sigma$-part of $\mathcal{Q}$ defined in [11] 2.3.

Let $\sigma$ be a finitely generated $\mathbb{Q}_{\geq 0}$-submonoid of $C_{\mathbb{Q}_{\geq 0}}$. Then, by abuse of notation, we denote $\overline{V}(\sigma \cap C)$ by $\overline{V}(\sigma)$ and $\widetilde{A}^{(\sigma \cap C)}$ by $\widetilde{A}^{(\sigma)}$.

Let $\Sigma$ be a fan in $C$, that is, a fan in $\mathrm{Hom}(\mathcal{S}^{\mathrm{gp}} \times X, \mathbb{Q})$ whose support is contained in $C \otimes \mathbb{Q}_{\geq 0}$. We define the subsheaf

$$\widetilde{A}^{(\Sigma)}$$

of $\widetilde{A}$ as the union of $\widetilde{A}^{(\sigma)}$ ($\sigma \in \Sigma$). This coincides with the pullback of $\mathcal{Q}^{(\Sigma)}$ by $\widetilde{A} \to \mathcal{Q}$, where

$$\mathcal{Q}^{(\Sigma)} = \bigcup_{\sigma \in \Sigma} \overline{V}(\sigma)$$

Next, assume that $\Sigma$ is stable under the action of $Y$ on $C$, where $y \in Y$ acts on $C$ by $(N, l) \mapsto (N, l + N(\langle -, y \rangle))$.

Then we define the subsheaf

$$A^{(\Sigma)}$$

of $A$ as the pullback of the image of $\mathcal{Q}^{(\Sigma)}$ in $\mathcal{Q}/\overline{Y}$ by $A \to \mathcal{Q}/\overline{Y}$.

We call these subsheaves $\Sigma$-*parts* of $\widetilde{A}$ and $A$, respectively.

**1.8 Remark.** In general, as was explained in [11] Remark 2.7, for a weak log abelian variety $A$ over an fs log scheme $S$, the above construction glues and the sheaf $\widetilde{A}$ exists globally on $S$ canonically. In the case of constant degeneration, it coincides with $G_{\log}^{(Y)}$ ([11] 1.4).

**1.9.** Assume that there is a homomorphism $\psi \colon \overline{Y} \to \overline{X}$ satisfying (1)–(3) in 1.4.1. Then we have a proper model $A^{(\psi)}$ of $A$ as follows.

Étale locally on the base $S$, take an $\mathcal{S}$-admissible pairing as in 1.7 and a homomorphism $\phi \colon Y \to X$ with finite cokernel such that

(1) $\langle \phi(y), z \rangle = \langle \phi(z), y \rangle$ in $\mathcal{S}^{\mathrm{gp}}$ for any $y, z \in Y$;
(2) $\langle \phi(y), y \rangle \in \mathcal{S}$ for any $y \in Y$

which induces $\psi$.

For each $y \in Y$, let

$$C(y) := \{(N, l) \in C \mid N(\langle \phi(z), z \rangle) + 2l(\phi(z)) \geq N(\langle \phi(y), y \rangle) + 2l(\phi(y)) \text{ for any } z \in Y\}.$$

Then the cones $C(y)$ for varying $y$ with their faces form a complete fan in $C$ (cf. [11] 3.1). We call this fan the *first standard* fan and denote it by $\Sigma_\phi$. As will be shown in 4.13, the local models $A^{(\Sigma_\phi)}$ glue into a proper model $A^{(\psi)}$ over $S$, which is described as follows.

Let $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\psi)} = \bigcup_{y \in \overline{Y}} U(y)$ as a sheaf, where

$$U(y) = \{h \in \mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m) \mid h(\psi(y))^2 \langle \psi(y), y \rangle | h(\psi(z))^2 \langle \psi(z), z \rangle \text{ for any } z \in \overline{Y}\}.$$

Let $A^{(\psi)} \subset A$ be the inverse image of $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\psi)}/\overline{Y} \subset \mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}/\overline{Y}$ under the canonical surjection $A \to \mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}/\overline{Y}$. Here, $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}$ is the subsheaf of $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ consisting of the sections $h \in \mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)(U)$ satisfying for any $u \in U$ and $x \in \overline{X}_{\overline{u}}$, there are $y, y' \in \overline{Y}_{\overline{u}}$ such that $\langle x, y\rangle_{\overline{u}} | h_{\overline{u}}(x) | \langle x, y'\rangle_{\overline{u}}$. When there are an $\mathcal{S}$-admissible pairing and $\phi$ at the beginning of this paragraph, this subsheaf is naturally isomorphic to $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$ (cf. [9] Remark 7.6.1).

The $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $A$ associated to $\psi$ (Proposition 1.6 (2)) has a canonical section on $A^{(\psi)}$ which is given on $U(y)$ by $\mathcal{Q} \to \mathbb{G}_{m,\log}/\mathbb{G}_m$ ; $h \mapsto h(\psi(y)^2)$ $\langle \psi(y), y\rangle$.

**1.10.** Let $p$ be a polarization on $A$.

As the image of $-p$ by the pullback to the diagonal $\mathrm{Biext}(A, A; \mathbb{G}_{m,\log}) \to H^1(A, \mathbb{G}_{m,\log})$ (1.5), we get an element $L_{-p} (= L_p^{-1})$ of $H^1(A, \mathbb{G}_{m,\log})$.

On the other hand, the image of $-p$ in $\mathrm{Biext}(A, A; \mathbb{G}_{m,\log}/\mathbb{G}_m)$ corresponds via Proposition 1.6 (1) to a homomorphism $\psi \colon \overline{Y} \to \overline{X}$ which satisfies the conditions (1)–(3) in 1.4.1. The $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $A$ corresponding to $\psi$ (Proposition 1.6 (2)) coincides with the associated $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor to the above $\mathbb{G}_{m,\log}$-torsor $L_{-p}$ on $A$.

In general, a $\mathbb{G}_{m,\log}$-torsor together with a section of the associated $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor gives a $\mathbb{G}_m$-torsor. Hence, from the above $L_{-p}$ and the canonical section of the associated $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $A^{(\psi)}$ explained in the last part of 1.9, we obtain a $\mathbb{G}_m$-torsor on $A^{(\psi)}$. Consider the invertible sheaf consisting of the sections of this $\mathbb{G}_m$-torsor.

The next is a main result in this paper and proved in 12.14.

**1.11 Theorem.** *Let $A$ be a polarized log abelian variety over an fs log scheme $S$. Let $\psi \colon \overline{Y} \to \overline{X}$ be the induced homomorphism. Then the above invertible sheaf on $A^{(\psi)}$ is relatively ample over the base $S$. In particular, $A^{(\psi)}$ is locally projective over $S$.*

**1.12.** In the above, we only discuss the first standard fan. There are other important special fans, which are introduced in Section 4, that is, the second standard fan and the fan associated to a star (4.6). These fans also produce projective models, as is shown similarly (cf. Theorem 9.1).

## 2. Main results, II (torsors)

In this section, we state our results on $\mathbb{G}_m$-torsors, $\mathbb{G}_{m,\log}$-torsors, and $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsors on a weak log abelian variety $A$. The main results are Theorem 2.2

and Proposition 2.4. The proofs of the propositions in this section will be given in later sections.

**2.1 Proposition.** *Let $A$ be a weak log abelian variety over an fs log scheme $S$. Assume that either $A$ is with constant degeneration or $A$ satisfies the condition* 1.4.1*. For $F = \mathbb{G}_m$, $\mathbb{G}_{m,\log}$, or $\mathbb{G}_{m,\log}/\mathbb{G}_m$, we have*

$$\mathcal{H}^0(A, F) = F.$$

*Here $F$ on the right-hand-side is regarded as a sheaf on $(\mathrm{fs}/S)_{\text{ét}}$, $F$ on the left-hand-side is regarded as a sheaf on the étale site $(\mathrm{fs}/A)_{\text{ét}}$ of pairs $(S', a)$, where $S'$ is an fs log scheme over $S$ and $a$ is an element of $A(S')$, and $\mathcal{H}^0(A, -)$ denotes the direct image functor from the category of sheaves on $(\mathrm{fs}/A)_{\text{ét}}$ to the category of sheaves on $(\mathrm{fs}/S)_{\text{ét}}$.*

This is proved after Proposition 12.1.

For a section $u$ of $A$, let

$$t_u \colon A \to A;\ v \mapsto uv$$

be the translation by $u$. (We denote the group law of $A$ multiplicatively.)

The cubic isomorphism of $\mathbb{G}_m$-torsors on abelian varieties ([12]) can be generalized as follows.

**2.2 Theorem** (Cubic isomorphism)**.** *Let $A$ be a weak log abelian variety over an fs log scheme $S$. We denote the group law of $A$ multiplicatively. Let $F = \mathbb{G}_m$, $\mathbb{G}_{m,\log}$, or $\mathbb{G}_{m,\log}/\mathbb{G}_m$. Let $L$ be an $F$-torsor on $A$.*

*Assume that one of the following conditions* (a)–(c) *is satisfied.*

(a) *$A$ is with constant degeneration.*

(b) *$A$ satisfies the condition* 1.4.1 *and $F = \mathbb{G}_{m,\log}/\mathbb{G}_m$.*

(c) *$A$ satisfies the condition* 1.4.1 *and $S$ is noetherian.*

*Then the following holds.*

(1) *There is a unique isomorphism*

$$\iota_{S,A,L} : s^*_{\{1,2,3\}}L \cdot s^*_{\{1,2\}}L^{-1} \cdot s^*_{\{1,3\}}L^{-1} \cdot s^*_{\{2,3\}}L^{-1} \cdot s^*_{\{1\}}L \cdot s^*_{\{2\}}L \cdot s^*_{\{3\}}L \cdot s^*_{\varnothing}L^{-1} \cong 1$$

*of $F$-torsors on $A \times A \times A$ whose pullback on $S$ by the zero section of $A \times A \times A$ is the evident one. Here for a subset $I$ of $\{1, 2, 3\}$, $s_I$ denotes the morphism $A \times A \times A \to A$ ; $(x_1, x_2, x_3) \mapsto \prod_{i \in I} x_i$ and $1$ on the right-hand-side denotes the trivial $F$-torsor on $A \times A \times A$. This isomorphism is functorial with respect to $(S, A, L)$. We have $\iota_{S,A,L\cdot L'} = \iota_{S,A,L} \cdot \iota_{S,A,L'}$ for $F$-torsors $L$ and $L'$ on $A$.*

(2) *For any $S' \to S$ and $a, b \in A(S')$, the $F$-torsor $t^*_{ab}L \cdot t^*_a L^{-1} \cdot t^*_b L^{-1} \cdot L$ on $A \times_S S'$ is trivial étale locally on $S'$.*

This is proved in 12.5.

**2.3 Proposition.** *Let $A$ be a weak log abelian variety over an fs log scheme. Assume that either $A$ is with constant degeneration or $A$ satisfies the condition* 1.4.1*. Let $F = \mathbb{G}_m$, $\mathbb{G}_{m,\log}$, or $\mathbb{G}_{m,\log}/\mathbb{G}_m$. Then we have*

$$\mathrm{Biext}(A, A; F) \xrightarrow{\cong} \mathrm{Hom}(A, \mathcal{E}xt(A, F)).$$

The case of constant degeneration of this is proved in 6.3. The general case is proved in 12.2.

Let $\mathcal{H}^i(A, -)$ ($i \in \mathbb{Z}$) be the right derived functor of $\mathcal{H}^0(A, -)$.

**2.4 Proposition.** *Let $A$ be a weak log abelian variety over an fs log scheme $S$. Let $F$ be either $\mathbb{G}_m$, $\mathbb{G}_{m,\log}$ or $\mathbb{G}_{m,\log}/\mathbb{G}_m$. Suppose that one of the conditions* (a)–(c) *in Theorem* 2.2 *is satisfied. Then we have an exact sequence*

$$0 \to \mathcal{E}xt(A, F) \to \mathcal{H}^1(A, F) \to \mathcal{B}iext_{\mathrm{sym}}(A, A; F)$$

*in which the last arrow is characterized by the following property. The composite $\mathcal{H}^1(A, F) \to \mathcal{B}iext_{\mathrm{sym}}(A, A; F) \to \mathcal{H}om(A, \mathcal{E}xt(A, F))$ is $L \mapsto (a \mapsto t_a^* L \cdot L^{-1})$.*

*The composite*

$$\mathcal{B}iext_{\mathrm{sym}}(A, A; F) \to \mathcal{H}^1(A, F) \to \mathcal{B}iext_{\mathrm{sym}}(A, A; F)$$

*is the multiplication by* 2*, where the first homomorphism is the pullback to the diagonal (*1.5*). Consequently, the cokernel of the last arrow of the exact sequence is killed by* 2*.*

This is proved in 12.6.

**2.5.** Let $A$ be a weak log abelian variety over an fs log scheme $S$. We consider more about $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsors on $A$. In the following, we assume that an admissible pairing

$$\langle\ ,\ \rangle \colon X \times Y \to \mathbb{G}_{m,\log}/\mathbb{G}_m$$

as in 1.1 exists globally on $S$, and fix such a pairing.

**2.6 Lemma.** *Let the notation be as in* 2.5*.*

(1) *The canonical homomorphism*

$$\overline{X} \to \mathcal{H}om(\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}, \mathbb{G}_{m,\log}/\mathbb{G}_m)$$

*is an isomorphism.*

(2) *There is a canonical injective homomorphism*

$$\mathcal{H}om(\overline{Y}, \mathbb{G}_{m,\log}/\mathbb{G}_m)/\overline{X} \to \mathcal{E}xt(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m).$$

(3) $\mathcal{H}om(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m) = 0$.
(4) *The canonical homomorphism*

$$\mathrm{Biext}(A/G, A/G; \mathbb{G}_{m,\log}/\mathbb{G}_m) \to \mathrm{Hom}(A/G, \mathcal{E}xt(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m))$$

*is an isomorphism.*

This is proved in 5.1.

**2.7.** By the homomorphisms in Lemma 2.6 (2) and (4) together with $A/G \cong \mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}/\overline{Y}$, we have a homomorphism

$$\mathrm{Hom}_{\langle\ ,\ \rangle}(\overline{Y}, \overline{X}) \to \mathrm{Hom}(\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}/\overline{Y}, \mathcal{H}om(\overline{Y}, \mathbb{G}_{m,\log}/\mathbb{G}_m)/\overline{X}))$$

$$\to \mathrm{Hom}(A/G, \mathcal{E}xt(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m)) \cong \mathrm{Biext}(A/G, A/G; \mathbb{G}_{m,\log}/\mathbb{G}_m).$$

This induces a homomorphism

$$\mathrm{Hom}_{\langle\ ,\ \rangle}(\overline{Y}, \overline{X}) \to \mathrm{Biext}_{\mathrm{sym}}(A/G, A/G; \mathbb{G}_{m,\log}/\mathbb{G}_m),$$

which is to be the first isomorphism in Proposition 1.6 (1).

## 3. Preliminaries

We gather general facts on biextensions which will be used later. For general references, see [7] Exp. VII and VIII, and [4].

Let $\mathcal{T}$ be a topos. Let $A$ and $F$ be commutative group objects of $\mathcal{T}$.

The following Lemmas 3.1–3.4 are well-known.

**3.1 Lemma.** *Assume* $\mathcal{H}^0(A, F) = F$. *Then* $\mathcal{H}om(A, F) = 0$.

**3.2 Lemma.** *Assume* $\mathcal{H}^0(A, F) = F$. *Then we have an exact sequence*

$$0 \to \mathcal{E}xt(A, F) \to \mathcal{H}^1(A, F) \to \mathcal{H}^0(A, \mathcal{H}^1(A, F)).$$

*Here the last arrow is* $L \mapsto (a \mapsto t_a^*(L) \cdot L^{-1})$, *where* $t_a$ *is the translation by* $a$.

**3.3 Lemma.** *Assume* $\mathcal{H}om(A, F) = 0$. *Then the canonical map* $\mathrm{Ext}(A, F) \to H^0(e, \mathcal{E}xt(A, F))$ *is an isomorphism. Here* $e$ *is the final object of* $\mathcal{T}$.

**3.4 Lemma.** *Assume* $\mathcal{H}om(A, F) = 0$. *Then the canonical map* $\mathrm{Biext}(A, A; F) \to \mathrm{Hom}(A, \mathcal{E}xt(A, F))$ *is an isomorphism.*

**3.5 Lemma.** *Assume* $\mathcal{H}^0(A, F) = F$. *Then the following three conditions are equivalent. Let* $L$ *be an* $F$*-torsor on* $A$.

(i) (Cubic isomorphism.) *There is a unique isomorphism*

$$\iota_{S,A,L} : s_{\{1,2,3\}}^* L \cdot s_{\{1,2\}}^* L^{-1} \cdot s_{\{1,3\}}^* L^{-1} \cdot s_{\{2,3\}}^* L^{-1} \cdot s_{\{1\}}^* L \cdot s_{\{2\}}^* L \cdot s_{\{3\}}^* L \cdot s_{\varnothing}^* L^{-1} \cong 1$$

*of $F$-torsors on $A \times A \times A$ whose pullback on $S$ by the zero section of $A \times A \times A$ is the evident one. Here, $s_{\{1,2,3\}}$ etc. are defined in the same way as in Theorem 2.2, and 1 is the trivial $F$-torsor.*

(ii) (i) *without the characterized uniqueness.*

(iii) *For any $S' \in \mathcal{T}$ and for any $a, b \in A(S')$, the $F$-torsor $t_{ab}^* L \cdot t_a^* L^{-1} \cdot t_b^* L^{-1} \cdot L$ on $A \times_S S'$ is isomorphic to 1 locally on $S'$.*

*Proof.* First we prove the equivalence of (i) and (ii). Let $N$ be the left-hand-side of $\iota_{S,A,L}$. If we have a cubic isomorphism $\iota \colon N \cong 1$ without the characterizing condition, we compose three isomorphisms $\iota$, the pullback of $e^*(\iota^{-1}) \colon 1 \cong e^* N$ ($e$ is the zero section of $A \times A \times A$), and the pullback of the evident isomorphism $e^* N \cong 1$ on $S$, and get the isomorphism having the characterizing property. Hence, (i) is equivalent to (ii).

We prove that (i) implies (iii). We call it the universal case the case where $S' = A \times A$, and $a, b \colon A \times A \to A$ are the first and the second projections. Assume (i). To prove (iii), it is enough to consider the universal case. In this case, the $F$-torsor $L_1 := t_{ab}^* L \cdot t_a^* L^{-1} \cdot t_b^* L^{-1} \cdot L$ is nothing but $s_{\{1,2,3\}}^* L \cdot s_{\{1,2\}}^* L^{-1} \cdot s_{\{1,3\}}^* L^{-1} \cdot s_{\{1\}}^* L$ on $A \times (A \times A) = A \times S'$, so that (i) means that $L_1$ is isomorphic to $0_{S'}^* L_1$, where $0_{S'}$ is the zero map $A \times S' \to A \times S'$ over $S'$. Since $0_{S'}$ factors through the zero section $e \colon S' \to A \times S'$, $L_1$ comes from $S'$. Hence, $L_1$ is locally trivial on $S'$.

Finally, assume that the universal case $S' = A \times A$ of (iii) is valid. Then, $L_1$ is trivial locally on $S'$. By the exact sequence $0 \to H^1(S', F) \to H^1(A \times S', F) \to H^0(S', \mathcal{H}^1(A \times S', F))$ (which is by the assumption; note that the last $A \times S'$ is regarded as an object over $S'$), it implies that $L_1$ is isomorphic to the pullback of some torsor $L_0$ on $S'$. Then, we have $e^* L_1 \cong L_0$ on $S'$ ($e$ is the zero section of $A$) and $0_{S'}^* L_1 \cong L_1$, which is a desired cubic isomorphism. □

**3.6.** Assume $\mathcal{H}^0(A, F) = F$ and assume that the equivalent conditions in Lemma 3.5 are satisfied. Then we have the following two facts.

(1) We have an exact sequence

$$0 \to \mathcal{E}xt(A, F) \to \mathcal{H}^1(A, F) \to \mathcal{H}om(A, \mathcal{E}xt(A, F)),$$

where the last arrow is $L \mapsto (a \mapsto t_a^*(L) \cdot L^{-1})$.

(2) The image of the last arrow in the above exact sequence is contained in $\mathcal{B}iext_{\mathrm{sym}}(A, A; F) \subset \mathcal{B}iext(A, A; F) \cong \mathcal{H}om(A, \mathcal{E}xt(A, F))$.

The first one is a direct consequence of Lemma 3.5 (iii). The second one is seen as follows. The induced map $\mathcal{H}^1(A, F) \to \mathcal{B}iext(A, A; F)$ sends $L$ to $(\mathrm{sum}^* L) \cdot (\mathrm{pr}_1^* L)^{-1} \cdot (\mathrm{pr}_2^* L)^{-1} \cdot (0^* L)$, which is symmetric.

**3.7.** Next, we consider the pullback to the diagonal $\mathcal{B}iext(A, A; F) \to \mathcal{H}^1(A, F)$.

Assume $\mathcal{H}^0(A, F) = F$ and assume that the equivalent conditions in 3.5 are satisfied. Then, the composite $\mathcal{B}iext(A, A; F) \to \mathcal{H}^1(A, F) \to \mathcal{H}om(A, \mathcal{E}xt(A, F))$

$\cong \mathcal{B}iext(A, A; F)$ is $x \mapsto xx^*$, where $x^*$ denotes the transpose of $x$. In fact, the above composite sends a biextension $L$ to $\mathrm{sum}^*\mathrm{diag}^*(L) \cdot \mathrm{pr}_1^*\mathrm{diag}^*(L)^{-1} \cdot \mathrm{pr}_2^*\mathrm{diag}^*(L)^{-1}$, which coincides with $L \cdot T^*(L)$ (see [4] 1.2.8). Here $T$ is the isomorphism $A \times A \to A \times A; (x, y) \mapsto (y, x)$.

Thus we have

**3.8 Lemma.** *Assume $\mathcal{H}^0(A, F) = F$ and assume that the equivalent conditions in* 3.5 *are satisfied. Then, we have an exact sequence*

$$0 \to \mathcal{E}xt(A, F) \to \mathcal{H}^1(A, F) \to \mathcal{B}iext_{\mathrm{sym}}(A, A; F),$$

*where the last arrow is characterized by the property that the composite $\mathcal{H}^1(A, F) \to \mathcal{B}iext_{\mathrm{sym}}(A, A; F) \to \mathcal{H}om(A, \mathcal{E}xt(A, F))$ is $L \mapsto (a \mapsto t_a^*L \cdot L^{-1})$. The composite*

$$\mathcal{B}iext_{\mathrm{sym}}(A, A; F) \to \mathcal{H}^1(A, F) \to \mathcal{B}iext_{\mathrm{sym}}(A, A; F)$$

*is the multiplication by* 2*. Consequently, the cokernel of the last arrow of the exact sequence is killed by* 2*.*

The facts in the next lemma are well-known (cf. [3]).

**3.9 Lemma.** *Let $A, B$ be commutative group objects on $\mathcal{T}$.*

(1) *Assume that* $\mathrm{Mor}(A^n, B) = \mathrm{Mor}(e, B)$ *for $n = 1, 2$, where $e$ is the initial object. Then* $\mathrm{Ext}(A, B) \to H^1(A, B)$ *is injective.*

(2) *Furthermore, assume that* $\mathrm{Mor}(A^3, B) = \mathrm{Mor}(e, B)$*. Then we have an exact sequence*

$$0 \to \mathrm{Ext}^1(A, B) \to H^1(A, B) \xrightarrow{a} H^1(A \times A, B),$$

*where* $a = \mathrm{sum}^* - \mathrm{pr}_1^* - \mathrm{pr}_2^*$ *(*$\mathrm{sum}, \mathrm{pr}_1, \mathrm{pr}_2 \colon A \times A \to A$*).*

**3.10 Lemma.** *Let $X$, $Y$ and $F$ be commutative group objects on $\mathcal{T}$. Let $\langle\ ,\ \rangle \colon X \times Y \to F$ be a bilinear map which is nondegenerate in the sense that the induced maps $X \to \mathcal{H}om(Y, F)$ and $Y \to \mathcal{H}om(X, F)$ are injective. Assume that $\mathcal{H}^0(\mathcal{H}om(X, F)/Y, F) = F$. Let $\mathcal{H}om_{\langle\ ,\ \rangle}(Y, X)$ be the subsheaf of $\mathcal{H}om(Y, X)$ consisting of the sections $\psi$ satisfying $\langle \psi(y), z\rangle = \langle \psi(z), y\rangle$ $(y, z \in Y)$.*

(1) *Let $\psi$ be a section of $\mathcal{H}om_{\langle\ ,\ \rangle}(Y, X)$. Then there is an action of the commutative group object $Y \times Y$ on the trivial $F$-torsor on $\mathcal{H}om(X, F) \times \mathcal{H}om(X, F)$ by*

$$(p, q, f) \mapsto (py, qz, p(\psi(z)^{-1})q(\psi(y)^{-1})\langle \psi(z), y\rangle^{-1} f)$$
$$((p, q, f) \in \mathcal{H}om(X, F) \times \mathcal{H}om(X, F) \times F),$$

*where $(y, z) \in Y \times Y$. Let*

$$B_\psi \in \mathcal{B}iext(\mathcal{H}om(X, F)/Y, \mathcal{H}om(X, F)/Y; F)$$

*be the biextension defined as the quotient of this trivial $F$-torsor by this action.*

(2) *The image of $B_\psi$ by the pullback to the diagonal (cf. 1.5)*

$$\mathcal{B}iext(\mathcal{H}om(X,F)/Y, \mathcal{H}om(X,F)/Y; F) \to \mathcal{H}^1(\mathcal{H}om(X,F)/Y, F)$$

*is described as a quotient of the trivial $F$-torsor on $\mathcal{H}om(X,F)$ by the action of $Y$, where $y \in Y$ acts by*

$$(p,f) \mapsto (py, p(\psi(y)^{-2})\langle\psi(y),y\rangle^{-1}f) \quad ((p,f) \in \mathcal{H}om(X,F) \times F).$$

(3) *The image of $\psi$ by the composite*

$$\begin{aligned}\mathcal{H}om_{\langle\,,\,\rangle}(Y,X) &\to \mathcal{H}om(\mathcal{H}om(X,F)/Y, \mathcal{H}om(Y,F)/X) \\ &\to \mathcal{H}om(\mathcal{H}om(X,F)/Y, \mathcal{E}xt(\mathcal{H}om(X,F)/Y,F))\end{aligned}$$

*coincides with the image of $B_\psi$ by the canonical map*

$$\begin{aligned}&\mathcal{B}iext(\mathcal{H}om(X,F)/Y, \mathcal{H}om(X,F)/Y; F) \\ &\to \mathcal{H}om(\mathcal{H}om(X,F)/Y, \mathcal{E}xt(\mathcal{H}om(X,F)/Y,F)).\end{aligned}$$

*Proof.* (1) Let $(y,z),(p,q,f)$ be as in the statement. Let $(y',z')$ be another section of $Y \times Y$. Then it sends $(py, qz, p(\psi(z)^{-1})q(\psi(y)^{-1})\langle\psi(z),y\rangle^{-1}f)$ to

$$(pyy', qzz', (py)(\psi(z')^{-1})(qz)(\psi(y')^{-1})\langle\psi(z'),y'\rangle^{-1}p(\psi(z)^{-1})q(\psi(y)^{-1})\langle\psi(z),y\rangle^{-1}f).$$

Since $(py)(\psi(z')) = p(\psi(z'))\langle\psi(z'),y\rangle$, $(qz)(\psi(y')) = q(\psi(y'))\langle\psi(y'),z\rangle$, and

$$\langle\psi(z'),y\rangle\langle\psi(y'),z\rangle\langle\psi(z'),y'\rangle\langle\psi(z),y\rangle = \langle\psi(zz'),yy'\rangle,$$

which is by $\langle\psi(y'),z\rangle = \langle\psi(z),y'\rangle$, this coincides with

$$(pyy', qzz', p(\psi(zz')^{-1})q(\psi(yy')^{-1})\langle\psi(zz'),yy'\rangle^{-1}f),$$

which completes the proof of (1).

(2) is deduced from (1) because, in the notation of (1), $(y,y)$ sends $(p,p,f)$ to

$$(py, py, p(\psi(y)^{-2})\langle\psi(y),y\rangle^{-1}f).$$

(3) Fix a section $\overline{p} \in \mathcal{H}om(X,F)/Y$ and take a lift $p \in \mathcal{H}om(X,F)$. Since

$$\mathcal{E}xt(\mathcal{H}om(X,F)/Y,F)) \subset \mathcal{H}^1(\mathcal{H}om(X,F)/Y,F)$$

(this is by $\mathcal{H}^0(\mathcal{H}om(X,F)/Y,F) = F$ and Lemma 3.2), it is enough to compare the image of $p \circ \psi$ by $\mathcal{H}om(Y,F) \to \mathcal{E}xt(\mathcal{H}om(X,F)/Y,F) \hookrightarrow \mathcal{H}^1(\mathcal{H}om(X,F)/Y,F)$ and the image of $\overline{p}$ by the map

$$\mathcal{H}om(X,F)/Y \to \mathcal{E}xt(\mathcal{H}om(X,F)/Y,F) \hookrightarrow \mathcal{H}^1(\mathcal{H}om(X,F)/Y,F)$$

induced by $B_\psi$. The former is the cokernel of $Y \to \mathcal{H}om(X,F) \times F$ which sends $z$ to $(z, p(\psi(z))^{-1}) = (z, p(\psi(z)^{-1}))$. The latter is the quotient of the trivial $F$-torsor on $\mathcal{H}om(X,F)$ by the action of $Y$, where $z \in Y$ acts by

$$(q,f) \mapsto (qz, p(\psi(z)^{-1})f) \quad (q \in \mathcal{H}om(X,F), f \in F).$$

They coincide with each other and we are done. □

## 4. Special models

In 1.9, we introduced the first standard fan. Here we define two more special fans, that is, the second standard fan and the fan associated to a star.

We start with some preliminary observations 4.1–4.4 on how various conditions are effective in yielding these fans. The definitions of the fans are in 4.6.

**4.1.** Let $S$ be an fs log scheme, and $A$ a weak log abelian variety over $S$. Let $s$ be a point of $S$, and we work at $s$ for a while. Let

$$C_s := \{(N,l) \in (M_{S,\overline{s}}/\mathcal{O}^\times_{S,\overline{s}})^\vee \times \mathrm{Hom}(\overline{X}_{\overline{s}}, \mathbb{Z}) \mid l((\overline{X}_{\overline{s}})_{\mathrm{Ker}(N)}) = 0\},$$

where $(-)^\vee = \mathrm{Hom}(-, \mathbb{N})$. Let $\psi \colon \overline{Y}_{\overline{s}} \to \overline{X}_{\overline{s}}$ be a homomorphism. For $y \in \overline{Y}_{\overline{s}}$, let

$$C_s(y) := \{(N,l) \in (M^{\mathrm{gp}}_{S,\overline{s}}/\mathcal{O}^\times_{S,\overline{s}})^\vee \times \mathrm{Hom}(\overline{X}_{\overline{s}}, \mathbb{Z}) \mid N(\langle \psi(z), z\rangle) + 2l(\psi(z)) \geq N(\langle \psi(y), y\rangle) + 2l(\psi(y)) \text{ for any } z \in Y_{\overline{s}}\}.$$

Assume that $\psi$ has a finite cokernel, that is, $\psi$ induces an isomorphism $\psi \colon \overline{Y}_{\overline{s}} \otimes \mathbb{Q} \to \overline{X}_{\overline{s}} \otimes \mathbb{Q}$. Under this assumption, we prove $C_s(y) \subset C_s$. It is sufficient to show $C_s(0) \subset C_s$. Assume $(N,l) \in C(0)$, that is,

(1) $N(\langle \psi(y), y\rangle) + 2l(\psi(y)) \geq 0$ for any $y \in Y_{\overline{s}}$.

What we have to see is $l((X_{\overline{s}})_{\mathrm{Ker}(N)}) = 0$. Let $x \in (X_{\overline{s}})_{\mathrm{Ker}(N)}$. Then $N(\langle x, y\rangle) = 0$ for any $y \in Y_{\overline{s}}$. We may assume $x = \psi(w)$ for some $w \in Y_{\overline{s}}$. By the case $y = w$ (resp. $y = -w$) of (1) and by $N(\langle \psi(w), w\rangle) = 0$, we have $l(\psi(w)) \geq 0$ (resp. $l(\psi(w)) \leq 0$). Hence $l(\psi(w)) = 0$.

**4.2.** Next, assume further that $1 | \langle \psi(y), y\rangle$ for all $y \in \overline{Y}_{\overline{s}}$ and that $\langle \psi(y), z\rangle = \langle \psi(z), y\rangle$ for all $y, z \in \overline{Y}_{\overline{s}}$. Then the union of the cones $C_s(y)$'s ($y \in \overline{Y}_{\overline{s}}$) cover $C_s$. To see this, first we prove the following lemma.

**4.3 Lemma.** *If $y \in \overline{Y}_{\overline{s}}$ and if $\langle \psi(y), y\rangle = 1$, then $y = 1$.*

*Proof.* Take any $z \in \overline{Y}_{\overline{s}}$. Let $N \colon M_{S,\overline{s}}/\mathcal{O}^\times_{S,\overline{s}} \to \mathbb{N}$ be any homomorphism. We have $0 \leq N(\langle \psi(y^m z), y^m z\rangle) = 2mN(\langle \psi(y), z\rangle) + N(\langle \psi(z), z\rangle)$. Varying $m$, we have $N(\langle\psi(y),z\rangle)=0$. Since this holds for all $N$, we have $\langle\psi(y),z\rangle=0$. Since $z$ is any element of $\overline{Y}_{\overline{s}}$, this implies $\psi(y)=1$. Hence $y=1$. □

**4.4.** We prove the union of the cones $C_s(y)$'s ($y\in\overline{Y}_{\overline{s}}$) contains $C_s$. We have to show that for each $(N,l)\in C_s$, the function $y\mapsto N(\langle\psi(y),y\rangle)+2l(\psi(y))$ ($y\in\overline{Y}_{\overline{s}}$) attains a minimum. Since $y\in(\overline{Y}_{\overline{s}})_{\mathrm{Ker}(N)}$ if and only if $\psi(y)\in(\overline{X}_{\overline{s}})_{\mathrm{Ker}(N)}$, this function can be regarded as a function on the quotient space $(\overline{Y}_{\overline{s}})/(\overline{Y}_{\overline{s}})_{\mathrm{Ker}(N)}$. It is enough to show that the first term $N(\langle\psi(y),y\rangle)$ is positive-definite on this quotient space. Let $y\in\overline{Y}_{\overline{s}}$. If $N(\langle\psi(y),y\rangle)=0$, then the argument in the proof of Lemma 4.3 shows that $N(\langle\psi(y),z\rangle)=0$ for all $z\in\overline{Y}_{\overline{s}}$. By the admissibility, it implies $\psi(y)\in(\overline{X}_{\overline{s}})_{\mathrm{Ker}(N)}$, and $y\in(\overline{Y}_{\overline{s}})_{\mathrm{Ker}(N)}$. This means that the term $N(\langle\psi(y),y\rangle)$ is positive-definite on the quotient space.

**4.5.** Now we stop the pointwise consideration.

Let $S$ be an fs log scheme, and $A$ a weak log abelian variety over $S$. Assume that there is a homomorphism $\psi\colon\overline{Y}\to\overline{X}$ satisfying (1)–(3) in 1.4.1. Then there always exist étale locally on $S$ the data $X,Y,\mathcal{S},\langle\ ,\ \rangle\colon X\times Y\to\mathcal{S}^{\mathrm{gp}}$, $\mathcal{S}\to M_S/\mathcal{O}_S^{\times}$, and $\phi$ for $(A,\psi)$ as in 1.7 and 1.9. In particular, $\phi$ satisfies the following two conditions.

(1) $\langle\phi(y),z\rangle=\langle\phi(z),y\rangle$ in $\mathcal{S}^{\mathrm{gp}}$ for any $y,z\in Y$;

(2) $\langle\phi(y),y\rangle\in\mathcal{S}$ for any $y\in Y$.

In the following, we assume that such data exist globally on $S$.

**4.6.** We give special fans in $C$.

First recall the first standard fan defined in 1.9.

This is determined by

$$C(y):=\{(N,l)\in C\mid N(\langle\phi(z),z\rangle)+2l(\phi(z))\geq N(\langle\phi(y),y\rangle)+2l(\phi(y))\text{ for any }z\in Y\},\ (y\in Y).$$

We remark that $C(y)$ is defined also as $\{(N,l)\in\mathcal{S}^{\vee}\times\mathrm{Hom}(X,\mathbb{Z})\mid\text{the same condition}\}$. This fact is proved similarly as in 4.1. The fact that the first standard fan is complete is proved similarly as in 4.4.

Next, for each $x\in X$, let

$$C(x):=\{(N,l)\in C\mid N(\langle w,\phi^{-1}(w)\rangle)+2l(w)\geq N(\langle x,\phi^{-1}(x)\rangle)+2l(x)\text{ for any }w\in X\}.$$

Here $\phi^{-1}(w)$ is taken in $Y\otimes\mathbb{Q}$ (so $N(\langle w,\phi^{-1}(w)\rangle)$ is defined over $\mathbb{Q}$). We remark that $C(x)$ is defined also as $\{(N,l)\in\mathcal{S}^{\vee}\times\mathrm{Hom}(X,\mathbb{Z})\mid\text{the same condition}\}$. The cones $C(x)$ for varying $x$ with their faces form a complete fan in $C$. The proofs are similar to the case of the first standard fan. We call this fan the *second standard* fan. This fan is used in [1] and [5].

Third, let $I$ be a finite subset of $X$ satisfying the following (i)–(iii):

(i) $I$ generates the abelian group $X$;

(ii) If $x \in I$, then $x^{-1} \in I$;

(iii) $1 \in I$.

Such an $I$ is called a *star* in $X$ ([13]). Assume further that $\langle \phi(y)\alpha, y\rangle \in \mathcal{S}$ for any $\alpha \in I$ and $y \in Y$ as in [13]. For $\alpha \in I$ and $y \in Y$, let

$$C(\alpha, y) := \{(N, l) \in C \mid N(\langle \phi(z)\beta, z\rangle) + l(\phi(z)^2\beta) \geq N(\langle \phi(y)\alpha, y\rangle) + l(\phi(y)^2\alpha) \text{ for any } \beta \in I \text{ and } z \in Y\}.$$

We remark that $C(\alpha, y)$ is defined also as $\{(N, l) \in \mathcal{S}^\vee \times \mathrm{Hom}(X, \mathbb{Z}) \mid$ the same condition$\}$. The cones $C(\alpha, y)$ for varying $(\alpha, y)$ ($\alpha \in I$, $y \in Y$) with their faces form a complete fan in $C$. The proofs are also similar. We call this fan the fan *associated to the star* $I$.

Further, we remark the following. As in [8] 5.4, we can construct more fans. Let $a\colon Y \to \mathcal{S}^{\mathrm{gp}}$ be a map satisfying $a(y) \in \mathcal{S}$ for all but finitely many $y \in Y$ and $a(yz) = a(y)a(z)\langle \phi(y), z\rangle$ for any $y, z \in Y$. Note that $y \mapsto \langle \phi(y), y\rangle$ gives an example of such an $a$ for the square $\phi^2$ instead of $\phi$ itself. (Cf. the verification of the cocycle condition in 4.8.) Then, as in [8] 5.4.3, we can define the associated first standard fan. The second standard fan and the fan associated to a star in this context also can be defined.

**4.7.** We prove that the above three fans are stable under the action of $Y$.

Let $\Sigma$ be either the first standard fan of $C$ (we call this the case 1), the second standard fan of $C$ (we call this the case 2) or the fan associated to a star $I$ in $X$ (we call this the case 3).

In the case 1 (resp. case 2, resp. case 3), for any $y, z \in Y$ (resp. $x \in X$ and $z \in Y$, resp. $y, z \in Y$ and $\alpha \in I$), we will see

$$z^*C(y) = C(z^{-1}y)$$

$$(\text{resp. } z^*C(x) = C(\phi(z)^{-1}x), \text{ resp. } z^*C(\alpha, y) = C(\alpha, z^{-1}y))$$

in the below, where $z^*$ denotes the action of $z$. These will show that $\Sigma$ is $Y$-stable.

**4.8.** To see the above equality, first consider the dual action of $Y$ on $\mathcal{S}_{\mathbb{Q}}^{\mathrm{gp}} \times X_{\mathbb{Q}}$ via which $y \in Y$ maps $(\mu, x) \in \mathcal{S}_{\mathbb{Q}}^{\mathrm{gp}} \times X_{\mathbb{Q}}$ to $(\langle x, y\rangle\mu, x)$. Note that it is indeed the dual of the action of $Y$ on $C^{\mathrm{gp}} = \mathrm{Hom}(\mathcal{S}^{\mathrm{gp}}, \mathbb{Z}) \times \mathrm{Hom}(X, \mathbb{Z})$ (1.7), which is seen as $N(\langle x, y\rangle\mu) + l(x) = N(\mu) + (l + N(\langle -, y\rangle))(x)$.

Let

$$c_y = (\langle \phi(y), y\rangle, \phi(y)^2) \in \mathcal{S}_{\mathbb{Q}_{\geq 0}} \times X_{\mathbb{Q}}$$

for each $y \in Y_{\mathbb{Q}}$. Then we have the following cocycle condition:

$$z^* c_y \cdot c_z = c_{zy},$$

where $z^*$ denotes the (dual) action of $z$. It is seen as

$$\begin{aligned} z^* c_y \cdot c_z &= (\langle \phi(y)^2, z\rangle \cdot \langle \phi(y), y\rangle, \phi(y)^2) \cdot (\langle \phi(z), z\rangle, \phi(z)^2) \\ &= (\langle \phi(y), z\rangle \cdot \langle \phi(z), y\rangle \cdot \langle \phi(y), y\rangle \cdot \langle \phi(z), z\rangle, \phi(y)^2\phi(z)^2) \\ &= (\langle \phi(zy), zy\rangle, \phi(zy)^2) = c_{zy}. \end{aligned}$$

Further, let $\alpha \in X$ and let $c_{\alpha,y} = c_y \cdot (\langle \alpha, y\rangle, \alpha) \in \mathcal{S}_{\mathbb{Q}_{\geq 0}} \times X_{\mathbb{Q}}$ for $y \in Y_{\mathbb{Q}}$. Then we have

$$z^* c_{\alpha,y} \cdot c_z = c_{\alpha,zy}.$$

This is deduced from the above cocycle condition together with the equality $z^*(\langle \alpha, y\rangle, \alpha) = (\langle \alpha, z\rangle \cdot \langle \alpha, y\rangle, \alpha) = (\langle \alpha, zy\rangle, \alpha)$.

**4.9.** In the case 1 (resp. case 2, resp. case 3), we prove

$$z^* C(y) = C(z^{-1}y)$$

$$(\text{resp. } z^* C(x) = C(\phi(z)^{-1}x), \text{ resp. } z^* C(\alpha, y) = C(\alpha, z^{-1}y)).$$

By the remark after the definition of $C(y)$ (resp. $C(x)$, resp. $C(\alpha, y)$), the cone $C(y)$ (resp. $C(x)$, resp. $C(\alpha, y)$) $\subset \mathrm{Hom}(\mathcal{S}^{\mathrm{gp}}, \mathbb{Z}) \times \mathrm{Hom}(X, \mathbb{Z})$ is the dual of the subset

$$S(y) := \left\{ \tfrac{c_{y'}}{c_y} \;\middle|\; y' \in Y \right\}$$
$$\left(\text{resp. } S(x) := \left\{ \tfrac{c_{\phi^{-1}(x')}}{c_{\phi^{-1}(x)}} \;\middle|\; x' \in X \right\}, \text{ resp. } S(\alpha, y) := \left\{ \tfrac{c_{\alpha',y'}}{c_{\alpha,y}} \;\middle|\; \alpha' \in I, y' \in Y \right\}\right)$$

of $\mathcal{S}^{\mathrm{gp}}_{\mathbb{Q}} \times X_{\mathbb{Q}}$. Hence $z^* C(y)$ (resp. $z^* C(x)$, resp. $z^* C(\alpha, y)$) is the dual cone of the subset $(z^{-1})^* S(y)$ (resp. $(z^{-1})^* S(x)$, resp. $(z^{-1})^* S(\alpha, y)$). By the cocycle condition in 4.8, we have

$$(z^{-1})^*(c_{y'}/c_y) = \left(\tfrac{c_{z^{-1}y'}}{c_{z^{-1}}}\right) \Big/ \left(\tfrac{c_{z^{-1}y}}{c_{z^{-1}}}\right) = \tfrac{c_{z^{-1}y'}}{c_{z^{-1}y}}$$
$$\Bigg(\text{resp. } (z^{-1})^*(c_{\phi^{-1}(x')}/c_{\phi^{-1}(x)}) = \tfrac{c_{z^{-1}\phi^{-1}(x')}}{c_{z^{-1}\phi^{-1}(x)}} = \tfrac{c_{\phi^{-1}(\phi(z^{-1})x')}}{c_{\phi^{-1}(\phi(z^{-1})x)}},$$
$$\text{resp. } (z^{-1})^*(c_{\alpha',y'}/c_{\alpha,y}) = \left(\tfrac{c_{\alpha',z^{-1}y'}}{c_{z^{-1}}}\right) \Big/ \left(\tfrac{c_{\alpha,z^{-1}y}}{c_{z^{-1}}}\right) = \tfrac{c_{\alpha',z^{-1}y'}}{c_{\alpha,z^{-1}y}}\Bigg).$$

Therefore, $(z^{-1})^* S(y)$ (resp. $(z^{-1})^* S(x)$, resp. $(z^{-1})^* S(\alpha, y)$) coincides with the subset

$$S(z^{-1}y) = \left\{ \tfrac{c_{y'}}{c_{z^{-1}y}} \;\middle|\; y' \in Y \right\}$$
$$\Bigg(\text{resp. } S(\phi(z)^{-1}x) = \left\{ \tfrac{c_{\phi^{-1}(x')}}{c_{\phi^{-1}(\phi(z^{-1})x)}} \;\middle|\; x' \in X \right\},$$
$$\text{resp. } S(\alpha, z^{-1}y) = \left\{ \tfrac{c_{\alpha',y'}}{c_{\alpha,z^{-1}y}} \;\middle|\; \alpha' \in I, y' \in Y \right\}\Bigg).$$

Taking the dual, we conclude

$$z^*C(y) = C(z^{-1}y)$$

$$(\text{resp. } z^*C(x) = C(\phi(z)^{-1}x), \text{ resp. } z^*C(\alpha, y) = C(\alpha, z^{-1}y)).$$

**4.10.** By the $Y$-stability of $\Sigma$ shown in the above, the model $\widetilde{A}^{(\Sigma)}$ is $Y$-stable. We can see this last fact by a different understanding as follows. In the case 1 (resp. case 2, resp. case 3), let $y \in Y$ (resp. $x \in X$, resp. $\alpha \in I$, $y \in Y$). Let

$$U(y)$$
$$(\text{resp. } U(x), \text{ resp. } U(\alpha, y))$$

be the $C(y)$-part (resp. $C(x)$-part, resp. $C(\alpha, y)$-part) of $\widetilde{A}$ (1.7). Then, as is shown below, this subfunctor of $\widetilde{A}$ satisfies the following.

$$z^*U(y) = U(z^{-1}y) \text{ for any } y, z \in Y$$
$$(\text{resp. } z^*U(x) = U(\phi(z)^{-1}x) \text{ for any } x \in X, z \in Y,$$
$$\text{resp. } z^*U(\alpha, y) = U(\alpha, z^{-1}y) \text{ for any } \alpha \in I,\ y, z \in Y),$$

which implies that $\widetilde{A}^{(\Sigma)}$ is $Y$-stable.

**4.11.** We prove the above formulas. As a preliminary, first observe that any element $(\mu, x)$ of $(\mathcal{S}^{\mathrm{gp}})_{\mathbb{Q}} \times X_{\mathbb{Q}}$ gives a section of the trivial $(\mathbb{G}_{m,\log}/\mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$-torsor over $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$ by $h \mapsto \mu h(x)$. This construction gives a homomorphism

$$(\mathcal{S}^{\mathrm{gp}})_{\mathbb{Q}} \times X_{\mathbb{Q}} \to \mathrm{Hom}(\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}, (\mathbb{G}_{m,\log}/\mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q});$$
$$(\mu, x) \mapsto (h \mapsto \mu h(x)).$$

Further, this construction of the sections is compatible with the action of $Y$ in the sense that for any $z \in Y$, the element $z^*(\mu, x)$ ($(\mu, x) \in \mathcal{S}^{\mathrm{gp}} \times X$) gives the section obtained by the pullback of the section that $(\mu, x)$ gives by the action of $z$ on the space $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$. This is seen as $(\langle x, z\rangle\mu)h(x) = \mu(h\langle -, z\rangle)(x)$.

**4.12.** By 4.11, the element $c_y$ ($y \in Y$) (resp. $c_y$ ($y \in Y_{\mathbb{Q}}$), resp. $c_{\alpha,y}$ ($\alpha \in I, y \in Y$)) in 4.8 gives a section

$$c(y) \text{ (resp. } c(y), \text{ resp. } c(\alpha, y)).$$

Since $C(y)$ (resp. $C(x)$, resp. $C(\alpha, y)$) is the dual of the subset $S(y)$ (resp. $S(x)$, resp. $S(\alpha, y)$) in 4.9, by definition, the $C(y)$-part (resp. $C(x)$-part, resp. $C(\alpha, y)$-part) of $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$ is the part where

$$c(y)|c(y') \text{ for any } y' \in Y$$
$$(\text{resp. } c(\phi^{-1}(x))|c(\phi^{-1}(x')) \text{ for any } x' \in X,$$
$$\text{resp. } c(\alpha, y)|c(\alpha', y') \text{ for any } \alpha' \in I, y' \in Y).$$

Hence, its translation by $z^*$ ($z \in Y$) is the part where $(z^{-1})^*c(y)|(z^{-1})^*c(y')$ for any $y' \in Y$ (resp. $(z^{-1})^*c(\phi^{-1}(x))|(z^{-1})^*c(\phi^{-1}(x'))$ for any $x' \in X$, resp. $(z^{-1})^*c(\alpha, y)|(z^{-1})^*c(\alpha', y')$ for any $\alpha' \in I, y' \in Y$). Here $(z^{-1})^*$ means the pullback by the action of $z^{-1}$ on the space $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$.

By the cocycle condition in 4.8 together with the facts in 4.11, we have

$$(z^{-1})^*c(y) = \tfrac{c(z^{-1}y)}{c(z^{-1})}$$
$$(\text{resp. } (z^{-1})^*c(\phi^{-1}(x)) = \tfrac{c(z^{-1}\phi^{-1}(x))}{c(z^{-1})}, \text{ resp. } (z^{-1})^*c(\alpha, y) = \tfrac{c(\alpha, z^{-1}y)}{c(z^{-1})}).$$

Therefore, the translation is the part where

$$\tfrac{c(z^{-1}y)}{c(z^{-1})} \Big| \tfrac{c(z^{-1}y')}{c(z^{-1})} \text{ for any } y' \in Y$$
$$(\text{resp. } \tfrac{c(z^{-1}\phi^{-1}(x))}{c(z^{-1})} \Big| \tfrac{c(z^{-1}\phi^{-1}(x'))}{c(z^{-1})} \text{ for any } x' \in X,$$
$$\text{resp. } \tfrac{c(\alpha, z^{-1}y)}{c(z^{-1})} \Big| \tfrac{c(\alpha', z^{-1}y')}{c(z^{-1})} \text{ for any } \alpha' \in I, y' \in Y),$$

which coincides with the part where

$$c(z^{-1}y)|c(z^{-1}y') \text{ for any } y' \in Y,$$
$$(\text{resp. } c(z^{-1}\phi^{-1}(x))|c(z^{-1}\phi^{-1}(x')) \text{ for any } x' \in X,$$
$$\text{resp. } c(\alpha, z^{-1}y)|c(\alpha', z^{-1}y') \text{ for any } \alpha' \in I, y' \in Y),$$

that is, the $C(z^{-1}y)$-part (resp. $C(\phi(z)^{-1}x)$-part, resp. $C(\alpha, z^{-1}y)$-part). Hence, we have the above formulas.

**4.13.** Let $S$ be an fs log scheme, and $A$ a weak log abelian variety over $S$. Let $\psi\colon \overline{Y} \to \overline{X}$ be a homomorphism satisfying (1)–(3) in 1.4.1. Then the above description of the subfunctor $U(y)$ (resp. $U(x)$, resp. $U(\alpha, y)$) also shows that this subfunctor can be defined stalkwise, defined only by $\psi$, and independent of the choices of $\phi$, so that the model $A^{(\Sigma)}$ glues globally on the base $S$. We denote this glued model on the base by $A^{(\psi)}$.

**4.14.** Here we describe the details of Remark 1.4.2.

First a homomorphism $\psi\colon \overline{Y} \to \overline{X}$ at a point of $S$ satisfying (1)–(3) in 1.4.1 induces $\psi$ satisfying (1)–(3) in 1.4.1 on some étale neighborhood (see 12.13 for the proof).

In particular, a log abelian variety satisfies the condition 1.4.1. In fact, for a log abelian variety $A$, by definition, there is a polarization $p$ on the associated log 1-motif at each point, and $p$ gives a $\psi$ satisfying (1)–(3) in 1.4.1 at that point. Hence $A$ satisfies the condition 1.4.1 by the above fact. On the other hand, a weak log abelian variety does not necessarily satisfy the condition 1.4.1 and a weak log abelian variety satisfying the condition 1.4.1 is not necessarily a log abelian variety. See examples in 4.15 below.

In [11] Corollary 9.5, we proved that any weak log abelian variety $A$ locally comes from a weak log abelian variety $A_0$ over some noetherian base. Another consequence of the above fact is that the above $A_0$ can be taken to satisfy the condition 1.4.1 whenever the original $A$ satisfies it.

**4.15.** Here we include examples mentioned in the above 4.14. That is, we give an example of weak log abelian variety which does not satisfy the condition 1.4.1 and an example of weak log abelian variety satisfying the condition 1.4.1 but not being a log abelian variety.

First, let $S = \big(\operatorname{Spec}\mathbb{C}, (\mathbb{N}^2 = \langle q_1, q_2\rangle)^a\big)$. Consider $\mathbb{G}_{m,\log} = \mathbb{G}_{m,\log,S}$ and regard $q_1, q_2$ as sections of it. Let $Y = \mathbb{Z}^2 = \langle e_1, e_2\rangle$, where $(e_i)_i$ is the canonical base. Then we claim that the weak log abelian variety on $S$ associated with the log 1-motif defined by $Y \to \mathbb{G}^2_{m,\log}; e_1 \mapsto (q_1q_2, 1), e_2 \mapsto (q_2, q_1q_2)$ does not satisfy the condition 1.4.1.

Second, let $S = \big(\operatorname{Spec}\mathbb{C}, (\mathbb{N} = \langle q\rangle)^a\big)$. Consider $\mathbb{G}_{m,\log} = \mathbb{G}_{m,\log,S}$ and regard $q$ as a section of it. Let $Y = \mathbb{Z}^2 = \langle e_1, e_2\rangle$, where $(e_i)_i$ is the canonical base. Then we claim that the weak log abelian variety on $S$ associated with the log 1-motif defined by $Y \to \mathbb{G}^2_{m,\log}; e_1 \mapsto (2q, 1), e_2 \mapsto (2, 2q)$ is not a log abelian variety but satisfies the condition 1.4.1.

We prove the above claims based on the fact that for any positive-definite real symmetric matrix $A$, the product $A\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is not symmetric.

First, we prove that the first example does not satisfy the condition 1.4.1. Let $X = \operatorname{Hom}(\mathbb{G}^2_m, \mathbb{G}_m) = \mathbb{Z}^2$. Then the associated pairing $X \times Y \to (\mathbb{N}^2)^{\mathrm{gp}}$ is represented by a pair of matrices $I := \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $S := \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. If the condition 1.4.1 is satisfied, there is another real square matrix $A$ whose determinant is not zero such that both $AI = A$ and $AS$ are symmetric and semipositive-definite. This is impossible by the above fact.

As for the second example, the pairing modulo $\mathbb{G}_m$ is the standard one and the identity $\psi\colon Y \to X = \operatorname{Hom}(\mathbb{G}^2_m, \mathbb{G}_m) = \mathbb{Z}^2$ satisfies the condition 1.4.1. We prove that, however, it is not a log abelian variety. For this, observe that the associated pairing $X \times Y \to \mathbb{G}_{m,\log}$ factors as $(I, S)\colon X \times Y \to \mathbb{Z}^2 = q^{\mathbb{Z}} \cdot 2^{\mathbb{Z}} \subset \mathbb{G}_{m,\log}$. If it is a log abelian variety, by definition, there is a polarization, which implies that there is another real square matrix $A$ whose determinant is not zero such that both $AI = A$ and $AS$ are symmetric and such that $A$ is positive-definite. This is impossible again by the above fact.

## 5. Special torsors

In the previous section, we define some special models of a weak log abelian variety $A$. Here we consider special $\mathbb{G}_m$-torsors on these models when a polarization on $A$ is given.

**5.1.** First we prove Lemma 2.6.

Let $A$ be a weak log abelian variety over an fs log scheme. Assume that either $A$ is with constant degeneration or $A$ satisfies the condition 1.4.1. Assume also that an admissible pairing

$$\langle\ ,\ \rangle\colon X \times Y \to \mathbb{G}_{m,\log}/\mathbb{G}_m$$

as in 1.1 is given globally on $S$.

We prove Lemma 2.6 (1). Let $F = \mathcal{H}om(\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}, \mathbb{G}_{m,\log}/\mathbb{G}_m)$. We prove that the natural map $\overline{X} \to F$ is injective. Let $T$ be an fs log scheme over $S$. Let $x \in \overline{X}(T)$. Assume that the image of $x$ in $F(T)$ is zero. Then $\langle x_{\bar{t}}, y\rangle_{\bar{t}} = 0$ for any $t \in T$ and $y \in \overline{Y}_{\bar{t}}$. Hence $x_{\bar{t}} = 0$ for any $t \in T$ and $x = 0$.

Next let $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})} \to \mathbb{G}_{m,\log}/\mathbb{G}_m = \mathcal{H}om(\mathbb{Z}, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ be a section of $F$. Then, we can prove that it is induced by some homomorphism $\mathbb{Z} \to \overline{X}$ by the same method as in [9] 7.26. Hence, $\overline{X} \to F$ is also surjective, which completes the proof of Lemma 2.6 (1).

We prove Lemma 2.6 (2) and (3). From the exact sequence

$$(*) \qquad 0 \to \overline{Y} \to \mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})} \to A/G \to 0$$

together with Lemma 2.6 (1), we obtain an exact sequence

$$\begin{aligned} 0 &\to \mathcal{H}om(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m) \\ &\to \overline{X} \to \mathcal{H}om(\overline{Y}, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to \mathcal{E}xt(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m). \end{aligned}$$

This gives the injection in Lemma 2.6 (2) and the vanishing Lemma 2.6 (3).

Lastly, Lemma 2.6 (4) is by Lemma 3.4 and Lemma 2.6 (3), which completes the proof of Lemma 2.6.

**5.2.** Let $A$ be a weak log abelian variety over an fs log scheme $S$. Let the notation be as in 1.1. Assume that we are given a homomorphism

$$\psi\colon \overline{Y} \to \overline{X}$$

satisfying the conditions (1)–(3) in 1.4.1.

In general, giving a $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $A/G = \overline{Y}\backslash \mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}$ is equivalent to giving a $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}$ endowed with an action of $\overline{Y}$ which is compatible with the canonical action of $\overline{Y}$ on $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}$.

Now, the $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor $\overline{L}_\psi$ on $A/G$ which is the image of $\psi$ by the homomorphism $\mathrm{Hom}_{\langle\ ,\ \rangle}(\overline{Y}, \overline{X}) \to \mathrm{Biext}_{\mathrm{sym}}(A/G, A/G; \mathbb{G}_{m,\log}/\mathbb{G}_m) \to H^1(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ induced by Lemma 2.6 (cf. 2.7), just proved, is described as follows. The corresponding $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}$ is the trivial

$\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor $\mathbb{G}_{m,\log}/\mathbb{G}_m$, and the action of $y \in \overline{Y}$ is given by the multiplication by $c(y)^{-1}$, where

$$c(y) := \langle \psi(y), y \rangle \cdot \psi(y)^2.$$

(In the situation where $\phi$ exists as in 4.5, this $c(y)$ is the image of the $c(\widetilde{y})$ in 4.11, where $\widetilde{y}$ is a lift of $y$.) This description is claimed in Proposition 1.6 (2) and is seen by Lemma 3.10.

Though the fact that the above description indeed gives an action of $\overline{Y}$ can be seen by Lemma 3.10, here we directly show this and observe that this fact relates to the cocycle condition in 4.8. The most explicitly, this fact is shown as follows. Let $y, z \in \overline{Y}$, $h \in \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$. Then we have

$$\begin{aligned}(yh)(\psi(z)^{-2})&\langle \psi(z), z\rangle^{-1} h(\psi(y)^{-2})\langle \psi(y), y\rangle^{-1} \\ &= h(\psi(zy)^{-2})\langle y, \psi(z)\rangle^{-2}\langle \psi(z), z\rangle^{-1}\langle \psi(y), y\rangle^{-1} \\ &= h(\psi(zy)^{-2})\langle \psi(zy), zy\rangle^{-1},\end{aligned}$$

which shows that the above gives an action.

This calculation is equivalent to

$$c(z)^{-1}(yh)c(y)^{-1}(h) = c(zy)^{-1}(h),$$

and to

$$(**) \qquad y^*(c(z)^{-1}) \cdot c(y)^{-1} = c(yz)^{-1} \text{ for } y, z \in \overline{Y}.$$

The last one can be checked also as

$$\begin{aligned} y^*(c(z)^{-1}) &= y^*(\langle \psi(z), z\rangle^{-1}) \cdot y^*(\psi(z)^{-2}) = \langle \psi(z), z\rangle^{-1} \cdot (\psi(z) \cdot \langle \psi(z), y\rangle)^{-2}, \\ y^*(c(z)^{-1}) \cdot c(y)^{-1} &= \langle \psi(z), z\rangle^{-1} \cdot \psi(z)^{-2} \cdot \langle \psi(z), y\rangle^{-2} \cdot \langle \psi(y), y\rangle^{-1} \cdot \psi(y)^{-2} \\ &= \langle \psi(yz), yz\rangle^{-1} \cdot \psi(yz)^{-2} = c(yz)^{-1},\end{aligned}$$

and also can be deduced from

$$y^*(c(z)^{-1}) \cdot c(y)^{-1} = c(yz)^{-1} \text{ for } y, z \in Y,$$

which already appeared in 4.12. Thus the fact that the above description gives an action relates to the cocycle condition in 4.8.

**5.2.1 Remark.** In the above, we adopt the sign convention that the boundary map $\mathcal{H}om(\overline{Y}, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to \mathcal{E}xt(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ associated to the short exact sequence $(*)$ in 5.1, which gives a part of the proof of Proposition 1.6, should be ($h \mapsto$ the extension obtained by the pushout by $h$ from $(*)$ in 5.1). Provided if we adopt the opposite convention ($h \mapsto$ the extension obtained by the pushout by $h^{-1}$ from $(*)$ in 5.1), then the resulting first isomorphism in Proposition 1.6 (1) would become the $(-1)$-times of ours and the action in Proposition 1.6 (2) would become $(h, \lambda) \mapsto (yh, h(\psi(y)^2)\langle\psi(y), y\rangle\lambda)$. Note that, to make our constructions and proofs more understandable, we adopt the similar convention in this paper except 8.5–8.12. See, for example, the proof of Lemma 3.10 (3). (In 8.5–8.12, the sign convention is not compatible with that in the other parts. Cf. Remark 8.5.1.)

**5.3.** Let the situation be as in 4.5. Let $\Sigma$ be as in 4.7, and $A^{(\Sigma)}$ the corresponding model of $A$.

Let $p$ be a polarization on $A$. Let $L_{-p}$ be the $\mathbb{G}_{m,\log}$-torsor which is the image of $-p$ by the pullback to the diagonal $\mathrm{Biext}(A, A; \mathbb{G}_{m,\log}) \to H^1(A, \mathbb{G}_{m,\log})$.

Assume that $\psi$ is compatible with $-p$ in the sense that the image of $\psi$ in $\mathrm{Biext}_{\mathrm{sym}}(A, A; \mathbb{G}_{m,\log}/\mathbb{G}_m)$ by the map

$$\mathrm{Hom}_{\langle\ ,\ \rangle}(\overline{Y}, \overline{X}) \to \mathrm{Biext}_{\mathrm{sym}}(A, A; \mathbb{G}_{m,\log}/\mathbb{G}_m)$$

in 2.7 coincides with the image of $-p$. Let $L_\psi$ be the $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor which is the image of this last image of $\psi$ by the pullback to the diagonal

$$\mathrm{Biext}_{\mathrm{sym}}(A, A; \mathbb{G}_{m,\log}/\mathbb{G}_m) \to H^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m).$$

Then $L_\psi$ is induced by $L_{-p}$.

In the case 1, let $n = 1$. In the case 2, let $n$ be an integer $\geq 1$ which kills the cokernel of $\phi\colon Y \to X$. In the case 3, let $n = 1$.

We define a special $\mathbb{G}_m$-torsor on $A^{(\Sigma)}$ which is inside the pullback of the $\mathbb{G}_{m,\log}$-torsor $L_{-p}^{\otimes n}$ on $A$ to $A^{(\Sigma)}$.

Giving such a $\mathbb{G}_m$-torsor is equivalent to giving a section of the $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $A^{(\Sigma)}$ induced by $L_\psi^{\otimes n}$. So, it is defined by a section of the $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $A^{(\Sigma)}/G = \overline{Y}\backslash \mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ induced by $L_\psi^{\otimes n}$, where $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ is defined similarly to $\mathcal{Q}^{(\Sigma)}$ in 1.7. Furthermore, it is given by a section of the trivial torsor $\mathbb{G}_{m,\log}/\mathbb{G}_m$ on $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ which is compatible with the action $c(y)^{-n}$ ($y \in \overline{Y}$) of $\overline{Y}$ (cf. 5.2). We define the last one as follows.

First consider the case 1. Let $s$ be the section of $\mathbb{G}_{m,\log}/\mathbb{G}_m$ on $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ which is given by $\langle\phi(z), z\rangle \cdot \phi(z)^2$ on the part of $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ corresponding to $C(z)$ ($z \in Y$). Then $s$ is compatible with the action of $\overline{Y}$ given by $c(y)^{-1}$ ($y \in \overline{Y}$), that is, we have $y^*(s) = s \cdot c(y)^{-1}$ for $y \in \overline{Y}$. Here $y^*$ is the pullback of the translation of $y$. We prove this. Since the action of $y$ sends the part of $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ corresponding to $C(z)$ ($z \in Y$) to the part corresponding to $C(z \cdot y^{-1})$ (cf. 4.12), on the part of $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$

corresponding to $C(z)$, $y^*(s)$ coincides with

$$\begin{aligned}&y^*(\langle\phi(z\cdot y^{-1}), zy^{-1}\rangle\cdot\phi(z\cdot y^{-1})^2)\\&=\langle\phi(z\cdot y^{-1}), zy^{-1}\rangle\cdot\phi(z\cdot y^{-1})^2\cdot\langle\phi(z\cdot y^{-1})^2, y\rangle\\&=\langle\phi(z), z\rangle\cdot\phi(z)^2\cdot\langle\phi(y), y\rangle^{-1}\cdot\phi(y)^{-2}\\&=s\cdot c(y)^{-1}.\end{aligned}$$

(This can be seen also by the equation $(**)$ in 5.2.)

Next consider the case 2. Let $s$ be the section of $\mathbb{G}_{m,\log}/\mathbb{G}_m$ on $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ which is given by $\langle x, \phi^{-1}(x^n)\rangle\cdot x^{2n}$ on the part of $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ corresponding to $C(x)$ $(x\in X)$. Then $s$ is compatible with the action of $\overline{Y}$ given by $c(y)^{-n}$ $(y\in\overline{Y})$, that is, we have $y^*(s)=s\cdot c(y)^{-n}$ for $y\in\overline{Y}$. We prove this. Since the action of $y$ sends the part of $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ corresponding to $C(x)$ to the part corresponding to $C(x\cdot\phi(y)^{-1})$ (cf. 4.12), on the part of $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ corresponding to $C(x)$, $y^*(s)$ coincides with

$$\begin{aligned}&y^*(\langle x\cdot\phi(y)^{-1}, \phi^{-1}(x^n)y^{-n}\rangle\cdot(x\cdot\phi(y)^{-1})^{2n})\\&=\langle x\cdot\phi(y)^{-1}, \phi^{-1}(x^n)y^{-n}\rangle\cdot(x\cdot\phi(y)^{-1})^{2n}\cdot\langle(x\cdot\phi(y)^{-1})^{2n}, y\rangle\\&=\langle x, \phi^{-1}(x^n)\rangle\cdot x^{2n}\cdot\langle\phi(y), y\rangle^{-n}\cdot\phi(y)^{-2n}\\&=s\cdot c(y)^{-n}.\end{aligned}$$

Lastly consider the case 3. Let $s$ be the section of $\mathbb{G}_{m,\log}/\mathbb{G}_m$ on $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ which is given by $\langle\phi(z)\alpha, z\rangle\cdot\phi(z)^2\alpha$ on the part of $\mathcal{H}om(\overline{X},\mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ corresponding to $C(\alpha, z)$ $(\alpha\in I,\ z\in Y)$. Then $s$ is compatible with the action of $\overline{Y}$ given by $c(y)^{-1}$ $(y\in\overline{Y})$, that is, we have $y^*(s)=s\cdot c(y)^{-1}$ for $y\in\overline{Y}$. We prove this. Since the action of $y$ sends the part of $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ corresponding to $C(\alpha, z)$ to the part corresponding to $C(\alpha, z\cdot y^{-1})$ (cf. 4.12), on the part of $\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$ corresponding to $C(\alpha, z)$, $y^*(s)$ coincides with

$$\begin{aligned}&y^*(\langle\phi(zy^{-1})\alpha, zy^{-1}\rangle\cdot\phi(zy^{-1})^2\alpha)\\&=\langle\phi(zy^{-1})\alpha, zy^{-1}\rangle\cdot\phi(zy^{-1})^2\alpha\cdot\langle\phi(zy^{-1})^2\alpha, y\rangle\\&=\langle\phi(z)\alpha, z\rangle\cdot\phi(z)^2\alpha\cdot\langle\phi(y), y\rangle^{-1}\phi(y)^{-2}\\&=s\cdot c(y)^{-1}.\end{aligned}$$

## 6. Case of constant degeneration, I

In Sections 6–8, we prove the results in Section 1 for weak log abelian varieties with constant degeneration. We also prove an important Theorem 8.5, which gives a description of $\mathcal{H}^1(A, \mathbb{G}_{m,\log})$.

In this section, we prove some basic results.

**6.1.** Let $A$ be a weak log abelian variety with constant degeneration over an fs log scheme $S$. Let $G$ be the semiabelian part and $[Y \to G_{\log}]$ be the corresponding log 1-motif. In particular, we have $A = G_{\log}^{(Y)}/Y$. Let $T$ and $B$ be the torus and abelian part of $G$, respectively. Thus we have an exact sequence $0 \to T \to G \to B \to 0$. Let $X = \mathcal{H}om(T, \mathbb{G}_m)$ and let $\langle\ ,\ \rangle \colon X \times Y \to \mathbb{G}_{m,\log}/\mathbb{G}_m$ be the induced canonical pairing. We have $\widetilde{A} = G_{\log}^{(Y)}$ in this case (cf. Remark 1.8).

The next proposition gives the proof of Proposition 2.1 in this case (i.e., in the case of constant degeneration).

**6.2 Proposition.** (1) *$H^0(A, \mathbb{G}_a) = H^0(S, \mathbb{G}_a)$ and $H^0(A, \mathbb{G}_m) = H^0(S, \mathbb{G}_m)$.*

(2) *$H^0(A, \mathbb{G}_{m,\log}) = H^0(S, \mathbb{G}_{m,\log})$ and $H^0(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) = H^0(S, \mathbb{G}_{m,\log}/\mathbb{G}_m)$.*

**6.3.** As a corollary, we see that Proposition 2.3 holds in the case of constant degeneration by Lemmas 3.1 and 3.4.

**6.4.** For the proof of Proposition 6.2, we first prove the following facts. Let $g \colon \widetilde{A} \to B$ be the canonical morphism. Then we have

**6.4.1.** $g_*\mathbb{G}_a = \mathbb{G}_a,\ g_*\mathbb{G}_m = \mathbb{G}_m$.

**6.4.2.** $g_*(\mathbb{G}_{m,\log}/\mathbb{G}_m) \simeq X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m)_B$ canonically.

**6.4.3.** $g_*(\mathbb{G}_{m,\log}) \simeq X \oplus \mathbb{G}_{m,\log,B}$ canonically.

*Proof.* Since $\widetilde{A}$ is a $T_{\log}^{(Y)}$-torsor on $B$, we have only to show the case where $B$ is trivial. In this case, first 6.4.2 is proved by [9] Proposition 7.9 (2). Second, [9] Proposition 7.9 (3) shows $g_*\mathbb{G}_a = \mathbb{G}_a$ and $g_*\mathbb{G}_m = \mathbb{G}_m$ (6.4.1). Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & X \oplus \mathbb{G}_{m,\log} & \longrightarrow & X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m) & \longrightarrow & 1 \\
 & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \\
1 & \longrightarrow & g_*\mathbb{G}_m & \longrightarrow & g_*(\mathbb{G}_{m,\log}) & \longrightarrow & g_*(\mathbb{G}_{m,\log}/\mathbb{G}_m), & &
\end{array}$$

from which we have 6.4.3. □

**6.5.** We prove Proposition 6.2. Note that for $F = \mathbb{G}_a, \mathbb{G}_m, \mathbb{G}_{m,\log}$, or $\mathbb{G}_{m,\log}/\mathbb{G}_m$, $H^0(A, F)$ is the $Y$-invariant part of $H^0(\widetilde{A}, F)$. For $F = \mathbb{G}_a$ or $\mathbb{G}_m$, 6.4.1 shows that $H^0(\widetilde{A}, F)$ coincides with $H^0(B, F) = H^0(S, F)$ proving Proposition 6.2 (1). Next we prove Proposition 6.2 (2). For $x \in X$ and $y \in Y$, we have

$$t_y^*(x \bmod \mathbb{G}_m) = (x \bmod \mathbb{G}_m) \cdot \langle x, y \rangle,$$

where we regard $x$ as a global section of $\mathbb{G}_{m,\log}/\mathbb{G}_m$ on $\widetilde{A}$. Since the pairing $\langle\ ,\ \rangle$ is nondegenerate, this shows that the $Y$-invariant part of $H^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ $= H^0(B, X) \oplus H^0(B, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ (6.4.2) coincides with $H^0(B, \mathbb{G}_{m,\log}/\mathbb{G}_m) = H^0(S, \mathbb{G}_{m,\log}/\mathbb{G}_m)$. Hence $H^0(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) = H^0(S, \mathbb{G}_{m,\log}/\mathbb{G}_m)$. Together with $H^0(A, \mathbb{G}_m) = H^0(S, \mathbb{G}_m)$, we also see $H^0(A, \mathbb{G}_{m,\log}) = H^0(S, \mathbb{G}_{m,\log})$ by localizing $S$.

**6.6 Proposition.** (1) $R^1g_*(\mathbb{G}_{m,\log}) = \{1\}$.

(2) $R^1g_*(\mathbb{G}_{m,\log}/\mathbb{G}_m) = \{1\}$.

*Proof.* The fact that $\widetilde{A}$ is a $T_{\log}^{(Y)}$-torsor on $B$ reduces (1) and (2) to

$$\mathcal{H}^1(T_{\log}^{(Y)}, \mathbb{G}_{m,\log}) = 0 \quad \text{and} \quad \mathcal{H}^1(T_{\log}^{(Y)}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = 0,$$

respectively. The former was seen by the argument in the fifth paragraph of [9] 7.17, which is a part of the proof of ibid. Theorem 7.3 (2). After replacing $\mathbb{G}_{m,\log}$ by $\mathbb{G}_{m,\log}/\mathbb{G}_m$, the same argument works and the latter is also seen. □

**6.7.** Let $\overline{H}$ be the sheaf of abelian groups on $(\mathrm{fs}/S)_{\text{ét}}$ defined by

$$\overline{H} = \{(\psi, a) \mid \psi \text{ is a homomorphism } Y \to X \text{ and } a \text{ is a map } Y \to \mathbb{G}_{m,\log}/\mathbb{G}_m \text{ such that } a(yz)a(y)^{-1}a(z)^{-1} = \langle \psi(y), z\rangle \text{ for all } y, z \in Y\}.$$

**6.8.** We define a natural homomorphism

$$\overline{H} \to \mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m).$$

Let $(\psi, a)$ be a section of $\overline{H}$. Consider the quotient of $\widetilde{A} \times \mathbb{G}_{m,\log}/\mathbb{G}_m$ by the action of $Y$ given by $(m, \lambda) \mapsto (ym, \lambda a(y)^{-1}m(\psi(y)^{-1}))$ $(y \in Y, m \in \widetilde{A}, \lambda \in \mathbb{G}_{m,\log}/\mathbb{G}_m)$, where the last $m$ represents the image of $m$ in $\widetilde{A}/G \cong \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$ by abuse of notation. Then this quotient gives a $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $A$, which is defined as the image of $(\psi, a)$.

Let $X \to \overline{H}$ be a homomorphism sending $x \in X$ to $(0, \langle x, -\rangle)$. It is injective and we identify $X$ with the image of this injection. Then the above homomorphism $\overline{H} \to \mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ factors through $\overline{H}/X$ because $m \mapsto (m, m(x^{-1}))$ gives a section of the torsor on $A$ associated to $x \in X$.

**6.9 Theorem.** *We have the following isomorphisms and the commutative diagram with exact rows.*

(1) $\mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)/X \xrightarrow{\cong} \mathcal{E}xt(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m) \xrightarrow{\cong} \mathcal{E}xt(A, \mathbb{G}_{m,\log}/\mathbb{G}_m)$.

(2) $\overline{H}/X \xrightarrow{\cong} \mathcal{H}^1(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m) \xrightarrow{\cong} \mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m)$.

(3)

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)/X & \longrightarrow & \overline{H}/X & \longrightarrow & \mathcal{H}om_{\langle\,,\,\rangle}(Y, X) \\
& \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
0 \longrightarrow & \mathcal{E}xt(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) & \longrightarrow & \mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) & \longrightarrow & \mathcal{B}iext_{\mathrm{sym}}(A, A; \mathbb{G}_{m,\log}/\mathbb{G}_m).
\end{array}$$

This is proved in 6.14–6.18.

**6.10.** We interpret the above homomorphism $\overline{H} \to \mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ in 6.8 in terms of $H^1(Y, H^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m))$.

First, a section of $\overline{H}$ can be regarded as a cocycle of the inhomogeneous cochain complex associated with the $Y$-module $X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m)_S$, where the action of $y \in Y$ on $X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m)_S$ is

$$(x, c) \mapsto (x, c\langle x, y\rangle).$$

In fact, a cocycle is a map $(\psi, a)\colon Y \to X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m)_S$ satisfying

$$(\psi(yz), a(yz)) = z(\psi(y), a(y)) \cdot (\psi(z), a(z)) = (\psi(y)\psi(z), a(y)\langle\psi(y), z\rangle a(z)),$$

which is nothing but a section of $\overline{H}$.

Next, $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)} \cong \widetilde{A}/G$ gives a natural homomorphism

$$X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m)_S \to \mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m),$$

which is in fact an isomorphism by 6.5. Hence there is a natural homomorphism

$$\overline{H} \to H^1(Y, X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m)_S) \to H^1(Y, \mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m)).$$

On the other hand, the canonical homomorphism

$$H^1(Y, \mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m)) \to \mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m)$$

is described as follows. Let $f(y)(m)$ $(y \in Y, m \in \widetilde{A})$ be a cocycle of the inhomogeneous cochain complex associated with the $Y$-module $\mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m)$. Consider the quotient of $\widetilde{A} \times \mathbb{G}_{m,\log}/\mathbb{G}_m$ by the action of $Y$ given by $(m, \lambda) \mapsto (ym, \lambda f(y)(m)^{-1})$ $(y \in Y, m \in \widetilde{A}, \lambda \in \mathbb{G}_{m,\log}/\mathbb{G}_m)$. Then this quotient gives a $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $A$, which is the image of $f$.

Therefore the composite of the above homomorphisms

$$\overline{H} \to H^1(Y, \mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m)) \to \mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m)$$

coincides with the homomorphism in 6.8.

**6.11 Lemma.** *Let $f\colon H \to S$ be a surjective morphism of schemes with geometrically connected fibers. Let $F$ be a sheaf of abelian groups on $S_{\text{ét}}$. Then we have the following.*

(1) *Assume that either $f$ is smooth or $f$ has a section. Then $f_*f^*F = F$.*

(2) *Assume that étale locally on $S$, there is a finite decomposition of $S$ by subschemes such that the restriction of $F$ to each subscheme is constant with torsion-free value. Assume further that there are a noetherian scheme $S_0$, a surjective smooth morphism $f_0\colon H_0 \to S_0$ of schemes with geometrically connected fibers, and a morphism $g\colon S \to S_0$ such that the base-changed morphism $f_0$ with respect to $g$ is isomorphic to $f$ over $S$. Then $R^1f_*f^*F = 0$.*

*Proof.* (1) Since $f$ is surjective, we see that the natural map $F \to f_* f^* F$ is injective. It suffices to show that $F(S) \to F(H)$ is surjective. Since the statement is étale local on $S$, in the case where $f$ is smooth, we may assume that there is a section $s\colon S \to H$ to $f$. Then it is enough to show that, for any element $a$ of $F(H)$, its pullback by $s \circ f$ coincides with $a$. To see this, we may assume that $S$ is the spectrum of an algebraically closed field. Then, $F$ is constant and the connectivity implies $F(H) = F(S)$, as desired.

(2) First we show that we may assume that $F$ is constant. Since the statement is étale local on $S$, we may assume that there is a decomposition of $S$ as in the assumption globally. Then we may assume that $F = j_! N_T$, where $j\colon T \to S$ is an immersion and $N_T$ is the constant sheaf with values in a torsion-free abelian group $N$. Replacing $S$ with the closure of $T$, we may assume that $j$ is an open immersion. Then, $j_! N_T$ injects to $N_S$, and by (1), $R^1 f_* f^* j_! N_T \to R^1 f_* f^* N_S$ is also injective. Hence, we may and will assume that $F$ is constant.

Next we show that we may assume that $S$ is noetherian. To see this, we may assume that $S$ is the projective limit of noetherian schemes $S_\lambda$ over $S_0$ whose transition morphisms are affine. If the conclusion holds for the base-changed morphism of $f_0$ over any $S_\lambda$, then it also holds for $f$. Thus, we may assume that $S$ is noetherian.

We show that we may assume further that $S$ is normal and integral. Let $S'$ be the disjoint union of the normalizations of the irreducible components of $S$. Let $p$ be the canonical surjection $S' \to S$. By (1), the homomorphism $R^1 f_* f^* F \to R^1 f_* f^* p_* p^* F$ is injective. Since $p$ is pro-finite, by the usual proper base change theorem and the fact $Rp_* = p_*$, we see that $R^1 f_* f^* p_* p^* F$ is equal to $p_* R^1 f_* f^* p^* F$, where we denote the base-changed morphisms by the same symbols. Hence we may assume that $S$ is normal and integral.

Further, we may assume that $S$ is strictly local. Then, $H$ is connected and normal because $f$ is smooth with connected fibers. Therefore, $H$ is irreducible and geometrically unibranch, and by [2] Exposé IX Proposition 3.6 and Remarques 3.7, we have $H^1(H_{\text{ét}}, F) = 0$, as desired. □

**6.12 Proposition.** *$\mathcal{H}^0(G, \mathbb{Z}) = \mathbb{Z}, \mathcal{H}^0(G, \mathbb{G}_{m,\log}/\mathbb{G}_m) = \mathbb{G}_{m,\log}/\mathbb{G}_m$, and $\mathcal{H}^1(G, \mathbb{Z}) = \mathcal{H}^1(G, \mathbb{G}_{m,\log}/\mathbb{G}_m) = 0$.*

*Proof.* As in the proof of [9] Proposition 7.22, this reduces to Lemma 6.11. Note that the sheaf $\mathbb{G}_{m,\log}/\mathbb{G}_m$ restricted in a small étale site and the morphism $G \to S$ satisfy the assumptions in Lemma 6.11 (1) and (2). □

**6.13 Proposition.** (1) $\mathcal{H}^1(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = \mathcal{H}^1(\widetilde{A}/G, \mathbb{G}_{m,\log}/\mathbb{G}_m) = 0$.

(2) $\mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = \mathcal{H}^0(\widetilde{A}/G, \mathbb{G}_{m,\log}/\mathbb{G}_m) = X \oplus \mathbb{G}_{m,\log}/\mathbb{G}_m$.

*Proof.* (1) First, by Proposition 6.6 (2) and 6.4.2, we have

$$\mathcal{H}^1(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = \mathcal{H}^1(B, g_*(\mathbb{G}_{m,\log}/\mathbb{G}_m)) = \mathcal{H}^1(B, X) \oplus \mathcal{H}^1(B, \mathbb{G}_{m,\log}/\mathbb{G}_m).$$

By Proposition 6.12, the last group vanishes.

Next, consider the inclusion $\mathcal{H}^1(\widetilde{A}/G, p_*(\mathbb{G}_{m,\log}/\mathbb{G}_m)) \subset \mathcal{H}^1(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = 0$, where $p$ is the projection $\widetilde{A} \to \widetilde{A}/G$. Since $p$ is a $G$-torsor in the category of étale sheaves, it is represented by $G$-torsors. Hence, $p_*(\mathbb{G}_{m,\log}/\mathbb{G}_m) = \mathbb{G}_{m,\log}/\mathbb{G}_m$ by Lemma 6.11 (1). Together with the above inclusion, we conclude $\mathcal{H}^1(\widetilde{A}/G, \mathbb{G}_{m,\log}/\mathbb{G}_m) = 0$.

(2) Again by 6.4.2, we have

$$\mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = \mathcal{H}^0(B, X) \oplus \mathcal{H}^0(B, \mathbb{G}_{m,\log}/\mathbb{G}_m) = X \oplus \mathbb{G}_{m,\log}/\mathbb{G}_m,$$

where the last equality is by Proposition 6.12.

Next, we have $\mathcal{H}^0(\widetilde{A}/G, \mathbb{G}_{m,\log}/\mathbb{G}_m) \subset \mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = X \oplus \mathbb{G}_{m,\log}/\mathbb{G}_m$. Further, there is the inverse map $X \oplus \mathbb{G}_{m,\log}/\mathbb{G}_m \to \mathcal{H}^0(\widetilde{A}/G, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ because $\widetilde{A}/G = \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$. Thus, $\mathcal{H}^0(\widetilde{A}/G, \mathbb{G}_{m,\log}/\mathbb{G}_m) = X \oplus \mathbb{G}_{m,\log}/\mathbb{G}_m$. □

**6.14.** We prove the part $\overline{H}/X \xrightarrow{\cong} \mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ of Theorem 6.9 (2). Consider the exact sequence

$$0 \to H^1(Y, H^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m)) \to H^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to H^1(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m).$$

By Proposition 6.13 (1), we have $\mathcal{H}^1(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = 0$. Hence

$$\mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) = H^1(Y, \mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m)) = H^1(Y, X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m)_S),$$

where the last equality is by Proposition 6.13 (2).

We compute $H^1(Y, X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m)_S)$ by the inhomogeneous cochain complex. As is seen in 6.10, the group of cocycles is $\overline{H}$. A coboundary is a map $Y \to X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m)_S$ sending $y \in Y$ to

$$y(x, c)/(x, c) = (x, c\langle x, y\rangle)/(x, c) = (0, \langle x, y\rangle)$$

for some $(x, c) \in X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m)_S$, which is nothing but a section of $X \subset \overline{H}$. Hence, we have $H^1(Y, X \oplus (\mathbb{G}_{m,\log}/\mathbb{G}_m)_S) \cong \overline{H}/X$.

**6.15 Proposition.** *Let* $\mathrm{Hom}_{\mathrm{adj}}(Y, X)$ *be the subgroup of* $\mathrm{Hom}(Y, X)$ *consisting of all homomorphisms* $p$ *such that there is another homomorphism* $p'$ *satisfying* $\langle p(y), z\rangle = \langle p'(z), y\rangle$ *for any* $y, z \in Y$. *We have the following.*

(1) $\mathrm{Hom}(Y, X) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}, \mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m))$.

(2) $\mathrm{Hom}_{\langle\ ,\ \rangle}(Y, X) \hookrightarrow \mathrm{Hom}_{\mathrm{adj}}(Y, X)$

$$\xrightarrow{\cong} \mathrm{Hom}(\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}/Y, \mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(X)}/X)$$

$$\xrightarrow{\cong} \mathrm{Hom}(\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}/Y, \mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)/X).$$

We remark that the image in $\mathrm{Hom}(Y, X)$ of $\mathrm{Hom}(\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}, \mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(X)})$ by the inverse isomorphism in (1) contains $\mathrm{Hom}_{\mathrm{adj}}(Y, X)$; but it does not necessarily coincide with $\mathrm{Hom}_{\mathrm{adj}}(Y, X)$ or $\mathrm{Hom}(Y, X)$.

*Proof.* (1) is by [9] Theorem 7.3 (3).

(2) We call the $p'$ in the statement is an *adjoint* of $p$. If $p$ is symmetric, $p$ itself is an adjoint of $p$. Hence we have the first inclusion.

Next we prove that the homomorphism $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)} \to \mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ corresponding to $p\colon Y \to X$ via the isomorphism in (1) factors through $\mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(X)}$ if $p$ has an adjoint $p'\colon Y \to X$ in the sense that:

$$(*) \qquad \text{For any } y, z \in Y, \text{ we have } \langle p(y), z\rangle = \langle p'(z), y\rangle.$$

Let $\varphi\colon X \to \mathbb{G}_{m,\log}/\mathbb{G}_m$ be any section of $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$. Since it belongs to $(Y)$-part, for any $y \in Y$, there are $y_1, y_2 \in Y$ such that

$$\langle p(y), y_1\rangle | \varphi(p(y)) | \langle p(y), y_2\rangle.$$

By $(*)$, this is equivalent to

$$\langle p'(y_1), y\rangle | \varphi(p(y)) | \langle p'(y_2), y\rangle,$$

which means that the homomorphism $\varphi \circ p\colon Y \to \mathbb{G}_{m,\log}$ belongs to the $(X)$-part of $\mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)$. Thus we have the canonical homomorphism

$$\mathrm{Hom}_{\mathrm{adj}}(Y, X) \to \mathrm{Hom}(\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}/Y, \mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(X)}/X).$$

To see that this is an isomorphism, apply [9] Theorem 7.6 (1) with $X' = Y$, $Y' = X$, and $\langle\ ,\ \rangle'$, where $\langle y, x\rangle' := \langle x, y\rangle$ $(x \in X, y \in Y)$.

To see the last isomorphism in (2), it is enough to show that any homomorphism

$$f\colon \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}/Y \to \mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)/X$$

comes from a homomorphism $p$ having an adjoint. Locally on the base, $f$ lifts to a homomorphism

$$\widetilde{f}\colon \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)} \to \mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m),$$

which sends $Y$ into $X$. This is possible because $\mathcal{E}xt(\widetilde{A}/G, X) = 0$, which comes from $\mathcal{E}xt(\widetilde{A}, X) = 0$ ([9] 7.23) and $\mathcal{H}om(G, X) = 0$. By (1), this $\widetilde{f}$ comes from a homomorphism $p\colon Y \to X$. It suffices to show that $p$ has an adjoint. Let $y$ be an element of $Y$. Since $\widetilde{f}$ sends $Y$ into $X$, there is an element $x$ of $X$ such that $\langle p(-), y\rangle = \langle x, -\rangle$. Let $p'(y) := x$. Then the map $p'$ makes a homomorphism from $Y$ to $X$, which is an adjoint of $p$. □

**6.16.** We prove the case $F = \mathbb{G}_{m,\log}/\mathbb{G}_m$ of the cubic isomorphism (Theorem 2.2) for this case of constant degeneration, which implies the exactness of the lower row in the diagram in Theorem 6.9 (3) by Lemma 3.8 and Proposition 2.1 (proved in Proposition 6.2). By Lemma 3.5, it is enough to show only Theorem 2.2 (2). (Notice that the functoriality and the last equality in Theorem 2.2 (1) are deduced from the uniqueness in the first statement of Theorem 2.2 (1).)

Let the notation be as in the statement. Let $u, v \in A(S')$. We identify $L$ with the class of a section $(\psi, a)$ of $\overline{H}$ modulo $X$ by 6.14. Then we have $t_u^*(\psi, a) = (\psi, au(\psi))$. Here $u(\psi)$ is the composite of $\psi\colon Y \to X$ and $u\colon X \to \mathbb{G}_{m,\log}/\mathbb{G}_m$ modulo $Y$ regarded as a section of $\mathcal{H}om(Y, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(X)}/X$. Thus $L \mapsto t_u^*L \cdot L^{-1}$ sends $(\psi, a)$ to $(1, u(\psi))$. Next, the same argument shows that $L \mapsto t_v^*L \cdot L^{-1}$ kills $(1, b)$ for any $b$ locally on $S'$. Hence the composite $L \mapsto t_{uv}^*L \cdot t_u^*L^{-1} \cdot t_v^*L^{-1} \cdot L$ kills the class of $L$ locally on $S'$, which completes the proof.

**6.17.** We continue to prove Theorem 6.9.

(a) The first homomorphism in Theorem 6.9 (1) is injective.

This is by Lemma 2.6 (2) which is already proved in 5.1.

(b) The second homomorphism in Theorem 6.9 (1) is also injective.

This is by $\mathcal{H}om(G, \mathbb{G}_{m,\log}/\mathbb{G}_m) = 0$ which is seen by [9] Lemma 6.1.1.

By (b), the natural homomorphism

$$\mathcal{H}om(A/G, \mathcal{E}xt(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m)) \to \mathcal{H}om(A, \mathcal{E}xt(A, \mathbb{G}_{m,\log}/\mathbb{G}_m))$$

is injective and hence

(c) $\mathcal{B}iext_{\mathrm{sym}}(A/G, A/G; \mathbb{G}_{m,\log}/\mathbb{G}_m) \to \mathcal{B}iext_{\mathrm{sym}}(A, A; \mathbb{G}_{m,\log}/\mathbb{G}_m)$ is also injective.

Further,

(d) The composite $\mathcal{H}om_{\langle\, ,\, \rangle}(Y, X) \to \mathcal{B}iext(A/G, A/G; \mathbb{G}_{m,\log}/\mathbb{G}_m)$ in 2.7 is injective.

In fact, the first homomorphism in the display in 2.7 is injective by Proposition 6.15 (2) and the second homomorphism in the display in 2.7 is also injective by (a). Hence we have (d).

The commutativity of the diagram in Theorem 6.9 (3) is by construction. By (c) and (d), the right vertical arrow of the diagram in Theorem 6.9 (3) is injective. Therefore, the left vertical arrow of the diagram in Theorem 6.9 (3) is surjective. Together with (a) and (b), we complete the proof of Theorem 6.9 (1).

We prove Theorem 6.9 (2). We already proved $\overline{H}/X \overset{\cong}{\to} \mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ in 6.14. The same proof works for $\overline{H}/X \overset{\cong}{\to} \mathcal{H}^1(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ if we know $\mathcal{H}^1(\widetilde{A}/G, \mathbb{G}_{m,\log}/\mathbb{G}_m) = 0$ and $\mathcal{H}^0(\widetilde{A}/G, \mathbb{G}_{m,\log}/\mathbb{G}_m) = X \oplus \mathbb{G}_{m,\log}/\mathbb{G}_m$. They are proved by Proposition 6.13 (1) and (2), respectively.

The remaining part is the surjectivity of the right vertical arrow of the diagram in Theorem 6.9 (3). This is a part of Proposition 1.6 (1), which will be proved in the next paragraph.

**6.18.** We prove Proposition 1.6 for the case of constant degeneration. We have already seen Proposition 1.6 (2) in 5.2. To see Proposition 1.6 (1), by (c) and (d) in the previous paragraph, it is enough to show that the map $\mathcal{H}om_{\langle\,,\,\rangle}(Y,X) \to \mathcal{B}iext_{\mathrm{sym}}(A,A;\mathbb{G}_{m,\log}/\mathbb{G}_m)$ is bijective, which also completes the proof of Theorem 6.9.

By Proposition 6.15 (2), we have

$$\mathcal{H}om_{\mathrm{adj}}(Y,X) \overset{\cong}{\to} \mathcal{H}om(\mathcal{H}om(X,\mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}/Y, \mathcal{H}om(Y,\mathbb{G}_{m,\log}/\mathbb{G}_m)/X)$$

(see Proposition 6.15 for the definition of $\mathcal{H}om_{\mathrm{adj}}(Y,X)$). By Theorem 6.9 (1) (already proved in 6.17), this is isomorphic to

$$\mathcal{H}om(A/G, \mathcal{E}xt(A/G,\mathbb{G}_{m,\log}/\mathbb{G}_m)).$$

Further, we prove

$$(*) \qquad \mathcal{H}om(A/G,\mathcal{E}xt(A/G,\mathbb{G}_{m,\log}/\mathbb{G}_m)) \cong \mathcal{H}om(A,\mathcal{E}xt(A,\mathbb{G}_{m,\log}/\mathbb{G}_m)).$$

First, we have $\mathcal{H}om(G,\mathbb{G}_{m,\log}/\mathbb{G}_m)=0$ ([9] Lemma 6.1.1) and $\mathcal{E}xt(G,\mathbb{G}_{m,\log}/\mathbb{G}_m) = 0$. (The latter is by Proposition 6.12 together with Lemma 3.2.) Hence, $\mathcal{E}xt(A,\mathbb{G}_{m,\log}/\mathbb{G}_m) = \mathcal{E}xt(A/G,\mathbb{G}_{m,\log}/\mathbb{G}_m)(= \mathcal{H}om(Y,\mathbb{G}_{m,\log}/\mathbb{G}_m)/X)$. Hence it is enough to show $\mathcal{H}om(G,\mathcal{H}om(Y,\mathbb{G}_{m,\log}/\mathbb{G}_m)/X) = 0$. By Proposition 6.12 together with Lemma 3.2, we have $\mathcal{E}xt(G,X) = 0$. On the other hand, $\mathcal{H}om(G,\mathcal{H}om(Y,\mathbb{G}_{m,\log}/\mathbb{G}_m))$ vanishes by [9] Lemma 6.1.1. Thus $\mathcal{H}om(G,\mathcal{H}om (Y,\mathbb{G}_{m,\log}/\mathbb{G}_m)/X) = 0$ and we have $(*)$.

By Proposition 2.3 (proved for the present case in 6.3), $\mathcal{H}om(A,\mathcal{E}xt(A,\mathbb{G}_{m,\log} /\mathbb{G}_m)) \cong \mathcal{B}iext(A,A;\mathbb{G}_{m,\log}/\mathbb{G}_m)$. Therefore, we have proved

$$\mathcal{H}om_{\mathrm{adj}}(Y,X) \cong \mathcal{B}iext(A,A;\mathbb{G}_{m,\log}/\mathbb{G}_m).$$

Taking the symmetric parts, we have the desired bijection. This completes the proof of the case of constant degeneration of Proposition 1.6 and also the proof of Theorem 6.9.

The following propositions will be used in the next section.

**6.19 Proposition.** *The image of any homomorphism $A \to \mathcal{E}xt(A,\mathbb{G}_{m,\log})$ is contained in the dual log abelian variety $A^*$ (1.2).*

*Proof.* By [9] Theorem 7.4 (3), we have $\mathcal{E}xt(A,\mathbb{G}_{m,\log}) \cong G^*_{\log}/X$. Any homomorphism $A \to G^*_{\log}/X$ induces a homomorphism $A/G \to \mathcal{H}om(Y,\mathbb{G}_{m,\log}/\mathbb{G}_m)/X$ by [9] Corollary 6.1.2, and its image is contained in $A^*/G^* = \mathcal{H}om(Y,\mathbb{G}_{m,\log}/\mathbb{G}_m)^{(X)}/ X$ by

$$\mathcal{H}om(A/G,\mathcal{H}om(Y,\mathbb{G}_{m,\log}/\mathbb{G}_m)/X) = \mathcal{H}om(A/G,\mathcal{H}om(Y,\mathbb{G}_{m,\log}/\mathbb{G}_m)^{(X)}/X)$$

(Proposition 6.15 (2)). Hence we see that $A \to G^*_{\log}$ factors through $G^{*(X)}_{\log}/X = A^*$, as desired. □

**6.20 Corollary.** *We have the following.*

$$\mathcal{H}om(A, A^*) \cong \mathcal{B}iext(A, A; \mathbb{G}_{m,\log}), \quad \mathcal{H}om_{\mathrm{sym}}(A, A^*) \cong \mathcal{B}iext_{\mathrm{sym}}(A, A; \mathbb{G}_{m,\log}).$$

*Here $\mathcal{H}om_{\mathrm{sym}}(A, A^*)$ is the sheaf of homomorphisms $A \to A^*$ which coincides with its dual $A = A^{**} \to A^*$.*

*Proof.* By Lemma 3.4 and $\mathcal{H}om(A, \mathbb{G}_{m,\log}) = 0$ ([9] Theorem 7.4 (4)), we have $\mathcal{H}om(A, \mathcal{E}xt(A, \mathbb{G}_{m,\log})) = \mathcal{B}iext(A, A; \mathbb{G}_{m,\log})$. Then, the first isomorphism is deduced from the previous proposition. Taking the symmetric parts, we obtain the second isomorphism. □

## 7. Case of constant total degeneration

In the next section, we will give a description of the group of $\mathbb{G}_{m,\log}$-torsors on a weak log abelian variety with constant degeneration. This section treats the special case where the abelian part $B = 0$. The reason why we treat the case $B = 0$ in this separated section is that in the case $B = 0$, the description becomes especially simple and becomes very similar to the classical theorem of Appel–Humbert which describes the group of $\mathbb{G}_m$-torsors on a complex torus (see 7.9), and hence this special case explains well the idea of proof and serves as a good introduction to the general case treated in the next section. In fact, this section is logically unnecessary as it is contained in the next section.

Let $A$ be a weak log abelian variety over an fs log scheme $S$ with constant degeneration such that the abelian part $B = 0$. Let the notation be as in 6.1 in the previous section. We further assume that the locally constant sheaves $X$ and $Y$ are constant. Let $\langle\langle\ ,\ \rangle\rangle : X \times Y \to \mathbb{G}_{m,\log}$ be the canonical pairing defined by $Y \to \widetilde{A} \subset \mathcal{H}om(X, \mathbb{G}_{m,\log})$.

**7.1.** Let $H$ be the sheaf of abelian groups on $(\mathrm{fs}/S)_{\text{ét}}$ defined by

$$H = \{(p, a) \mid p \text{ is a homomorphism } Y \to X \text{ and } a \text{ is a map } Y \to \mathbb{G}_{m,\log} \text{ such that } a(yz)a(y)^{-1}a(z)^{-1} = \langle\langle p(y), z\rangle\rangle \text{ for all } y, z \in Y\}.$$

**7.2.** We define a natural homomorphism

$$H \to \mathcal{H}^1(A, \mathbb{G}_{m,\log}).$$

Let $(p, a)$ be a section of $H$. Consider the quotient of $\widetilde{A} \times \mathbb{G}_{m,\log}$ by the action of $Y$ given by $(m, \lambda) \mapsto (ym, \lambda a(y)^{-1} m(p(y)^{-1}))$ $(y \in Y, m \in \widetilde{A}, \lambda \in \mathbb{G}_{m,\log})$, where the last $m$ is regarded as a section of $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)} \cong \widetilde{A}$. Then this quotient gives a $\mathbb{G}_{m,\log}$-torsor on $A$, which is defined as the image of $(p, a)$.

Let $X \to H$ be a homomorphism sending $x \in X$ to $(0, \langle\langle x, -\rangle\rangle)$. It is injective and we identify $X$ with the image of this injection. Then the above homomorphism $H \to \mathcal{H}^1(A, \mathbb{G}_{m,\log})$ factors through $H/X$ because $m \mapsto (m, m(x^{-1}))$ gives a section of the torsor on $A$ associated to $x \in X$.

**7.3.** We interpret the above homomorphism $H \to \mathcal{H}^1(A, \mathbb{G}_{m,\log})$ in 7.2 in terms of $H^1(Y, H^0(\widetilde{A}, \mathbb{G}_{m,\log}))$.

First, a section of $H$ can be regarded as a cocycle of the inhomogeneous cochain complex associated with the $Y$-module $X \oplus \mathbb{G}_{m,\log,S}$, where the action of $y \in Y$ on $X \oplus \mathbb{G}_{m,\log,S}$ is

$$(x, c) \mapsto (x, c\langle\langle x, y\rangle\rangle).$$

In fact, a cocycle is a map $(p, a)\colon Y \to X \oplus \mathbb{G}_{m,\log,S}$ satisfying

$$(p(yz), a(yz)) = z(p(y), a(y)) \cdot (p(z), a(z)) = (p(y)p(z), a(y)\langle\langle p(y), z\rangle\rangle a(z)),$$

which is nothing but a section of $H$.

Next, $\mathcal{H}om(X, \mathbb{G}_{m,\log})^{(Y)} \cong \widetilde{A}$ gives a natural homomorphism

$$X \oplus \mathbb{G}_{m,\log,S} \to \mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log}).$$

Hence there is a natural homomorphism

$$H \to H^1(Y, X \oplus \mathbb{G}_{m,\log,S}) \to H^1(Y, \mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log})).$$

On the other hand, the canonical homomorphism

$$H^1(Y, \mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log})) \to \mathcal{H}^1(A, \mathbb{G}_{m,\log})$$

is described as follows. Let $f(y)(m)$ $(y \in Y, m \in \widetilde{A})$ be a cocycle of the inhomogeneous cochain complex associated with the $Y$-module $\mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log})$. Consider the quotient of $\widetilde{A} \times \mathbb{G}_{m,\log}$ by the action of $Y$ given by $(m, \lambda) \mapsto (ym, \lambda f(y)(m)^{-1})$ $(y \in Y, m \in \widetilde{A}, \lambda \in \mathbb{G}_{m,\log})$. Then this quotient gives a $\mathbb{G}_{m,\log}$-torsor on $A$, which is the image of $f$.

Therefore the composite of the above homomorphisms

$$H \to H^1(Y, \mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log})) \to \mathcal{H}^1(A, \mathbb{G}_{m,\log})$$

coincides with the homomorphism in 7.2.

**7.4 Theorem.** *The natural homomorphism*

$$H/X \xrightarrow{\cong} \mathcal{H}^1(A, \mathbb{G}_{m,\log})$$

*in* 7.2 *is an isomorphism.*

*Proof.* Consider the exact sequence

$$0 \to H^1(Y, H^0(\widetilde{A}, \mathbb{G}_{m,\log})) \to H^1(A, \mathbb{G}_{m,\log}) \to H^1(\widetilde{A}, \mathbb{G}_{m,\log}).$$

By Proposition 6.6 (1), we have $\mathcal{H}^1(\widetilde{A}, \mathbb{G}_{m,\log}) = 0$. Hence

$$\mathcal{H}^1(A, \mathbb{G}_{m,\log}) = H^1(Y, \mathcal{H}^0(\widetilde{A}, \mathbb{G}_{m,\log})) = H^1(Y, X \oplus \mathbb{G}_{m,\log,S}),$$

where the last equality is by 6.4.3.

We compute $H^1(Y, X \oplus \mathbb{G}_{m,\log,S})$ by the inhomogeneous cochain complex. As is seen in 7.3, the group of cocycles is $H$. A coboundary is a map $Y \to X \oplus \mathbb{G}_{m,\log,S}$ sending $y \in Y$ to

$$y(x,c)/(x,c) = (x, c\langle\langle x, y\rangle\rangle)/(x,c) = (0, \langle\langle x, y\rangle\rangle)$$

for some $(x,c) \in X \oplus \mathbb{G}_{m,\log,S}$, which is nothing but a section of $X \subset H$. Hence, we have $H^1(Y, X \oplus \mathbb{G}_{m,\log,S}) \cong H/X$. □

**7.5.** We prove the cubic isomorphism (Theorem 2.2) for this case of the constant total degeneration. By Lemma 3.5, it is enough to show only Theorem 2.2 (2).

Let the notation be as in the statement. The case $F = \mathbb{G}_{m,\log}/\mathbb{G}_m$ is already proved in 6.16. We prove the case $F = \mathbb{G}_{m,\log}$ which implies the case $F = \mathbb{G}_m$ because the natural homomorphism $\mathcal{H}^1(A, \mathbb{G}_m) \to \mathcal{H}^1(A, \mathbb{G}_{m,\log})$ is injective by Proposition 2.1 (proved in Proposition 6.2). The proof is parallel to that for the case $F = \mathbb{G}_{m,\log}/\mathbb{G}_m$ in 6.16.

Let $u, v \in A(S')$. We identify $L$ with the class of a section $(p, a)$ of $H$ modulo $X$ via the isomorphism in Theorem 7.4. Then we have $t_u^*(p,a) = (p, au(p))$. Here $u(p)$ is the composite of $p\colon Y \to X$ and $u\colon X \to \mathbb{G}_{m,\log}$ modulo $Y$ regarded as a section of $\mathcal{H}om(Y, \mathbb{G}_{m,\log})^{(X)}/X$. Thus $L \mapsto t_u^*L \cdot L^{-1}$ sends $(p,a)$ to $(1, u(p))$. Next, the same argument shows that $L \mapsto t_v^*L \cdot L^{-1}$ kills $(1, b)$ for any $b$ locally on $S'$. Hence the composite $L \mapsto t_{uv}^*L \cdot t_u^*L^{-1} \cdot t_v^*L^{-1} \cdot L$ kills the class of $L$ locally on $S'$, which completes the proof.

**7.6.** Define a subgroup sheaf $\mathcal{H}om_{\langle\langle\,,\,\rangle\rangle}(Y, X)$ of $\mathcal{H}om(Y, X)$ by

$$\mathcal{H}om_{\langle\langle\,,\,\rangle\rangle}(Y, X) := \{p \in \mathcal{H}om(Y, X) \mid \langle\langle p(y), z\rangle\rangle = \langle\langle p(z), y\rangle\rangle \text{ for all } y, z \in Y\}.$$

Then we have an exact sequence

$$0 \to \mathcal{H}om(Y, \mathbb{G}_{m,\log}) \to H \to \mathcal{H}om_{\langle\langle\,,\,\rangle\rangle}(Y, X),$$

where the definitions of the arrows are as follows:

$$\mathcal{H}om(Y, \mathbb{G}_{m,\log}) \to H;\ a \mapsto (1, a),$$
$$H \to \mathcal{H}om_{\langle\langle\,,\,\rangle\rangle}(Y, X);\ (p, a) \mapsto p.$$

**7.7 Proposition.** *We have a commutative diagram with exact rows*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}om(Y, \mathbb{G}_{m,\log})/X & \longrightarrow & H/X & \longrightarrow & \mathcal{H}om_{\langle\langle\,,\,\rangle\rangle}(Y, X) \\ & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ 0 & \longrightarrow & \mathcal{E}xt(A, \mathbb{G}_{m,\log}) & \longrightarrow & \mathcal{H}^1(A, \mathbb{G}_{m,\log}) & \longrightarrow & \mathcal{H}om_{\mathrm{sym}}(A, A^*) \end{array}$$

*in which all the vertical arrows are isomorphisms. Here the last arrow in the lower row sends $L \in \mathcal{H}^1(A, \mathbb{G}_{m,\log})$ to $A \to A^*$; $u \mapsto t_u^*(L) \cdot L^{-1}$. (Cf. Corollary* 6.20 *for the definition of $\mathcal{H}om_{\mathrm{sym}}(A, A^*)$.)*

**7.8.** We prove Proposition 7.7. By Lemma 3.8, the part of the cubic isomorphism proved in 7.5 and the part of Proposition 2.1 proved in Proposition 6.2 (2), we have the exact sequence

$$0 \to \mathcal{E}xt(A, \mathbb{G}_{m,\log}) \to \mathcal{H}^1(A, \mathbb{G}_{m,\log}) \to \mathcal{B}iext_{\mathrm{sym}}(A, A; \mathbb{G}_{m,\log}).$$

By Corollary 6.20, the last group is naturally isomorphic to $\mathcal{H}om_{\mathrm{sym}}(A, A^*)$. Thus we obtain the lower row in the diagram.

The upper row is clearly exact and the right vertical homomorphism is defined by the definition of the group $\mathcal{H}om_{\langle\langle\,,\,\rangle\rangle}(Y, X)$. The commutativity of the diagram is by construction. The middle vertical isomorphism is by Theorem 7.4. The rest is to show that the right vertical homomorphism is an isomorphism. By Proposition 6.15 (2), we have an isomorphism

$$\mathcal{H}om_{\mathrm{adj}}(Y, X) \xrightarrow{\cong} \mathcal{H}om(A/G, A^*/G^*)$$

(see Proposition 6.15 for the definition of $\mathcal{H}om_{\mathrm{adj}}(Y, X)$). We can regard $\mathcal{H}om_{\mathrm{sym}}$ $(A, A^*)$ as a subsheaf of $\mathcal{H}om(A/G, A^*/G^*)$ because any homomorphism $A$ to the torus $G^*$ is trivial by [9] Theorem 7.4 (4). Since its corresponding subsheaf of $\mathcal{H}om_{\mathrm{adj}}(Y, X)$ is $\mathcal{H}om_{\langle\langle\,,\,\rangle\rangle}(Y, X)$, the right vertical homomorphism is an isomorphism.

**7.9.** We remark that the above Proposition 7.7 is similar to the classical theorem of Appell–Humbert (see Chapter I, Section 2 of the textbook [12] of Mumford) concerning the analytic presentation of the Picard group of a complex torus.

## 8. Case of constant degeneration, II

Here we prove all the results described in Sections 1–2 in the case of constant degeneration that are not proven so far.

Let the notation be as in the beginning of Section 6. For simplicity, unless otherwise stated, we assume in this section that $X$ and $Y$ are constant, not only locally constant, though the results in this section can be generalized to the case of locally constant $X$ and $Y$.

**8.1.** For $x \in X$, define a $\mathbb{G}_m$-torsor $E(x)$ on $B$ to be the pushout of $\mathbb{G}_m \overset{x}{\leftarrow} T \to G$ in the category of sheaves of abelian groups on $(\mathrm{fs}/S)_{\text{ét}}$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0 \\ & & {\scriptstyle \text{by } x}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & E(x) & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

For $x \in X$ and $y \in Y$, we define an isomorphism of $\mathbb{G}_{m,\log}$-torsors on $B$

$$\langle\langle x, y\rangle\rangle\colon E(x)_{\log} \overset{\simeq}{\to} t_y^*(E(x)_{\log})$$

as follows. Here $E(x)_{\log}$ denotes the $\mathbb{G}_{m,\log}$-torsor on $B$ obtained from $E(x)$, and $t_y^*$ denotes the pullback by the translation $t_y\colon B \to B$ by the image of $y$ in $B$ under the canonical homomorphism $Y \to B$.

Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{\log} & \longrightarrow & G_{\log} & \longrightarrow & B & \longrightarrow & 0 \\ & & {\scriptstyle \text{by } x}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{G}_{m,\log} & \longrightarrow & E(x)_{\log} & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

Note that $Y$ is embedded in $G_{\log}$. We have a commutative diagram

$$\begin{array}{ccc} E(x)_{\log} & \xrightarrow{\simeq} & E(x)_{\log} \\ \downarrow & & \downarrow \\ B & \underset{t_y}{\xrightarrow{\simeq}} & B, \end{array}$$

where the upper horizontal arrow is the translation by the image of $y$. This diagram defines the desired isomorphism $\langle\langle x, y\rangle\rangle\colon E(x)_{\log} \overset{\simeq}{\to} t_y^*(E(x)_{\log})$.

This isomorphism $\langle\langle x, y\rangle\rangle$ has the following properties 8.1.1–8.1.4.

**8.1.1.** $\langle\langle x, y\rangle\rangle \bmod \mathbb{G}_m = \langle x, y\rangle$.

Here, since the $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor obtained from $E(x)$ is trivial, the isomorphism of $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsors obtained from $\langle\langle x, y\rangle\rangle$ is regarded as a section of $\mathbb{G}_{m,\log}/\mathbb{G}_m$, and we denote this section of $\mathbb{G}_{m,\log}/\mathbb{G}_m$ by $\langle\langle x, y\rangle\rangle \bmod \mathbb{G}_m$.

**8.1.2.** $\langle\langle xx', y\rangle\rangle = \langle\langle x, y\rangle\rangle \cdot \langle\langle x', y\rangle\rangle$ for $x' \in X$.

Here the product on the right-hand-side is defined because the $\mathbb{G}_m$-torsor $E(xx')$ is the product of the $\mathbb{G}_m$-torsors $E(x)$ and $E(x')$.

**8.1.3.** $\langle\langle x, yy'\rangle\rangle = t_{y'}^*(\langle\langle x, y\rangle\rangle) \circ \langle\langle x, y'\rangle\rangle$ for $y' \in Y$.

**8.1.4.** Assume $B = \{1\} = S$ and identify $E(x)$ with $\mathbb{G}_m = \mathbb{G}_{m,S}$ in the natural way. Then $\langle\langle x, y\rangle\rangle$, regarded as a section of $\mathbb{G}_{m,\log}$, coincides with the image of $y$ under the composite $Y \to G_{\log} = T_{\log} \xrightarrow{x} \mathbb{G}_{m,\log}$.

**8.2.** Let $H$ be the sheaf of abelian groups on $(\mathrm{fs}/S)_{\text{ét}}$ associated to the presheaf classifying triples $(L, p, a)$ of a $\mathbb{G}_m$-torsor $L$ on $B$, a homomorphism $p\colon A \to A^*$, and an isomorphism

$$a(y)\colon L_{\log} \cdot E(p(y))_{\log} \xrightarrow{\cong} t_y^*(L_{\log})$$

of $\mathbb{G}_{m,\log}$-torsors on $B$ given for any $y \in Y$ that satisfy the following conditions 8.2.1 and 8.2.2. Here $L_{\log}$ denotes the $\mathbb{G}_{m,\log}$-torsor on $B$ associated to $L$ and $p(y)$ denotes the image of $y$ in $X$ under the homomorphism $Y \to X$ induced by $p$ (by abuse of notation).

**8.2.1.** The homomorphism $B \to B^*$; $u \mapsto t_u^*(L) \cdot L^{-1}$ coincides with the map induced by $p$.

This condition for $u = y \in Y$ shows that $L \cdot E(p(y)) \cong t_y^* L$ locally on the base $S$.

**8.2.2.** For any $y, z \in Y$, we have

$$a(yz) = t_z^*(a(y)) \circ a(z) \cdot \langle\langle p(y), z\rangle\rangle.$$

The meaning of the equation in 8.2.2 is as follows: $a(yz)$ is an isomorphism $L_{\log} \cdot E(p(yz))_{\log} \cong t_{yz}^*(L_{\log})$. On the other hand, by using

$$E(p(yz)) = E(p(y)) \cdot E(p(z)),$$

we have the composite isomorphism

$$L_{\log} \cdot E(p(yz))_{\log} \cong L_{\log} \cdot E(p(z))_{\log} \cdot E(p(y))_{\log} \cong t_z^*(L_{\log}) \cdot t_z^*(E(p(y))_{\log}) \cong t_{yz}^*(L_{\log}),$$

where the second isomorphism is given by $a(z) \cdot \langle\langle p(y), z\rangle\rangle$ and the last isomorphism is given by $t_z^*(a(y))$. The equation in 8.2.2 means that these two isomorphisms coincide.

**8.3.** We construct a homomorphism

$$H \to \mathcal{H}^1(A, \mathbb{G}_{m,\log}).$$

Let $g\colon \widetilde{A} \to B$ be the canonical morphism. Let $(L, p, a)$ be a triple satisfying 8.2.1 and 8.2.2. We show that we have a canonical action of $Y$ on $g^*(L_{\log})$ which is compatible with the action of $Y$ on $\widetilde{A}$ (by translation), and hence $g^*(L_{\log})$ descends to a $\mathbb{G}_{m,\log}$-torsor on $A = \widetilde{A}/Y$. This defines the desired homomorphism $H \to \mathcal{H}^1(A, \mathbb{G}_{m,\log})$.

For $x \in X$, the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{\log}^{(Y)} & \longrightarrow & \widetilde{A} & \longrightarrow & B & \longrightarrow & 0 \\ & & {\scriptstyle \text{by } x}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{G}_{m,\log} & \longrightarrow & E(x)_{\log} & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

shows:

**8.3.1.** The $\mathbb{G}_{m,\log}$-torsor $g^*(E(x)_{\log})$ on $\widetilde{A}$ is canonically trivial.

We have isomorphisms

$$g^*(L_{\log}) \cong g^*(L_{\log}) \cdot g^*(E(p(y))_{\log}) \cong t_y^* g^*(L_{\log})$$

of $\mathbb{G}_{m,\log}$-torsors on $\widetilde{A}$, where the first isomorphism is by 8.3.1 applied to $x = p(y)$, the second isomorphism is induced by $a(y)$, and $t_y$ denotes the translation by $y$ regarded as a section of $\widetilde{A}$. Thus we have an isomorphism for each $y \in Y$

**8.3.2.** $g^*(L_{\log}) \cong t_y^*(g^*(L_{\log}))$ for $y \in Y$.

As is easily checked, the condition 8.2.2 shows that the isomorphisms 8.3.2 for $y \in Y$ give an action of $Y$ on $g^*(L_{\log})$ which is compatible with the action of $Y$ on $\widetilde{A}$.

**8.4.** We have an exact sequence

**8.4.1.** $0 \to G_{\log}^* \to H \to \mathcal{H}om(A, A^*)$,

where the homomorphism $H \to \mathcal{H}om(A, A^*)$ is defined by $(L, p, a) \mapsto p$, and the homomorphism $G_{\log}^* \to H$ is defined as follows. For an exact sequence $0 \to \mathbb{G}_m \to L \to B \to 0$ and a homomorphism $s\colon Y \to L_{\log}$ such that the composite $Y \to L_{\log} \to B$ coincides with the canonical map $Y \to B$, the map $G_{\log}^* \to H$ sends the class of the pair $(L, s)$ to the class of the triple $(L, 1, a)$, where 1 denotes the trivial homomorphism and the isomorphism $a(y)\colon L_{\log} \xrightarrow{\simeq} t_y^*(L_{\log})$ is defined by the following commutative diagram:

$$\begin{array}{ccc} L_{\log} & \xrightarrow[s(y)]{\simeq} & L_{\log} \\ \downarrow & & \downarrow \\ B & \xrightarrow[t_y]{\simeq} & B. \end{array}$$

Here the upper horizontal arrow is the translation by $s(y)$.

The exactness of the sequence 8.4.1 is proved as follows. The problem is to show that a triple $(L, 1, a)$ in the kernel of $H \to \mathcal{H}om(A, A^*)$ comes from $G_{\log}^*$. By the classical exact sequence

$$0 \to \mathcal{E}xt(B, \mathbb{G}_m) \to \mathcal{H}^1(B, \mathbb{G}_m) \to \mathcal{H}om(B, B^*)$$

and 8.2.1, $L$ comes from $\mathcal{E}xt(B, \mathbb{G}_m)$ locally on $S$. For $y \in Y$, the restriction of the isomorphism $a(y)$ to the origin of $B$ induces a map

**8.4.2.** $\mathbb{G}_{m,\log,S} = L_{\log}|_{\{1\}} \xrightarrow{\cong} L_{\log}|_{\{y\}} \subset L_{\log}$.

Here $L_{\log}|_{\{y\}}$ (resp. $L_{\log}|_{\{1\}}$) denotes the $\mathbb{G}_{m,\log}$-torsor on $S$ obtained as the pull-back of $L_{\log}$ under $y\colon S \to B$ (resp. under the origin $S \to B$ of $B$), and $L|_{\{1\}}$ is identified with $\mathbb{G}_{m,\log,S}$ since $L$ comes from $\mathrm{Ext}(B, \mathbb{G}_m)$. Denote the image of $1 \in \mathbb{G}_{m,\log,S}$ in $L_{\log}$ under the map 8.4.2 by $s(y)$. It is easily seen that $(L, 1, a)$ comes from $(L, s) \in G^*_{\log}$.

**8.5 Theorem.** (1) *If $L$ is a $\mathbb{G}_{m,\log}$-torsor on $A$, the map*

$$A \to \mathcal{H}^1(A, \mathbb{G}_{m,\log});\ u \mapsto t_u^*(L) \cdot L^{-1}$$

*is a homomorphism, and the image of this map is contained in $A^*$. Note that*

$$A^* \subset \mathcal{E}xt(A, \mathbb{G}_{m,\log}) \subset \mathcal{H}^1(A, \mathbb{G}_{m,\log}).$$

(2) *We have a commutative diagram with exact rows*

$$\begin{array}{ccccccc} 0 & \longrightarrow & G^*_{\log}/X & \longrightarrow & H/X & \longrightarrow & \mathcal{H}om_{\mathrm{sym}}(A, A^*) \\ & & \downarrow\wr & & \downarrow\wr & & \| \\ 0 & \longrightarrow & \mathcal{E}xt(A, \mathbb{G}_{m,\log}) & \longrightarrow & \mathcal{H}^1(A, \mathbb{G}_{m,\log}) & \longrightarrow & \mathcal{H}om_{\mathrm{sym}}(A, A^*) \end{array}$$

*in which the vertical arrows are isomorphisms. Here the last arrow in the lower row sends $L \in \mathcal{H}^1(A, \mathbb{G}_{m,\log})$ to $A \to A^*$; $u \mapsto t_u^*(L) \cdot L^{-1}$. (Cf. Corollary* 6.20 *for the definition of $\mathcal{H}om_{\mathrm{sym}}(A, A^*)$.)*

**8.5.1 Remark.** Note that in the case of total degeneration, the left and the middle vertical arrows in the diagram in (2) reduce to $(-1)$-times of the corresponding ones in Proposition 7.7. See Remark 5.2.1 for the left one. As for the middle one, note that the $\mathbb{G}_{m,\log}$-torsor on $A$ associated to $(p, a) \in H$ in Section 7 coincides with the one associated to $(\mathbb{G}_m, -p, a^{-1}) \in H$ in this section, where $-p\colon A \to A^*$ is induced by $(-1)$-times of $p$ and $a^{-1}(y)\colon \mathbb{G}_{m,\log} \to \mathbb{G}_{m,\log}$ $(y \in Y)$ is the multiplication by $a(y)^{-1}$.

Before proving this theorem, we prove the following proposition.

**8.6 Proposition.** *We have a bijection to $\mathcal{H}^1(A, \mathbb{G}_{m,\log})$ from the sheaf of the isomorphism classes of a $\mathbb{G}_m$-torsor $L$ over $B$ paired with an action of $Y$ on the $\mathbb{G}_{m,\log}$-torsor $g^*(L_{\log})$ on $\widetilde{A}$ which is compatible with the action of $Y$ on $\widetilde{A}$. See* 8.2 *for $L_{\log}$.*

*Proof.* We prove the bijectivity by giving an inverse map. Let $\mathfrak{L}$ be a $\mathbb{G}_{m,\log}$-torsor on $A$. We construct strict étale locally on $S$, a $\mathbb{G}_m$-torsor $L$ and an action of $Y$ on $g^*(L_{\log})$. The exact sequence

$$0 \to H^1(B, \mathbb{G}_{m,\log}) \to H^1(\widetilde{A}, \mathbb{G}_{m,\log}) \to H^0(B, R^1g_*(\mathbb{G}_{m,\log}))$$

and Proposition 6.6 (1) prove

$$H^1(B, \mathbb{G}_{m,\log}) \xrightarrow{\cong} H^1(\widetilde{A}, \mathbb{G}_{m,\log}).$$

By

$$\mathcal{H}^0(B, \mathbb{G}_{m,\log}/\mathbb{G}_m) = \mathbb{G}_{m,\log}/\mathbb{G}_m,\ \mathcal{H}^1(B, \mathbb{G}_{m,\log}/\mathbb{G}_m) = \{1\},$$

we have

$$\mathcal{H}^1(B, \mathbb{G}_m) \xrightarrow{\cong} \mathcal{H}^1(\widetilde{A}, \mathbb{G}_{m,\log}).$$

This proves that $\mathfrak{L}$ comes strict étale locally on $S$, from a $\mathbb{G}_m$-torsor $L$ on $B$ such that $g^*(L_{\log})$ is endowed with an action of $Y$ which is compatible with the action of $Y$ on $\widetilde{A}$. □

**8.7.** We show that the homomorphism $H \to \mathcal{H}^1(A, \mathbb{G}_{m,\log})$ defined in 8.3 is surjective. Let $\mathfrak{L}$ be a $\mathbb{G}_{m,\log}$-torsor on $A$. We construct locally on $S$, a triple $(L, p, a)$ in $H$ which produces $\mathfrak{L}$.

First by Proposition 8.6, $\mathfrak{L}$ comes locally on $S$, from a $\mathbb{G}_m$-torsor $L$ on $B$ such that $g^*(L_{\log})$ is endowed with an action of $Y$ which is compatible with the action of $Y$ on $\widetilde{A}$.

We next define a homomorphism $\psi\colon Y \to X$ as a preparation for the definitions of $p$ and $a$. Since the $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $\widetilde{A}$ obtained from $g^*(L_{\log})$ is trivial, the action of $y \in Y$ on $g^*(L_{\log})$ defines an element of the group

$$H^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = X \oplus H^0(B, \mathbb{G}_{m,\log}/\mathbb{G}_m) \quad (6.4.2).$$

Define $\psi(y) \in X$ to be the first projection of this element.

We define an isomorphism

$$a(y)\colon L_{\log} \cdot E(\psi(y))_{\log} \xrightarrow{\cong} t_y^*(L_{\log})$$

for $y \in Y$. Recall that for $x \in X$, $g^*(E(x)_{\log})$ has a canonical section (8.3.1). Since the $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $\widetilde{A}$ obtained from $g^*(E(x))$ is trivial, the canonical section of $g^*(E(x)_{\log})$ induces an element of $H^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m) = X \oplus H^0(B, \mathbb{G}_{m,\log}/\mathbb{G}_m)$, and it is seen easily that this element coincides with $(x, 1)$. Consider the isomorphism of $\mathbb{G}_{m,\log}$-torsors on $\widetilde{A}$

**8.7.1.** $g^*(E(\psi(y))_{\log}) \xrightarrow{\cong} t_y^* g^*(L_{\log}) \cdot g^*(L_{\log})^{-1}$

which sends the canonical section of $g^*(E(\psi(y))_{\log})$ to the section of $t_y^*g^*(L_{\log}) \cdot g^*(L_{\log})^{-1}$ defined by the action of $Y$ on $g^*(L_{\log})$. Since the canonical section of $g^*(E(\psi(y))_{\log})$ and the above section of $t_y^*g^*(L_{\log}) \cdot g^*(L_{\log})^{-1}$ induce the same element $\psi(y)$ of $X = H^0(\widetilde{A}, \mathbb{G}_{m,\log}/\mathbb{G}_m)/H^0(B, \mathbb{G}_{m,\log}/\mathbb{G}_m)$, the isomorphism 8.7.1 comes from an isomorphism

$$E(\psi(y))_{\log} \xrightarrow{\cong} t_y^*(L_{\log}) \cdot L_{\log}^{-1}$$

on $B$ which is determined by 8.7.1 uniquely. This gives the desired $a(y)$.

Next we define a homomorphism $p\colon \widetilde{A} \to G_{\log}^*$. Let $u$ be a section of $\widetilde{A}$ given on an fs log scheme $U$ over $S$. We define a section $p(u)$ of $G_{\log}^*$ on $U$. By replacing $U$ by $S$, we consider the case $U = S$. Locally on $S$, the $\mathbb{G}_m$-torsor $t_u^*(L) \cdot L^{-1}$ on $B$ comes from $\mathrm{Ext}(B, \mathbb{G}_m)$. We assume that $t_u^*(L) \cdot L^{-1}$ comes from $\mathrm{Ext}(B, \mathbb{G}_m)$. For $y \in Y$, the action of $Y$ on $g^*(L_{\log})$ induces an isomorphism $g^*(L_{\log}) \xrightarrow{\cong} t_y^*g^*(L_{\log})$ and it induces

$$t_u^*g^*(L_{\log}) \cdot g^*(L_{\log})^{-1} \xrightarrow{\cong} t_u^*t_y^*g^*(L_{\log}) \cdot t_y^*g^*(L_{\log})^{-1}.$$

By taking the pullback of this isomorphism by the origin $S \to \widetilde{A}$ of $\widetilde{A}$, we have an isomorphism

$$(t_u^*(L_{\log}) \cdot L_{\log}^{-1})|_{\{1\}} \xrightarrow{\cong} (t_u^*(L_{\log}) \cdot L_{\log}^{-1})|_{\{y\}}.$$

Since $t_u^*(L) \cdot L^{-1}$ comes from $\mathrm{Ext}(B, \mathbb{G}_m)$, the left-hand-side is the canonically trivial $\mathbb{G}_{m,\log}$-torsor on $S$, and hence we obtain a section $s(y)$ of the right-hand-side. This gives a homomorphism $s\colon Y \to (t_u^*(L) \cdot L^{-1})_{\log}$ such that the composite $Y \to (t_u^*(L) \cdot L^{-1})_{\log} \to B$ coincides with the canonical map $Y \to B$. We define $p(u)$ to be the section $(t_u^*(L) \cdot L^{-1}, s)$ of $G_{\log}^*$.

Since $\mathrm{Hom}(\widetilde{A}, G_{\log}^*) \cong \mathrm{Hom}(G, G^*)$ ([9] Theorem 7.3 (3)), the image of the homomorphism $p\colon \widetilde{A} \to G_{\log}^*$ is contained in $(G_{\log}^*)^{(X)} = \widetilde{A^*}$. It can be seen that the following diagram is commutative.

$$\begin{array}{ccc} Y & \longrightarrow & \widetilde{A} \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \longrightarrow & \widetilde{A^*}. \end{array}$$

Hence $p$ induces a homomorphism $A \to A^*$, and $a(y)$ for $y \in Y$ is regarded as an isomorphism $L_{\log} \cdot E(p(y))_{\log} \xrightarrow{\cong} t_y^*(L_{\log})$.

We can check that the conditions 8.2.1 and 8.2.2 so that the triple $(L, p, a)$ is a section of $H$, and it is easily seen that $\mathfrak{L} \in \mathcal{H}^1(A, \mathbb{G}_{m,\log})$ comes from $(L, p, a)$.

**8.8.** We prove Theorem 8.5.

First we prove Theorem 8.5 (1). Consider the diagram

$$\begin{array}{ccc} H & \longrightarrow & \mathcal{H}om(A, A^*) \\ \big\downarrow & & \big\downarrow \cap \\ \mathcal{H}^1(A, \mathbb{G}_{m,\log}) & \longrightarrow & \mathcal{M}or(A, \mathcal{H}^1(A, \mathbb{G}_{m,\log})), \end{array}$$

where $\mathcal{M}or$ means the sheaf of morphisms (which are not necessarily homomorphisms of sheaves of groups) and the lower horizontal arrow is defined by sending the class of a $\mathbb{G}_{m,\log}$-torsor $L$ on $A$ to the morphism $u \mapsto t_u^*(L) \cdot L^{-1}$. It is easily seen that this diagram is commutative. Since the left vertical arrow is surjective as we have already shown, this diagram shows that $u \mapsto t_u^*(L) \cdot L^{-1}$ is in fact a homomorphism from $A$ to $A^*$. This proves Theorem 8.5 (1).

(2) We prove that the surjective homomorphism $H \to \mathcal{H}^1(A, \mathbb{G}_{m,\log})$ induces an isomorphism $H/X \xrightarrow{\cong} \mathcal{H}^1(A, \mathbb{G}_{m,\log})$. It is easy to see that the diagram

$$\begin{array}{ccc} G^*_{\log} & \longrightarrow & H \\ \big\downarrow & & \big\downarrow \\ \mathcal{E}xt(A, \mathbb{G}_{m,\log}) & \longrightarrow & \mathcal{H}^1(A, \mathbb{G}_{m,\log}). \end{array}$$

is commutative, where the left vertical arrow is a surjection defined by [9] Theorem 7.4 (3). By this and the commutativity of the diagram

$$\begin{array}{ccc} H & \longrightarrow & \mathcal{H}om(A, A^*) \\ \big\downarrow & & \big\| \\ \mathcal{H}^1(A, \mathbb{G}_{m,\log}) & \longrightarrow & \mathcal{H}om(A, A^*), \end{array}$$

and by the exactness of the sequence 8.4.1, the kernel of $H \to \mathcal{H}^1(A, \mathbb{G}_{m,\log})$ comes from $G^*_{\log}$ and hence coincides with $\mathrm{Ker}(G^*_{\log} \to \mathcal{E}xt(A, \mathbb{G}_{m,\log})) = X$ ([9] Theorem 7.4 (3)).

We have shown that the lower row in Theorem 8.5 (2) without "sym" is isomorphic to the upper row in Theorem 8.5 (2) without "sym". Hence the exactness of the upper row in Theorem 8.5 (2) without "sym" shows that the lower row in Theorem 8.5 (2) without "sym" is exact. By the argument in the proof of 3.6 (2), the image of the last map in the lower row is contained in the symmetric part. Hence the same holds for the upper row. This completes the proof of Theorem 8.5 (2).

**8.9.** Proof of the cubic isomorphism (Theorem 2.2) for the case of constant degeneration. As in 6.16, it suffices to show only (2) of Theorem 2.2. First, we already proved the case $F = \mathbb{G}_{m,\log}/\mathbb{G}_m$ in 6.16. Next, (1) of Theorem 8.5

proves (2) of Theorem 2.2 for $F = \mathbb{G}_{m,\log}$. By Proposition 6.2, the canonical homomorphism $\mathcal{H}^1(A, \mathbb{G}_m) \to \mathcal{H}^1(A, \mathbb{G}_{m,\log})$ is injective. Hence, (2) of Theorem 2.2 for $F = \mathbb{G}_m$ reduces to that for $F = \mathbb{G}_{m,\log}$.

By Lemma 3.8, we prove Proposition 2.4 in the case of constant degeneration.

The next is a complement of Proposition 2.4 in the case of constant degeneration. The proof is straightforward.

**8.10 Proposition.** *The composite*

$$\mathcal{H}om_{\mathrm{sym}}(A, A^*) \cong \mathcal{B}iext_{\mathrm{sym}}(A, A; \mathbb{G}_{m,\log}) \to \mathcal{H}^1(A, \mathbb{G}_{m,\log}) \cong H/X$$

*(where the first isomorphism is by Corollary* 6.20 *and the next arrow is the pullback to the diagonal) is described as* $p \mapsto$ *the class of* $(L_{p_B}, p^2, a)$*, where* $L_{p_B}$ *and* $a$ *are defined as in* 8.11 *and* 8.12 *below. In particular, the cokernel of* $\mathcal{H}^1(A, \mathbb{G}_{m,\log}) \to \mathcal{H}om_{\mathrm{sym}}(A, A^*)$ *is killed by* 2*. (Cf. Corollary* 6.20 *for the definition of* $\mathcal{H}om_{\mathrm{sym}}(A, A^*)$*.)*

**8.11.** The definition of $L_{p_B}$ in Proposition 8.10 is as follows. Let $p_B \colon B \to B^*$ be the homomorphism induced by $p$. Regard $p_B$ as a biextension of $B \times B$ by $\mathbb{G}_m$. By 1.5, we get a $\mathbb{G}_m$-torsor $L_{p_B}$ on $B$.

**8.12.** The definition of $a$ in Proposition 8.10 is as follows. For simplicity, write $L_{p_B}$ as $L$. The commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & B \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p_B} \\ X & \longrightarrow & B^* \end{array}$$

gives for any $y$ of $Y$ the isomorphism

$$E(p(y)) \cdot t_y^*(E(p(y))) \xrightarrow{\cong} t_y^*(L) \cdot L^{-1},$$

which induces

$$t_y^*(E(p(y))_{\log}) \xrightarrow{\cong} E(p(y))_{\log}^{-1} \cdot t_y^*(L_{\log}) \cdot L_{\log}^{-1}.$$

By composing it with

$$\langle\langle p(y), y \rangle\rangle \colon E(p(y))_{\log} \xrightarrow{\cong} t_y^*(E(p(y))_{\log}),$$

we obtain an isomorphism

$$a(y) \colon L_{\log} \cdot E(p(y))_{\log}^2 \xrightarrow{\cong} t_y^*(L_{\log}).$$

**8.13.** In the rest of this section, we prove Theorem 1.11 in the case of constant degeneration. Let the notation be as in there (cf. Remark 5.2.1). Let $\widetilde{A}^{(\psi)}$ be the inverse image of $A^{(\psi)}$ by $\widetilde{A} \to A$.

First we explain the case of constant total degeneration (cf. Section 7), that is, the case where $A$ is with constant degeneration and the abelian part is trivial. Let $\langle\langle\ ,\ \rangle\rangle : X \times Y \to \mathbb{G}_{m,\log}$ be the canonical pairing.

Then $\widetilde{A}^{(\psi)}$ is $\mathrm{Proj}(R)^{\mathrm{sat}}$ for the following graded ring $R$ over $\mathcal{O}_S$ with an fs log structure as follows. Here $(-)^{\mathrm{sat}}$ is the saturation.

Let $E$ be the subring of sheaves of $\mathcal{O}_S[X \times M_S, \theta]$, where $\theta$ is an indeterminate, generated over $\mathcal{O}_S[M_S]$ by local sections of the form $s(y) := \psi(y)^2 \otimes \langle\langle \psi(y), y\rangle\rangle\theta$ ($y \in Y$). Let $R = E \otimes_{\mathcal{O}_S[M_S]} \mathcal{O}_S$. Then $R$ is quasi-coherent as an $\mathcal{O}_S$-module. The log structure is as follows. Let $y \in Y$ and let $E_{(s(y))}$ be the part of $E_{s(y)}$ of degree zero. We endow $\mathrm{Spec}(E_{(s(y))})$ with a fine log structure defined as a subsheaf of $X \times M_S^{\mathrm{gp}}$ generated by $\frac{s(z)}{s(y)}$ ($z \in Y$). This glues into a fine log structure on $\mathrm{Proj}(E)$. We endow $\mathrm{Proj}(R)$ with the pullback log structure.

By definition, the pullback to $\widetilde{A}^{(\psi)}$ of the special invertible sheaf on $A^{(\psi)}$, defined in 1.10, is $\mathcal{O}(1)$. In this case, the relative ampleness over $S$ of the special invertible sheaf on $A^{(\psi)}$ follows from this, for example, by Nakai's criterion, which completes the proof of Theorem 1.11 in the case of constant total degeneration.

**8.14.** Consider the case of constant degeneration in general.

In this case, over $B$, $\widetilde{A}^{(\psi)}$ is $\mathrm{Proj}(R)^{\mathrm{sat}}$ for the following graded ring $R$ over $\mathcal{O}_B$ with an fs log structure as follows.

The morphism $G \to B$ is affine, and we can write $G = \mathrm{Spec}(\bigoplus_{x\in X} \mathcal{L}(x))$ over $B$. Here $\mathcal{L}(x)$ is an invertible $\mathcal{O}_B$-module defined as the part on which $T$ acts via $x$ (we have $\mathcal{L}(x) \otimes \mathcal{L}(x') = \mathcal{L}(xx')$). Let $E$ be the subring of sheaves of $(\bigoplus_{x\in X} \mathcal{L}(x)) \otimes_{\mathcal{O}_B} \mathcal{O}_B[M_S, \theta]$ on $B$, where $\theta$ is an indeterminate, generated over $\mathcal{O}_B[M_S]$ by local sections of the form $b \otimes a\theta$, where $a \in M_S$ such that the class of $a$ in $M_S/\mathcal{O}_S^\times$ coincides with $\langle \psi(y), y\rangle$ and $b$ belongs to $\mathcal{L}(\psi(y)^2)$ ($y \in Y$). Let $R = E \otimes_{\mathcal{O}_B[M_S]} \mathcal{O}_B$. Then $R$ is quasi-coherent as an $\mathcal{O}_B$-module. The log structure is as follows. Let $y \in Y$. Let $b \otimes a\theta$ be a section of $E$, where $a \in M_S$ such that the class of $a$ in $M_S/\mathcal{O}_S^\times$ coincides with $\langle \psi(y), y\rangle$ and $b$ is a local generator of $\mathcal{L}(\psi(y)^2)$. Let $E_{(b\otimes a\theta)}$ be the part of $E_{b\otimes a\theta}$ of degree zero. We endow $\mathrm{Spec}(E_{(b\otimes a\theta)})$ with a fine log structure defined as a submonoid sheaf of the structure ring generated by $\frac{b'\otimes a'\theta}{b\otimes a\theta}$, where $a' \in M_S$ such that the class of $a'$ in $M_S/\mathcal{O}_S^\times$ coincides with $\langle \psi(z), z\rangle$ and $b'$ is a local generator of $\mathcal{L}(\psi(z)^2)$ ($z \in Y$). This glues into a fine log structure on $\mathrm{Proj}(E)$. We endow $\mathrm{Proj}(R)$ with the pullback log structure.

The pullback to $\widetilde{A}^{(\psi)}$ of the special invertible sheaf on $A^{(\psi)}$ defined in 1.10 is $\mathcal{O}(1) \otimes L_{p_B}$. Then the relative ampleness over $S$ of the pullback is by the relative ampleness over $S$ of $L_{p_B}$ and the relative ampleness over $B$ of $\mathcal{O}(1)$. The relative ampleness over $S$ of the special invertible sheaf on $A^{(\psi)}$ follows from this by Nakai's criterion.

**8.15.** In the above, assume that there are $\mathcal{S}, \langle\ ,\ \rangle\colon X \times Y \to \mathcal{S}^{\mathrm{gp}}$, $\mathcal{S} \to M_S/\mathcal{O}_S^\times$, and $\phi = \psi$ as in 1.7–1.9 (cf. 4.5). Then, $E$ is the subring of sheaves of $(\bigoplus_{x\in X} \mathcal{L}(x)) \otimes_{\mathcal{O}_B} \mathcal{O}_B[\mathcal{S}, \theta]$ on $B$ generated over $\mathcal{O}_B[\mathcal{S}]$ by local sections of the form $b \otimes \langle \phi(y), y\rangle\theta$, where $b$ belongs to $\mathcal{L}(\phi(y)^2)$ ($y \in Y$). We have $R = E \otimes_{\mathcal{O}_B[\mathcal{S}]} \mathcal{O}_B$. The log structure is described as follows. Let $y \in Y$. Let $E'$ be the subring of $\bigoplus_{x\in X} \mathcal{L}(x) \otimes \mathbb{Z}[\mathcal{S}^{\mathrm{gp}}]$ generated by $\mathcal{L}(\phi(zy^{-1})^2) \otimes \langle \phi(z), z\rangle\langle \phi(y), y\rangle^{-1}$ ($z \in Y$). The induced log structure on $\mathrm{Spec}(E')$ is the one defined as a submonoid sheaf of the structure ring generated by $b \otimes \langle \phi(z), z\rangle\langle \phi(y), y\rangle^{-1}$ ($z \in Y$), where $b$ is a local generator of $\mathcal{L}(\phi(zy^{-1})^2)$.

**8.16.** For the other special fans introduced in Section 4, the proof of the analogue of Theorem 1.11 is parallel. Note that, for a general fan $\Sigma$ in $C$ stable under the action of $Y$, $\widetilde{A}^{(\sigma)}$ in $\widetilde{A}^{(\Sigma)}$ is $\mathrm{Spec}(E' \otimes_{\mathcal{O}_B[\mathcal{S}]} \mathcal{O}_B)$, where $E'$ is the subring of $\bigoplus_{x\in X} \mathcal{L}(x) \otimes \mathbb{Z}[\mathcal{S}^{\mathrm{gp}}]$ generated by $\mathcal{L}(x) \otimes s$ ($s \in \mathcal{S}^{\mathrm{gp}}$) such that $N(s) + l(x) \geq 0$ for all $(N, l) \in \sigma$. The induced log structure on $\mathrm{Spec}(E')$ is the one defined as a submonoid sheaf of the structure ring generated by $b \otimes s$ ($s \in \mathcal{S}^{\mathrm{gp}}$), where $b$ is a local generator of $\mathcal{L}(x)$ for some $x \in X$ satisfying $N(s) + l(x) \geq 0$ for all $(N, l) \in \sigma$.

The translation by $y \in Y$ sends $\mathcal{L}(x)$ to $\mathcal{L}(x)\langle x, y\rangle$ ($x \in X$).

**8.17 Remark.** We continue to assume that $A$ is with constant degeneration. We can give an alternative proof of this constant degeneration case of Theorem 1.11 by constructing the sections of $L$ by theta series (or theta function) in the same way as in [8] 5.4.

## 9. Projectivity of special models

In this section, we prove the following theorem on the existence of a projective model of a polarized log abelian variety $A$. This implies the main result Theorem 1.11 of this paper under the assumption that a polarization on $A$ corresponds to a symmetric homomorphism from $\overline{Y}$ to $\overline{X}$ of $A$, which is guaranteed by Proposition 1.6. In the last section, we will prove this Proposition 1.6, which completes the proof of Theorem 1.11 (cf. 12.14).

**9.1 Theorem.** *Let the notation and the assumption be as in* 4.5. *Assume that the image of $\psi$ in* $\mathrm{Biext}_{\mathrm{sym}}(A/G, A/G; \mathbb{G}_{m,\log}/\mathbb{G}_m)$ *by the homomorphism in* 2.7 *lifts to a polarization on $A$. Let $\Sigma$ be either the first standard fan, or the second standard fan or the fan associated to a star in $X$* (4.6)*. Then $A^{(\Sigma)}$ is represented* *by an fs log scheme over $S$ whose underlying scheme is projective over $S$.*

To show this, we use the following proposition.

**9.2 Proposition.** *Let $f\colon X \to S$ be a proper algebraic space over a locally noetherian scheme $S$. Let $F$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, for $s \in S$ and for any $q$, $(R^q f_* F)_s$ is an $\mathcal{O}_{S,s}$-module of finite type and its completion is naturally isomorphic to*

$$\varprojlim_i H^q(f^{-1}(s), F \otimes_{\mathcal{O}_S} (\mathcal{O}_{S,s}/m^i_{S,s})),$$

*where $m_{S,s}$ is the maximal ideal of $\mathcal{O}_{S,s}$.*

The proof of this proposition is parallel to the case of schemes ([6] Proposition (4.2.1); see [15] Theorem 57.20.5).

**9.3.** We prove Theorem 9.1. We may assume that the base is noetherian. By [11] Theorem 8.1 and [11] Theorem 17.1, we already know that $A^{(\Sigma)}$ is a proper fs log algebraic space over $S$.

Let $f\colon A^{(\Sigma)} \to S$ be the structure morphism. Let $L$ be the invertible sheaf on $A^{(\Sigma)}$ consisting of the sections of the special $\mathbb{G}_m$-torsor on $A^{(\Sigma)}$ in 5.3. It is enough to show that $L$ is relatively ample over $S$. This is equivalent to that, for any coherent sheaf $F$ on $A^{(\Sigma)}$, there is an integer $n_0$ such that for any $q > 0$ and any $n > n_0$, $R^q f_*(F \otimes L^{\otimes n}) = 0$ (cf. [6] Proposition (2.6.1)). Hence, it is reduced by Proposition 9.2 to the case where $M_S/\mathcal{O}_S^\times$ is locally constant. Thus we may assume that we are in the case of constant degeneration. Under this assumption, the relative ampleness of $L$ is by 8.14 in the case of the first standard fan and the other two cases are similar.

## 10. Wide fans

In this section, we introduce the wideness of fans. The model associated to a wide fan can recover the original weak log abelian variety (cf. Section 11).

**10.1.** Let the situation be as in 1.7. Let $\Sigma$ be a fan which is stable under the action of $Y$. We say that $\Sigma$ is *wide* if the following condition holds:

There is a $\sigma \in \Sigma$ such that : If $(N, \ell) \in C$, we have $(N, \varepsilon\ell) \in \sigma$ for any $\varepsilon \in \mathbb{Q}$ such that $|\varepsilon|$ is sufficiently small.

At present, we cannot show that any weak log abelian variety has a wide fan, and so at present, we cannot recover a weak log abelian variety from a proper model in general.

On the other hand, as is shown below, a weak log abelian variety satisfying the condition 1.4.1 has a complete wide fan.

**10.2 Lemma.** *Assume that $\Sigma$ is wide. Let $\ell$ be a prime number.*

(1) $\mathcal{Q} = \bigcup_{n\geq 0} \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\ell^n\Sigma)}$. *Here the fan $\ell^n\Sigma$ is the set of cones $\{\ell^n\tau \mid \tau \in \Sigma\}$ (cf.* [11] 2.2*).*

(2) *If $\ell$ is invertible on the base $S$, then $\widetilde{A} = \bigcup \ell^n \widetilde{A}^{(\Sigma)}$ as két sheaves.*

(3) *If $\ell$ is invertible on the base $S$, then $A = \bigcup \ell^n A^{(\Sigma)}$ as két sheaves.*

*Proof.* First note that the sheaves $\widetilde{A}$ and $A$ are két sheaves by [11] Theorem 5.1. From this, we see that their $\Sigma$-parts are also két sheaves.

(1) Let $\sigma$ be as in the definition of the wideness. Then $C = \bigcup \ell^n\sigma$, and we have the desired equality.

(2) By (1), $\widetilde{A} = \bigcup \widetilde{A}^{(\ell^n\Sigma)}$. Since the homomorphism $\ell\colon \widetilde{A}^{(\Sigma)} \to \widetilde{A}^{(\ell\Sigma)}$ of két sheaves is surjective (cf. [11] 18.6, 18.10, Lemma 18.10.11), we have $\widetilde{A}^{(\ell^n\Sigma)} = \ell^n\widetilde{A}^{(\Sigma)}$.

(3) is by (2). □

**10.3 Proposition.** *Let $A$ be a weak log abelian variety satisfying the condition* 1.4.1*. Assume that there are data including a homomorphism $\phi\colon Y \to X$ as in* 1.7–1.9 *(they always exist étale locally (cf.* 4.5*)). Then both the first and the second standard fans are wide.*

*Proof.* We consider the case of the first standard fan. The second is similar.

It is sufficient to prove that for $(N, \ell) \in C$, if $|\varepsilon|$ is sufficiently small, then $(N, \varepsilon\ell)$ belongs to the $\mathbb{Q}_{\geq 0}$-span of $C(0) = \{(N, \ell) \mid N(\langle\phi(y), y\rangle) + \ell(\phi(y)) \geq 0$ for all $y \in Y\}$ (that is, $N(\langle\phi(y), y\rangle) + \varepsilon\ell(\phi(y)) \geq 0$ for all $y \in Y$). By replacing $Y$ by $Y/Y_\sigma$, where $\sigma$ is the face of $\mathrm{Hom}(\mathcal{S}, \mathbb{N})$ such that $N$ is in the interior of $\sigma$, we may assume that $N(\langle\phi(-), -\rangle)$ is positive-definite. Thus the statement to see is that if we have a positive-definite quadratic form $Q$ on $\mathbb{R}^n$ and a linear form $h : \mathbb{R}^n \to \mathbb{R}$, then if $\varepsilon > 0$ is sufficiently small, we have $Q(y) \geq \varepsilon h(y)$ for any $y \in \mathbb{Z}^n$. We can assume $Q(t_1, \dots, t_n) = t_1^2 + \cdots + t_n^2$. Let $h(t_1, \dots, t_n) = \sum_i 2a_it_i$. Then $Q(t) - \varepsilon h(t) = \sum_i (t_i - \varepsilon a_i)^2 - \varepsilon^2 \sum_i a_i^2$. If $\varepsilon$ is such that $|\varepsilon a_i| \leq 1/2$ for any $i$, the minimum of $\sum_i (t_i - \varepsilon a_i)^2$ is given in the case $t_i = 0$ for any $i$, and in the case $t = 0$, $Q(t) - \varepsilon h(t)$ becomes $Q(0) - \varepsilon h(0) = 0$. □

## 11. Presentation of a weak log abelian variety by proper models as a sheaf for the étale topology

Let $A$ be a weak log abelian variety over an fs log scheme $S$ satisfying the condition 1.4.1. Let $\overline{X}$ and $\overline{Y}$ be those associated to $A$.

The aim of this section is to prove the following proposition, which roughly says that $A$ can be covered by proper models and which will be used in the next section.

**11.1 Proposition.** *There are fs log schemes $S'$ and $P$ over $S$ and morphisms $P \to S'$ and $P \to A$ over $S$ having the following properties.*

(i) *$S' \to S$ is strict, étale and surjective.*

(ii) *As a morphism of sheaves on* (fs/$S$)*, $P \to A$ is surjective.*

(iii) *$P$ is proper over $S'$.*

(iv) *$P \times_A P$ is represented by an fs log scheme over $S$, and the morphism $P \times_A P \to S' \times_S S'$ is proper.*

Assume that we are given $X \times Y \to \mathcal{S}^{\mathrm{gp}}$ and $\mathcal{S} \to M_S/\mathcal{O}_S^\times$ as in 1.7.

**11.2 Lemma.** *Let $\ell$ be a prime number which is invertible on the base $S$. Then the following holds étale locally on $S$. There are schemes $S_n$ over $S$ for $n \geq 0$ with $S_0 = S$, strict étale surjective morphisms $S_{n+1} \to S_n$ for $n \geq 0$, and subgroup sheaves $H_n \subset G[\ell^n] \times_S S_n$ and finite subsets $Y_n \subset Y$ for $n \geq 0$, satisfying the following conditions.*

(i) *$H_n$ and the quotients $((A[\ell^n] \times_S S_n)/H_n)_{\text{két}}$ as két sheaves are represented by finite flat group schemes (which are strict) over $S_n$. Below, this quotient sheaf on $S_n$ is denoted simply by $A[\ell^n]/H_n$ by abuse of notation.*

(ii) *Any fiber of $H_n$ contains $T[\ell^n] \times_S S_n$ in the fiber of $G[\ell^n] \times_S S_n$, where $T$ is the torus part of the fiber of $G$.*

(iii) *$\ell H_{n+1} \subset H_n \times_{S_n} S_{n+1}$.*

(iv) *There is an isomorphism of sheaves $Y_n \cong A[\ell^n]/H_n$ for which the composite $Y_n \cong A[\ell^n]/H_n \to \overline{Y}/\ell^n\overline{Y}$ is the canonical projection.*

*Proof.* When $Y$ is trivial, $S_n = S$, $H_n = G[\ell^n]$ and $Y_n = Y$ satisfy the desired conditions. Hence we may and will assume that $Y$ is not trivial. We proceed inductively. Assume that there are $S_i, H_i$, and $Y_i$ for $i \leq n$ satisfying the conditions. Let $s \in S_n$ and we work around $s$. Consider the két stalk $(H_n)_{\bar{s}(\text{két})}$ of $H_n$ at $s$. By [11] Proposition 18.1 (3), $A[\ell^{n+1}]$ is két locally constant. Let $H'_{n+1}$ be the két locally constant subsheaf of $A[\ell^{n+1}] \times_S S_n$ whose stalk coincides with the pullback of $(H_n)_{\bar{s}(\text{két})}$ by the stalk of $\ell \colon G[\ell^{n+1}] \times_S S_n \to G[\ell^n] \times_S S_n$. By [11] Proposition 18.11, the geometric log fundamental group at $s$ acts trivially both on the két stalk of $H'_{n+1}$ and the két stalk of the két quotient $(A[\ell^{n+1}] \times_S S_n)/H'_{n+1}$. Hence, there is a strict étale neighborhood $S_{n+1}$ of $s$ such that the pullback $H_{n+1} \subset A[\ell^{n+1}] \times_S S_{n+1}$ of $H'_{n+1}$ is constant and the két quotient is also constant. This $S_{n+1}$ and $H_{n+1}$ satisfy the conditions (i) for $n+1$, (ii) for $n+1$, and (iii). Finally, we can take a subset $Y_{n+1}$ of $Y$ satisfying the condition (iv) for $n+1$ because $Y$ is infinite. □

**11.3 Lemma.** *Let $H_n$ be as in Lemma* 11.2*. Let $A_n := (A/H_n)_{\text{két}}$ be the quotient as a két sheaf on $S_n$. Here and hereafter we denote the base change $A \times_S S_n$ simply by $A$ by abuse of notation. Then $A_n$ is a weak log abelian variety, and the map*

*$A_n \to A$ ; $x \mapsto \ell^n x$ is a surjection of sheaves for the étale topology.*

*Proof.* Let

$$G' = G/H_n, \quad \overline{X}' = \ell^n \overline{X}, \quad \overline{Y}' = \overline{Y}.$$

Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & G & \to & A & \to & \mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}/\overline{Y} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & G' & \to & A_n & \to & (\mathcal{H}om(\overline{X}', \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y}')} \otimes \mathbb{Q})/\overline{Y}' & & \end{array}$$

on $S_n$. Here $\otimes\mathbb{Q}$ appeared from the két localization.

The multiplication by $\ell^n : A \to A$ factors as $A \to A_n \to A$ uniquely. By this and by

$$\ell^n \colon (\mathcal{H}om(\overline{X}', \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y}')} \otimes \mathbb{Q})/\overline{Y}' \cong (\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})} \otimes \mathbb{Q})/\ell^n\overline{Y},$$

we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \to & G' & \to & A_n & \to & (\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})} \otimes \mathbb{Q})/\ell^n\overline{Y} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & G & \to & A & \to & (\mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})} \otimes \mathbb{Q})/\overline{Y}. \end{array}$$

From this, we have an exact sequence

$$0 \to G' \to A_n \to \mathcal{H}om(\overline{X}, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\overline{Y})}/\ell^n\overline{Y} \to 0.$$

By using this, we can see that $A_n$ is a weak log abelian variety over $S_n$. (The fact that $A_n$ is separated follows from the fact that $A_n \to A$ is, relatively, étale locally isomorphic to $Y_n$.) □

**11.4.** We prove Proposition 11.1 till 11.6.

We use $H_n$, $S_n$, and $A_n$ to construct $P$, $S'$ in 11.6.

Let $\Sigma$ be a wide complete fan (Proposition 10.3).

Let $H_n$, $S_n$, and $A_n$ be as in Lemma 11.2 and Lemma 11.3.

Let $P_n$ be the két sheaf associated to the quotient $(A^{(\Sigma)} \times_S S_n)/H_n$ as a sheaf for the két topology. Then $P_n$ is the model of $A_n$ corresponding to a fan $\Sigma_n$ defined as follows.

We understand the $A_n$-version of $(X, Y)$ of $A$ as $(X, \ell^n Y)$, not as $(\ell^n X, Y)$. Let $\Sigma_n = \{\sigma_n \mid \sigma \in \Sigma\}$, where $\sigma_n = \{(N, \ell^n f) \mid (N, f) \in \sigma\}$. Then $\Sigma_n$ is a complete fan for $A_n$. Hence $P_n$ is proper over $S_n$.

We have a surjection of étale sheaves $\coprod_{n \geq 0} P_n \to A$ on (fs/$S$), whose $n$-th part is $x \mapsto \ell^n x$.

**11.5.** The sheaf $P_m \times_A P_n$ is understood as follows. Assume $m \geq n$ (the case $m \leq n$ is similar).

Note that $A/H_m \times_A A/H_n \cong A/H_m \times A[\ell^n]/H_n$ ; $(a,b) \mapsto (a, \ell^{m-n}a - b)$ over $S_m \times_S S_n$, where the quotients are the két quotients.

Then we have $P_m \times_A P_n = \coprod_{y \in Y_n} P_{m,n,y} \times_S S_n$, where $P_{m,n,y}$ is a proper model of $A_m$ corresponding to the fan $\Sigma(m,n,y)$ defined as follows. We regard $(X, \ell^m Y)$ as the $A_m$-version of $(X,Y)$ of $A$.

The fan $\Sigma(m,n,y)$ is the one consisting of all faces of all cones

$$(\sigma,\tau)_{m,n,y} = \{(N,f) \in \sigma_m \mid (N,f_y) \in \tau_n\}$$

$(\sigma, \tau \in \Sigma)$, where $f_y$ is defined by $f_y(x) = f(x) + N(\langle x,y\rangle)$.

Hence $P_m \times_A P_n$ is proper over $S_m \times_S S_n$.

$P_m \times_A P_n$ is also described, by using quotients of sheaves for két topology, as $\coprod_{y\in Y_n} \{a \in A^{(\Sigma)}/H_m \mid \ell^{m-n}a + y \in A^{(\Sigma)}/H_n\}$.

**11.6.** Let $S' = \coprod_n S_n$, $P = \coprod_n P_n$. So, $P \times_A P = \coprod_{m,n} P_m \times_A P_n$. Then Proposition 11.1 holds for these $S'$ and $P$, which is seen by the above arguments. This completes the proof of Proposition 11.1.

**11.7 Example.** Let $A$ be the log Tate elliptic curve $\mathbb{G}_{m,\log}^{(q)}/q^{\mathbb{Z}}$ and let $\Sigma$ be the wide fan corresponding to the intervals $[q^{m-1/2}, q^{m+1/2}]$ $(m \in \mathbb{Z})$. Then, for any $n \geq 1$, $P_n$ is the model associated to the fan corresponding to the intervals $[q^{\ell^n(m-1/2)}, q^{\ell^n(m+1/2)}]$ $(m \in \mathbb{Z})$ of log Tate elliptic curve $\mathbb{G}_{m,\log}^{(q)}/q^{\ell^n\mathbb{Z}}$.

**11.8 Remark.** The above proof shows, in particular, that for a wide complete fan $\Sigma$, the morphism $A^{(\Sigma)} \to A$ is represented by proper log étale morphisms which are surjective (as a morphism of schemes, not as a morphism of sheaves). This justifies the sentence that "a log abelian variety is a proper object."

**11.9 Remark.** The group structure of $A$ is also recovered as follows from the model associated with a wide complete fan $\Sigma$. Hence the group object is recovered from data given by fs log schemes.

Let $X' = X \times X$ and $Y' = Y \times Y$. Take the admissible pairing $X' \times Y' \to \mathcal{S}$ induced by $X \times Y \to \mathcal{S}$ as the pairing for $A \times A$. Let $C' \subset \mathrm{Hom}(\mathcal{S},\mathbb{Z}) \times \mathrm{Hom}(X',\mathbb{Z}) = \mathrm{Hom}(\mathcal{S},\mathbb{Z}) \times \mathrm{Hom}(X,\mathbb{Z}) \times \mathrm{Hom}(X,\mathbb{Z})$ be the $A \times A$-version of the cone $C$ in 1.7 of $A$. Let $\Sigma'$ be the fan in $C'$ defined to be the set of all faces of the cones $(\sigma,\tau,\rho) := \{(N,\ell_1,\ell_2) \in C' \mid (N,\ell_1) \in \sigma, (N,\ell_2) \in \tau, (N, \ell_1+\ell_2) \in \rho\}$ given for each $\sigma,\tau,\rho \in \Sigma$. Then $\Sigma'$ is complete.

Let $P'_n$ be the quotient $(A \times A)^{(\Sigma')}/(H_n \times H_n)$ as a két sheaf. It is the model of $A_n \times A_n$ corresponding to the fan $\Sigma'$. We have a surjection of étale sheaves $\coprod_n P'_n \to A \times A$ whose $n$-th part is $x \mapsto \ell^n x$. We have $P'_n \to P_n$ induced by the group law $A_n \times A_n \to A_n$. The map $\coprod_n P'_n \to \coprod_n P_n$ induces the group law of $A$. The morphism between relation parts $\coprod_n P'_n \times_{A\times A} \coprod_n P'_n \to \coprod_n P_n \times_A \coprod_n P_n$

is given by using $P'_{m,n,(y_1,y_2)} \to P_{m,n,y_1+y_2}$ ; $(a,b) \mapsto a+b$, where $P'_{m,n,(y_1,y_2)}$ is the $(A \times A)$-version of $P_{m,n,y}$ in 11.5 for $(y_1, y_2)$. It commutes with the two projections. This is shown by the easily seen fact that the map $C' \to C$ sends $((\sigma_1, \tau_1, \rho_1), (\sigma_2, \tau_2, \rho_2))_{m,n,(y_1,y_2)}$ into $(\rho_1, \rho_2)_{y_1+y_2}$ ($\sigma_i, \tau_i, \rho_i \in \Sigma$ for $i = 1, 2$).

## 12. $\mathbb{G}_m$-torsors, $\mathbb{G}_{m,\log}$-torsors, and $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsors on weak log abelian varieties

In this section, we study $\mathbb{G}_m$-torsors, $\mathbb{G}_{m,\log}$-torsors, and $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsors on a weak log abelian variety. In particular, we prove the cubic isomorphism, and the results concerning such torsors stated in Section 2. Every torsor here is regarded with respect to the usual étale topology.

**12.1 Proposition.** *Let $A$ be a weak log abelian variety over an fs log scheme $S$ satisfying the condition* 1.4.1. *Let $F = \mathbb{G}_a, \mathbb{G}_m, \mathbb{G}_{m,\log}$, or $\mathbb{G}_{m,\log}/\mathbb{G}_m$. Then, we have $H^0(A, F) = H^0(S, F)$.*

*Proof.* By the zero section, we have a surjection $H^0(A, F) \to H^0(S, F)$. We prove that this surjection is injective. The case $F = \mathbb{G}_{m,\log}$ is reduced to the cases $F = \mathbb{G}_m$ and $F = \mathbb{G}_{m,\log}/\mathbb{G}_m$. But, since $H^0(S, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to \prod_{s\in S} H^0(\overline{s}, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ is injective, the latter case $F = \mathbb{G}_{m,\log}/\mathbb{G}_m$ is reduced to the case of constant degeneration (Proposition 6.2 (2)). Thus it suffices to treat the cases where $F = \mathbb{G}_a$ and $\mathbb{G}_m$. The case $F = \mathbb{G}_m$ is reduced to the case $F = \mathbb{G}_a$.

Let $f\colon A \to \mathbb{G}_a$ be a section. Assuming $f = 0$ on the zero section, we prove $f = 0$ on the whole $A$.

Since the problem is étale local on the base, using the condition 1.4.1, we may and will assume that there are data including a homomorphism $\phi\colon Y \to X$ as in 1.7–1.9. Then, we can take a complete wide fan $\Sigma$ (Proposition 10.3). Let $P$ be the associated proper model. Then $P$ contains $G$ as a sheaf. Since $P$ is proper, $\mathcal{H}^0(P, \mathbb{G}_a)$ is coherent as an $\mathcal{O}_S$-module. Hence $f|_G$ is contained in a subring $R$ of $\mathcal{H}^0(G, \mathbb{G}_a)$ over $\mathcal{O}_S$ such that $\operatorname{Spec}(R) \to S$ is finite. Since $G \to S$ is smooth and geometrically connected, such an $R$ coincides with $\mathcal{O}_S$. Hence $f|_G$ is 0. Further, let $p + G$ be any $G$-orbit in $A$, where $p\colon T \to A$ ($T \in (\mathrm{fs}/S)$) is a section of $A$ and $+$ means the addition of $A$. Then $p + P \supset p + G$. Since $p + G$ is isomorphic to $G$ and $p + P$ is isomorphic to $P$, similarly as above, $f|_{p+G}$ is constant. Thus $f$ induces a morphism $A/G \to \mathbb{G}_a$. Let $\widetilde{f}\colon \widetilde{A}/G \to A/G \to \mathbb{G}_a$ be the composite morphism. Take a finitely generated subcone $\sigma$ of $C$ such that $n\sigma$ ($n \ge 1$) cover $C$. (Such a cone exists. Indeed, the cone $\sigma = C(1)$ in [8] 3.4.9 satisfies the condition, as is seen in ibid. 3.4.10.) Then, $\overline{V}(n\sigma)$ ($n \ge 1$) cover $\widetilde{A}/G$. By [11] Lemma 9.10, the restriction of $\widetilde{f}$ to $\overline{V}(n\sigma)$ for each $n \ge 1$ is constant and hence is 0. Thus we conclude $\widetilde{f} = 0$ and $f = 0$. □

This proposition together with Proposition 6.2 completes the proof of Proposition 2.1.

**12.2.** We prove Proposition 2.3. This is a consequence of Lemma 3.1, Lemma 3.4 and Proposition 12.1 just proved.

**12.3 Lemma.** *Let $S$ be an fs log scheme, and $A$ a weak log abelian variety over $S$ satisfying the condition* 1.4.1. *Let $F = \mathbb{G}_m, \mathbb{G}_{m,\log}$, or $\mathbb{G}_{m,\log}/\mathbb{G}_m$.*

(1) *The canonical homomorphism* $\mathrm{Ext}(A, F) \to H^1(A, F)$ *is injective.*

(2) *For $\theta \in H^1(A, F)$, the following are equivalent*:

(i) $\theta \in \mathrm{Ext}(A, F)$;

(ii) $\mu^*(\theta) = \mathrm{pr}_1^*(\theta) + \mathrm{pr}_2^*(\theta)$ *in $H^1(A \times_S A, F)$, where $\mu\colon A \times_S A \to A$ denotes the sum of $A$.*

We remark that the condition (ii) in (2) implies that $e^*(\theta) = 0$ in $H^1(S, F)$, where $e\colon S \to A$ denotes the zero section of $A$.

*Proof.* This follows from Lemma 3.9 and Proposition 12.1. □

We have more results for $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsors.

**12.4 Proposition.** *Let $S$ be an fs log scheme whose underlying scheme is strictly local. Let $s$ be the closed point of $S$ endowed with the inverse image log structure from $S$. Let $A$ be a weak log abelian variety over $S$ satisfying the condition* 1.4.1. *Then,*

$$H^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to H^1(A \times_S s, \mathbb{G}_{m,\log}/\mathbb{G}_m)$$

*is injective.*

*Proof.* Let $L$ be a $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor on $A$. Assume that the pullback of $L$ (denoted by the same notation) to $A_s := A \times_S s$ has a section. Take a covering by proper models $P \to A$ by Proposition 11.1. (Here we use the assumption 1.4.1.) Consider the commutative diagram

$$\begin{array}{ccccc} H^0(A, L) & \to & H^0(P, L) & \rightrightarrows & H^0(P \times_A P, L) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(A_s, L) & \to & H^0(P_s, L) & \rightrightarrows & H^0(P_s \times_{A_s} P_s, L) \end{array}$$

with exact rows, where $P_s := P \times_S s$. Since the right and the middle vertical arrows are isomorphisms by the proper base change theorem and by the fact that the pullback of $L$ as a $\mathbb{G}_{m,\log}/\mathbb{G}_m$-torsor with respect to a strict morphism coincides with the pullback of $L$ as a sheaf of sets. Hence $L$ has a section on $A$. □

**12.5.** We prove the cubic isomorphism stated in Theorem 2.2. By Lemma 3.5 and Proposition 2.1 (proved after Proposition 12.1), it is enough to show the first statement of Theorem 2.2 (1) (cf. 6.16).

The case of constant degeneration (a) is shown in 8.9.

We prove the case (b). It is valid at each geometric point by (a). Then, by Proposition 12.4, it holds if the base is strictly local (cf. Lemma 3.5 (i) $\Leftrightarrow$ (ii)). Then it holds étale locally. Since the isomorphisms are canonical, they glue. This completes the case (b).

We prove the case (c). First consider the case of $\mathbb{G}_m$. By the case (a), we already have the cubic isomorphism on every log locus (including the nonreduced ones). We may assume that $S$ is affine. Let $R = \Gamma(S, \mathcal{O}_S)$. For an fs log scheme $T$ charted by an fs monoid $P$ and a face $F$ of $P$, we temporarily call the subset $\{t \in T \mid$ the set of the elements of $P$ which are invertible at $t$ is $F\}$ of $T$ the $F$-locus. For a point $s \in S$, let $r_s$ be the number of the faces $F$ of $P := (M_S/\mathcal{O}_S^\times)_{\bar{s}}$ such that the $F$-locus of the strict localization of $S$ at $s$ is not empty. We proceed by induction on $r := \sup_{s \in S} r_s$. We may assume that there are an fs chart $P$ for $S$ and a point $s \in S$ such that the induced homomorphism $P \to (M_S/\mathcal{O}_S^\times)_{\bar{s}}$ is bijective and such that the closure of each nonempty $F$-locus, where $F$ is a face of $P$, contains $s$. If the log of $S$ is not locally constant, then there is a nontrivial face $F$ of $P$ such that the $F$-locus $S'$ is not empty. Take a nontrivial element $f$ of $F$. Then $r$ for $\mathrm{Spec}(R[\frac{1}{f}])$ is strictly less than that for $R$ because this open subscheme does not intersect with the $\{0\}$-locus, where $0$ is the unit element of $P$. Further, for each $n \geq 1$, $r$ for $\mathrm{Spec}(R/f^nR)$ is also strictly less than that for $R$ because this subscheme does not intersect with the locus $S'$. Hence, by the induction hypothesis, the cubic isomorphisms exist on both. Let $\hat{R} = \varprojlim_n (R/f^nR)$. We see that the cubic isomorphism exists on $\mathrm{Spec}(\hat{R})$ by GAGF as follows. Take a covering by proper models $P \to A$ by Proposition 11.1. Then the cubic isomorphisms on $\mathrm{Spec}(R/f^nR)$ $(n \geq 1)$ induce a formal isomorphism on $P \otimes_R \hat{R}$, which gives the algebraic isomorphism by the classical GAGF. The last isomorphism descends to $A \otimes_R \hat{R}$.

Further, since $R$ is noetherian, the natural sequence

$$0 \to R \to R[\tfrac{1}{f}] \oplus \hat{R} \to \hat{R}[\tfrac{1}{f}] \to 0$$

of $R$-modules is exact (cf. [9] Proposition 9.9 with $R' = \hat{R}$ and $F = \mathbb{G}_a$). Since $\hat{R}[\frac{1}{f}]$ is flat over $R$, via this exact sequence, the cubic isomorphisms on $\mathrm{Spec}(R[\frac{1}{f}])$ and on $\mathrm{Spec}(\hat{R})$ yield the cubic isomorphism on $\mathrm{Spec}(R)$, which completes the proof of the case of $\mathbb{G}_m$.

Next consider the case of $\mathbb{G}_{m,\log}$. Let $N$ be the left-hand-side of the cubic isomorphism. We want to prove $N \cong \mathbb{G}_{m,\log}$. By the case (b), we see that $N$

comes from a $\mathbb{G}_m$-torsor $N_0$ over $A$. This $N_0$ has a canonical section on each log locus by the case (a). By the same argument in the case of $\mathbb{G}_m$, these canonical sections glue into a global one. Thus $N_0$ has a canonical section and $N$ also.

**12.6.** We prove Proposition 2.4. This is by Lemma 3.8, Proposition 2.1, and Theorem 2.2.

Till 12.12, we prove Proposition 1.6. First, we note the following fact, which will be used in the proof of Proposition 12.8.

**12.7 Proposition.** *Let $A$ be a weak log abelian variety over an fs log scheme $S$. Then $A$ is locally of finite presentation.*

*Proof.* This is because $G$ and $\mathcal{Q}/\overline{Y}$ are locally of finite presentation and the canonical homomorphism $\varinjlim H^1(S_\lambda, G) \to H^1(\varprojlim S_\lambda, G)$ is injective, where $(S_\lambda)_\lambda$ is a cofiltered projective system of affine fs log schemes over $S$ whose transition morphisms are strict. □

**12.8 Proposition.** *Let $A$ be a weak log abelian variety over an fs log scheme $S$ satisfying the condition* 1.4.1*. Let $(S_\lambda)_\lambda$ be a cofiltered projective system of affine fs log schemes over $S$ whose transition morphisms are strict. Let $S_\infty = \varprojlim_\lambda S_\lambda$. Let $A_\lambda$ (for any $\lambda$) and $A_\infty$ be the base-changed object of $A$ over $S_\lambda$ and $S_\infty$ respectively. Let $F = \mathbb{G}_m, \mathbb{G}_{m,\log}$, or $(\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\text{két}}$, where the last one is the két sheafification of $\mathbb{G}_{m,\log}/\mathbb{G}_m$. Then the following holds.*

(1) $\varinjlim_\lambda \operatorname{Ext}_{\text{két}}(A_\lambda, F) = \operatorname{Ext}_{\text{két}}(A_\infty, F)$.

(2) $\varinjlim_\lambda \operatorname{Biext}_{\text{két}}(A_\lambda, A_\lambda; F) = \operatorname{Biext}_{\text{két}}(A_\infty, A_\infty; F)$.

(3) $\varinjlim_\lambda \operatorname{Biext}_{\text{sym,két}}(A_\lambda, A_\lambda; F) = \operatorname{Biext}_{\text{sym,két}}(A_\infty, A_\infty; F)$.

(4) *Assume that $S_\infty$ is noetherian if $F = \mathbb{G}_m$ or $\mathbb{G}_{m,\log}$. Then*

$$\varinjlim_{\lambda} H^1_{\text{két}}(A_\lambda, F) = H^1_{\text{két}}(A_\infty, F).$$

**12.8.1 Remark.** We remark the following.

(1) The proof below also shows the following: In (4) and in the cases of $\mathbb{G}_m$ and $\mathbb{G}_{m,\log}$, the canonical homomorphism from the left-hand-side to the right-hand-side is injective under the assumption that each $S_\lambda$ is noetherian instead of that $S_\infty$ is noetherian.

(2) Under the same assumption as in Proposition 12.8, we have $\varinjlim_\lambda H^0(A_\lambda, F) = H^0(A_\infty, F)$ for $F = \mathbb{G}_m, \mathbb{G}_{m,\log}$, $\mathbb{G}_{m,\log}/\mathbb{G}_m$, and $(\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\text{két}}$. This is reduced to the case of $A = S$ by Proposition 12.1.

*Proof.* To see (1) (resp. (2), resp. (3), resp. (4)), it is enough to show that we can describe a (két) extension (resp. a biextension, resp. a symmetric biextension, resp. a torsor) by some finite data (more precisely, a finite number of fs log schemes of finite presentation over the base and a finite number of morphisms between them, which can be spread out) as follows.

To this end, we may assume that there are a homomorphism from an fs monoid $\mathcal{S} \to M_S/\mathcal{O}_S^\times$, an admissible pairing $X \times Y \to \mathcal{S}^{\mathrm{gp}}$ and exact sequences $0 \to G \to A \to \mathcal{Q}/\overline{Y} \to 0$ and $0 \to G \to \widetilde{A} \to \mathcal{Q} \to 0$ as in 1.7. Further, we may assume that there is a prime number $\ell$ which is invertible on the base.

Take a finitely generated subcone $\sigma$ in $C$ such that $\bigcup_n \ell^n\sigma = C$, where $\ell^n\sigma$ is defined in [11] 2.2 (cf. the proof of Proposition 12.1). Then, we have $\bigcup_n \overline{V}(\ell^n\sigma) = \mathcal{Q}$. Let $I := \widetilde{A}^{(\sigma)} \subset \widetilde{A}$ be the part corresponding to $\sigma$, that is, the pullback of $\overline{V}(\sigma)$. Let $J = \widetilde{A}^{(\ell^{-1}\sigma)} \subset I$. We have a surjection of két sheaves $J \to I$ ; $x \mapsto x^\ell$ (cf. [11] Theorem 5.1). Note that here it is essential for us to work with két topology.

To show (1), it is enough to see $\varinjlim_\lambda \mathrm{Ext}(\widetilde{A}_\lambda, F) = \mathrm{Ext}(\widetilde{A}_\infty, F)$ in this setting. In fact, giving an extension $E$ on $A$ is equivalent to giving an extension $\widetilde{E}$ on $\widetilde{A}$ endowed with a homomorphism $Y \to \widetilde{E}$ which lifts the homomorphism $Y \to \widetilde{A}$. So, by taking care of $Y$ by Proposition 12.7, we see that if we can treat an extension on $\widetilde{A}$, then we can do with an extension on $A$ also.

As for extensions on $\widetilde{A}$, there is a categorical equivalence between the category of the extensions on $\widetilde{A}$ and that of the torsors $L$ on $I$ endowed with an isomorphism $i\colon \mu^*(L) \cong \mathrm{pr}_1^*(L)\mathrm{pr}_2^*(L)$ on $J \times J$ satisfying a certain set of conditions, where $\mu$, $\mathrm{pr}_1$, and $\mathrm{pr}_2$ are the map $J \times J \to I$ induced by the summation, the first projection, and the second projection, respectively. We explain the above-mentioned set of conditions. Let $(L, i)$ be the pair as above. Then, using the existence of the isomorphism $i$, we can show that $L^{\otimes \ell}$ restricted to $J$ descends to $J/G[\ell]$, and canonically isomorphic to $L$ via $\ell\colon J/G[\ell] \cong I$. Hence, we can glue the push by $\ell^n$-multiplication of $L^{\otimes \ell^n}$ on $I/G[\ell^n]$ to $\widetilde{A}^{(\ell^n\sigma)}$ and get a torsor on $\widetilde{A}$. The above-mentioned set of conditions are the ones for this torsor on $\widetilde{A}$ to make an extension. For example, for the associativity, we impose the condition that the composite of isomorphisms $\mathrm{pr}_1^*(L)\mathrm{pr}_2^*(L)\mathrm{pr}_3^*(L) \cong (\mu, \mathrm{pr}_3)^*\mu^*(L) = (\mathrm{pr}_1, \mu)^*\mu^*(L) \cong \mathrm{pr}_1^*(L)\mathrm{pr}_2^*(L)\mathrm{pr}_3^*(L)$ on $K \times K \times K$, where $K = \widetilde{A}^{(\ell^{-2}\sigma)}$, is the identity. We have to impose more compatibilities, though we do not write down here. Thus we can describe an extension on $\widetilde{A}$ by finite data in the above sense so that the desired statement (1) follows.

We proceed to (2). As for the biextension, more complicated but similar argument works. There is a categorical equivalence between the category of the biextensions on $\widetilde{A}$ and that of the torsors $L$ on $I$ endowed with an isomorphism $i\colon \mu_{12}^*(L) \cong \mathrm{pr}_{13}^*(L)\mathrm{pr}_{23}^*(L)$ on $J \times J \times I$ and an isomorphism $j\colon \mu_{23}^*(L) \cong \mathrm{pr}_{12}^*(L)\mathrm{pr}_{13}^*(L)$ on $I \times J \times J$ satisfying a certain set of conditions, where $\mu_{ab}$ and $\mathrm{pr}_{ab}$ are the maps induced by the summation of $a$-th component and $b$-th component and by the projections to $a$-th and $b$-th components, respectively. From these isomorphisms, we can see that $L^{\otimes \ell}$ restricted to $J \times J$ descends to $J/G[\ell] \times J/G[\ell]$. Hence $L^{\otimes \ell^2}$ also descends. The descended $L^{\otimes \ell^2}$ is isomorphic to the original one via $\ell$-multiplication. Hence, we can glue the push by $\ell^n$-multiplication of $L^{\otimes \ell^{2n}}$ on $I/G[\ell^n]$ to $\widetilde{A}^{(\ell^n \sigma)}$ and get a torsor on $\widetilde{A}$. The above-mentioned set of conditions are the ones for this torsor on $\widetilde{A}$ to make a biextension. In particular, besides the conditions on the associativity etc. with respect to $i$ or $j$, which are similar to the case of the extension, we need the compatibility of $i$ and $j$, that is, that the composite of isomorphisms $\mathrm{pr}_{13}^*(L)\mathrm{pr}_{14}^*(L)\mathrm{pr}_{23}^*(L)\mathrm{pr}_{24}^*(L) \cong (\mu, \mathrm{pr}_3, \mathrm{pr}_4)^*\mu_{23}^*(L) = (\mathrm{pr}_1, \mathrm{pr}_2, \mu)^*\mu_{12}^*(L) \cong \mathrm{pr}_{13}^*(L)\mathrm{pr}_{14}^*(L)\mathrm{pr}_{23}^*(L)\mathrm{pr}_{24}^*(L)$ on $J \times J \times J \times J$ is the identity. We omit here the other conditions. Thus we can describe a biextension by finite data.

(3) is reduced to (2) by taking symmetric parts.

Finally we show (4). We may assume that each $S_\lambda$ is noetherian. First we claim that the natural homomorphism $\varinjlim_\lambda H^i_{\text{két}}(S_\lambda, F) \to H^i_{\text{két}}(S_\infty, F)$ is bijective for any $i$. (Below, only the cases $i = 1, 2$ are necessary.) Since $(\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\text{két}}$ is compatible with the pullback with respect to a strict morphism, the case where $F = (\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\text{két}}$ is reduced to the standard property of két cohomology. In the other cases where $F = \mathbb{G}_m$ or $\mathbb{G}_{m,\log}$, since $\varinjlim(S_\infty \to S_\lambda)^*F \xrightarrow{\sim} F$, the map concerned is bijective again by the standard property of két cohomology, which completes the proof of the claim.

To prove (4), we describe a torsor by finite data. To this end, the cubic isomorphism is crucial. By the cubic isomorphism, we already proved Proposition 2.4 (12.6), which implies the két version of it, that is, we have an exact sequence

$$0 \to \mathcal{E}xt_{\text{két}}(A, F) \to \mathcal{H}^1_{\text{két}}(A, F) \xrightarrow{p} \mathcal{B}iext_{\text{sym},\text{két}}(A, A; F).$$

The pullback to the diagonal $d$ satisfies $p \circ d = 2$. Let $L$ be any torsor. Since on the part of the biextension the pullback $n\colon A \to A$ acts by the multiplication by $n^2$, the pullback $2^*L$ satisfies $p(2^*L) = 4p(L)$. Hence $p(2^*L - d(2p(L))) = 4p(L) - 4p(L) = 0$ and $2^*L - d(2p(L))$ comes from an extension. Therefore, if any extension and biextension can be written by some finite data, then, $2^*L$ can be also. Then, by applying the kummer log flat descent to $2\colon A \to A$, we can also write the original $L$ by finite data. Thus (1)–(3) together with the above claim imply (4). □

Next we prove a non-két variant of (4) of the above proposition.

**12.9 Lemma.** *Let the notation and the assumption be as in Proposition* 12.8.

*Let $F' = \mathbb{G}_m, \mathbb{G}_{m,\log}$, or $\mathbb{G}_{m,\log}/\mathbb{G}_m$. In the cases of $\mathbb{G}_m$ and $\mathbb{G}_{m,\log}$, assume further that $S_\infty$ is noetherian. In the case of $\mathbb{G}_{m,\log}/\mathbb{G}_m$, assume further that* *there are two distinct primes which are invertible on $S_\infty$. Then the natural map*

$$\varinjlim_{\lambda} H^1_{\text{ét}}(A_\lambda, F') \to H^1_{\text{ét}}(A_\infty, F')$$

*is injective.*

*Proof.* By Proposition 12.8 (4), it is enough to show the injectivity of $H^1_{\text{ét}}(A, F') \to H^1_{\text{ét}}(A, \varepsilon_*(F')_{\text{két}})$, where $(F')_{\text{két}}$ is the két sheafification of $F'$. Since $\mathbb{G}_m$ and $\mathbb{G}_{m,\log}$ are két sheaves, these cases are proved.

As for the case of $\mathbb{G}_{m,\log}/\mathbb{G}_m$, since we have not yet proved that $H^1_{\text{ét}}(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to H^1_{\text{ét}}(A, \varepsilon_*((\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\text{két}}))$ is injective, we argue as follows. By assumption, we may assume that there are two distinct primes $\ell_1$ and $\ell_2$ which are invertible on all $S_\lambda$. For an integer $n$ invertible on all $S_\lambda$, we consider the $n$-két topology, which is an intermediate one between két topology and the usual étale topology. Roughly, it admits only the két covering consisting of $n$-két morphisms; a két morphism is $n$-két if and only if the cokernel of each stalk of the homomorphism between $M/\mathcal{O}^\times$ is killed by a power of $n$. See the remark below for a more precise definition of $n$-két topology. Then, the $n$-két variant of Proposition 12.8 (4) holds by the same proof. Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[1/\ell_1] \oplus \mathbb{Z}[1/\ell_2] \to \mathbb{Z}[1/\ell_1\ell_2] \to 0.$$

By tensoring it with $\mathbb{G}_{m,\log}/\mathbb{G}_m$ on $A$, we have an exact sequence

$$\begin{aligned} 0 \to \mathbb{G}_{m,\log}/\mathbb{G}_m &\to \mathbb{G}_{m,\log}/\mathbb{G}_m \otimes \mathbb{Z}[1/\ell_1] \oplus \mathbb{G}_{m,\log}/\mathbb{G}_m \otimes \mathbb{Z}[1/\ell_2] \\ &\to \mathbb{G}_{m,\log}/\mathbb{G}_m \otimes \mathbb{Z}[1/\ell_1\ell_2] \to 0 \end{aligned}$$

on $A$. Since $H^0_{\text{ét}}(A, -)$ of the last exact sequence coincides with that on $S$ (cf. 6.4.2), by using $\varepsilon_{\ell_i *}((\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\ell_i\text{-két}}) = \mathbb{G}_{m,\log}/\mathbb{G}_m \otimes \mathbb{Z}[1/\ell_i]$, where $\varepsilon_{\ell_i}$ is the projection from the $\ell_i$-két site to the usual étale site, we obtain an injective homomorphism $H^1_{\text{ét}}(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to H^1_{\text{ét}}(A, \varepsilon_{\ell_1 *}((\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\ell_1\text{-két}})) \oplus H^1_{\text{ét}}(A, \varepsilon_{\ell_2 *}((\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\ell_2\text{-két}}))$ and hence an injective homomorphism $H^1_{\text{ét}}(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to H^1_{\ell_1\text{-két}}(A, (\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\ell_1\text{-két}}) \oplus H^1_{\ell_2\text{-két}}(A, (\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\ell_2\text{-két}})$. Thus, the desired injectivity reduces to $\ell_i$-két variant of Proposition 12.8 (4). □

**12.10 Remark.** We give a more precise definition of $n$-két topology used in the above proof. Let $n$ be a positive integer. A morphism $f\colon T \to S$ of fs log schemes is *$n$-két* if it is kummer, log étale, and for any $t \in T$, the cokernel of the homomorphism $(M_S/\mathcal{O}_S)^{\text{gp}}_{\overline{f(t)}} \to (M_T/\mathcal{O}_T)^{\text{gp}}_{\bar{t}}$ is killed by a power of $n$. A két covering $(T_i \to S)_i$ is called an *$n$-két covering* if every $T_i \to S$ is $n$-két. Then the $n$-két coverings give a topology of the big site of $S$, which we call the *$n$-két topology.*

As an application of Lemma 12.9, we have

**12.11 Proposition.** *Let $A$ be a weak log abelian variety over an fs log scheme $S$. Assume that $A$ satisfies the condition* 1.4.1*. Then the natural homomorphism*

$$H^0(S, R^1f_*(\mathbb{G}_{m,\log}/\mathbb{G}_m)) \to \prod_{s\in S} H^0(\overline{s}, R^1(f\times_S \overline{s})_*(\mathbb{G}_{m,\log}/\mathbb{G}_m))$$

*is injective, where $f$ is the structure morphism of $A$ over $S$.*

*Proof.* This is reduced to Proposition 12.4 by Lemma 12.9. □

**12.12.** We prove Proposition 1.6. Proposition 1.6 (2) is already seen in 5.2. We prove Proposition 1.6 (1). We explain how we can prove it if we have a proof of the usual étale variant of Proposition 12.8 (3). After that, since we have not yet proved it in actual, we explain how to modify the proof.

By Proposition 2.3, it is enough to show that the canonical homomorphism

$$\mathcal{H}om_{\langle\ ,\ \rangle}(\overline{Y},\overline{X}) \to \mathcal{B}iext_{\mathrm{sym}}(A,A;\mathbb{G}_{m,\log}/\mathbb{G}_m)$$

given by 2.7 is an isomorphism. Assume that, for the present, the usual étale variant of Proposition 12.8 (3) holds. Let $F_1 = \mathcal{H}om_{\langle\ ,\ \rangle}(\overline{Y},\overline{X})$ and $F_2 = \mathcal{B}iext_{\mathrm{sym}}(A,A;\mathbb{G}_{m,\log}/\mathbb{G}_m)$. We prove that $F_1(T) \to F_2(T)$ is an isomorphism for any fs log scheme $T$ over $S$. Since $F_1$ and $F_2$ are locally of finite presentation (here we use the usual étale variant of Proposition 12.8 (3) for $F_2$), we may assume that the underlying scheme of $T$ is the spectrum of a strictly local ring $(R,m)$. Since we have $F_1(R/m) \cong F_2(R/m)$ by the case of constant degeneration (6.18), it is enough to show the following two statements.

(∗) $F_1(R) \to F_1(R/m)$ is bijective.

(∗∗) $F_2(R) \to F_2(R/m)$ is injective.

We prove (∗). Let $s$ be the closed point of $\operatorname{Spec} R$. First $F_1(R) \to F_1(R/m)$ is injective because the map $\overline{Y} \to \overline{X}$ is determined by $\overline{Y}_{\overline{s}} \to \overline{X}_{\overline{s}}$ (for $\overline{Y}_{\overline{s}} \to \overline{Y}_{\overline{t}}$ is surjective for any generization $t$ of $s$). Next, using the nondegeneracy of the pairing $\overline{X}\times\overline{Y} \to M_S/\mathcal{O}_S^\times$, we prove the surjectivity of $F_1(R) \to F_1(R/m)$ as follows. Let $\psi\colon \overline{Y}_{\overline{s}} \to \overline{X}_{\overline{s}}$ be a homomorphism which is compatible with $\langle\ ,\ \rangle$. It suffices to prove that, for any generization $t$ of $s$, the map $\overline{Y}_{\overline{s}} \to \overline{X}_{\overline{s}} \to \overline{X}_{\overline{t}}$ factors through $\overline{Y}_{\overline{s}} \to \overline{Y}_{\overline{t}}$. Let $y$ be in the kernel of $\overline{Y}_{\overline{s}} \to \overline{Y}_{\overline{t}}$. Then we have $\langle\psi(z),y\rangle = 0$ at $t$ for any $z\in\overline{Y}_{\overline{s}}$. Hence $\langle\psi(y),z\rangle = 0$ at $t$ for any $z$. By the nondegeneracy, $\psi(y)=0$ at $t$ follows.

We prove (∗∗). First we prove that the natural homomorphism $f\colon F_2 \to \mathcal{H}^1(A\times A,\mathbb{G}_{m,\log}/\mathbb{G}_m)$ is injective. By Proposition 2.3 (proved in 12.2) and Proposition 2.4 (proved in 12.6), we have $F_2 \subset \mathcal{H}om(A,\mathcal{E}xt^1(A,\mathbb{G}_{m,\log}/\mathbb{G}_m)) \subset$

$\mathcal{H}om(A, \mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m)) \subset \mathcal{H}^0(A, \mathcal{H}^1(A, \mathbb{G}_{m,\log}/\mathbb{G}_m))$. Since this inclusion factors through $f$, the homomorphism $f$ is injective. Hence, $(**)$ is reduced to the injectivity of $\mathcal{H}^1(A \times A, \mathbb{G}_{m,\log}/\mathbb{G}_m)(R) \to \mathcal{H}^1(A \times A, \mathbb{G}_{m,\log}/\mathbb{G}_m)(R/m)$. This is by Proposition 12.4 because $A \times A$ satisfies the condition 1.4.1.

We explain how to modify the above argument. Since the statement is local, we may assume that there are two distinct primes $\ell_1$ and $\ell_2$ which are invertible on $S$. Let $n$ be either $\ell_1$, $\ell_2$, or $\ell_1\ell_2$. If we replace $F_1$ by $F_1[1/n] := F_1 \otimes \mathbb{Z}[1/n]$ and $F_2$ by its $n$-két version $F_2^n$ (cf. the proof for Lemma 12.9), then, since the $n$-két variant of Proposition 12.8 (3) holds and since the $n$-két variant of Proposition 12.4 also holds, we can obtain the proof of $F_1[1/n] \cong F_2^n$ by modifying the above proof. (Here we remark that, instead of strict localization, we have to work with strict $n$-két localization. Then we lost noetherianness and everything should be thought as limits. Further, in 6.4.2, $X$ should be replaced by $X \otimes \mathbb{Z}[1/n]$; then the same proof in Section 6 shows the $n$-két version of the isomorphism for the residue field.)

Now consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_1 & \longrightarrow & F_1[1/\ell_1] \oplus F_1[1/\ell_2] & \longrightarrow & F_1[1/\ell_1\ell_2] \\ & & \downarrow & & \| & & \| \\ & & F_2 & \longrightarrow & F_2^{\ell_1} \oplus F_2^{\ell_2} & \longrightarrow & F_2^{\ell_1\ell_2}. \end{array}$$

The upper row is exact. The lower row is a complex and the first arrow is injective. Then the left vertical arrow is bijective. (Hence, $F_2^n \cong F_1[1/n] \cong F_2[1/n]$.) Here the injectivity of the first arrow in the lower row is reduced to that of $H^1_{\text{ét}}(A, \mathbb{G}_{m,\log}/\mathbb{G}_m) \to H^1_{\ell_1\text{-két}}(A, (\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\ell_1\text{-két}}) \oplus H^1_{\ell_2\text{-két}}(A, (\mathbb{G}_{m,\log}/\mathbb{G}_m)_{\ell_2\text{-két}})$ in the proof of Proposition 12.9. This completes the proof of the fact that $\mathcal{H}om_{\langle\, ,\, \rangle}(\overline{Y}, \overline{X}) \to \mathcal{B}iext_{\text{sym}}(A, A; \mathbb{G}_{m,\log}/\mathbb{G}_m)$ is an isomorphism.

Next, we prove that $\text{Biext}_{\text{sym}}(A/G, A/G; \mathbb{G}_{m,\log}/\mathbb{G}_m) \to \text{Biext}_{\text{sym}}(A, A; \mathbb{G}_{m,\log}/\mathbb{G}_m)$ is injective, which completes the proof of Proposition 1.6 (1). By Proposition 2.3 and Lemma 2.6 (4) (proved in 5.1), we reduce the desired injectivity to that of $\text{Hom}(A/G, \mathcal{E}xt(A/G, \mathbb{G}_{m,\log}/\mathbb{G}_m)) \to \text{Hom}(A, \mathcal{E}xt(A, \mathbb{G}_{m,\log}/\mathbb{G}_m))$. This is reduced to the vanishing of $\mathcal{H}om(G, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ ([9] Lemma 6.1.1).

**12.13.** The statement at the beginning of 4.14 is reduced to that the sheaf $F = \mathcal{H}om_{\langle\, ,\, \rangle}(\overline{Y}, \overline{X})$ has the property $F(R) = F(R/m)$ for any strictly local ring $(R, m)$, which we already showed in 12.12 $(*)$.

**12.14.** We prove Theorem 1.11. As we explained in the beginning of Section 9, this is implied by Theorem 9.1 and Proposition 1.6.

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