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ON SOLUTIONS OF $x'' = t^{-2}x^{1+\alpha}$ WITH $\alpha < 0$

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Abstract. As a continuation work, we consider a second order nonlinear differential equation denoted in the title. We show the domains of its solutions and have analytical expressions valid in the neighbourhoods of the ends of these domains. In this way, we clarify asymptotic behaviour of all solutions. We then find a regulary varying solution and represent this with the help of an asymptotic expansion of a solution of a nonlinear differential equation with an irregular singular point.

1. Introduction

In the papers [8, 9, 11, 13, 14, 16, 17, 18], we treat a second order nonlinear differential equation

$$x'' = t^{\alpha\lambda-2}x^{1+\alpha} \quad ({}' = d/dt) \tag{E}$$

where t, x are positive variables and α, λ are real parameters. As stated in those papers, it is valuable to solve this, for this can be applied to many other fields and many authors treat this or its general types (see [2, 4, 5, 6, 7, 19, etc.]). Also, (E) has a remarkable property that (E) can be transformed into a first order rational differential equation. This property was first discovered in [8] and using this property, we have investigated asymptotic behaviour of all solutions of (E).

However, in the case $\alpha < 0$ we treated only the following cases:

$$\text{(i)}\ \ \lambda_0 < \alpha < 0,\ \lambda < -1,\ \lambda > 0, \qquad \text{(ii)}\ \ \alpha = \lambda_0,\ \lambda < -1,\ \lambda > 0$$

where $\lambda_0 = -(2\lambda+1)^2/4\lambda(\lambda+1)$. So we here consider the case $\alpha < 0$, $\lambda = 0$, namely

$$x'' = t^{-2}x^{1+\alpha} \quad (\alpha < 0). \tag{E_0}$$

In this case, the transformation and the rational differential equation which we

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use have singular forms, and so we cannot treat this case and another case simultaneously.

We state the asymptotic behaviour of all solutions of (E_0) as our theorems in Section 2. The proofs of these are carried out in Section 3. In these proofs, (E_0) is shown to admit a slowly varying solution. For expressing this solution, we use an asymptotic expansion of a solution of a nonliner differential equation in the complex domain with an irregular singlular point. Here we say that a function $x(t)$ is regularly varying of index k if

$$\lim_{t\to\infty} \frac{x(\gamma t)}{x(t)} = \gamma^k$$

for all $\gamma > 0$ and in particular, slowly varying if $k = 0$ ([1]). In this paper, the solution is said to be slowly varying as $t \to +0$ instead of $t \to \infty$.

For completing those proofs, it is sufficient to follow the discussion of [14] and so we state only the outlines, except the discussion on the slowly varying solution.

Finally, we note that the asymptotic behaviour of the solutions of (E) of the case $\lambda = -1$ is known directly from our conclusions via the transformation $(t, x) \to (1/t, x/t)$ of Panayotounakos and Sotiropoulos ([7]) and that this equation has a regularly varying solution of index 1.

2. Statement of our theorems

Let us consider (E_0). First, put

$$y = x^\alpha, \quad z = ty'. \tag{T}$$

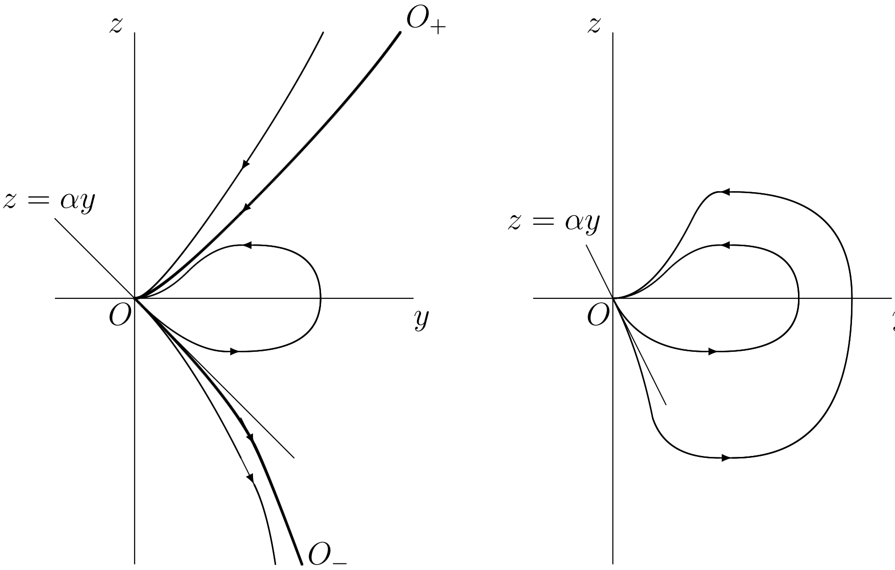
Then we have $y > 0$, for x is a positive variable. Also, we get

$$\frac{dz}{dy} = \frac{(\alpha-1)z^2 + \alpha yz + \alpha^2 y^3}{\alpha yz} \tag{R}$$

which is rewritten as a two dimensional autonomous system

$$\frac{dy}{ds} = \alpha yz \quad \frac{dz}{ds} = (\alpha-1)z^2 + \alpha yz + \alpha^2 y^3 \tag{S}$$

where s is a parameter. (R), (S) are the same as (2.10), (2.11) of [14] respetively. The critical point of (S) is only the origin in the region $y \geq 0$. As shown below, the phase portrait of (S) is as in Figure.



The case $-2 < \alpha < 0$ The case $\alpha \leq -2$

Figure The phase portrait of (S)

In this figure, $O_{\pm}$ denote the unique orbits such that

$$z = \pm\alpha\sqrt{\frac{2}{\alpha+2}}y^{3/2}\left(1+\sum_{n=1}^{\infty} z_n y^{-n/2}\right) \tag{2.1}$$

in the neighbourhood of $y = \infty$, where z_n are constants. The orbits lying above O_+ or below O_- are represented as

$$z = \Gamma^{-1}y^{(\alpha-1)/\alpha}\left\{1+\sum_{m+n>0} z_{mn}y^{-(1/2)m+((\alpha+2)/2\alpha)n}\right\} \tag{2.2}$$

in the neighbourhood of $y = \infty$, where Γ, z_{mn} are constants. If the orbit tends to the origin in $z > 0$, then this orbit is represented as

$$z = -\alpha y^2\left(1+\sum_{n=1}^{N-1} z_n y^n + O(y^N)\right) \quad \text{as } y \to 0 \tag{2.3}$$

where N is any integer (≥ 2) and z_n are constants, and if the orbit tends to the origin in $z < 0$, then as

$$z = \alpha y\left[1+\sum_{m+n>0} w_{mn}y^m\left\{y^{-1/\alpha}(h\log y + C)\right\}^n\right] \tag{2.4}$$

in the neighbourhood of $y = 0$, where w_{mn}, h, C are constants and $h = 0$ if $-1/\alpha \notin \mathbb{N}$.

Now, given an initial condition

$$x(t_0) = A,\ x'(t_0) = B \quad (0 < t_0 < \infty,\ 0 < A < \infty,\ -\infty < B < \infty), \tag{I}$$

we denote the solution of the initial value problem (E_0), (I) as $x = x(t)$. From $x(t)$ we have a solution $z = z(y)$ of (R) with the initial condition

$$z(y_0) = z_0 \tag{2.5}$$

where

$$y_0 = A^\alpha,\ z_0 = \alpha t_0 A^{\alpha-1} B,$$

for we get

$$y = x^\alpha,\ z = \alpha t x^{\alpha-1} x'$$

from (T) and $(y, z) = (y_0, z_0)$ if $t = t_0$. Also, from $x(t)$ we have an orbit (y, z) of (S) passing (y_0, z_0). Conversely, via (T) we get the solution $x(t)$ of (E_0), (I) from the solution $z(y)$ of (R) with (2.5) or from the orbit (y, z) of (S) passing (y_0, z_0).

So, taking (t_0, A, B) of (I) and depending on where (y_0, z_0) lies, we state the asymptotic behaviour of $x(t)$. First, suppose $-2 < \alpha < 0$.

Theorem 1. *If $(y_0, z_0) \in O_+$, then $x(t)$ is defined for $0 < t < \omega_+$ (here and hereafter, ω_+ denotes some positive constant). Also, $x(t)$ is represented as the following asymptotic expansion:*

$$\begin{aligned} x(t) = (\alpha \log t + C)^{-1/\alpha} \Bigg[1 + \sum_{0<m+n\le N} & x_{mn} (\alpha \log t + C)^{-m} \\ \times \left\{ (\alpha \log t + C)^{-1} \log(\alpha \log t + C) \right\}^n & + O((\alpha \log t)^{-N}) \Bigg] \end{aligned} \tag{2.6}$$

as $t \to +0$, where C, x_{mn} are constants, N is any integer (≥ 2), and $x_{N0} = 0$, and as

$$x(t) = \left\{ \frac{2(\alpha+2)\omega_+^2}{\alpha^2} \right\}^{1/\alpha} (\omega_+ - t)^{-2/\alpha} \left\{ 1 + \sum_{n=1}^{\infty} x_n (\omega_+ - t)^n \right\} \tag{2.7}$$

in the neighbourhood of $t = \omega_+$, where x_n are constants.

Note that (2.6) is the slowly varying solution of (E_0).

Theorem 2. *If (y_0, z_0) lies in the region above O_+, then $x(t)$ is defined for $0 < t < \omega_+$. Moreover $x(t)$ is represented as (2.6) as $t \to +0$, and as*

$$x(t) = K(\omega_+ - t)\left\{1 + \sum_{m+n>0} x_{mn} \times (\omega_+ - t)^{-(\alpha/2)m} (\omega_+ - t)^{((\alpha+2)/2)n}\right\} \tag{2.8}$$

in the neighbourhood of $t = \omega_+$, where K, x_{mn} are constants.

Theorem 3. *If (y_0, z_0) lies in the region between O_+ and O_-, then $x(t)$ is defined for $0 < t < \infty$. Also, $x(t)$ is expressed as (2.6) as $t \to +0$, and as*

$$x(t) = Kt\left(1 + \sum_{m+n>0} x_{mn} t^{\alpha m - n}\right) \quad \text{if } -1/\alpha \notin \mathbb{N} \tag{2.9}$$

$$x(t) = Kt\left(1 + \sum_{k=1}^{\infty} t^{\alpha k} p_k(\log t)\right) \quad \text{if } -1/\alpha \in \mathbb{N} \tag{2.10}$$

in the neighbourhood of $t = \infty$, where K, x_{mn} are constants and p_k are polynomials with $\deg p_k \le [-\alpha k]$.

Theorem 4. *If $(y_0, z_0) \in O_-$, then $x(t)$ is defined for $\omega_- < t < \infty$ (here and hereafter, ω_- denotes some positive constant). Moreover $x(t)$ is represented as*

$$x(t) = \left\{\frac{2(\alpha+2)\omega_-^2}{\alpha^2}\right\}^{1/\alpha} (t - \omega_-)^{-2/\alpha} \left\{1 + \sum_{n=1}^{\infty} x_n (t - \omega_-)^n\right\} \tag{2.11}$$

in the neighbourhood of $t = \omega_-$ where x_n are constants, and as (2.9), (2.10) in the neighbourhood of $t = \infty$.

Theorem 5. *If (y_0, z_0) lies in the region below O_-, then $x(t)$ is defined for $\omega_- < t < \infty$. Also, in the neighbourhood of $t = \omega_-$, $x(t)$ is represented as*

$$x(t) = K(t - \omega_-)\left\{1 + \sum_{m+n>0} x_{mn} \times (t - \omega_-)^{-(\alpha/2)m} (t - \omega_-)^{((\alpha+2)/2)n}\right\} \tag{2.12}$$

where K, x_{mn} are constants, and in the neighbourhood of $t = \infty$, as (2.9), (2.10).

Next, suppose $\alpha \le -2$. Then we have the following:

Theorem 6. *The conclusion of Theorem 3 follows for all (y_0, z_0).*

3. Proofs of our theorems

For the proofs, let us follow the discussion of [14]. First, put

$$w = y^{-2}z$$

in (R). Then we have

$$\frac{dw}{dy} = \frac{\alpha^2 + \alpha w - (\alpha+1)yw^2}{\alpha y^2 w}. \tag{3.1}$$

If γ denotes an accumulation point of a solution of (3.1) as $y \to 0$, then we get

$$\gamma = -\alpha, \pm\infty$$

from Lemma 3.1 of [14]. We here postpone the discussion on the case $\gamma = -\alpha$. If $\gamma = \pm\infty$, then we have (2.4) in the neighbourhood of $y = 0$ from Lemma 2.5 of [12] and the proof of Lemma 3.3 of [14] (and eventually we have $\gamma = -\infty$). If we follow the discussion of Section 3 of [9], then from (2.4) we get (2.9), (2.10) in the neighbourhood of $t = \infty$.

Next, let us consider the case when y does not tend to 0. Following Section 4 of [14], we then conclude that $y \to \infty$, $z \to \pm\infty$ as s tends to the end of the domain of the orbit (y, z). Also, if we put $y = 1/\eta$, $z = 1/\zeta$ and $\theta = \eta^{-3/2}\zeta$, $\xi = \eta^{1/2}$, then we have

$$\xi\frac{d\theta}{d\xi} = -\frac{\alpha+2}{\alpha}\theta + 2\xi\theta^2 + 2\alpha\theta^3. \tag{3.2}$$

Let δ be an accumulation point of a solution of (3.2) as $\xi \to 0$ (namely, $y \to \infty$). If $-2 < \alpha < 0$, then we get

$$\delta = 0, \pm\rho \qquad \left(\rho = \frac{1}{\alpha}\sqrt{\frac{\alpha+2}{2}}\right)$$

and if $\alpha \le -2$, then there exists no δ, namely y always tends to 0 as s tends to the end of the domain of (y, z) (see the proof of Lemma 7.1 of [16]).

Suppose $-2 < \alpha < 0$. If $\delta = 0$, then from Section 5 of [9] we have (2.2) and from this, (2.7), (2.11) if $z > 0$, $z < 0$ respectively. In fact, y is increasing in t as $z > 0$ and decreasing in t as $z < 0$ from $z = ty'$ of (T). Also if $\delta = \pm\rho$, then from Lemma 2.5 of [12] and the proof of Lemma 4.4 of [14] we get (2.1) and from this, (2.8), (2.12) if $z > 0$, $z < 0$, respectively. Here we draw the phase portrait of (S) as in Figure from considering γ, δ of (3.1), (3.2) and the direction of the orbit of (S) (see Section 4 of [14]).

Finally, let us discuss the case $\gamma = -\alpha$ in (3.1). In this case, put $\theta = w + \alpha$. Then we have

$$y^2 \frac{d\theta}{dy} = (\alpha+1)y - \frac{\theta}{\alpha} + \cdots \tag{3.3}$$

and $\theta \to 0$ as $y \to 0$.

We really have such θ from the orbits tending to the origin in $z > 0$. So, we consider the nonlinear differential equation

$$x^{\sigma+1} \frac{dy}{dx} = f(x, y) \tag{3.4}$$

with an irregular singular point, more generally. Here x, y are complex variables, $\sigma \in \mathbb{N}$, and $f(x, y)$ is a holomorphic function in the neighbourhood of $(x, y) = (0, 0)$ such that

$$f(x, y) = ax + \lambda y + \sum_{j+k \geq 2} a_{jk} x^j y^k$$
$$(a,\ \lambda,\ a_{jk} : \text{constants},\ \lambda \neq 0).$$

Lemma. *If* (3.4) *has a solution* $y(x)$ *tending to* 0 *as* $x \to 0$ *along a line* ℓ *in a sector*

$$-\frac{\pi}{2} < \sigma \arg x - \arg \lambda < \frac{\pi}{2},$$

then we have

$$y(x) = \sum_{j=1}^{N-1} p_j x^j + O(x^N) \quad (p_j : \textit{constants})$$

as $x \to 0$ *along* ℓ. *Here* N *is any integer* (≥ 2).

Proof. We first follow the discussion of [3] (Sections 4, 6 of Chapter III) and use a formal transformation

$$y = \sum_{j+k>0} p_{jk} x^j \eta^k \quad (p_{jk} : \text{constants with } p_{01} = 1)$$

which reduces (3.4) to

$$x^{\sigma+1} \eta' = \eta \sum_{j=0}^{\sigma} \alpha_j x^j$$

where $\alpha_0 = \lambda$, α_j are constants and $' = d/dx$. From this, we have

$$\eta = C x^{\alpha_\sigma} e^{\Lambda(x)} \quad \left(C : \text{a constant},\ \Lambda(x) = \sum_{j=0}^{\sigma-1} \frac{\alpha_j}{j-\sigma} x^{j-\sigma} \right).$$

Next, we put

$$y = z + P_N(x,\eta), \quad P_N(x,\eta) = \sum_{j+k<N} p_{jk} x^j \eta^k$$

and have

$$x^{\sigma+1} z' = g_N(x,\eta,z) \tag{3.5}$$

where

$$g_N(x,\eta,z) = f(x, z+P_N(x,\eta)) - \sum_{j+k<N} j p_{jk} x^{j+\sigma} \eta^k - \sum_{j+k<N} k p_{jk} x^j \eta^k \sum_{J=0}^{\sigma} \alpha_J x^J.$$

Also, if we put

$$g_N(x,\eta,z) = \sum_{j+k+l>0} b_{jkl} x^j \eta^k z^l$$

then we have

$$b_{001} = \lambda, \ b_{jk0} = 0 \quad (j+k<N),$$

for (3.5) has a formal solution

$$z = \sum_{j+k \geq N} p_{jk} x^j \eta^k.$$

Now, let us start our discussion. For sufficiently small $|x|$ we have

$$\frac{\pi}{2} < \arg \Lambda(x) < \frac{3}{2}\pi,$$

that is,

$$\operatorname{Re} \Lambda(x) < 0,$$

since ℓ lies in $-\pi/2 < \sigma \arg x - \arg \lambda < \pi/2$. Hence as $x \to 0$ along ℓ we have

$$\eta \to 0, \quad x^{-1}\eta \to 0.$$

On the other hand, we get $z = z(x) \to 0$ as $x \to 0$ along ℓ from the assumption $y(x) \to 0$. Therefore we have

$$g_N(x,\eta,z) = \hat{\lambda}(x) z + h_N(x)$$

where $\hat{\lambda}(x)$ is a holomorphic function of $x \neq 0$, $h_N(x)$ is that of x, and

$$\hat{\lambda}(x) \to \lambda, \quad \frac{h_N(x)}{x^N} \to b_{N00} \quad \text{as } x \to 0 \text{ along } \ell.$$

Hence from (3.5) we have

$$z = e^{\int_{x_0}^{x} \hat{\lambda}(x) x^{-\sigma-1} dx} \left(\int_{x_0}^{x} h_N(x) x^{-\sigma-1} e^{-\int_{x_0}^{x} \hat{\lambda}(x) x^{-\sigma-1} dx} dx + C \right)$$

where x_0 is a nonzero constant on ℓ, C is a constant, and the paths of integration are a segment connecting 0 and x_0. Therefore if we put $x = re^{i\theta}$, $x_0 = r_0 e^{i\theta_0}$, $\lambda = \rho e^{i\omega}$ then θ is constant and

$$\operatorname{Re} \int_{x_0}^{x} \hat{\lambda}(x) x^{-\sigma-1} dx = \int_{r_0}^{r} \tilde{\lambda}(r,\theta) r^{-\sigma-1} dr$$

where

$$\tilde{\lambda}(r,\theta) = \operatorname{Re} \hat{\lambda}(x) \cos \sigma\theta + \operatorname{Im} \hat{\lambda}(x) \sin \sigma\theta.$$

Hence we have

$$\begin{aligned} |z| &\le e^{\operatorname{Re} \int_{x_0}^{x} \hat{\lambda}(x) x^{-\sigma-1} dx} \left(\left| \int_{r_0}^{r} |h_N(x)| r^{-\sigma-1} e^{-\operatorname{Re} \int_{r_0}^{r} \hat{\lambda}(x) x^{-\sigma-1} dx} dr \right| + |C| \right) \\ &= e^{\int_{r_0}^{r} \tilde{\lambda}(r,\theta) r^{-\sigma-1} dr} \left(\left| \int_{r_0}^{r} |h_N(x)| r^{-\sigma-1} e^{-\int_{r_0}^{r} \tilde{\lambda}(r,\theta) r^{-\sigma-1} dr} dr \right| + |C| \right). \end{aligned}$$

Since $\hat{\lambda}(x) \to \lambda$, we get

$$\tilde{\lambda}(r,\theta) \to \rho \cos(\omega - \sigma\theta) \quad \text{as } r \to 0$$

and

$$\tilde{\lambda}(r,\theta) > \frac{\rho}{2} \cos(\omega - \sigma\theta) > 0$$

for sufficiently small r. Here, suppose $r < r_0$. Then for sufficiently small r_0 we have

$$\int_{r_0}^{r} \tilde{\lambda}(r,\theta) r^{-\sigma-1} dr < \left\{ \frac{\rho}{2} \cos(\omega - \sigma\theta) \right\} \left(-\frac{r^{-\sigma}}{\sigma} + \frac{r_0^{-\sigma}}{\sigma} \right) \to -\infty \quad \text{as } r \to 0.$$

Hence from l'Hospital's theorem we get

$$\frac{\int_{r_0}^{r} |h_N(x)| r^{-\sigma-1} e^{-\int_{r_0}^{r} \tilde{\lambda}(r,\theta) r^{-\sigma-1} dr} dr}{r^N e^{-\int_{r_0}^{r} \tilde{\lambda}(r,\theta) r^{-\sigma-1} dr}} \to -\frac{b_{N00}}{\rho \cos(\omega - \sigma\theta)},$$

which implies

$$z = O(x^N) \quad \text{as } x \to 0 \text{ along } \ell.$$

Therefore it completes the proof to put $p_j = p_{j0}$, for $x^{-1}\eta \to 0$ as $x \to 0$ along ℓ.

□

If we put $\lambda = -1/\alpha (> 0)$, $\sigma = 1$, $x = y (> 0)$, $y = \theta$ in (3.4), then we have (3.3) and may take $\sigma \arg x - \arg \lambda = 0$. Hence from this lemma, we get

$$\theta = \sum_{n=1}^{N-1} \theta_n y^n + O(y^N) \quad \text{as } y \to 0$$

where θ_n are constants and $\theta_1 = \alpha(\alpha+1)$. Since $\theta = w + \alpha$, $w = y^{-2} z$, we have

$$w = -\alpha + \sum_{n=1}^{N-1} \theta_n y^n + O(y^N)$$

and (2.3). Also, from (T) we get

$$ty' = -\alpha y^2 \left(1 + \sum_{n=1}^{N-1} \tilde{\theta}_n y^n + O(y^N)\right) \quad \left(\tilde{\theta}_n = -\frac{\theta_n}{\alpha}\right).$$

Solving this, we have

$$\begin{aligned} -y^{-1} + (\alpha+1)\log y + \sum_{n=2}^{N-1} \hat{\theta}_n y^{n-1} + O(y^{N-1}) \\ = -\alpha \log t - C \quad (\hat{\theta}_n, C : \text{constants}) \end{aligned}$$

and from this,

$$y \left\{ 1 + \sum_{0 < m+n \leq N} a_{mn} y^m (y \log y)^n \right\} (1 + O(y^N)) = (\alpha \log t + C)^{-1}$$

where a_{mn} are constants. As $y \to +0$, we get $t \to +0$ and

$$\frac{y}{(\alpha \log t)^{-1}} \to 1.$$

Hence we have

$$O(y^N) = O((\alpha \log t)^{-N})$$

and

$$\begin{aligned} y \left\{ 1 + \sum_{0 < m+n \leq N} a_{mn} y^m (y \log y)^n \right\} \\ = (\alpha \log t + C)^{-1} \{1 + O((\alpha \log t)^{-N})\}. \end{aligned} \tag{3.6}$$

Taking the logarithms of both sides, we get

$$\begin{aligned} \log y + \sum_{m+n>0} \tilde{a}_{mn} y^m (y \log y)^n = -\log(\alpha \log t + C) \\ + O((\alpha \log t)^{-N}) \quad (\tilde{a}_{mn} : \text{constants}) \end{aligned} \tag{3.7}$$

and multiplying (3.6) with (3.7),

$$\begin{aligned} & y \log y + \sum_{m+n>1} \hat{a}_{mn} y^m (y \log y)^n \\ & = -(\alpha \log t + C)^{-1} \log(\alpha \log t + C) + O((\alpha \log t)^{-N-1} \log(\alpha \log t)) \\ & \qquad (\hat{a}_{mn} : \text{constants}). \end{aligned} \tag{3.8}$$

Note that the left hand sides of (3.6), (3.8) are functions of y, $y \log y$. Then applying the inverse function theorem to (3.6), (3.8), we determine y, $y \log y$ and in particular we have

$$\begin{aligned} y = (\alpha \log t + C)^{-1} \Bigg[1 + & \sum_{0<m+n\le N} b_{mn} (\alpha \log t + C)^{-m} \\ & \times \{(\alpha \log t + C)^{-1} \log(\alpha \log t + C)\}^n + O((\alpha \log t)^{-N}) \Bigg] \\ & (b_{mn} : \text{constants}). \end{aligned}$$

This implies (2.6) from (T). Now the proofs are complete.

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