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GAPS OF BEANS FUNCTIONS OF GRAPHS OVER INTERVALS BOUNDED BY UNIT FRACTIONS

By

Kengo Enami and Seiya Negami

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Abstract. The *beans function* $B_G(x)$ of a connected graph G is defined as the maximum number of points on G such that any pair of points have distance at least $x > 0$. This is a decreasing and left continuous function and has many discontinuous points $\lambda_{n,1}, \ldots, \lambda_{n,k}$ over the interval $(1/(n+1), 1/n]$ in general. We call these points the *gaps* of $B_G(x)$. We shall show that any gap $B_G(x)$ is a rational number. By the recursive formula given in [3], we can show that $\lambda_{n,i} = \lambda_{1,i}/(1+(n-1)\lambda_{1,i})$ easily. So, we shall focus on the interval $(1/2, 1]$ and show that $\lambda_{1,1} \in (1/2, 3/5]$ and $\lambda_{1,k} \in [3/4, 1)$ if G has at least three vertices and is not isomorphic to the star $K_{1,|V(G)|-1}$. Furthermore, we shall find another gap of $B_G(x)$ in $[2/3, 3/4)$ with a suitable condition for G, and discuss the sharpness of these results, considering some examples.

Introduction

Our graph G is a simple graph and we regard it as a 1-dimensional simplicial complex each of whose edges has a unit length. Thus, if G is connected, then we can define the distance $d_G(p, q)$ between any two points p and q on G; they may be not only points located at vertices, but also intermediate points lying along edges.

Negami [4] has introduced a function $B_G : \mathbb{R}_+ \to \mathbb{N}$, called the *beans function* of G, as the maximum number of points on G any pair of which have distance at least $x > 0$, where $\mathbb{R}_+$ stands for the set of positive real numbers and $\mathbb{N}$ is the set of natural numbers. He also has given the following upper and lower bounds for the values of $B_G(x)$ taken over the interval $A_n = (1/(n+1), 1/n]$:

Theorem 1. (Negami [4]) *Let G be a connected graph and let n be a natural number. If $x \in A_n$, then:*

$$n \cdot |E(G)| + |V(G)| - 1 \geq B_G(x) \geq n \cdot |E(G)| + \alpha$$

where $\alpha = 1$ if G is a tree, and $\alpha = 0$ otherwise .

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It is easy to see that the lower bound is attained by $x = 1/n$ for any $n \geq 1$. On the other hand, Negami [4] has shown an example of a graph G whose beans function takes the value $n \cdot |E(G)| + |V(G)| - 1$ as the maximum of $B_G(x)$ over $A_n = (1/(n+1), 1/n]$. Furthermore, Enami [2] has already proved that the upper bound is attained by a suitable value of $x > 1/(n+1)$ for any connected graph G. Thus, we should investigate what happens inside the interval A_n.

Notice that $\bigcup_{n=1}^{\infty} A_n = (0, 1]$ and that $B_G(x)$ is decreasing and left continuous. Since $B_G(x)$ takes discrete values in $\mathbb{N}$, it has finitely many discontinuous points $\lambda_{n,1}, \ldots, \lambda_{n,k_n} \in A_n - \{1/n\}$ and takes a constant value over each of the segments separated by these points. We call each of $\lambda_{n,i}$'s a *gap* of $B_G(x)$ in A_n. In particular, $B_G(x)$ takes the value $n \cdot |E(G)| + |V(G)| - 1$ over $(1/(n+1), \lambda_{n,1}]$ and $n \cdot |E(G)| + \alpha$ over $(\lambda_{n,k_n}, 1/n]$ by the facts described in the previous paragraph.

In fact, the authors have established the following recursive formula for $B_G(x)$:

Theorem 2. (Enami and Negami [3]) *For any positive real number $x \leq 1$ and a natural number $k \geq 1$, we have:*

$$B\left(\frac{x}{1+kx}\right) = B_G(x) + k|E(G)|$$

The function $f_k(x) = x/(1 + kx)$ sends any value in A_n to a value in A_{n+k} bijectively. Thus, this formula enables us to determine all values of $B_G(x)$ only by knowing its values over the interval $A_1 = (1/2, 1]$. For examples, the gap $\lambda_{n,i}$ corresponds to $\lambda_{1,i}$ and we have $\lambda_{n,i} = \lambda_{1,i}/(1 + (n-1)\lambda_{1,i})$ and $k_n = k_1$ for any natural number n. That is, the number of gaps over $(1/(n+1), 1/n]$ is a constant, say k. To simplify the notations below, we set $\lambda_i = \lambda_{1,i}$. To know all gaps of $B_G(x)$, it suffices to focus on the interval $(1/2, 1]$.

In this paper, we shall discuss the *first left gap* λ_1 and the *first right gap* λ_k in $A_1 = (1/2, 1]$ and estimate where they are located in A_1, as given in the following two theorems:

Theorem 3. *Let G be a connected graph which has at least three vertices and is not isomorphic to the star $K_{1,|V(G)|-1}$. Then the first left gap λ_1 of $B_G(x)$ lies in $(1/2, 3/5]$ and $B_G(x) = |E(G)| + |V(G)| - 1$ for $x \in (1/2, \lambda_1]$.*

Theorem 4. *Let G be a connected graph which has at least three vertices and is not isomorphic to the star $K_{1,|V(G)|-1}$. Then the firsr right gap λ_k of $B_G(x)$ lines in $[3/4, 1)$ and $B_G(\lambda_k) > |E(G)| + \alpha$, where $\alpha = 1$ if G is tree and $\alpha = 0$ otherwise.*

If we restrict the graph G slightly, we can find one more gap of $B_G(x)$, as follows. A set of edges in a graph G each pair of which have no common end is called a *matching* in G and we denote the maximum size of matchings in G by $\mu(G)$.

THEOREM 5. *Let G be a connected graph which has the minimum degree at least 2. Then there is a gap λ of $B_G(x)$ with $2/3 \leq \lambda \leq 3/4$, and $B_G(\lambda) \geq |E(G)| + \mu(G) > B_G(x)$ for $x > \lambda$.*

(i) *If there is no odd cycle of length at most $2k-1$ in G, then $\lambda \leq (2k+1)/(3k+1)$.*
(ii) *If G is bipartite, then $\lambda = 2/3$.*

First, we shall discuss a combinatorial criterion for a positive real number to be one of gaps of $B_G(x)$ and prove that any gap is a rational number in Section 1. Next, focusing on the interval $(1/2, 1]$, we shall prove the above theorems in Sections 2 and 3, and discuss their sharpness with examples in Section 4. Our terminology is quite standard and can be found in [1].

1. Gaps of functions

Let $x > 0$ be a positive real number. A set S of points on G is called an *x-set* if any pair of points in S have distance at least x. In particular, if S has the maximum size among all x-sets, then we call S a *maximum x-set*. That is, $B_G(x) = |S|$ for any maximum x-set in G.

Let $\lambda < 1$ be a positive real number and let S be a λ-set in G. A cycle C in G is said to be *λ-full* for S if it contains exactly $|C|/\lambda$ point in S, where $|C|$ stands for the length of C and is equal to the number of edges of C. If C is a λ-full cycle for S, then the points in S are located at equal intervals of length λ along C. Thus, λ must be a rational number.

Similarly, a path Q in G is said to be *λ-full* for S if it contains exactly $|Q|/\lambda+1$ points in S, where $|Q|$ denotes the length of Q, which is equal to the number of edges in Q. If Q is a λ-full path for S, then two points in S are located at both ends of Q and the other points in S divide into $|Q|/\lambda$ segements of length λ. This implies that λ must be a rational number in this case, too.

Using these notions, we can establish a criterion for a positive real number $\lambda < 1$ to be a gap of $B_G(x)$, as follows.

THEOREM 6. *Let G be a connected graph and let $\lambda < 1$ be a positive real number. The number λ is one of a gap of $B_G(x)$ if and only if there is either a λ-full cycle or a λ-full path for any maximum λ-set S.*

Proof. First, we shall show the sufficiency. Suppose that λ is not a gap of $B_G(x)$. Then $B_G(\lambda) = B_G(\lambda + \varepsilon)$ for a sufficiently small positive number $\varepsilon > 0$. This implies that any maximum $(\lambda + \varepsilon)$-set S in G is also a maximum λ-set in G. Then any cycle and any path in G cannot be λ-full for this λ-set S since any pair of points in S lying along it have distance at least $\lambda + \varepsilon > \lambda$.

Next, we shall show the necessity. Let S be a maximum λ-set in G and suppose that there is neither a λ-full cycle nor a λ-full path for S. Furthermore, we may assume that we have chosen S to minimize the number of pairs of points in S which have distance exactly λ.

Assume that there is still a pair of points s_0 and s_1 in S which have distance exactly λ. Try to find a sequence of points $s_0, s_1, \dots$ in S such that $s_{i-1} \neq s_{i+1}$ and that $d_G(s_{i-1}, s_i) = \lambda$ for $i \geq 1$. If such a sequence included a cycle, then it would be a λ-full cycle for S, contrary to our assumption. Thus, the sequence of points $s_0, s_1, \dots$ runs along a path in G, and will stop at a point s_t.

Extend such a sequence toward both directions and take a maximal one. Then we may assume that a path $Q = v_0 v_1 \cdots v_k$ in G contains $s_0, s_1, \dots, s_t$, where each of v_i's is a vertex in G. The point s_0 lies on the edge $v_0 v_1$ and s_t lies on the edge $v_{k-1} v_k$. Since there is no λ-full path for S in G, at least one of s_0 and s_k is not located at the ends of Q, Thus, we may assume that s_k is an intermediate point of the edge $v_{k-1} v_k$.

By the maximality of the sequence of points $s_0, s_1, \dots, s_t$, we have $d_G(s, s_t) > \lambda$ for any point s in S placed around v_k except s_t and s_{t-1}; s may or may not lie on $v_{k-1} v_k$. Then we can move s_t slightly toward v_k to obtain another λ-set S'. Since $|S'| = |S|$, this new λ-set S' also is maximum, but the number of pairs of points in S' which have distance exactly λ would be less than S, contrary to the assumption on S.

Therefore, S contains no pair of points which have distance exactly λ and hence the minimum distance taken over all pairs in S is greater than λ, say $\lambda + \varepsilon$ with a suitable positive real number $\varepsilon > 0$. Thus, S is a $(\lambda + \varepsilon)$-set in G and we have $B_G(\lambda + \varepsilon) = B_G(\lambda)$. This implies that λ is not a gap of $B_G(x)$. □

Corollary 7. *Any gap of $B_G(x)$ of a connected graph G is a rational number.*

Proof. If λ is a gap of $B_G(x)$ of G, then there is either a λ-full cycle or a λ-full path for any maximum λ-set S in G. Let L be the length of such a cycle or a path. The points in S lying along it divide it into several segments of length λ, say M segments. Therefore, $\lambda = L/M$ must be a rational number. □

2. First left and first right gaps

Given an x-set S in G, we decompose S into a disjoint union $\bigcup_{e\in E(G)} S_e$ so that each edge contains the points in S_e. If a point p in S is located at a vertex v, then we choose only one of edges incident to v, say e, and consider that p belongs to S_e and not to the others.

Given an x-set S and its decomposition $\bigcup_{e\in E(G)} S_e$, we set $E_i = \{e \in E(G) : |S_e| = i\}$. If $x > 1/2$, then it is clear that each edge contains at most two points in S and hence $E(G) = E_0 \cup E_1 \cup E_2$. Thus, if S is a maximum x-set for $x > 1/2$, then we have:

$$|S| = B_G(x) = |E_1| + 2|E_2| = |E(G)| + |E_2| - |E_0|$$

We shall use these notations in our arguments below.

Proof of Theorem 3. Suppose that $x > 1/2$ and that $B_G(x) \geq |E(G)|+|V(G)|-1$. Since the equality holds by Theorem 1, we have $|S| = B_G(x) = |E(G)|+|V(G)|-1 = |E(G)|+|E_2|-|E_0|$ and hence $|E_2|-|E_0| = |V(G)|-1$ for a maximum x-set S.

Assume that the subgraph $\langle E_2\rangle$ induced by E_2 includes a cycle. Such a cycle C contains exactly $2|C|$ points in S and those points divide C into $2|C|$ segments. This implies that $x \leq |C|/2|C| = 1/2$, contrary to our assumption. Therefore, $\langle E_2\rangle$ includes no cycle and hence $|E_2| \leq |V(G)|-1$. This implies that $|E_2| = |V(G)| - 1$, $|E_0| = 0$ And that $T = \langle E_2\rangle$ is a spanning tree of G.

Assume that there is a path of length 3 in the spanning tree T. Then the path contains exactly six points in S, possibly located at its ends. These six points divide the path into at least five segments. This implies that $x \leq 3/5$.

On the other hand, if T includes no path of length 3, then T must be isomorphic to the star $K_{1,s}$ with $s \geq 2$ since G has at least three vertices. By the assumption in the theorem, G does not coincide with T and hence there is an edge of G not belonging to T, which should belong to E_1, and it joints two vertices of degree 1 in T. Then there is a cycle of length 3 consisting of two edges in E_2 and one edge in E_1, and exactly five points in S divide the cycle into five segments. This implies that $x \leq 3/5$, again.

Therefore, if $x > 3/5$, then $B_G(x) < |E(G)| + |V(G)| - 1$. Since Enami's result [2] guarantees the existence of a real number $\lambda > 1/2$ such that $B_G(\lambda) = |E(G)| + |V(G)| - 1$, The supremum of such λ's should be the first left gap λ_1 and $\lambda_1 \leq 3/5$. □

Note that if we can construct a 3/5-set S such that $E_0 = \emptyset$ and that E_2 forms a spanning tree of G, then $B_G(3/5) = |E(G)|+|V(G)|-1$ and we can determine the first left gap λ_1 of $B_G(x)$; $\lambda_1 = 3/5$.

COROLLARY 8. *If a connected graph G has a spanning tree isomorphic to $K_{1,s}$ with $s \geq 2$, but is not isomorphic to $K_{1,s}$, then $\lambda_1 = 3/5$.*

Proof. Let T be a spanning tree in G which is isomorphic to $K_{1,s}$ and let $e_1, \ldots, e_s$ be its edges with a unique common end v. Put two points p_i and q_i on each edge e_i so that $d_G(p_i, v) = 3/10$ and $d_G(q_i, v) = 9/10$, and take the midpoints of all other edges. Then these points form a 3/5-set S and we have $E_2 = \{e_1, \ldots, e_s\}$, $E_1 = E(G) - E_2$ and $E_0 = \emptyset$. The set S satisfies the desired condition. □

Proof of Theorem 4. Let G be a connected graph with at least three veritces. First suppose that G is not a tree. Then $B_G(1) = |E(G)|$ by Theorem 1. It suffice to show that $B_G(3/4) \geq |E(G)| + 1$.

Choose one edge $e = uv$ of G with endpoints u and v, and take the following points as those in S:

(i) Two points p and q lying on e with $d_G(p, u) = 1/8$ and $d_G(q, v) = 1/8$
(ii) A point r lying on each edge incident to u (or v) with $d_G(r, u) = 5/8$ (or $d_G(r, v) = 5/8$)
(iii) The midpoints of all edges other than e and edges incident to u or v

The set S consists of these $|E(G)| + 1$ points and it is easy to see that it forms a 3/4-set. This implies that $B_G(3/4) \geq |E(G)| + 1$.

Secondly suppose that G is a tree not isomorphic to the star $K_{1,|V(G)|-1}$. Then there is a path of length at least 3 which joins a pair of vertices of degree 1, say u_1 and u_2. Place one point at u_i and another point p_i on the unique edge incident to u_i with $d_G(u_i, p_i) = 3/4$ for $i = 1, 2$. Add one point at the midpoint of each edge incident to neither u_1 nor u_2. Then it is easy to see that these $|E(G)| + 2$ points form a 3/4-set and hence $B_G(3/4) \geq |E(G)| + 2 = |E(G)| + 1 + \alpha$. Therefore, the theorem follows in this case, too. □

If a connected graph G has exactly one or two vertices, then G is isomorphic to either K_1 or K_2. It is clear that $B_{K_1}(x) = 1$ for all $x > 0$ and that $B_{K_2}(x) = 2$ for all $x \in A_1$. In either case, there is no gap for $B_G(x)$ over A_1 and hence the theorem does not hold for them. On the other hand, it is not so difficult to determine $B_{K_{1,s}}(x)$ over A_1.

$$B_{K_{1,s}}(x) = \begin{cases} 2s & (1/2 < x \leq 2/3) \\ s+1 & (2/3 < x \leq 1) \end{cases}$$

Thus, there is only one gap of $B_{K_{1,s}}(x)$ over A_1. These graphs K_1, K_2 and $K_{1,s}$ must be excluded by the assumptions in the theorems.

3. Another gap

Here we shall discuss basic facts on maximum matchings in a connected graph. It is easy to see that if $\mu(G) = 1$, then either G has at most three vertices, or G is isomorphic to $K_{1,s}$ for $s \geq 3$.

Lemma 9. *Let G be a connected graph having the minimum degree at least 2 and let $M \subset E(G)$ be any maximum matching with $|M| = \mu(G)$. Then there is a cycle $C = e_1 \cdots e_m$, given as a sequence of edges, such that e_{2i} belongs to M and e_{2i-1} does not belong to M for $i \geq 1$; that is, if k is even, then C is an alternating cycles, but otherwise, the pair of consecutive edges e_1 and e_k do not belong to M.*

Proof. Let $Q = v_0 v_1 \cdots v_k$ be a longest alternating path for M with edges $e_i = v_{i-1}v_i$ for $i = 1, \dots, k$. Since G has the minimum degree at least 2, there is at least one edge incident to v_k other than e_k, say $e' = v_k v_{k+1}$. If e_k belongs to M, then e' does not belong to M. In this case, if v_{k+1} did not contain in $\{v_0, v_1, \dots, v_{k-2}\}$, then we could extend the alternating path Q, contrary to the assumption on Q. Thus, v_{k+1} coincides with one of $v_0, \dots, v_{k-2}$ and we find such a cycle described in the lemma with suitable change of indexes.

Now assume that e_k does not belong to M. If we can choose one of edges in M as $e' = v_k v_{k+1}$, then v_{k+1} should coincide with v_0 and e_1 does not belong to M since Q is the longest and since each of $v_1, \dots, v_{k-2}$ is covered by M. In this case, we find an alternating cycles of even length.

Finally, we may assume that v_k is incident to no edge in M and also that v_0 is incident to no edge in M, considering the extension of Q from v_0. However, we could find another matching in G larger than M, exchanging edges along Q. This is contrary to the maximality of M and hence this is not the case. □

Proof of Theorem 5. Suppose that $x > 1/2$ and that $B_G(x) \geq |E(G)| + \mu(G)$. Then there is a maximum x-set S such that $|S| = B_G(x) = |E(G)| + |E_2| - |E_0| \geq |E(G)| + \mu(G)$ and hence we have $|E_2| - |E_0| \geq \mu(G)$.

If E_0 is not empty, then $|E_2| > \mu(G)$ and hence E_2 cannot be a matching and contains a pair of edges having a common end u, say uv_1 and uv_2. Then these two edges form a path $v_1 u v_2$ and contains four points in S, possibly located at the ends of this path. Thus, the four points divide the path of length 2 into at least three segments. This implies that $x \leq 2/3$.

Now assume that E_0 is empty. Then $|E_2| \geq \mu(G)$. If $|E_2| > \mu(G)$, or if E_2 is not a matching, then the same argument as in the previous works for this case and we conclude that $x \leq 2/3$. Thus, we may assume that E_2 is a maximum matching and has exactly $\mu(G)$ edges.

By Lemma 9, there is a cycle $C = e_1 \cdots e_m$ such that e_{2i} belongs to E_2 and e_{2i+1} belongs to E_1. If m is an odd number $2k+1$, then C contains $3k+1$ points in S and these points divide C into $3k+1$ segments. Then we find a segment of length at most $(2k+1)/(3k+1)$ among them. This implies that $x \leq (2k+1)/(3k+1)$. On the other hand, if m is an even number $2k$, then C is an alternating cycle for E_2 and contains $3k$ points in S. This implies that $x \leq 2k/3k = 2/3$, as well as in the previous.

Notice that if $k < h$, then $2/3 < (2h+1)/(3h+1) < (2k+1)/(3k+1) \leq 3/4$ and the second tends to $2/3$ if $h \to \infty$. Therefore, if there is no odd cycle of length at most $2k-1$ in G, then we find either an odd cycle of length at least $2k+1$ or an even cycle. Thus, we have $x \leq (2k+1)/(3k+1)$ in the former case while $x \leq 2/3 < (2k+1)/(3k+1)$ in the latter case. If G is bipartite, that is, if there is no odd cycle in G, then the latter case happens and we have $\leq 2/3$. These imply that there is a positive real number λ with $2/3 \leq \lambda \leq 3/4$ such that $|E(G)| + \mu(G) > B_G(x)$ for $x > \lambda$.

Now, take a maximum matching $M = \{e_1, \ldots, e_m\}$ in G and put $e_i = u_i v_i$. Place two points p_i and q_i on each e_i so that $d_G(p_i, u_i) = d_G(q_i, v_i) = 1/6$ for $i = 1, \ldots, m$ and take the midpoints of all other edges, not belonging to M. It is easy to see that these $|E(G)| + \mu(G)$ points form a $2/3$-set. This implies that $B_G(2/3) \geq |E(G)| + \mu(G)$. Thus, the infimum of such λ's given in the above will be a gap of $B_G(x)$ which lies in $[2/3, 3/4]$. □

Corollary 10. *If G has the minimum degree at least 2 and contains no triangle, then there is a gap λ of $B_G(x)$ with $2/3 \leq \lambda \leq 5/7$, which is different from its first left and first right gaps.*

Proof. Since there is no odd cycle of length $3 = 2\cdot 2 - 1$ by the assumption, we can take 2 as k in Theorem 5 and obtained the upper bound $(2\cdot 2+1)/(3\cdot 2+1) = 5/7$ for λ. Since $5/7 < 3/4$, λ is different to λ_1 and λ_k. □

4. Examples

Here we shall discuss the sharpness of our main theorems, considering concrete examples of beans functions of graphs.

Example 1. The beans functions of the cycle C_m and the path P_m having m vertices for $x \leq 1$ are:

$$B_{C_m}(x) = \left\lfloor \frac{m}{x} \right\rfloor, \quad B_{P_m}(x) = \left\lfloor \frac{m}{x} \right\rfloor + 1 \quad (0 < x \leq 1)$$

We can rewrite these formula as:

$$B_{C_m}(x) = n, \quad B_{P_m}(x) = n+1 \quad (m/(n+1) < x \le m/n)$$

Substituting $n = 2m-1$ and $n = m$, we obtain the following:

$$B_{C_m}(x) = \begin{cases} 2m-1 & (1/2 < x \le m/(2m-1)) \\ m & (m/(m+1) < x \le 1) \end{cases}$$

$$B_{P_m}(x) = \begin{cases} 2m & (1/2 < x \le m/(2m-1)) \\ m+1 & (m/(m+1) < x \le 1) \end{cases}$$

Thus, the first left gap is $\lambda_1 = m/(2m-1)$ and tends to $1/2$ if $m \to \infty$ while the first right gap is $\lambda_{m-1} = m/(m+1)$ and tends to 1 if $m \to \infty$. Since each of these beans functions decreases by 1 when it passed one gap, there are exactly $m-1$ gaps over A_1.

Example 2. The perfect form of the beans function of the complete graph K_m of order $m \ge 3$ has been given in [3]. In particular, it can be presented as below for $x \in (1/2, 1]$. Thus, it has exactly three gaps.

$$B_{K_m}(x) = \begin{cases} \dfrac{m(m-1)}{2} + m - 1 & (1/2 < x \le 3/5) \\ \dfrac{m(m-1)}{2} + \left\lfloor \dfrac{m}{2} \right\rfloor & (3/5 < x \le 2/3) \\ \dfrac{m(m-1)}{2} + 1 & (2/3 < x \le 3/4) \\ \dfrac{m(m-1)}{2} & (3/4 < x \le 1) \end{cases}$$

All of the critical values given in Theorem 3, 4 and 5 appear as three gaps in the above. Notice that this formula can be expressed as:

$$B_{K_m}(x) = \begin{cases} |E(K_m)| + |V(G)| - 1 & (1/2 < x \le 3/5) \\ |E(K_m)| + \mu(K_m) & (3/5 < x \le 2/3) \\ |E(K_m)| + 1 & (2/3 < x \le 3/4) \\ |E(K_m)| & (3/4 < x \le 1) \end{cases}$$

These values in cases appear as the bounds for $B_G(x)$ given in the theorems.

One might wonder if $B_G(2/3) = |E(G)| + \mu(G)$, following our argument in the proof of Theorem 5. The next lemma will deny it:

Lemma 11. *Let G be a bipartite connected graph which has the minimum degree at least 2 and let $V(G) = X \cup Y$ be its bipartition. If $|X| \geq |Y|$, then $B_G(2/3) \geq |E(G)| + |X| > B_G(x)$ for $x > 2/3$.*

Proof. We call each vertex in X a *black vertex* and one in Y a *white vertex*. Put a point at each black vertex in X and put a point p on each edge incident to each white vertex u in Y so that $d_G(p,u) = 1/3$. Then these points form a 2/3-set and they are $|E(G)| + |X|$ in number. Thus, we have $B_G(2/3) \geq |E(G)| + |X|$.

Now we shall show that if $B_G(x) \geq |E(G)| + |X|$, then $x \leq 2/3$. Let S be a maximum x-set and suppose that $B_G(x) \geq |E(G)| + |X|$. Then we have $|E_2| - |E_0| \geq |X|$. If $|E_2| > |X|$ or if E_2 is not a matching in G, then there is a vertex v in G such that two edges e_1 and e_2 in E_2 incident to v. These edges form a path of length 2 and contains four points in S, which divide the path into at least three segments. This implies that $x \leq 2/3$. Thus, we may assume that E_2 is a matching with $|E_2| = |X|$ and hence E_0 is empty. Since $|X| \geq |Y|$, E_2 must be a *perfect matching*, that is, E_2 covers all vertices, making pairs of black and white vertices.

Since any perfect matching is a maximum matching and G is a bipartite graph with the minimum degree at least 2, then there is an alternating cycle C for E_2 in G by Lemma 9. If C has length $2k$, then it contains $3k$ points in S. This implies that $x \leq 2/3$. Therefore, if $x > 2/3$, then $B_G(x) < |E(G)| + |X|$. □

Example 3. Consider the complete bipartite graph $K_{s,t}$ with $s, t \geq 2$. It is clear that if $s < t$, then $\mu(K_{s,t}) = s$ and we can use the independent set of size t as X in the above lemma.

$$B_{K_{s,t}}(2/3) \geq |E(K_{s,t})| + t = |E(K_{s,t})| + \mu(K_{s,t}) + (t - s)$$

Thus, $B_{K_{s,t}}(2/3)$ can be arbitrarily larger than $|E(K_{s,t})| + \mu(K_{s,t})$ if the difference between the sizes of two independent sets, $t - s$, is arbitrarily large.

Our arguments for Theorem 5 do not cover the case when G is a tree since we assume that G has the minimum degree at least 2. However, very similar arguments will work for trees.

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Kengo Enami
Graduate School of Environment and Information Sciences,
Yokohama National University,
79-2 Tokiwadai, Hodogaya-Ku, Yokohama 240-8501,
Japan.
E-mail: `enami-kengo-ps@ynu.jp`

Seiya Negami
Faculty of Environment and Information Sciences,
Yokohama National University,
79-2 Tokiwadai, Hodogaya-Ku, Yokohama 240-8501,
Japan.
E-mail: `negami-seiya-vj@ynu.ac.jp`