Bilevel Programming Approaches to One-Shot Decision Problems and Their Applications

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Preface

The main topic of this dissertation is one-shot (one-time) decision problems involving risk or uncertainty. The one-shot decision problem refers in particular to the situation where the decision is made only once. This sort of decision problems is often encountered in social production and living activities that are concerned with short-term benefits. Considering the one-time features of such problems, we propose new decision approaches based on the one-shot decision theory (OSDT). Unlike existing ways, the OSDT-based decision approach obtains a decision based on the most appropriate scenario (state) for the decision-maker. OSDT-based decision models are bilevel programming problems with maximin or minimax lower level programs. Since their lower level problems are non-smooth and sometimes even non-convex, traditional solution methods for bilevel optimization are not applicable to them directly. The second purpose of this dissertation is to provide efficient optimization methods to solve these special bilevel programs. As applications, we utilize the proposed approach to analyze a single-item newsvendor problem and a multi-item production planning problem.

The dissertation is mainly composed of three parts. The first part is shown in Chapter 2, which presents modeling processes. We first overview existing decision approaches including the expected value approach, the maximax approach and the maximin approach, and then present the OSDT-based decision approach. Specifically, the OSDT-based decision approach obtains an optimal decision by the following two-step process: in the first step, for each feasible decision, the decision-maker examines all possible states and then chooses an appropriate one as the focus point with considering the probability of its occurrence and the payoff associated with it; in the second step, based on the focus points of all feasible decisions, the decision-maker determines the best decision by considering which one coupled with its focus point can generate the highest payoff. OSDT-based decision models are behavioral models in which, with different preferences for focus points, decision-makers may make different decisions.

The second part is presented in Chapter 3, in which we develop new solution methods to OSDT-based decision models. In fact, the OSDT-based decision model is a bilevel programming problem where the upper level program is used to determine the optimal decision based on its focus point and the lower level program is used to seek this focus point. Since these bilevel programs have non-smooth and non-convex lower level programs, they are difficult to deal with. In this part, we propose new solution methods to these special bilevel programs by reformulating them into general single-level optimization problems, such that they can be solved via the commonly used optimization methods or software.

The third part consists of Chapters 4 and 5, in which OSDT-based newsvendor models for an innovative product and OSDT-based production planning models for multiple short life-cycle products are built, respectively. We apply the proposed reformulation methods to these specific models and provide corresponding single-level equivalent models for them. The effectiveness of the proposed methods is also examined by numerical examples.

One-shot decision problems are important managerial decision problems. Bilevel Programs are important constrained optimization problems. We hope that the results obtained in this dissertation will be helpful to advance the research in these two fields.

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Contents

Chapter 1

Introduction

In many decision problems, we often encounter a situation where decision-makers have one and only one opportunity to make a decision. Such decision problems are called one-shot (one-time) decision problems. The one-shot decision problem is an important decision problem, which arises in various areas of social production and living activities interested in short-term benefits. In general, a decision problem contains a set of alternative actions and each of which corresponds to a set of possible states of nature. For each alternative, one and only one state will happen in the future, resulting in an outcome associated with that decision.

When making decisions in practice, decision-makers may face three different decision conditions: certainty, risk and uncertainty. Certainty is for the situation where the state is unique for each alternative. In this situation, since the outcomes of all alternatives are accurately known, the decision-makers can choose the best decision or at least they can choose an alternative that generates the best outcome. Obviously, certainty is an ideal condition for decision-making. However, in practice, lots of decision problems always involve risks or uncertainties. More specifically, risk involves the situation where the probability of every possible state is known, under which the decision-makers can exactly calculate the probabilities of all possible outcomes. Uncertainty is for the situation where the decision-makers know all possible states related to every alternative but they do not obtain exact probabilities of them due to limited information.

Different decision situations, especially involving risks and uncertainties, require different decision theories, among which the expected utility theory of von Neumann and Morgenstern (1944) and the subjective expected utility theory of Savage (1954) have almost been regarded as a normative theory for rational choice under risk and uncertainty, respectively. However, plenty of hypothetical experiments show that people's preferences systematically violate the axioms of these two decision theories, such as the independence in the expected utility theory and the sure-thing principle in the subjective expected utility theory; see, e.g., Allais (1953), Ellsberg (1961), Kahneman and Tversky (1979), and Starmer (2000).

From the aspect of mathematical optimization, decision problems under risk or uncertainty are usually modeled using the expected value: a weighted average of all possible outcomes (or utilities defined over the outcomes) for each alternative where objective or subjective probabilities are used as weights. We can easily understand that if the process repeats over a great large number of times in the same decision circumstances, then the expected value based decision will lead to the largest average outcome. However, this largest expected value may not be obtained in the short term or if the decision is made only once. In other words, for a one-shot decision problem, it is less justifiable to use the expected value to evaluate the decision.

Another two commonly used decision approaches, especially for situations involving uncertainty, are based on the maximax and maximin criterion: the maximax approach evaluates each decision by the maximum possible outcome associated with that decision, whereas the maximin approach considers the minimum possible outcome. The maximax approach is appropriate for an optimistic decision-maker who is often attracted by the best results. The maximin approach would be suitable for a pessimistic decision-maker who is always worried about the worst results. Clearly, the maximax approach might be too daring whereas the maximin approach might be too conservative in the sense that the former only focuses on the best scenario but the latter only considers the worst scenario no matter what will happen in the future. Besides maximax and maximin criterions, the minimax regret criterion, Hurwicz criterion and Laplace criterion are also widely used in applications. Since the minimax regret approach is to minimize the worst-case regret (difference or ratio of the outcomes) rather than to minimize the outcome itself, it is sometimes not as pessimistic as the standard maximin approach. Likewise, to achieve a compromise between the optimism of the maximax criterion and the pessimism of the maximin criterion, Hurwicz criterion is proposed that is a weighted average of these two extremes. Laplace criterion applies to the situation where the decision-maker is completely ignorant about the probability of all states. In this case, the probability of each state is usually considered to be equal and the decision-maker chooses a decision by maximizing the expected outcome.

In recent years, Guo (2011) proposed the one-shot decision theory (OSDT) for one-shot decision problems under risk or uncertainty, and applied it to several problems in business and management; see, e.g., Guo (2010a), Guo (2010b), Guo and Ma (2014). Based on the OSDT, Wang and Guo (2017) built a behavioral model for explaining the anomalies in the first-price sealed-bid auctions. To the best of our knowledge, this is the first time to provide a theoretical explanation for throwing-away phenomenon for such auction problems. Recently, Guo (2017) advanced the OSDT and axiomatized the focus theory of choice (FTC) for one-shot decision-making under risk or uncertainty. OSDT and FTC are based on two basic assumptions (axioms): a decision-maker can choose the most attractive scenario (state) for him/her from among all possible scenarios for each action; a decision-maker can choose the most preferred action by comparing the salient states of all actions. These assumptions are intuitively appealing and there are indeed growing evidences supporting them; see, e.g., Bordalo et al. (2012), Busse et al. (2013), Orquin and Loose (2013), Stewart et al. (2016). OSDT and FTC are behavioral decision theories that can explain many puzzling phenomena in psychology and economics, such as the St. Petersburg, Allais and Ellsberg paradoxes.

In this dissertation, we propose new decision approaches to one-shot decision problems based on the OSDT. Different from traditional decision models, the OSDT-based decision model is a bilevel programming problem that obtains an optimal decision by the following two steps: (1) for each feasible decision variable (or vector) given by the upper level problem, the lower level problem examines every possible realization of the random variable (or vector) with considering the probability of this realization and the payoff associated with it and then chooses one as a focus point of this decision; (2) based on the focus points of all feasible decisions, the upper level problem determines an optimal decision by considering which one coupled with its focus point can generate the highest payoff. We consider two types of behaviors of the decision-maker choosing focus points: one is choosing the scenario which has a relatively high probability and can bring about a relatively high payoff, such focus points are called active focus points; the other is choosing the scenario which has a relatively high probability but can lead to a relatively low payoff, such focus points are called passive focus points. With different preferences for choosing the focus points, the decision-makers may make different decisions.

Bilevel program is a special constrained optimization problem, whose constraints or part of constraints are defined by another optimization problem. Since bilevel program is a non-convex optimization problem with an implicitly determined feasible set, to solve it or find its optimality conditions, the problem has to be reformulated as a single-level optimization problem. Since OSDT-based decision models have non-smooth and non-convex lower level programs, traditional reformulation methods are not applicable to them. In this dissertation, we propose new reformulation methods to these special bilevel programs by transforming them into general single-level optimization problems. We consider two models with one-dimensional lower level variables and multi-dimensional lower level variables, respectively. The reformulated models are more tractable than original bilevel optimization models so that they can be solved with the commonly used optimization methods or software. Finally, we apply the OSDT-based decision approach to a single-item newsvendor problem and a multi-item production planning problem.

The reminder of this dissertation is organized as follows. In Chapter 2, we propose bilevel programming approaches to one-shot decision problems with the OSDT and, moreover, we compare the OSDT-based decision approaches with several existing decision approaches including the maximax approach, the maximin approach and the expected value approach. In Chapter 3, we provide new methods to the proposed bilevel optimization models, by which these bilevel models can be equivalently reformulated as tractable single-level optimization problems. In Chapters 4 and 5, we build OSDT-based newsvendor models for an innovative product and OSDT-based production planning models for multiple short life-cycle products, respectively. We apply the proposed reformulation methods to these specific models and give corresponding single-level equivalent models for them. In particular, for OSDT-based production planning models, we consider two types of constraints of the lower level problems, that is, cuboid constraints and ellipsoidal constraints. Preliminary numerical experiments and computational discussions are also given in these two chapters. Finally, we conclude the research of this dissertation in Chapter 6.

Chapter 2

Bilevel Programming Approaches to One-Shot Decision Problems

2.1 Research Problem

Consider a decision problem with uncertainty where the state is characterized as a real-valued random vector and the action is represented by a real-valued decision vector. In this dissertation, we use bold $\boldsymbol{\xi}$ to represent the random vector in order to distinguish it from its possible realization ξ. More precisely, $\boldsymbol{\xi} = \xi$ means that there exists a random event ω such that its outcome corresponds to the value ξ. Denoting by x the decision vector, then we can model such an uncertain decision problem as the following stochastic optimization problem:

$$\max_{x} \ f(x, \boldsymbol{\xi}) \quad \text{s.t. } x \in X. \tag{2.1}$$

Here $X \subset R^n$ represents the feasible region of decision vectors and $n \geq 1$; $\Xi \subset R^m$ stands for the range of all possible values that the random vector $\boldsymbol{\xi}$ can realize and $m \geq 1$; $f: R^n \times R^m \rightarrow R$ denotes the original payoff function (or a utility function defined over the payoff). We assume throughout that the probability density (or mass) function $\rho: \Xi \rightarrow R_+$ is given for continuous (or discrete) random vector $\boldsymbol{\xi}$. Usually, the probability distribution can be estimated from objective historical data or is exogenously given to represent a decision-maker's subjective degree of belief on the occurrence of each possible realization.

In this dissertation, we are interested in resolving (2.1) with a situation where there is only one chance to make a decision before knowing the real realization of the random vector. Since $\boldsymbol{\xi}$ is a random vector, the meaning of "max" is not clear at all. In other words, (2.1) is not a well-defined problem. Therefore, it is necessary to revise the modeling process and develop suitable decision approaches to (2.1).

2.2 Existing Decision Approaches and Optimization Models

In this section, we briefly review existing three kinds of decision approaches to (2.1) and use a simple example to illustrate their differences. The first approach is based on the expected value, which indicates that the best choice of x is a global optimal solution of the following optimization problem:

$$\max_{x} \ \mathbb{E}\{f(x, \boldsymbol{\xi})\} \quad \text{s.t. } x \in X, \tag{2.2}$$

where $\mathbb{E}\{\cdot\}$ represents the expectation operator with respect to the distribution of $\boldsymbol{\xi}$.

Setting $F(x) \coloneqq \mathbb{E}\{f(x, \boldsymbol{\xi})\}$, we can understand that (2.2) is a deterministic mathematical programming problem. If F(x) has a closed form, then (2.2) can be solved using suitable optimization methods. Now let us consider a sample example as follows:

$$\max_{x} \ \boldsymbol{\xi_1} x_1 + \boldsymbol{\xi_2} x_2 \quad \text{s.t. } 3x_1 + x_2 \leq 9,\ x_1 + 2x_2 \leq 8,\ x_1, x_2 \in N. \tag{2.3}$$

Suppose that $\boldsymbol{\xi_1}$ and $\boldsymbol{\xi_2}$ are independent, discrete random variables and the probability distributions of $\boldsymbol{\xi_1}$ and $\boldsymbol{\xi_2}$ are given by $\rho(\boldsymbol{\xi_1} = 1) = 0.6$, $\rho(\boldsymbol{\xi_1} = 13) = 0.4$ and $\rho(\boldsymbol{\xi_2} = 3) = 0.3$, $\rho(\boldsymbol{\xi_2} = 4) = 0.7$, respectively. We can obtain that the joint probability distribution of the random vector $\boldsymbol{\xi} \coloneqq (\boldsymbol{\xi}_1; \boldsymbol{\xi}_2)$ as $\rho\big(\boldsymbol{\xi} = (1;3)\big) = 0.18$, $\rho\big(\boldsymbol{\xi} = (1;4)\big) = 0.42$, $\rho\big(\boldsymbol{\xi} = (13;3)\big) = 0.12$, $\rho\big(\boldsymbol{\xi} = (13;4)\big) = 0.28$, and the expectation of $\boldsymbol{\xi}$ as $\bar{\boldsymbol{\xi}} = (5.8; 3.7)$. Further, we can obtain the optimal solutions corresponding to four scenarios of the problem (2.3), that is, $x^*\big((1;3)\big) = (0;4)$, $x^*\big((1;4)\big) = (0;4)$, $x^*\big((13;3)\big) = (3;0)$ and $x^*\big((13;4)\big) = (3;0)$ while the optimal solution determined by (2.2) is $x^*\big(\bar{\boldsymbol{\xi}}\big) = (2;3)$ with $\bar{\boldsymbol{\xi}} = (5.8; 3.7)$. Clearly, $\bar{\boldsymbol{\xi}}$ does not correspond to any realization (scenario) of $\boldsymbol{\xi}$. Although $x^*\big(\bar{\boldsymbol{\xi}}\big)$ is not an optimal solution at all no matter which scenario happens, by the Law of Large Numbers, if the process repeats over a large number of times, then the solution of (2.2) will be optimal on average. Indeed, in this case, it makes sense to talk about the expected value. However, for one-shot decision problems, using expected values to evaluate decisions might not be reasonable.

The maximax approach is another common decision approach for solving (2.1), which indicates that the best choice of x is a global optimal solution of the following maximax optimization problem:

$$\max_{x} \left\{\max_{\xi}\ f(x,\xi) : \xi \in \Xi\right\} \quad \text{s.t.}\ \ x \in X. \tag{2.4}$$

Setting $F_u(x) \coloneqq \max_{\xi\in\Xi}\ f(x,\xi)$, we can see that (2.4) is a deterministic mathematical programming problem. If $F_u(x)$ has a closed form, then (2.4) can be solved using suitable optimization methods. Considering the example (2.3), we have $F_u(x) = 13x_1 + 4x_2$. That is, the best scenario is $\xi_u = (13;4)$ and $x^*(\xi_u) = (3;0)$ is obtained by (2.4) as an optimal solution to correspond to this scenario. The maximax approach is an important decision criterion assuming that the best scenario will happen whatever action is taken. This approach is appropriate for an optimistic decision-maker who is often attracted by the best results.

Another common decision approach for solving (2.1) is called the maximin approach, which indicates that the best choice of x is a global optimal solution of the following maximin optimization problem:

$$\max_{x} \left\{ \min_{\xi} \ f(x, \xi) : \xi \in \Xi \right\} \quad \text{s.t. } x \in X. \tag{2.5}$$

Similarly, setting $F_l(x) \coloneqq \min_{\xi \in \Xi} f(x, \xi)$, we know that (2.5) is a deterministic mathematical programming problem. If $F_l(x)$ has a closed form, then (2.5) can be solved using suitable optimization methods. Considering the example (2.3), we have $F_l(x) = x_1 + 3x_2$. That is, the worst scenario is $\xi_l = (1; 3)$ and $x^*(\xi_l) = (0; 4)$ is obtained by (2.5) as an optimal solution to correspond to this scenario. The maximin approach is another important decision criterion which assumes that the worst scenario will appear whatever action is taken. Opposite to the maximax approach, the maximin approach would be suitable for a pessimistic decision-maker who is always worried about the worst results.

2.3 Bilevel Programming Approaches based on One-Shot Decision Theory

We have already noted that for a different realization ξ of the random vector $\boldsymbol{\xi}$, $f(x, \xi)$ can be quite different from $\mathbb{E}\{f(x, \boldsymbol{\xi})\}$. Therefore, it makes no sense to consider using the expected value approach to solve a one-shot decision problem. We also noted that the maximax approach might be too daring whereas the maximin approach might be too conservative in the sense that the former only considers the best scenario but the latter only takes into account the worst scenario regardless of their probabilities.

In this section, we remodel the one-shot decision problem (2.1) where x should be chosen before a real realization (a scenario) of $\boldsymbol{\xi}$ is known. Note that the decision-maker has only one opportunity to make a decision before the scenario reveals and only one scenario will appear. Considering the one-time features of such problems, we propose new decision approaches to (2.1) by the following two steps (Guo, 2011). In the first step, for each feasible decision, the decision-maker examines every possible realization of the random vector with considering the probability of its occurrence and the payoff associated with it and chooses one as a focus point of that decision. In the second step, the decision-maker determines such a decision as the optimal one that generates the maximum payoff with its focus points.

We consider two types of behaviors of the decision-maker choosing focus points: one is choosing the scenario (one realization of the random vector) which has a relatively high probability and can bring about a relatively high payoff as the active focus point; the other is choosing the scenario which has a relatively high probability but can lead to a relatively low payoff as the passive focus point. We formulate these two types of focus points as follows.

(a) The active focus point of x, denoted as $\xi^1(x)$, is a non-dominated optimal solution of the following two-objective optimization problem:

$$\left(\max_{\xi} \rho(\xi),\ \max_{\xi} f(x,\xi)\right) \quad \text{s.t. } \xi \in \Xi. \tag{2.6}$$

(b) The passive focus point of x, denoted as $\xi^2(x)$, is a non-dominated optimal solution of the following two-objective optimization problem:

$$\left(\max_{\xi} \rho(\xi), \min_{\xi} f(x,\xi)\right) \quad \text{s.t. } \xi \in \Xi. \tag{2.7}$$

(2.6) and (2.7) reflect the optimistic and pessimistic attitudes to evaluate the scenario, respectively. We call the decision-maker who takes into account the active focus points or passive focus points as an active decision-maker or passive decision-maker, respectively.

The decision-maker considers the focus point as his/her most appropriate scenario for each feasible decision and then chooses the optimal decision that generates the highest payoff when its focus point occurs. It should be noted that for each $x \in X$, we only consider one non-dominated solution of (2.6) or (2.7) as an active or passive focus point of x. In other words, we do not take into account the frontier of (2.6) or (2.7). Based on the above considerations, we formulate this decision approach for these two kinds of decision-makers as follows.

(c) The optimal decision for active decision-makers is a global optimal solution of the following optimization problem:

$$\max_{x} f(x, \xi^1(x)) \quad \text{s.t. } x \in X. \tag{2.8}$$

(d) The optimal decision for passive decision-makers is a global optimal solution of the following optimization problem:

$$\max_{x} f(x, \xi^2(x)) \quad \text{s.t. } x \in X. \tag{2.9}$$

2.3.1 Bilevel Optimization Model with Active Focus Points

Lemma 2.1 The maximax optimization model (2.4) is a special case of (2.8).

Proof. For any $x \in X$, we have $F_u(x) = \max_{\xi \in \Xi} f(x, \xi)$. Given $\xi_u(x) \in \operatorname{argmax}_{\xi \in \Xi} f(x, \xi)$, it is easy to verify that $\xi_u(x)$ is Pareto optimal for the two-objective optimization problem (2.6) due to $f(x, \xi_u(x)) = F_u(x)$. In other words, if the best scenario $\xi_u(x)$ is chosen by the decision-maker as the active focus point of every feasible $x \in X$, then (2.8) with these active focus points reduces to the maximax optimization problem (2.4). ■

Lemma 2.1 clarifies the relationship between (2.8) and (2.4). We can easily observe that the maximax approach completely ignores probabilities. In what follows, we give other approach which can obtain a non-dominated optimal solution of the problem (2.6) with considering the payoff and the probability simultaneously. To this end, we give two definitions as follows.

Definition 2.1 (Relative Likelihood Function) Let $\rho: \Xi \to R_+$ be the original probability density (or mass) function for continuous (or discrete) random vector $\boldsymbol{\xi}$ and π be a function defined from set Ξ to set $[0,1]$. We call $\pi(\xi)$ as the relative likelihood degree of ξ if it satisfies

$$\max_{\xi\in\Xi} \pi(\xi) = 1,$$

and, for any $\xi_1, \xi_2 \in \Xi$,

$$\pi(\xi_1) > \pi(\xi_2) \iff \rho(\xi_1) > \rho(\xi_2) \text{ and } \pi(\xi_1) = \pi(\xi_2) \iff \rho(\xi_1) = \rho(\xi_2).$$

Definition 2.2 (Satisfaction Function) Let $f: X\times\Xi \to R$ be the original payoff function and u be a function defined from set $X\times\Xi$ to set $[0,1]$. We call $u(x,\xi)$ as the satisfaction level of $\xi \in \Xi$ for $x \in X$ if it satisfies

$$\max_{(x;\xi)\in X\times\Xi} u(x,\xi) = 1,$$

and, for any $x \in X$, for any $\xi_1, \xi_2 \in \Xi$,

$$u(x,\xi_1) > u(x,\xi_2) \iff f(x,\xi_1) > f(x,\xi_2) \text{ and } u(x,\xi_1) = u(x,\xi_2) \iff f(x,\xi_1) = f(x,\xi_2).$$

The relative likelihood degree and the satisfaction level are used to represent the relative positions of the probability and the payoff, respectively. In fact, there are many ways that we can use to choose π and u satisfying Definitions 2.1 and 2.2, respectively.

For example, we can use

$$\pi(\xi) = \frac{\rho(\xi) - \rho_l}{\rho_u - \rho_l}, \tag{2.10}$$

where ρ_u and ρ_l are the upper and lower bounds of ρ in Ξ, respectively, that is,

$$\rho_u := \max_{\xi\in\Xi} \rho(\xi) \text{ and } \rho_l := \min_{\xi\in\Xi} \rho(\xi).$$

If the random vector follows a log-concave distribution, i.e., $\log(\rho)$ is a concave function, we can use the following relative likelihood function:

$$\pi(\xi) = \frac{\log(\rho(\xi)) - \log(\rho_l)}{\log(\rho_u) - \log(\rho_l)}. \tag{2.11}$$

Similarly, we often use the following satisfaction level function:

$$u(x, \xi) = \frac{f(x, \xi) - f_l}{f_u - f_l}, \tag{2.12}$$

where f_u and f_l are the upper and lower bounds of f in X×Ξ, respectively, that is,

$$f_u := \max_{(x;\xi)\in X\times\Xi} f(x, \xi) \quad \text{and} \quad f_l := \min_{(x;\xi)\in X\times\Xi} f(x, \xi).$$

With π and u, we give another non-dominated solution of the problem (2.6) as follows:

$$\max_{\xi\in\Xi} \ \min\{\pi(\xi), u(x, \xi)\}. \tag{2.13}$$

Since the minimal function between $\pi(\xi)$ and $u(x, \xi)$ can be expressed as

$$\min\{\pi(\xi), u(x, \xi)\} = \frac{1}{2}\{\pi(\xi) + u(x, \xi) - |\pi(\xi) - u(x, \xi)|\},$$

it is straightforward that using (2.13) we can find out a scenario ξ which simultaneously makes π and u large. In other words, comparing with $\pi(\xi) + u(x, \xi)$ and $\pi(\xi) * u(x, \xi)$, (2.13) can avoid obtaining ξ with a large $\pi(\xi)$ but a small $u(x, \xi)$ or a small $\pi(\xi)$ but a large $u(x, \xi)$ because it tries to make $|\pi(\xi) - u(x, \xi)|$ as smaller as possible.

With (2.13), we formulate (2.8) as the following optimization model with active focus points.

Bilevel optimization model with active focus points (Model A):

$$\max_{(x;\xi)} f(x, \xi) \qquad \text{s.t. } x \in X, \xi \in \Lambda^1(x), \tag{2.14}$$

where $\Lambda^1(x)$ (which is related to x) denotes the set of global optimal solutions of the following maximin optimization problem:

$$\max_{\xi\in\Xi} \min\{\pi(\xi), u(x, \xi)\}. \tag{2.15}$$

Clearly, Model A is a bilevel optimization model where the upper level problem (2.14) is used to find the optimal decision for maximizing the payoff on a specific scenario associated with it; the lower level problem (2.15) is used to seek this scenario which has a relatively high probability and can cause a relatively high payoff. Considering the example (2.3), by using (2.10) and (2.12) we can obtain that the optimal decision is $x^* = (3;0)$ and its active focus point is $\xi^1(x^*) = (13;4)$. It means that the active decision-maker chooses $(3;0)$ as the best decision based on the scenario $(13;4)$. The relative likelihood degree of $\xi^1(x^*)$ is 0.5333 and it can lead to the satisfaction level of 1 with x^*.

2.3.2 Bilevel Optimization Model with Passive Focus Points

Lemma 2.2 The maximin optimization model (2.5) is a special case of (2.9).

Proof. For any $x \in X$, we have $F_l(x) = \min_{\xi\in\Xi} f(x,\xi)$. Given $\xi_l(x) \in \operatorname{argmin}_{\xi\in\Xi} f(x,\xi)$, it is easy to verify that $\xi_l(x)$ is Pareto optimal for the two-objective optimization problem (2.7) due to $f\big(x,\xi_l(x)\big) = F_l(x)$. In other words, if the worst scenario $\xi_l(x)$ is chosen by the decision-maker as the passive focus point of every feasible $x \in X$, then (2.9) with these passive focus points reduces to the maximin optimization problem (2.5). ■

Lemma 2.2 clarifies the relationship between (2.9) and (2.5). It follows that the maximin approach completely ignores probabilities. With π and u, we give another non-dominated solution of the problem (2.7) as follows:

$$\min_{\xi\in\Xi} \max\{1-\pi(\xi), u(x,\xi)\}. \tag{2.16}$$

Since the maximal function between $1-\pi(\xi)$ and $u(x,\xi)$ can be expressed as

$$\max\{1-\pi(\xi), u(x,\xi)\} = \frac{1}{2}\{1-\pi(\xi)+u(x,\xi)+|1-\pi(\xi)-u(x,\xi)|\},$$

we know that using (2.16) we can find out a scenario ξ with a relatively larger π and a relatively smaller u.

With (2.16), we formulate (2.9) as the following optimization model with passive focus points.

Bilevel optimization model with passive focus points (Model B):

$$\max_{(x;\xi)} f(x,\xi) \quad \text{s.t. } x \in X, \xi \in \Lambda^2(x), \tag{2.17}$$

where $\Lambda^2(x)$ (which is related to x) denotes the set of globally optimal solutions of the following minimax optimization problem:

$$\min_{\xi \in \Xi} \max\{1 - \pi(\xi), u(x,\xi)\}. \tag{2.18}$$

Clearly, Model B is a bilevel optimization model where the upper level problem (2.17) is used to find the optimal decision for maximizing the payoff on a specific scenario associated with it; the lower level problem (2.18) is used to find out this scenario which has a relatively high probability and can cause a relatively low payoff. Considering the example (2.3), by using (2.10) and (2.12) we can obtain that the optimal decision is $x^* = (0;4)$ whose passive focus point is $\xi^2(x^*) = (1;4)$. It means that the passive decision-maker chooses $(0;4)$ as the best decision based on the scenario $(1;4)$. The relative likelihood degree of $\xi^2(x^*)$ is 1 and it can lead to the satisfaction level of 0.4103 with x^*.

2.4 Concluding Remarks

In this chapter, we build OSDT-based decision models for one-shot decision problems where the obtained optimal decisions are based on the most appropriate scenarios for the decision-maker with considering the payoff and the probability. OSDT-based decision models (Model A and Model B) are some bilevel programming problems with maximin or minimax lower level problems. Comparing with the maximax approach and the maximin approach, although Model

A and Model B utilize the ‘maximin’ and ‘minimax’ operators, they also incorporate the probability, and so they simply eliminate the possibility of obtaining extreme results, that is, too optimistic results from the maximax approach or too conservative results from the maximin approach. We provide a fundamentally new idea to deal with one-shot decision problems involving uncertainty. Based on different assumptions, we can utilize different decision approaches to obtain a decision.

Chapter 3

Solution Methods to the Proposed Bilevel Optimization Models

3.1 Introduction

Bilevel programming problem (BLPP) is a special constrained optimization problem whose constraints have another optimization problem. BLPP has two kinds of decision variables, referred to as the upper level variable and the lower level variable. The general formulation is

$$\max_{(x;\,y)} F(x,y) \quad \text{s.t. } x \in X, y \in S(x), \tag{3.1}$$

where $S(x)$ (which is related to x) denotes the set of global optimal solutions of the following optimization problem:

$$\max_{y} G(x,y) \quad \text{s.t.}\, y \in Y(x), \tag{3.2}$$

where $X \subset R^n$, $Y(x) \subset R^m$ for any $x \in X$, and $F, G: R^n \times R^m \to R$ are functions.

BLPP (3.1)-(3.2) represents an optimistic optimization approach in which the follower (lower level) is assumed to be cooperative and the leader (upper level) is allowed to choose the most suitable element from the set of global optimal solutions of the lower level problem. On the contrary, a pessimistic optimization approach deals with the case that the follower may be non-cooperative. In this case, the leader cannot decide which of the best responses is implemented by the follower so that he/she chooses a decision that performs best in the case that the worst follower response happens, that is, solving the following pessimistic BLPP:

$$\max_{x \in X} \quad \min_{y \in S(x)} F(x, y). \tag{3.3}$$

BLPP has been always an important research area. It was initially introduced by von Stackelberg (1952) for modeling a duopoly market. From then on, a great number of contributions including theories, algorithms and applications for BLPPs have been made by researchers; see, e.g., Allende and Still (2013), Bard (1998), Colson et al. (2005), Dempe (2002), Dempe and Zemkoho (2013), Dempe et al. (2012), Dempe and Zemkoho (2014), Lin et al. (2014), Ye and Zhu (1995), Ye and Zhu (2010). Since BLPP is a non-convex optimization problem with an implicitly determined feasible set, in order to solve it or find its optimality conditions, the problem has to be reformulated as a single-level optimization problem.

When the lower level program is a convex optimization problem and Salter's condition holds for it, a common method to BLPP is to replace the lower level program by its Karush-Kuhn-Tucker (KKT) condition and then solve a mathematical program with equilibrium constraints (MPEC) or mathematical program with complementarity constrains (MPCC). However, solving an MPEC is still difficult because its constraints fail to satisfy the standard constraint qualifications, such as the most commonly used Mangasarian-Fromovitz constraint qualification (MFCQ); see, e.g., [Proposition 1.1, Ye (1997)]. Even under some convexity conditions on the function F and the set X, MPEC is still not easy to be solved due to the non-convexities that occur in the Lagrangean or complementarity constraints. In other words, applying the classical optimization theories and algorithms in nonlinear programs to an MPCC directly may not be valid. To remedy this unusual structural optimization problem, several variants of stationary conditions including the strong (S-), Mordukhovich (M-) and Clarke (C-) stationary conditions have been proposed and various optimization methods have been studied; see, e.g., Pang et al. (1996), Facchinei et al. (1999), Ye (2005), Fletcher et al. (2006), Guo et al. (2015), Lin and Fukushima (2005), Scholtes (2001), Zhu and Lin (2016).

When the lower level program is a non-convex optimization problem or the Slater's condition is violated for some upper level variables, the BLPP may not be equivalent to the model reformulated by the KKT approach; see, Dempe et al. (2012). Another appealing way to BLPP is based on the so-called optimal value function of the lower level program. For any x given by the upper level, we define the optimal value function of the lower level problem (3.2) as

$$V(x) := \max_{y \in Y(x)} G(x, y), \tag{3.4}$$

then BLPP (3.1)-(2.2) can be reformulated as the following optimization problem:

$$\max_{(x;\, y)} F(x, y) \quad \text{s.t. } V(x) - G(x, y) \leq 0, x \in X, y \in Y(x). \tag{3.5}$$

This approach was first introduced by Outrata (1990) for obtaining a numerical solution and subsequently used by Ye and Zhu (1995) for obtaining necessary optimality conditions. Recently, Lin et al. (2014) used this approach to solve a simple BLPP where the constraint set of the lower level program does not depend on x, that is, $Y(x) \equiv Y$. Xu and Ye (2014) proposed a smoothing projected gradient algorithm for solving (3.4) by using some smooth functions to approximate the optimal value function. In fact, it is not difficult to see that this approach does not require the lower level problem (3.2) to be a convex optimization. But it is significantly difficult to design effective numerical algorithms to solve the reformulated model (3.5) due to the existence of an implicitly determined constraint function. Due to inherent mathematical difficulties, most papers for a BLPP assume that the lower level is a smooth convex optimization problem so that its KKT reformulation can be solved by using appropriate methods for MPCC.

However, to the best of our knowledge, there is only a few papers dealing with BLPPs with some special non-smooth lower level programs; see, Li et al. (2014), Solodov (2007). Recently, Dempe and Zemkoho (2014) extended the KKT condition based single-level reformulation to non-smooth BLPPs and discussed various stationarity conditions. We have already observed from Chapter 2 that the proposed models are some special BLPPs with maximin or minimax lower level problems. Since the lower level problems are non-smooth and sometimes even non-convex, they are difficult to be solved. In this paper, we focus on proposing solvable single-level reformulations of these non-smooth BLPPs.

The remainder of this chapter is organized as follows. In Section 3.2, we use the KKT condition based method to convert these BLPPs into MPECs. We show the condition which guarantees the global optimal solutions of the reformulated models and the original BLPPs are equivalent. In Section 3.3, we propose other two types of single-level reformulations based on some relatively weak conditions, the proposed models are easily to be solved with the commonly used optimization methods or software. In Section 3.4, some conclusions are given.

3.2 Traditional Solution Methods.

In this section, we consider using traditional KKT condition based methods to solve the BLPP (2.14)-(2.15) and the BLPP (2.17)-(2.18). To this end, we assume that Ξ takes the form of

$$\Xi := \{\xi \in R^m : g(\xi) \leq 0\}, \tag{3.6}$$

where $g: R^m \rightarrow R^J$ is a vector-valued function with $J \geq 1$. The following definition will be used.

Definition 3.1 (Stephen 2004) Let C be a convex set and let $F: C \rightarrow R$ be a continuous function.

1) F is called concave if for all $x, y \in C$ and $\lambda \in [0,1]$, we have

$$F(\lambda * x + (1-\lambda) * y) \geq \lambda * F(x) + (1-\lambda) * F(y).$$

2) F is called quasi-concave if for all $x, y \in C$ and $\lambda \in [0,1]$, we have

$$F(\lambda * x + (1-\lambda) * y) \geq \min\{F(x), F(y)\}.$$

3) F is called (quasi-) convex if $-F$ is (quasi-) concave.

First, let us consider the BLPP (2.14)-(2.15). Since the minimal function is not always differentiable, even though all the functions inside are smooth, the lower level problem (2.15) is a typical non-smooth optimization problem for any x given by the upper level problem (2.14). In order to remove the obstacle caused by non-smoothness, by introducing a new auxiliary variable y, we can transform the problem (2.15) into the following optimization problem:

$$\max_{(\xi;\, y)} y \quad \mathrm{s.t.}\ y - \pi(\xi) \le 0, y - u(x,\xi) \le 0, g(\xi) \le 0, y \in R. \tag{3.7}$$

Assumption 3.1 Functions $\pi(\cdot)$, $u(x,\cdot)$ and $-g(\cdot)$ are differentiable and concave for all $x \in X$.

With Assumption 3.1, we know that (3.7) is a differentiable convex optimization problem. It is known that Slater's condition is an important regularity condition (or constraint qualification) for a convex optimization, which ensures that the KKT-type first-order condition is both necessary and sufficient for a solution to be optimal [Section 5.2.3, Stephen (2004)].

Assumption 3.2 Ξ takes the form of (3.6), and there exists $\xi_0 \in \Xi$ such that

$$g_j(\xi_0) < 0, \quad \forall\, j = 1, \cdots, J. \tag{3.8}$$

With Assumptions 3.1 and 3.2, we know that solving the problem (3.7) is equivalent to solving its first-order KKT condition, that is, solving the following system of equalities and inequalities:

$$\begin{cases} (1-\alpha) * \pi'(\xi) + \alpha * u'_\xi(x,\xi) - \sum_{j=1}^{J} \beta_j * g'_j(\xi) = 0, \\ 0 \le 1-\alpha \perp y - \pi(\xi) \le 0, \\ 0 \le \alpha \perp y - u(x,\xi) \le 0, \\ 0 \le \beta_j \perp g_j(\xi) \le 0,\ j = 1, \cdots, J, \end{cases} \tag{3.9}$$

where $\alpha \in R$, $\beta \in R^J$ and $0 \le a \perp b \le 0$ means that $a \ge 0, b \le 0$ and $a * b = 0$. Setting

$$h_1(x,\xi,\alpha,\beta) := (1-\alpha) * \pi'(\xi) + \alpha * u'_\xi(x,\xi) - \textstyle\sum_{j=1}^{J} \beta_j * g'_j(\xi),$$

and

$$H_1(\alpha,\beta) := (1-\alpha; \alpha; \beta), \quad H_2(x,\xi,y) := \big(y - \pi(\xi); y - u(x,\xi); -g(\xi)\big),$$

then we can reformulate the BLPP (2.14)-(2.15) into the following MPEC:

$$\max_{(x;\xi;\alpha;\beta;y)} f(x,\xi) \quad \mathrm{s.t.}\ h_1(x,\xi,\alpha,\beta) = 0, 0 \le H_1(\alpha,\beta) \perp H_2(x,\xi,y) \ge 0, x \in X. \tag{3.10}$$

Next, let us consider the BLPP (2.17)-(2.18). Similarly, by introducing a new auxiliary variable z, we can transform the problem (2.18) into the following optimization problem:

$$\min_{(\xi;\, z)} z \quad \mathrm{s.t.}\ 1-\pi(\xi) \le z, u(x,\xi) \le z, g(\xi) \le 0, z \in R. \tag{3.11}$$

Assumption 3.3 Functions $\pi(\cdot)$, $-u(x,\cdot)$ and $-g(\cdot)$ are differentiable and concave for all $x \in X$.

Clearly, Assumption 3.3 ensures that (3.11) is a differentiable convex optimization problem. With Assumptions 3.3 and 3.2, we know that solving the problem (3.11) is equivalent to solving its first-order KKT condition, that is, solving the following system of equalities and inequalities:

$$\begin{cases} (1-\gamma) * \pi'(\xi) - \gamma * u'_\xi(x,\xi) - \sum_{j=1}^{J} \eta_j * g'_j(\xi) = 0, \\ 0 \le 1-\gamma \perp 1-\pi(\xi) - z \le 0, \\ 0 \le \gamma \perp u(x,\xi) - z \le 0, \\ 0 \le \eta_j \perp g_j(\xi) \le 0,\ j = 1, \cdots, J, \end{cases} \tag{3.12}$$

where $\gamma \in R$, $\eta \in R^J$. Setting

$$h_2(x,\xi,\gamma,\eta) := (1-\gamma) * \pi'(\xi) - \gamma * u'_\xi(x,\xi) - \sum_{j=1}^{J} \eta_j * g'_j(\xi),$$

and

$$H_3(\gamma,\eta) := (1-\gamma;\gamma;\eta), H_4(x,\xi,z) := \big(z+\pi(\xi)-1;\ z-u(x,\xi); -g(\xi)\big),$$

then we can reformulate the BLPP (2.17)-(2.18) as the following MPEC:

$$\max_{(x;\xi;\gamma;\eta;z)} f(x,\xi) \quad \mathrm{s.t.}\ h_2(x,\xi,\gamma,\eta) = 0, 0 \le H_3(\gamma,\eta) \perp H_4(x,\xi,z) \ge 0, x \in X. \tag{3.13}$$

3.3 New Reformulation Methods

We have shown in Section 3.1 that BLPP (2.14)-(2.15) can be equivalently reformulated as the MPEC (3.10) with some assumptions. Although (3.10) is a single-level optimization problem, solving it is still difficult due to the existence of complementarity constraints. In fact, we can see that the reformulated model (3.10) has a combinatorial structure, That is, the feasible region of (3.10) is a union of lots of pieces, which makes it hard to solve effectively. Another defect is that, to ensure the equivalence between the BLPP (2.14)-(2.15) and the MPEC (3.10), the functions π and u are required to be concave with respect to the lower level variable. Such requirement seems to be too strict in applications. The above difficulties also exist in the problem (3.13). In this section, we provide new reformulation methods to solve the BLPP (2.14)-(2.15) and the BLPP (2.17)-(2.18) under some weaker conditions, the reformulated models are relatively easier to solve than (3.10) and (3.13), respectively.

3.3.1 Case of One-Dimensional Lower Level Variables

In this section, we consider the case where Ξ is a bounded convex subset of R, that is,

$$\Xi = [\xi_l, \xi_u] \subset R \text{ with } \xi_l < \xi_u,$$

where ξ_l and ξ_u are the lower and upper bounds of Ξ, respectively.

Assumption 3.4 $\pi(\cdot)$ and $u(x,\cdot)$ are quasi-concave functions for all $x \in X$; $\pi(\xi_l) = \pi(\xi_u) = 0$ holds and there exists $\xi_c \in (\xi_l, \xi_u)$ satisfying $\pi(\xi_c) = 1$.

(1) Equivalent Model of the BLPP (2.14)-(2.15)

Theorem 3.1 With Assumption 3.4, the global optimal solutions of the BLPP (2.14)-(2.15) are equivalent to the ones of the following single-level optimization problem:

$$\max_{(x;\,\xi)} f(x,\xi) \quad \text{s.t. } u(x,\xi) - \pi(\xi) \le 0, x \in X, \xi \in \Xi. \tag{3.14}$$

Proof. We divide the difference of $u(x,\xi)$ and $\pi(\xi)$ into two cases, that is $u(x,\xi) \geq \pi(\xi)$ and $u(x,\xi) < \pi(\xi)$. If the global optimal solutions of the BLPP (2.14)-(2.15) satisfy the first case, that is, $u(x,\xi) \geq \pi(\xi)$, then (2.15) is equivalent to the following optimization problem:

$$\max_{\xi} \ \pi(\xi) \quad \text{s.t.}\, u(x,\xi) - \pi(\xi) \geq 0, \xi \in \Xi. \tag{3.15}$$

According to Definitions 2.1 and 2.2, we have $0 \leq u(x,\xi) \leq \max_{\xi\in\Xi} \pi(\xi) = 1$. Together with Assumption 3.4, it is trivial to check that the inequality constraint of (3.15) is actually an equality constraint at the global optimal solutions. Combined with the second case, we can understand that the global optimal solutions of the lower level problem (2.15) must satisfy $u(x,\xi) \leq \pi(\xi)$, which implies the BLPP (2.14)-(2.15) is equivalent to the following optimization problem:

$$\max_{(x;\,\xi)} \ f(x,\xi) \quad \text{s.t.}\, x \in X, \xi \in \operatorname{argmax}_{\xi\in\Xi} \{u(x,\xi) \mid \ u(x,\xi) \leq \pi(\xi)\}. \tag{3.16}$$

Considering Definition 2.2, we know that problems (3.16) and (3.14) are equivalent. ■

In fact, it is trivial to proof that, for any $x \in X$, the minimal function $\min\{\pi(\xi), u(x,\xi)\}$ is also quasi-concave under Assumption 3.4, which satisfies $\min\{\pi(\xi), u(x,\xi)\} \in [0,1]$. In this case, the lower level problem (2.15) is a non-smooth quasi-convex optimization problem. In the following, we will give another equivalent form by considering the first order optimality condition. To this end, we give another assumption as follows.

Assumption 3.5 (1) Function $\pi(\cdot)$ is quasi-concave and differentiable, $u(x,\cdot)$ is concave and differentiable for all $x \in X$; (2) $\pi(\xi_l) = \pi(\xi_u) = 0$ holds, and there exists $\xi_c \in (\xi_l, \xi_u)$ such that $\pi(\xi_c) = 1$, and $\pi'(\xi) \neq 0$ for all $\xi \in (\xi_l, \xi_c) \cup (\xi_c, \xi_u)$.

With Assumption 3.5, we have $\min\{\pi(\xi_l), u(x,\xi_l)\} = \min\{\pi(\xi_u), u(x,\xi_u)\} = 0$, so that solving (2.15) is equivalent to finding out $\xi \in (\xi_l, \xi_u)$ which satisfies the following first-order optimality condition:

$$0 \in \partial_{\xi} \min\{\pi(\xi), u(x,\xi)\}, \tag{3.17}$$

where $\partial_\xi \min\{\pi(\xi), u(x,\xi)\}$ denotes the subdifferential of $\min\{\pi(\xi), u(x,\xi)\}$ in variable ξ. Further, we can rewrite the condition (3.17) as:

$$\begin{cases} \pi'(\xi) = 0, & \text{if } \pi(\xi) - u(x,\xi) < 0; \\ u'_\xi(x,\xi) = 0, & \text{if } \pi(\xi) - u(x,\xi) > 0; \\ \pi'(\xi) * u'_\xi(x,\xi) \le 0, & \text{if } \pi(\xi) - u(x,\xi) = 0. \end{cases} \tag{3.18}$$

In fact, it is trivial to verify that the first item of (3.18) doesn't hold under Assumption 3.5 so that the set of solutions of (2.15) can be rewritten as

$$\begin{aligned} S_1(x) := \{\xi \in (\xi_l, \xi_u) \mid u'_\xi(x,\xi) = 0, \pi(\xi) > u(x,\xi); \\ \text{or } \pi'(\xi) * u'_\xi(x,\xi) \le 0, \pi(\xi) = u(x,\xi)\}. \end{aligned} \tag{3.19}$$

In the following, we consider a new optimization problem:

$$\max_{(x;\,\xi)} \ f(x,\xi) \quad \text{s.t. } x \in X, \xi \in \bar{S}_1(x), \tag{3.20}$$

where

$$\bar{S}_1(x) := \{\xi \in (\xi_l, \xi_u) \mid \pi'(\xi) * u'_\xi(x,\xi) \le 0, \pi(\xi) - u(x,\xi) \ge 0\}. \tag{3.21}$$

Actually, for any $x \in X$, it is easy to check that $S_1(x)$ is a proper subset of $\bar{S}_1(x)$, and the difference of them can be obtained as:

$$\bar{S}_1(x) - S_1(x) = \{\xi \in (\xi_l, \xi_u) \mid \pi'(\xi) * u'_\xi(x,\xi) \le 0, u'_\xi(x,\xi) \ne 0, \pi(\xi) > u(x,\xi)\}. \tag{3.22}$$

The following theorem shows the relation between the BLPP (2.14)-(2.15) and (3.20)-(3.21).

Theorem 3.2 With Assumption 3.5, the global optimal solutions of the BLPP (2.14)-(2.15) and the problem (3.20)-(3.21) are equivalent.

Proof. To prove this theorem, it suffices to show that, for any $x \in X$, it holds that

$$f(x,\bar{\xi}) > f(x,\tilde{\xi}), \quad \forall\, \bar{\xi} \in S_1(x), \tilde{\xi} \in \bar{S}_1(x) - S_1(x). \tag{3.23}$$

Let us prove (3.23) in what follows. First of all, it follows from (3.19) and (3.22) that

$$u'_\xi(x,\bar{\xi}) = 0, \pi(\bar{\xi}) - u(x,\bar{\xi}) > 0, \text{or } \pi'(\bar{\xi}) * u'_\xi(x,\bar{\xi}) \le 0, \pi(\bar{\xi}) - u(x,\bar{\xi}) = 0,$$

and

$$\pi'(\tilde{\bar{\xi}}) * u'_{\xi}(x, \tilde{\bar{\xi}}) \leq 0, u'_{\xi}(x, \tilde{\bar{\xi}}) \neq 0, \pi(\tilde{\bar{\xi}}) - u(x, \tilde{\bar{\xi}}) > 0,$$

respectively.

In the following, we discuss in details. In the case that $u'_{\xi}(x, \bar{\xi}) = 0$ and $\pi(\bar{\xi}) - f(x, \bar{\xi}) > 0$. If $u'_{\xi}(x, \tilde{\bar{\xi}}) < 0$ and $\pi(\tilde{\bar{\xi}}) - u(x, \tilde{\bar{\xi}}) > 0$ hold, then $u(x, \xi)$ is a decreasing function in variable ξ at the interval $[\bar{\xi}, \tilde{\bar{\xi}}]$, which implies (3.23). If $u'_{\xi}(x, \tilde{\bar{\xi}}) > 0$ and $\pi(\tilde{\bar{\xi}}) - u(x, \tilde{\bar{\xi}}) > 0$ hold, then $u(x, \xi)$ is an increasing function in variable ξ at the interval $[\tilde{\xi}, \bar{\xi}\,]$, which implies (3.23).

In the case that $\pi'(\bar{\xi}) = 0$ and $\pi(\bar{\xi}) - u(x, \bar{\xi}) = 0$ hold, we can obtain $\pi(\bar{\xi}) = u(x, \bar{\xi}) = 1$. Combined with $u(x, \tilde{\bar{\xi}}) < \pi(\tilde{\bar{\xi}}) \leq 1$, we have (3.23).

In the case that $u'_{\xi}(x, \bar{\xi}) = 0$ and$\pi(\bar{\xi}) - u(x, \bar{\xi}) = 0$ hold, we know $\bar{\xi} = \text{argmax}_{\xi \in \Xi}\, u(x, \xi)$. Combined with $u'_{\xi}(x, \tilde{\bar{\xi}}) \neq 0$, we have (3.23).

In the case that $u'_{\xi}(x, \bar{\xi}) < 0$, $\pi'(\bar{\xi}) > 0$ and $\pi(\bar{\xi}) - u(x, \bar{\xi}) = 0$, If $u'_{\xi}(x, \tilde{\bar{\xi}}) < 0$, $\pi'(\tilde{\bar{\xi}}) \geq 0$ and $\pi(\tilde{\bar{\xi}}) - u(x, \tilde{\bar{\xi}}) > 0$ hold, then $u(x, \xi)$ is a decreasing function in variable ξ at the interval $[\bar{\xi}, \tilde{\bar{\xi}}]$ which implies (3.23). If $u'_{\xi}(x, \tilde{\bar{\xi}}) > 0$, $\pi'(\tilde{\bar{\xi}}) \leq 0$ and $\pi(\tilde{\bar{\xi}}) - u(x, \tilde{\bar{\xi}}) > 0$ hold, then we have $u'_{\xi}(x, \tilde{\bar{\xi}}) > 0 > u'_{\xi}(x, \bar{\xi})$ and $\pi'(\tilde{\bar{\xi}}) \leq 0 < \pi'(\bar{\xi})$, which conflict with the definition of these two functions.

In the case that $u'_{\xi}(x, \bar{\xi}) > 0$, $\pi'(\bar{\xi}) < 0$ and $\pi(\bar{\xi}) - u(x, \bar{\xi}) = 0$, if $u'_{\xi}(x, \tilde{\bar{\xi}}) > 0$, $\pi'(\tilde{\bar{\xi}}) \leq 0$ and $\pi(\tilde{\bar{\xi}}) - u(x, \tilde{\bar{\xi}}) > 0$ hold, then $u(x, \xi)$ is an increasing function in variable ξ at the interval $[\tilde{\bar{\xi}}, \bar{\xi}]$, which implies (3.23). If $u'_{\xi}(x, \tilde{\bar{\xi}}) < 0$, $\pi'(\tilde{\bar{\xi}}) \geq 0$ and $\pi(\tilde{\bar{\xi}}) - u(x, \tilde{\bar{\xi}}) > 0$ hold, it is easy to verify that this result conflicts with the definitions of these two functions.

From the above discussion, we know that (3.23) holds for any $x \in X$, which implies that the global optimal solutions of problems (2.14)-(2.15) and (3.20)-(3.21) are equivalent. ■

Considering $u(x, \xi) \leq \pi(\xi)$ and $\pi(\xi_l) = \pi(\xi_u) = 0$, we obtain the following theorem.

Theorem 3.3 With Assumption 3.5, the global optimal solutions of the BLPP (2.14)-(2.15) are equivalent to the ones of the following single-level optimization problem:

$$\max_{(x;\,\xi)} u(x,\xi) \quad \text{s.t.}\ \pi'(\xi) * u'_{\xi}(x,\xi) \le 0, u(x,\xi) - \pi(\xi) \le 0, x \in X, \xi \in \Xi. \tag{3.24}$$

(2) Equivalent Model of the BLPP (2.17)-(2.18)

From Assumption 3.4, it is easy to check that $1 - u(x,\xi)$ is quasi-convex continuous in variable ξ, and for any $x \in X$, there exist $\xi_x \in \Xi$ such that $1 - u(x,\xi_x) = \pi(\xi_x)$. In what follows, we translate the BLPP (2.17)-(2.18) into a minimax optimization problem.

Theorem 3.4 With Assumption 3.4, the global optimal solutions of the BLPP (2.17)-(2.18) are equivalent to the ones of the following continuous minimax optimization problem:

$$\min_{x \in X} \max_{\xi \in \Xi} \{-u(x,\xi) \mid 1 - u(x,\xi) - \pi(\xi) = 0\}. \tag{3.25}$$

Proof. Suppose that $(x^*;\xi^*)$ is a global optimal solution of the BLPP (2.17)-(2.18), that is, $\xi^* \in \Lambda^2(x^*)$. Let us prove that, under Assumption 3.4, solution (x^*,ξ^*) must satisfy:

$$\pi(\xi) = 1 - u(x,\xi). \tag{3.26}$$

If $\pi(\xi^*) < 1 - u(x^*,\xi^*)$, then (2.18) is equivalent to the following optimization problem:

$$\max_{\xi \in \Xi} \pi(\xi) \quad \text{s.t.}\ \pi(\xi) < 1 - u(x^*,\xi).$$

Considering the continuity of $\pi(\xi)$ and $u(x^*,\xi)$, there must be a point $\tilde{\xi} \in (\xi^*,\xi_c)$ or $\tilde{\xi} \in (\xi_c,\xi^*)$ such that $\pi(\tilde{\xi}) < 1 - u(x^*,\tilde{\xi})$ and $\pi(\tilde{\xi}) > \pi(\xi^*)$. Clearly, the results are not consistent with the fact of $\xi^* \in \Lambda^2(x^*)$.

If $1 - u(x^*,\xi^*) < \pi(\xi^*)$, then (2.18) is equivalent to the following optimization problem:

$$\max_{\xi \in \Xi} 1 - u(x^*,\xi) \quad \text{s.t.}\ \pi(\xi) > 1 - u(x^*,\xi).$$

Considering Assumption 3.4, the function $1-u(x^*,\xi)$ is quasi-convex continuous in variable ξ so that there must be a point $\tilde{\xi} \in [\xi_l, \xi^*)$ or $\tilde{\xi} \in (\xi^*, \xi_u]$ which satisfies $1-u(x^*,\tilde{\xi}) < \pi(\tilde{\xi})$ and $1-u(x^*,\tilde{\xi}) > 1-u(x^*,\xi^*)$. The results are also not consistent with $\xi^* \in \Lambda^2(x^*)$. As a result, the global optimal solutions of the BLPP (2.17)-(2.18) must satisfy (3.26). Thus, the global optimal solutions of the BLPP (2.17)-(2.18) are equivalent to the ones of the following maximin optimization problem:

$$\max_{x\in X} \min_{\xi\in\bar{S}_2(x)} f(x,\xi), \tag{3.27}$$

where $\bar{S}_2(x)$ denotes the set of solutions of the following problem:

$$\max_{\xi\in\Xi} -u(x,\xi) \quad \text{s.t. } \pi(\xi) = 1-u(x,\xi). \tag{3.28}$$

In the following, we prove that the problems (3.27)-(3.28) and (3.25) are equivalent. First of all, we define the optimal value function of (3.28) as follows:

$$V(x) := \max_{\xi\in\Xi}\{-u(x,\xi) \mid 1-u(x,\xi) = \pi(\xi)\},$$

then we can obtain that

$$\min_{\xi\in\bar{S}_2(x)} u(x,\xi) = -V(x),$$

which implies that the problem (3.27) is equivalent to

$$\max_{x\in X} -V(x). \tag{3.29}$$

Obviously, the global optimal solutions of (3.29) and (3.25) are equivalent. ■

Since for any $x \in X$, we have $1-u(x,\xi_l)-\pi(\xi_l) \geq 0$, $1-u(x,\xi_c)-\pi(\xi_c) \leq 0$, $1-u(x,\xi_u)-\pi(\xi_u) \geq 0$, solving the equation $1-u(x,\xi)-\pi(\xi) = 0$ is equivalent to finding out $\xi \in \{\xi_1, \xi_2\}$ such that

$$1-u(x,\xi_1)-\pi(\xi_1) = 0, \xi_1 \in [\xi_l, \xi_c]; 1-u(x,\xi_2)-\pi(\xi_2) = 0, \xi_2 \in [\xi_c, \xi_u].$$

Obviously, the points ξ_1, ξ_2 depend on x. Let $\Xi_1 := [\xi_l, \xi_c]$, $\Xi_2 := [\xi_c, \xi_u]$, the minimax optimization problem (3.25) can be rewritten as

$$\begin{array}{ll} \min & \max\{-u(x,\xi_1), -u(x,\xi_2)\} \\ \text{s.t.} & x \in X, \xi_1 \in \Xi_1, \xi_2 \in \Xi_2, \\ & 1 - u(x,\xi_i) - \pi(\xi_i) = 0,\ i = 1,2. \end{array} \tag{3.30}$$

By introducing an auxiliary variable z, we can further transform (3.30) into the following optimization problem:

$$\begin{array}{ll} \min & z \\ \text{s.t.} & x \in X, \xi_1 \in \Xi_1, \xi_2 \in \Xi_2, z \in R, \\ & 1 - u(x,\xi_i) - \pi(\xi_i) = 0, i = 1,2, \\ & -u(x,\xi_1) \le z, -u(x,\xi_2) \le z. \end{array} \tag{3.31}$$

Theorem 3.5 Suppose that $(x^*; \xi_1^*; \xi_2^*; z^*)$ is a global optimal solution of the problem (3.31), then $(x^*; \xi^*)$ is still global optimal to the BLPP (2.17)-(2.18) with $u(x^*, \xi^*) = -z^*$.

3.3.2 Case of Multi-Dimensional Lower Level Variables

In this section, we consider the case where Ξ is a bounded convex subset of R^m with $m > 1$. We assume that Ξ takes the following form:

$$\Xi := \{\xi \in R^m : g(\xi) \le 0\} \text{ with } m > 1, \tag{3.32}$$

where $g: R^m \to R^J$ is a vector-valued function with $J \ge 1$.

(1) Equivalent Model of the BLPP (2.14)-(2.15)

In the following, we reformulate the BLPP (2.14)-(2.15) into a solvable single-level optimization problem. To this end, we give an assumption as follows.

Assumption 3.6 Function $\pi(\cdot)$ is quasi-concave and continuous; there exists $\xi_c \in \Xi$ such that

$$\pi(\xi_c) = \max_{\xi \in \Xi} \pi(\xi) \ge u(x,\xi), \quad \forall\ (x;\xi) \in X \times \Xi.$$

Theorem 3.6 Suppose that $(x^*; \xi^*)$ is a global optimal solution of the following problem:

$$\max_{(x;\,\xi)} f(x, \xi) \quad \text{s.t.}\, u(x, \xi) - \pi(\xi) \le 0, g(\xi) \le 0, x \in X, \xi \in R^m. \tag{3.33}$$

then it is still global optimal for the BLPP (2.14)-(2.15) under Assumption 3.6.

Proof. Suppose that $\xi(x)$ is a global optimal solution of the lower level problem (2.15) where $x \in X$, then we have

$$\min\{\pi(\xi(x)), u(x, \xi(x))\} \ge \min\{\pi(\xi), u(x, \xi)\}, \quad \forall\, \xi \in \Xi. \tag{3.34}$$

In fact, $\xi(x)$ must satisfy the following condition:

$$u(x, \xi(x)) - \pi(\xi(x)) \le 0. \tag{3.35}$$

The reason is as follows.

If (3.35) does not hold, then we have $\pi(\xi(x)) = \min\{\pi(\xi(x)), u(x, \xi(x))\}$. Considering (3.34), if $\pi(\xi) < u(x, \xi)$, we can obtain that

$$\pi(\xi(x)) = \min\{\pi(\xi(x)), u(x, \xi(x))\} \ge \min\{\pi(\xi), u(x, \xi)\} = \pi(\xi). \tag{3.36}$$

Clearly, if $\pi(\xi(x)) < u(x, \xi(x))$, $w(x)$ is still global optimal for the following problem:

$$\max_{\xi} \pi(\xi) \quad \text{s.t. } \pi(\xi) - u(x, \xi) < 0, \xi \in \Xi. \tag{3.37}$$

From Assumption 3.6, we know that $\pi(\xi)$ is a quasi-concave function and $\pi(\xi_c) = \max\{\pi(\xi): \xi \in \Xi\} = 1$. Together with $\pi(\xi) \in [0, 1]$ and $u(x, \xi) \in [0, 1]$, we can obtain that

$$\pi(\xi_c) - u(x, \xi_c) = 1 - u(x, \xi_c) \ge 0. \tag{3.38}$$

It is clear that $\xi(x) \ne \xi_c$ if $\pi(\xi(x)) < u(x, \xi(x))$. Since $\pi(\xi)$ is quasi-concave and Ξ is a bounded set, there must exist $\xi_x = (1 - \sigma) * \xi_c + \sigma * \xi(x) \in \Xi$ where $0 < \sigma < 1$ such that

$$\pi(\xi_x) > \pi(\xi(x)) \text{ and } \pi(\xi_x) - u(x, \xi_x) < 0, \tag{3.39}$$

which conflict with the assumption that $\xi(x)$ is a global optimal solution of (3.37). Hence, (3.35) is true. Thus, we can rewrite (2.15) as the following optimization problem:

$$\max_{\xi} u(x, \xi) \quad \text{s.t. } u(x, \xi) - \pi(\xi) \le 0, \xi \in \Xi. \tag{3.40}$$

Since $u(x, \xi)$ is the normalized profit function, we know that solving the BLPP (2.14)-(3.40) is equivalent to solving the following single-level optimization problem:

$$\max_{(x;\xi)} f(x, \xi) \quad \text{s.t. } x \in X, \xi \in \Xi, u(x, \xi) - \pi(\xi) \leq 0. \tag{3.41}$$

Combined with (3.32), we can rewrite (3.41) as (3.33). Clearly, if (x^*, ξ^*) is a global optimal solution of the problem (3.33), then it is still global optimal for the BLPP (2.14)-(2.15). ■

(2) Equivalent Model of the BLPP (2.17)-(2.18)

In the following, we reformulate the BLPP (2.17)-(2.18) into a solvable single-level optimization problem. To this end, we give another assumption as follows.

Assumption 3.7 For any $x \in X$, we assume that function $u(x,\cdot)$ is affine and continuous, function $\pi(\cdot)$ is concave and continuously differentiable, and there exists $\xi_0 \in \Xi$ such that

$$1 - \pi(\xi_0) - u(x, \xi_0) < 0 \text{ and } g(\xi_0) < 0.$$

Theorem 3.7 Suppose that $(x^*; \xi^*; \lambda^*; \tau^*)$ is a global optimal solution of the following problem:

$$\max_{(x;\, \xi;\, \lambda;\, \tau)} L(x, \xi, \lambda, \tau) \quad \text{s.t. } G(x, \xi, \lambda, \tau) = 0,\ g(\xi) \leq 0,\ x \in X,\ \lambda, \tau \geq 0, \tag{3.42}$$

where $L(x, \xi, \lambda, \tau)$ and $G(x, \xi, \lambda, \tau)$ are respectively given by

$$L(x, \xi, \lambda, \tau) := (1 - \tau) * u(x, \xi) + \tau * \big(1 - \pi(\xi)\big) + \lambda^T g(\xi), \tag{3.43}$$

and

$$G(x, \xi, \lambda, \tau) := (1 - \tau) * u'_\xi(x, \xi) - \tau * \pi'(\xi) + \textstyle\sum_{j=1}^{J} \lambda_j * g'_j(\xi), \tag{3.44}$$

then (x^*, ξ^*) is still global optimal for the BLPP (2.17)-(2.18) under Assumption 3.7.

Proof. Suppose that $\xi(x)$ is a global optimal solution of (2.18) where $x \in X$, then we have

$$\max\{1 - \pi\big(\xi(x)\big), u\big(x, \xi(x)\big)\} \leq \max\{1 - \pi(\xi), u(x, \xi)\},\ \ \forall\, \xi \in \Xi. \tag{3.45}$$

We divide the difference of $1 - \pi\big(\xi(x)\big)$ and $u\big(x, \xi(x)\big)$ into the following two cases:

$$1-\pi\big(\xi(x)\big)-u\big(x,\xi(x)\big)>0 \text{ and } 1-\pi\big(\xi(x)\big)-u\big(x,\xi(x)\big)\le 0. \tag{3.46}$$

In fact, the first case of (3.46) does not hold. The reason is as follows.

If the first case of (3.46) holds, that is, $\pi\big(\xi(x)\big)<1-u\big(x,\xi(x)\big)$, then we have

$$1-\pi\big(\xi(x)\big)=\max\{1-\pi\big(\xi(x)\big),u\big(x,\xi(x)\big)\}\le \max\{1-\pi(\xi),u(x,\xi)\}, \tag{3.47}$$

which implies that $\xi(x)$ is still global optimal for the following optimization problem:

$$\min_{\xi} 1-\pi(\xi) \quad \mathrm{s.t.}\ 1-\pi(\xi)>u(x,\xi),\xi\in\Xi. \tag{3.48}$$

From Assumption 3.7 as well as Definition 2.1, we know that $\pi(\xi)$ is a concave function and there exists $\xi_c\in\Xi$ satisfying $\pi(\xi_c)=\max\{\pi(\xi):\xi\in\Xi\}=1$. Together with $\pi(\xi)\in[0,1]$ and $u(x,\xi)\in[0,1]$, we have

$$1-\pi(\xi_c)=0\le u\big(x,\xi(x)\big). \tag{3.49}$$

Clearly, $\xi(x)\neq\xi_c$ if $1-\pi\big(\xi(x)\big)>u\big(x,\xi(x)\big)$. Since functions $\pi(\xi)$ and $u(x,\xi)$ are concave and affine for variable ξ, respectively, there is $\xi_x=(1-\sigma)*\xi_c+\sigma*\xi(x)\in\Xi$ where $0<\sigma<1$ such that

$$1-\pi(\xi_x)<1-\pi\big(\xi(x)\big) \ \text{ and } \ 1-\pi(\xi_x)>u(x,\xi_x), \tag{3.50}$$

which conflict with the assumption that $\xi(x)$ is a global optimal solution of (3.48). Therefore, solving problem (2.18) is equivalent to solving the following optimization problem:

$$\min_{\xi} u(x,\xi) \quad \mathrm{s.t.}\, 1-\pi(\xi)-u(x,\xi)\le 0, g(\xi)\le 0. \tag{3.51}$$

Since $u(x,\xi)$ is the normalized profit function, we can rewrite the BLPP (2.17)-(2.18) as the following maximin optimization problem:

$$\max_{x\in X}\left\{\min_{\xi}\ u(x,\xi):\ 1-\pi(\xi)-u(x,\xi)\le 0, g(\xi)\le 0\right\}. \tag{3.52}$$

From Assumption 3.7, we know that solving (3.51) is to solve a convex optimization problem. In addition, for any $x\in X$, Slater's constraint qualification holds for the problem (3.51). From the strongly dual theory for a convex optimization problem [Stephen (2004), Section 5.2], we know that solving problem (3.51) can be equivalent to solving its duality problem and, moreover, they have the same optimal value. Let us give the Lagrange function of problem (3.51) as follows

$$\mathrm{L}(x,\xi,\lambda,\tau) := u(x,\xi) + \tau * \big(1 - \pi(\xi) - u(x,\xi)\big) + \lambda^T g(\xi), \tag{3.53}$$

where $\lambda \in R_+^J$ and $\tau \in R_+$. Clearly, $L(x,\xi,\lambda,\tau)$ is convex about the variable ξ for any fixed $\lambda \geq 0$ and $\tau \geq 0$. Thus, we can express its dual problem as

$$\max_{(\lambda;\tau)} \ L(x,\xi,\lambda,\tau) \quad \text{s.t.}\, G(x,\xi,\lambda,\tau) = 0, \lambda \geq 0, \tau \geq 0, \tag{3.54}$$

where

$$G(x,\xi,\lambda,\tau) := L'_\xi(x,\xi,\lambda,\tau). \tag{3.55}$$

(3.54) is also called the Wolfe duality problem. Further, we can rewrite (43) as (24), (45) as (25). Since the strongly dual condition holds for the problem (3.51), we know that solving (3.52) is equivalent to solving the problem (3.42). Clearly, if $(x^*;\xi^*;\lambda^*;\tau^*)$ is a global optimal solution of the problem (3.42), then $(x^*;\xi^*)$ is still global optimal for the BLPP (2.17)-(2.18). ■

3.4 Concluding Remarks

In this chapter, we study various equivalent single-level models for BLPPs with a maximin or minimax lower level program. Note that using the traditional KKT condition based method requires that the lower level program is always a convex optimization problem and satisfies the Slater’s condition for each upper level variable. Such requirements seem to be too strict in applications. Note further that the reformulated model based on the KKT method is still hard to be solved because the feasible region is disconnected and necessarily non-convex. We propose new reformulation methods to these special non-smooth BLPPs with some weaker assumption conditions. These equivalent single-level models can be easily solved by commonly used nonlinear optimization algorithms.

Chapter 4

Applications to Single-Item Newsvendor Problems

4.1 Introduction

Newsvendor problems, sometimes called newsboy problems, are renowned as typical single-period problems. These problems are characterized by predetermined prices and uncertain demands for the products associated with them. For each item, if its order exceeds its real demand, then the decision-maker (a buyer) will face the cost of salvaging the remaining units at the end of the period; otherwise he/she will lose the sales. Very often the purchasing lead-time of products is long and hence the decision-maker usually has one and only one opportunity to make an order for these products at the beginning of a single-period. Lots of products have the above-mentioned characteristics, especially in fresh food, fast fashion and service industries, so newsvendor problems receive more and more attention in the real world.

The newsvendor problem is a typical one-shot decision problem. Since the actual end-of-period profit cannot be viewed at the beginning of a single-period, the classic newsvendor model suggests that the optimal order can be taken by maximizing the expected profit. Unfortunately, researchers observed that maximizing the expected profit is not consistent with the actions of many decision-makers. Subsequently, various variants of the classic newsvendor model have been proposed for different situations in the past half century; please refer to the review articles (Khouja, 1999; Qin, 2011) and the references therein for more details on them.

However, as far as we know, almost all expansions assume that the objective function is to maximize the probability of achieving a target profit or the expected utility. In this chapter, we apply the OSDT-based decision approaches to a traditional single-item newsvendor problem. Different from traditional models, the OSDT-based newsvendor model intuitively portrays a person's psychological process of decision-making.

The remainder of this chapter is organized as follows. In Section 4.2, OSDT-based newsvendor models for an innovative product are built. In Section 4.3, we apply the proposed reformulation methods to OSDT-based newsvendor models and give two specific single-level equivalent models for them. A numerical example is used to show the effectiveness of the proposed solution methods. Finally, we conclude our research in Section 4.4.

4.2 One-Shot Decision Theory Based Newsvendor Models

Consider a retailer who orders a product prior to the selling season. The demand is characterized as a random variable $\boldsymbol{\xi}$. Suppose that this random variable follows a truncated normal distribution with mean μ and variance σ^2, i.e., $\boldsymbol{\xi}\sim\mathcal{N}(\mu,\sigma^2)$, and the feasible region is $\Xi \coloneqq [\mu-\xi_0,\mu+\xi_0] \subset R$ with $\xi_0 > 0$. Denote x as the retailer's order quantity, w as the unit wholesale price, r as the unit revenue with $r > w > 0$. Any excess product can be salvaged at the unit salvage price $s_0 > 0$. If there is a shortage, the unit opportunity cost is $s_u > 0$. If the realization of the random variable is ξ, then the profit function of the retailer can be given as

$$p(x,\xi) \coloneqq \begin{cases} r*\xi+(x-\xi)*s_0-w*x, & \text{if } \xi < x, \\ (r-w)*x-s_u*(\xi-x), & \text{if } \xi \geq x. \end{cases}$$

Because the set of uncertain demand is Ξ, a reasonable order quantity should also lie in this region. Given an order quantity $x \in \Xi$, the function for evaluating the order quantity with considering the regret of the retailer can be given as

$$f(x,\xi) := -\left(p(x,\xi) - p_u(x)\right)^2,$$

where $p_u(x)$ denotes the highest profit for an order quantity x, that is,

$$p_u(x) = \max_{\xi\in\Xi} p(x,\xi) = (r-w) * x.$$

Further, we can obtain

$$f(x,\xi) := \begin{cases} -(r-s_0)^2 * (x-\xi)^2, & \text{if } \xi < x; \\ -s_u^2 * (x-\xi)^2, & \text{if } \xi \geq x. \end{cases} \tag{4.1}$$

Using (2.10), we have the following relative likelihood function:

$$\pi(\xi) = \frac{\rho(\xi)-\rho_l}{\rho_u-\rho_l}, \tag{4.2}$$

where $\rho(\xi)$ denotes the original probability density function,

$$\rho_u = \frac{1}{\sqrt{2\pi} * \sigma} \text{ and } \rho_l = \frac{1}{\sqrt{2\pi} * \sigma} * \exp\left(-\frac{\xi_0^2}{2\sigma^2}\right).$$

Using (2.12), we have the following satisfaction function:

$$u(x,\xi) = \frac{f(x,\xi)-f_l}{f_u-f_l}, \tag{4.3}$$

where f_l and f_u are the lower and upper bounds of $f(x,\xi)$ in $\Xi\times\Xi$, respectively. Clearly, the highest value is $f_u = 0$, that is, the demand is equal to the order quantity. The lowest value is

$$f_l = \min\{-4(r-s_0)^2 * \xi_0^2, -4s_u^2 * \xi_0^2\}.$$

With (4.1)-(4.3), we build OSDT-based newsvendor models as follows.

The OSDT-based newsvendor model with active focus points:

$$\max_{(x;\xi)} f(x,\xi) \quad \text{s.t. } x \in \Xi, \xi \in \Lambda^1(x), \tag{4.4}$$

where $\Lambda^1(x)$ denotes the set of global optimal solutions of the following optimization problem:

$$\max_{\xi\in\Xi} \min\{\pi(\xi), u(x,\xi)\}. \tag{4.5}$$

The OSDT-based newsvendor model with passive focus points:

$$\max_{(x;\xi)} f(x,\xi) \quad \text{s.t. } x \in \Xi, \xi \in \Lambda^2(x), \tag{4.6}$$

where $\Lambda^2(x)$ denotes the set of global optimal solutions of the following optimization problem:

$$\min_{\xi\in\Xi} \max\{1-\pi(\xi), u(x,\xi)\}. \tag{4.7}$$

Clearly, for Model (4.4)-(4.5), the upper level problem (4.4) is used to find the optimal order on a specific scenario associated with it; the lower level problem (4.5) is used to seek this scenario which has a relatively high relative likelihood degree and can cause a relatively high satisfaction level. Likewise, for Model (4.6)-(4.7), the upper level problem (4.6) is used to find the optimal order on a specific scenario associated with it; the lower level problem (4.7) is used to seek this scenario which has a relatively high relative likelihood degree and can cause a relatively low satisfaction level.

4.3 Solutions to the Proposed Newsvendor Models and Numerical Examples

We have already proposed single-level reformulation methods to solve OSDT-based decision models in Chapter 3. In this section, we apply these methods to OSDT-based newsvendor models (4.4)-(4.5) and (4.6)-(4.7). Considering (4.2), it is easy to prove that $\pi(\cdot)$ is quasi-concave and continuously differentiable and, moreover, it satisfies $\pi(\mu)$=1 and $\pi(\mu-\xi_0)=\pi(\mu+\xi_0)=0$. Considering (4.3), it is easy to prove that $u(x,\cdot)$ is quasi-concave for all $x \in \Xi$. Hence, Assumption 3.4 holds.

By Theorem 3.1, we know that solving the BLPP (4.4)-(4.5) becomes equivalent to solving the following single-level optimization problem:

$$\max_{(x;\,\xi)} f(x,\xi) \quad \text{s.t. } u(x,\xi) - \pi(\xi) \le 0, x \in [\mu - \xi_0, \mu + \xi_0], \xi \in [\mu - \xi_0, \mu + \xi_0], \tag{4.8}$$

where $f(x,\xi)$, $\pi(\xi)$ and $u(x,\xi)$ are given by (4.1), (4.2) and (4.3), respectively.

By Theorems 3.4 and 3.5, we know that solving the BLPP (4.6)-(4.7) becomes equivalent to solving the following single-level optimization problem:

$$\begin{array}{ll} \min & z \\ \text{s.t.} & x \in [\mu - \xi_0, \mu + \xi_0], \xi_1 \in [\mu - \xi_0, \mu], \xi_2 \in [\mu, \mu + \xi_0], z \in R, \\ & 1 - u(x,\xi_i) - \pi(\xi_i) = 0, i = 1, 2, \\ & -u(x,\xi_1) \le z, -u(x,\xi_2) \le z. \end{array} \tag{4.9}$$

where $f(x,\xi)$, $\pi(\xi)$ and $u(x,\xi)$ are defined by (4.1), (4.2) and (4.3), respectively.

We demonstrate the proposed solution methods with the following example. A sports clothing store, located in Tokyo, Japan, is planning to order a new fashion sportswear before the selling season. The unit wholesale price w, the unit revenue r, the unit salvage price s_0 and the unit opportunity cost s_u are 6, 9, 4 and 3 (thousand JPY), respectively. The demand is distributed normally with mean 500 and variance 200^2. The range of the possible demand is $[200, 800]$.

By using (4.1), we have

$$f(x,\xi) := \begin{cases} -25(x-\xi)^2, & \text{if } \xi < x, \\ -9(x-\xi)^2, & \text{if } \xi \ge x, \end{cases}$$

where $x, \xi \in [200,800]$. The highest value is $f_u = 0$ and the lowest value is $f_l = -9000000$.

By using (4.2), the relative likelihood function is

$$\pi(\xi) := \frac{\exp(-(\xi - 500)^2/80000) - \exp\ (-9/8)}{1 - \exp(-9/8)}.$$

By using (4.3), the satisfaction function is

$$u(x,\xi) = \begin{cases} -\dfrac{1}{600^2}(x-\xi)^2 + 1, & \text{if } \xi < x, \\ -\dfrac{1}{1000^2}(x-\xi)^2 + 1, & \text{if } \xi \ge x. \end{cases}$$

In our experiments, we utilize the interior-point algorithm from Global Optimization Toolbox of MATLAB 7.10.0 to solve the reformulated models (4.8) and (4.9). The numerical results are listed in the following table.

Table 4-1: Numerical results for the newsvendor example

Model	x^*	ξ^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$
(4.8)	500	500	1.0000	1.0000
(4.9)	432	230/770	0.1146	0.8854

Table 4-1 shows that the optimal order quantity of the active retailer is equal to its focus point. Interestingly, it implies that the active retailer has a confidence that he/she can sell what he/she optimally orders. The optimal order quantity of the active retailer is more than the one of the passive retailer, that is in perfect accordance with the situations occurred in the real world.

4.4 **Concluding Remarks**

In this chapter, we apply the OSDT-based decision approaches to a single-item newsvendor problem and build OSDT-based newsvendor models. The proposed newsvendor models are BLPPs with one-dimensional lower level variables. We utilize the proposed single-level reformulation methods to solve them and use a numerical example to show their effectiveness.

Chapter 5

Applications to Multi-Item Production Planning Problems

5.1 Introduction

Production planning problems are fundamental and important managerial decision problems in various industries, such as agricultural industry, manufacturing industry, entertainment industry, etc. Since production planning problems in the real world invariably include some unknown parameters, uncertainty is a main factor that affects the effectiveness of the obtained plan. Uncertainty involved in production planning problems can generally be categorized into two major types: system uncertainty and environmental uncertainty. The former includes uncertainties within the production processes, such as quality uncertainty, operation yield uncertainty and so on; the latter involves uncertainties beyond the production processes, such as demand uncertainty, supply uncertainty and so on (Ho 1989). The uncertainty in production planning problems was first examined by Dantzig (1955). From then on, a considerable amount of research (e.g., Aouam and Brahimi 2013; Graves 2011; Gyulai and Pfeiffer 2016; Higle and Kempf 2011; Kazemi et al. 2010; Shi et al. 2011; Sodhi and Tang 2009; Tang et al. 2012) and surveys have appeared in the production planning literature, including those of Mula et al. (2006) and Wazed et al. (2010).

From the aspect of mathematical optimization, production planning problems involving uncertainty can be mainly modeled by the following approaches. The first approach is stochastic programming where the uncertain parameters can be characterized by random variables whose probability distributions are known or can be estimated (see, e.g., Bornapour and Hooshmand 2015; Koca et al. 2015; Nasiri et al. 2014; Tempelmeier and Hilger 2015). Specifically, there are two kinds of methods to deal with stochastic programming problems: chance-constrained methods and recourse methods. The chance-constrained methods ensure that the optimal solution makes the probability of a certain constraint being satisfied above a certain level (see, e.g., Charnes and Cooper 1959; Miller and Wagner 1965; Nemirovski and Shapiro 2006; Pagnoncelli et al. 2009). The recourse methods are mainly used in two-stage (or multi-stage) problems: in the first stage, a feasible solution is chosen before observing the random parameters; in the second stage, upon a realization of the random parameters, further decisions are allowed to avoid the infeasibility of constraints (see, e.g., Birge and Louveaux 1997; Shapiro et al. 2009). The second approach is robust optimization to deal with the uncertain parameters which are known to reside in (bounded) uncertainty sets (see, e.g., Alem and Morabito 2012; Alvarez and Vera 2014; Aouam and Brahimi 2013; Ardjmand et al. 2016; Carvalho et al. 2016). According to different decision environments, this kind of problem can be subdivided as static robust optimization or adjustable robust optimization. Static robust optimization is for the case that all decision variables represent here-and-now decisions, that is, they should be made before the actual parameters are observed. Adjustable robust optimization is for the case that the part of the decision variables must be determined before the realization of the uncertain parameters, while the others can be adjusted after some parts of the uncertain parameters are revealed (see, e.g., Ben-Tal et al. 2004; Gorissen and Den Hertog 2013). Ben-Tal et al. (2009) give a comprehensive overview of robust optimization theory and applications. In addition, distributionally robust optimization approach is also widely used for the case that the uncertain parameters can be characterized by random variables but the probability distributions are not fully known due to the lack of enough historical data. Distributionally robust optimization provides an alternative way to overcome the conservativeness of the robust optimization without requiring exact specifications of the probability distributions (see, e.g., Hanasusanto et al. 2015; Wiesemann et al. 2014; Zymler et al. 2013).

In this chapter, we consider a production planning problem for innovative products as defined by Fisher (1997). According to Fisher, an innovative product has a higher profit margin, an intrinsically unpredictable demand and a short life cycle. In addition, for such an innovative product, the procurement lead-time is usually longer than the selling season so that there is often only one opportunity to produce goods before the season. One typical example is fashion clothes which are characterized by volatile and unpredictable demands, short life cycles and long supply processes (Sen 2008). Fashion items are sold punctually in a short period and generally not replenished so that they are called as “one-shot” items (Thomassey 2010). Hence, the production planning problem for such products is typically a one-shot decision problem. In this chapter, we apply the OSDT-based decision approaches to a multi-item production planning problem with short life-cycle products. For the sake of simplicity, we only consider the market uncertainty which is characterized by a random vector of unit profits of innovative products.

The remainder of this chapter is organized as follows. In Section 5.2, OSDT-based production planning models are built. In Section 5.3, we apply the proposed reformulation methods to these models and give specific single-level equivalent models for them. In particular, we consider two types of constraints, namely, cuboid constraints and ellipsoidal constraints. In Section 5.4, two numerical examples are used to show the effectiveness of the proposed reformulation methods, a comparison with other methods is made, and the managerial insights are gained. Finally, we conclude our research in Section 5.5.

5.2 One-Shot Decision Theory Based Production Planning Models

Consider a manufacturer who is making a production planning for multiple innovative products with short life-cycles under market uncertainty. The production quantity of product i is the decision variable x^i $(i = 1, \cdots, n)$, the unit profit of product i is characterized as a random variable $\boldsymbol{\xi}^{\mathbf{i}}$. The realization of the random vector $\boldsymbol{\xi} = (\boldsymbol{\xi}^{\mathbf{1}}; \cdots; \boldsymbol{\xi}^{\mathbf{n}})$ is denoted by $\xi = (\xi^1; \cdots; \xi^n)$ whose feasible region is denoted by $\Xi \subset R^n$. $X := \{x \in R^n : Ax \leq b, x \geq 0\}$ represents the constraints of available resources, such as time, materials, etc.

Usually, such a production planning problem can be modeled as the following stochastic optimization problem:

$$\max_{x} \boldsymbol{\xi}^T x \quad \text{s.t.} \, Ax \leq b, x \geq 0. \tag{5.1}$$

Due to the short life cycles of these products, one and only one realization of a random vector of the unit profits of these products (a scenario) will appear in the future and the manufacturer has only one opportunity to determine the production levels before the scenario reveals. Clearly, such a production planning problem is a one-shot decision problem. However, (5.1) is not well defined since $\boldsymbol{\xi}$ is a random vector.

In the following, we remodel the problem (5.1) with the one-shot decision theory. Since the normal distribution is one of the most frequently used probability distributions, we assume hereafter that $\boldsymbol{\xi}$ follows a (truncated) normal distribution with mean vector $\mu = (\mu^1; \cdots; \mu^n) \in R^n$ and covariance matrix $\Sigma \in R^{n \times n}$, that is, $\boldsymbol{\xi} \sim \mathcal{N}(\mu, \Sigma)$. Using (2.11) and (2.12), we can give two formulas as follows:

$$\pi(\xi) := \frac{1}{\log(\rho_u) - \log(\rho_l)} \big(\log(\rho(\xi)) - \log(\rho_l)\big), \tag{5.2}$$

where ρ denotes the original probability density function and

$$u(x,\xi) := \frac{1}{f_u - f_l}(\xi^T x - f_l), \tag{5.3}$$

where $\rho_l \coloneqq \min\{\rho(\xi): \xi \in \Xi\}$ and $\rho_u \coloneqq \max\{\rho(\xi): \xi \in \Xi\}$ are the lower and upper bounds of $\rho(\xi)$ in Ξ, $f_l = \min\{\xi^T x: x \in X, \xi \in \Xi\}$ and $f_u = \max\{\xi^T x: x \in X, \xi \in \Xi\}$ are the lower and upper bounds of $\xi^T x$ in $X \times \Xi$, respectively. Instead of directly normalizing the original density function, we utilize (5.2) as the relative likelihood function because it is a concave function and hence it is computationally tractable.

With (5.2)-(5.3), we build OSDT-based production planning models as follows.

The OSDT-based production planning model with active focus points:

$$\max_x \ \xi^T x \quad \text{s.t. } Ax \le b, x \ge 0, \xi \in \Lambda^1(x), \tag{5.4}$$

where $\Lambda^1(x)$ denotes the set of global optimal solutions of the following optimization problem:

$$\max_{\xi \in \Xi} \ \min\{\pi(\xi), u(x,\xi)\}. \tag{5.5}$$

The OSDT-based production planning model with passive focus points:

$$\max_x \ \xi^T x \quad \text{s.t. } Ax \le b, x \ge 0, \xi \in \Lambda^2(x), \tag{5.6}$$

where $\Lambda^2(x)$ denotes the set of global optimal solutions of the following optimization problem:

$$\min_{\xi \in \Xi} \ \max\{1 - \pi(\xi), u(x,\xi)\}. \tag{5.7}$$

Clearly, for Model (5.4)-(5.5), the upper level problem (5.4) is used to find the optimal production plan for maximizing the total profit on a specific scenario associated with it; the lower level problem (5.5) is used to seek this scenario which has a relatively high relative likelihood degree and can cause a relatively high satisfaction level. Likewise, for Model (5.6)-(5.7), the upper level problem (5.6) is used to find the optimal production plan for maximizing the total profit on a specific scenario associated with it; the lower level problem (5.7) is used to seek this scenario which has a relatively high relative likelihood degree and can cause a relatively low satisfaction level.

5.3 Solutions to the Proposed Production Planning Models.

We have already proposed single-level reformulation methods to solve OSDT-based decision models in Chapter 3. In this section, we apply these methods to OSDT-based production planning models (5.4)-(5.5) and (5.6)-(5.7).

5.3.1 Case of Cuboid Constraints

In this subsection, we consider the case where Ξ is a cuboid constraint, i.e.,

$$\Xi = \left\{\xi \in R^n : \mu^i - k\sigma^i \leq \xi^i \leq \mu^i + k\sigma^i,\ \forall\, i = 1, \cdots, n\right\} \text{ with } k > 0, \tag{5.8}$$

where $\sigma^i = \sqrt{\Sigma(i,i)}$ represents the standard deviation of the random variable $\boldsymbol{\xi}^{\mathbf{i}}$ for $i = 1, \cdots, n$. It follows from (5.8) that $\mu - k\sigma$ and $\mu + k\sigma$ are the lower and upper bounds of Ξ, respectively.

Property 5.1 Suppose that Ξ takes the form of (5.8), then $\pi(\xi)$ is a quadratic concave function:

$$\pi(\xi) = 1 - \frac{1}{k^2\sigma^T\Sigma^{-1}\sigma} * (\xi - \mu)^T\Sigma^{-1}(\xi - \mu). \tag{5.9}$$

Property 5.1 means that the mean vector has the maximal relative likelihood degree 1, i.e., $\pi(\mu) = 1$, and the vector which has a longer Euclidean distance from the mean vector will have a less relative likelihood degree. Considering Property 5.1, we know that Assumption 3.6 holds. By (5.3) and (5.9), we can easily to verify that Assumption 3.7 is also satisfied.

By Theorem 3.6, we know that solving the BLPP (5.4)-(5.5) becomes equivalent to solving the following single-level optimization problem:

$$\begin{array}{ll} \max & \xi^T x \\ \text{s.t.} & F_1(x,\xi) \le 0, Ax \le b, \\ & x \in R_+^n, \xi \in [\mu - k\sigma, \mu + k\sigma], \end{array} \tag{5.10}$$

where F_1 is defined as

$$F_1(x,\xi) = \frac{\xi^T x - f_l}{f_u - f_l} + \frac{(\xi-\mu)^T \Sigma^{-1} (\xi-\mu)}{k^2 \sigma^T \Sigma^{-1} \sigma} - 1.$$

By Theorem 3.7, we know that solving the BLPP (5.6)-(5.7) becomes equivalent to solving the following single-level optimization problem:

$$\begin{array}{ll} \max & L_1(x,\xi,\lambda,\eta,\tau) \\ \text{s.t.} & G_1(x,\xi,\lambda,\eta,\tau) = 0, Ax \le b, \\ & x \in R_+^n, \xi \in [\mu - k\sigma, \mu + k\sigma], \lambda \in R_+^n, \eta \in R_+^n, \tau \in R_+, \end{array} \tag{5.11}$$

where L_1 and G_1 are respectively defined as

$$L_1(x,\xi,\lambda,\eta,\tau) = (1-\tau) * \frac{\xi^T x - f_l}{f_u - f_l} + \tau * \frac{(\xi-\mu)^T \Sigma^{-1} (\xi-\mu)}{k^2 \sigma^T \Sigma^{-1} \sigma} + \lambda^T(\xi - \mu - k\sigma) + \eta^T(\mu - k\sigma - \xi),$$

and

$$G_1(x,\xi,\lambda,\eta,\tau) = (1-\tau) * \frac{x}{f_u - f_l} + 2\tau * \frac{\Sigma^{-1}(\xi-\mu)}{k^2 \sigma^T \Sigma^{-1} \sigma} + \lambda - \eta.$$

5.3.2 Case of ellipsoidal constraints

In this subsection, we consider the case where Ξ is an ellipsoidal constraint, i.e.,

$$\Xi = \left\{\xi \in R^n : g(\xi) = \frac{1}{2}(\xi-\mu)^T \Sigma^{-1} (\xi-\mu) - r^2 \le 0\right\} \text{ with } r > 0. \tag{5.12}$$

Further, we have

$$\nabla g(\xi) = \Sigma^{-1}(\xi - \mu).$$

Property 5.2 Suppose that Ξ takes the form of (5.12), then $\pi(\xi)$ is a quadratic concave function:

$$\pi(\xi) = 1 - \frac{1}{2r^2} * (\xi-\mu)^T \Sigma^{-1} (\xi-\mu). \tag{5.13}$$

Property 5.2 means that the mean vector has the maximal relative likelihood degree 1, i.e., $\pi(\mu) = 1$, and the vector which has a longer Euclidean distance from the mean vector will have a less relative likelihood degree. Considering Property 5.2, we know that Assumption 3.6 holds. By (5.3) and (5.12), we can easily to verify that Assumption 3.7 is also satisfied.

By Theorem 3.6, we know that solving the BLPP (5.4)-(5.5) becomes equivalent to solving the following single-level optimization problem:

$$\begin{aligned} \max \quad & \xi^T x \\ \text{s.t.} \quad & F_2(x,\xi) \le 0, \frac{1}{2}(\xi-\mu)^T\Sigma^{-1}(\xi-\mu) - r^2 \le 0, \\ & Ax \le b, x \in R_+^n, \xi \in R^n, \end{aligned} \tag{5.14}$$

where F_2 is defined as

$$F_2(x,\xi) = \frac{\xi^T x - f_l}{f_u - f_l} + \frac{(\xi-\mu)^T\Sigma^{-1}(\xi-\mu)}{2r^2} - 1.$$

By Theorem 3.7, we know that solving the BLPP (5.6)-(5.7) becomes equivalent to solving the following single-level optimization problem:

$$\begin{aligned} \max \quad & L_2(x,\xi,\lambda,\tau) \\ \text{s.t.} \quad & G_2(x,\xi,\lambda,\tau) = 0,\ \frac{1}{2}(\xi-\mu)^T\Sigma^{-1}(\xi-\mu) - r^2 \le 0, \\ & Ax \le b, x \in R_+^n, \xi \in R^n, \lambda \in R_+, \tau \in R_+, \end{aligned} \tag{5.15}$$

where L_2 and G_2 are respectively defined as

$$L_2(x,\xi,\lambda,\tau) = (1-\tau) * \frac{\xi^T x - f_l}{f_u - f_l} + \tau * \frac{(\xi-\mu)^T\Sigma^{-1}(\xi-\mu)}{2r^2} + \lambda * \left(\frac{1}{2}(\xi-\mu)^T\Sigma^{-1}(\xi-\mu) - r^2\right),$$

and

$$G_2(x,\xi,\lambda,\tau) = (1-\tau) * \frac{x}{f_u - f_l} + \tau * \frac{\Sigma^{-1}(\xi-\mu)}{r^2} + \lambda * \Sigma^{-1}(\xi-\mu).$$

5.4. Numerical Experiments and Computational Discussions

In order to illustrate the proposed methods, let us consider a numerical example as follows. An apparel manufacturer is planning to produce four types of new fashion clothes for the coming summer season. For producing fashion clothes 1, 2, 3 and 4, four kinds of resources, that is, A, B, C and D are needed. The available amounts of A, B, C and D are 1500, 2250, 1100 and 1300 units, respectively. The amounts of resource A needed for producing one unit fashion clothes 1, 2, 3 and 4 are 2, 3, 3 and 2 units, respectively; the amounts of resource B needed for producing one unit fashion clothes 1, 2, 3 and 4 are 2, 3, 4 and 5 units, respectively; the amounts of resource C needed for producing one unit fashion clothes 1, 2, 3 and 4 are 3, 2, 2 and 1 units, respectively; the amounts of resource D needed for producing one unit fashion clothes 1, 2, 3 and 4 are 1, 2, 2 and 3 units, respectively. Hence, the feasible set of production levels is

$$\mathrm{X} = \{\mathrm{x} \in \mathrm{R}_+^4 : \mathrm{Ax} \leq \mathrm{b}\}. \tag{5.16}$$

where

$$\mathrm{A} = \begin{pmatrix} 2 & 3 & 3 & 2 \\ 2 & 3 & 4 & 5 \\ 3 & 2 & 2 & 1 \\ 1 & 2 & 2 & 3 \end{pmatrix} \text{ and } \mathrm{b} = \begin{pmatrix} 1500 \\ 2250 \\ 1100 \\ 1300 \end{pmatrix}.$$

5.4.1 Numerical Examples with Cuboid Constraints

In this subsection, we consider the case where Ξ takes the form of (5.8), that is,

$$\Xi := [\mu - \mathrm{k}\sigma, \mu + \mathrm{k}\sigma] \subset \mathrm{R}^4 \text{ with } \mathrm{k} > 0, \tag{5.17}$$

It follows from (2.2) that the expected value based production planning model is

$$\max\ \mu^{\mathrm{T}}\mathrm{x} \quad \mathrm{s.t.}\ \mathrm{Ax} \leq \mathrm{b}, \mathrm{x} \geq 0. \tag{5.18}$$

It follows from (2.4) that the maximax approach based production planning model is

$$\max\ (\mu + k\sigma)^T x \quad \text{s.t.}\, Ax \le b, x \ge 0. \tag{5.19}$$

It follows from (2.5) that the maximin approach based production planning model is

$$\max\ (\mu - k\sigma)^T x \quad \text{s.t.}\, Ax \le b, x \ge 0. \tag{5.20}$$

We utilize the interior-point algorithm from Global Optimization Toolbox of MATLAB 7.10.0 to solve (5.10) and (5.11), and use the simplex algorithm to solve (5.18), (5.19) and (5.20).

Case I: $k = k_0, \Sigma = \Sigma_0$ and $\mu = \mu_0$ where

$$k_0 = 2,\ \Sigma_0 = \begin{pmatrix} 2500 & 1250 & 1250 & 1250 \\ 1250 & 2500 & 1250 & 1250 \\ 1250 & 1250 & 2500 & 1250 \\ 1250 & 1250 & 1250 & 2500 \end{pmatrix} \text{ and } \mu_0 = \begin{pmatrix} 150 \\ 200 \\ 200 \\ 150 \end{pmatrix}.$$

The numerical results are shown in the following tables.

Table 5-1. Solutions of (5.10) for Case I

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 191.5182 \\ 244.4839 \\ 244.4839 \\ 197.4494 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 150.0000 \\ 150.0000 \\ 200.0000 \end{pmatrix}$	0.7999	0.7999	131990

Table 5-2. Solutions of (5.11) for Case I

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 91.9217 \\ 133.4775 \\ 133.4775 \\ 88.3976 \end{pmatrix}$	$\begin{pmatrix} 84.9330 \\ 180.1356 \\ 180.1324 \\ 124.6650 \end{pmatrix}$	0.5945	0.4055	66915

Table 5-3. Solutions of (5.18) for Case I

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*,\xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix}150.0000\\200.0000\\200.0000\\150.0000\end{pmatrix}$	$\begin{pmatrix}100.0000\\186.1648\\113.8352\\200.0000\end{pmatrix}$	1.0000	0.6364	105000

Table 5-4. Solutions of (5.19) for Case I

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*,\xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix}250.0000\\300.0000\\300.0000\\250.0000\end{pmatrix}$	$\begin{pmatrix}100.0000\\200.0000\\100.0000\\200.0000\end{pmatrix}$	0.0000	1.0000	165000

Table 5-5. Solutions of (5.20) for Case I

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*,\xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix}50.0000\\100.0000\\100.0000\\50.0000\end{pmatrix}$	$\begin{pmatrix}0.0000\\236.7627\\263.2373\\0.0000\end{pmatrix}$	0.0000	0.3030	50000

For examining how the solutions change with k for Case I, we consider the following case.

Case II: $k = 1.5 * k_0$, $\Sigma = \Sigma_0$ and $\mu = \mu_0$.

The obtained results are shown in the following tables.

Table 5-6. Solutions of (5.10) for Case II

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*,\xi^*)$	${\xi^*}^T x^*$

$\begin{pmatrix} 213.5324 \\ 281.6845 \\ 254.4564 \\ 222.6085 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 300.0000 \\ 0.0000 \\ 200.0000 \end{pmatrix}$	0.7712	0.7712	150380

Table 5-7. Solutions of (5.11) for Case II

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 77.1232 \\ 113.8610 \\ 113.8608 \\ 75.6643 \end{pmatrix}$	$\begin{pmatrix} 78.1756 \\ 193.6479 \\ 193.6497 \\ 90.8779 \end{pmatrix}$	0.7077	0.2923	57003

Table 5-8. Solutions of (5.18) for Case II

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 150.0000 \\ 200.0000 \\ 200.0000 \\ 150.0000 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 186.1648 \\ 113.8352 \\ 200.0000 \end{pmatrix}$	1.0000	0.5385	105000

Table 5-9: Solutions of (5.19) for Case II

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 300.0000 \\ 350.0000 \\ 350.0000 \\ 300.0000 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 202.1351 \\ 97.8649 \\ 200.0000 \end{pmatrix}$	0.0000	1.0000	195000

Table 5-10. Solutions of (5.20) for Case II

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 0.0000 \\ 50.0000 \\ 50.0000 \\ 0.0000 \end{pmatrix}$	$\begin{pmatrix} 0.0000 \\ 271.1877 \\ 228.8123 \\ 0.0000 \end{pmatrix}$	0.0000	0.1282	25000

We examine how the solutions change with Σ for Case I. We consider the following case III.

Case III: $\mathrm{k} = \mathrm{k}_0$, $\Sigma = 1.44 * \Sigma_0$ and $\mu = \mu_0$.

The obtained results are shown in the following tables.

Table 5-11. Solutions of (5.10) for Case III

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 201.8225 \\ 255.5241 \\ 255.5241 \\ 209.2257 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 150.0000 \\ 150.0000 \\ 200.0000 \end{pmatrix}$	0.7835	0.7835	138680

Table 5-12. Solutions of (5.11) for Case III

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 85.3928 \\ 124.8230 \\ 124.8230 \\ 82.7798 \end{pmatrix}$	$\begin{pmatrix} 81.5170 \\ 186.9661 \\ 186.9659 \\ 107.5849 \end{pmatrix}$	0.6467	0.3533	62542

Table 5-13. Solutions of (5.18) for Case III

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 150.0000 \\ 200.0000 \\ 200.0000 \\ 150.0000 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 186.1648 \\ 113.8352 \\ 200.0000 \end{pmatrix}$	1.0000	0.5932	105000

Table 5-14. Solutions of (5.19) for Case III

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$

$\begin{pmatrix}270.0000\\320.0000\\320.0000\\270.0000\end{pmatrix}$	$\begin{pmatrix}100.0000\\200.0000\\100.0000\\200.0000\end{pmatrix}$	0.0000	1.0000	177000

Table 5-15. Solutions of (5.20) for Case III

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*,\xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix}30.0000\\80.0000\\80.0000\\30.0000\end{pmatrix}$	$\begin{pmatrix}0.0000\\258.4314\\241.5686\\0.0000\end{pmatrix}$	0.0000	0.2260	40000

Table 5-16. The scenarios associated with the optimal solutions in Cases I and II

	(5.20)	(5.11)	(5.18)	(5.10)	(5.19)
I	$\begin{pmatrix}50.0000\\100.0000\\100.0000\\50.0000\end{pmatrix}$	$\begin{pmatrix}91.9217\\133.4775\\133.4775\\88.3976\end{pmatrix}$	$\begin{pmatrix}150.0000\\200.0000\\200.0000\\150.0000\end{pmatrix}$	$\begin{pmatrix}191.5182\\244.4839\\244.4839\\197.4494\end{pmatrix}$	$\begin{pmatrix}250.0000\\300.0000\\300.0000\\250.0000\end{pmatrix}$
II	$\begin{pmatrix}0.0000\\50.0000\\50.0000\\0.0000\end{pmatrix}$	$\begin{pmatrix}77.1232\\113.8610\\113.8608\\75.6643\end{pmatrix}$	$\begin{pmatrix}150.0000\\200.0000\\200.0000\\150.0000\end{pmatrix}$	$\begin{pmatrix}213.5324\\281.6845\\254.4564\\222.6085\end{pmatrix}$	$\begin{pmatrix}300.0000\\350.0000\\350.0000\\300.0000\end{pmatrix}$

Table 5-17. The scenarios associated with the optimal solutions in Cases I and III

	(5.20)	(5.11)	(5.18)	(5.10)	(5.19)
I	$\begin{pmatrix}50.0000\\100.0000\\100.0000\\50.0000\end{pmatrix}$	$\begin{pmatrix}91.9217\\133.4775\\133.4775\\88.3976\end{pmatrix}$	$\begin{pmatrix}150.0000\\200.0000\\200.0000\\150.0000\end{pmatrix}$	$\begin{pmatrix}191.5182\\244.4839\\244.4839\\197.4494\end{pmatrix}$	$\begin{pmatrix}250.0000\\300.0000\\300.0000\\250.0000\end{pmatrix}$
III	$\begin{pmatrix}30.0000\\80.0000\\80.0000\\30.0000\end{pmatrix}$	$\begin{pmatrix}85.3928\\124.8230\\124.8230\\82.7798\end{pmatrix}$	$\begin{pmatrix}150.0000\\200.0000\\200.0000\\150.0000\end{pmatrix}$	$\begin{pmatrix}201.8225\\255.5241\\255.5241\\209.2257\end{pmatrix}$	$\begin{pmatrix}270.0000\\320.0000\\320.0000\\270.0000\end{pmatrix}$

Table 5-18. The optimal total profit for Cases I and II

	(5.20)	(5.11)	(5.18)	(5.10)	(5.19)
I	50000	66915	105000	131990	165000
II	25000	57003	105000	150380	195000

Table 5-19. The optimal total profit for Cases I and III

	(5.20)	(5.11)	(5.18)	(5.10)	(5.19)
I	50000	66915	105000	131990	165000
III	40000	62542	105000	138680	177000

5.4.2 Numerical Examples with Cuboid Constraints

In this subsection, we consider the case where Ξ takes the form of (5.12), that is,

$$\Xi = \left\{\xi \in R^4 : g(\xi) = \frac{1}{2}(\xi - \mu)^T \Sigma^{-1}(\xi - \mu) - r^2 \leq 0\right\} \text{ with } r > 0. \tag{5.21}$$

It follows from (2.2) that the expected value based production planning model is

$$\max_{x} \ \mu^T x \quad \text{s.t.}\, Ax \leq b, x \geq 0. \tag{5.22}$$

It follows from (2.4) that the maximax approach based production planning model is

$$\max_{(x;\xi)} \ \xi^T x \quad \text{s.t.}\, Ax \leq b, x \geq 0, g(\xi) \leq 0. \tag{5.23}$$

It follows from (2.5) that the maximin approach based production planning model is

$$\max_{x} \left\{\min_{\xi} \xi^T x : g(\xi) \leq 0\right\} \quad \text{s.t.}\, Ax \leq b, x \geq 0. \tag{5.24}$$

Using the strongly dual theory to (5.24), we can transform it into the following problem:

$$\max_{(x;\xi;\lambda)} \ \xi^T x \quad \text{s.t.}\, Ax \leq b, x \geq 0, \lambda \geq 0, g(\xi) = 0, x + \lambda * \Sigma^{-1}(\xi - \mu) = 0. \tag{5.25}$$

We utilize the interior-point algorithm from Global Optimization Toolbox of MATLAB 7.10.0 to solve (5.12), (5.13), (5.23) and (5.25), use the simplex algorithm to solve (5.22).

Case I: $\mathrm{r} = \mathrm{r}_0$, $\Sigma = \Sigma_0$ and $\mu = \mu_0$ where

$$\mathrm{r}_0 = 2,\ \Sigma_0 = \begin{pmatrix} 2500 & 1250 & 1250 & 1250 \\ 1250 & 2500 & 1250 & 1250 \\ 1250 & 1250 & 2500 & 1250 \\ 1250 & 1250 & 1250 & 2500 \end{pmatrix} \text{ and } \mu_0 = \begin{pmatrix} 150 \\ 200 \\ 200 \\ 150 \end{pmatrix}.$$

The numerical results are shown in the following tables.

Table 5-20. Solutions of (5.12) for Case I

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*,\xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 195.9639 \\ 259.0964 \\ 239.3976 \\ 202.5301 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 300.0000 \\ 0.0000 \\ 200.0000 \end{pmatrix}$	0.7844	0.7844	137831

Table 5-21. Solutions of (5.13) for Case I

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*,\xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 88.4210 \\ 128.8371 \\ 128.8371 \\ 85.3855 \end{pmatrix}$	$\begin{pmatrix} 82.9979 \\ 184.0042 \\ 184.0042 \\ 114.9895 \end{pmatrix}$	0.6325	0.3675	64570

Table 5-22. Solutions of (5.22) for Case I

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*,\xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 150.0000 \\ 200.0000 \\ 200.0000 \\ 150.0000 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 186.1648 \\ 113.8352 \\ 200.0000 \end{pmatrix}$	1.0000	0.6364	105000

Table 5-23. Solutions of (5.23) for Case I

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 248.9949 \\ 327.2792 \\ 384.8528 \\ 263.1371 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 300.0000 \\ 0.0000 \\ 200.0000 \end{pmatrix}$	0.0000	1.0000	175711

Table 5-24. Solutions of (5.25) for Case I

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 53.0560 \\ 79.5840 \\ 79.5840 \\ 53.0561 \end{pmatrix}$	$\begin{pmatrix} 62.2845 \\ 208.4773 \\ 208.4772 \\ 62.2837 \end{pmatrix}$	0.0000	0.2265	39792

For examining how the solutions change with r for Case I, we consider the following case.

Case II: $r = 1.2 * r_0$, $\Sigma = \Sigma_0$ and $\mu = \mu_0$.

The obtained results are shown in the following tables.

Table 5-25. Solutions of (5.12) for Case II

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 207.1906 \\ 273.5308 \\ 249.0205 \\ 215.3607 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 300.0000 \\ 0.0000 \\ 200.0000 \end{pmatrix}$	0.7682	0.7682	145850

Table 5-26. Solutions of (5.13) for Case II

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$

$\begin{pmatrix} 81.8363 \\ 120.1086 \\ 120.1086 \\ 79.7196 \end{pmatrix}$	$\begin{pmatrix} 79.9678 \\ 190.0645 \\ 190.0645 \\ 99.8388 \end{pmatrix}$	0.6831	0.3169	60160

Table 5-27. Solutions of (5.22) for Case II

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 150.0000 \\ 200.0000 \\ 200.0000 \\ 150.0000 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 186.1648 \\ 113.8352 \\ 200.0000 \end{pmatrix}$	1.0000	0.5385	105000

Table 5-28: Solutions of (5.23) for Case II

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 268.7939 \\ 352.7351 \\ 301.8234 \\ 285.7645 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 300.0000 \\ 0.0000 \\ 200.0000 \end{pmatrix}$	0.0000	1.0000	189853

Table 5-29. Solutions of (5.25) for Case II

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 36.4944 \\ 54.7416 \\ 54.7416 \\ 36.4944 \end{pmatrix}$	$\begin{pmatrix} 51.6320 \\ 215.5786 \\ 215.5786 \\ 51.6323 \end{pmatrix}$	0.0000	0.1442	27371

We examine how the solutions change with Σ for Case I. We consider the following Case III.

Case III: $r = r_0$, $\Sigma = 1.69 * \Sigma_0$ and $\mu = \mu_0$.

The obtained results are shown in the following tables.

Table 5-30. Solutions of (5.12) for Case III

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 212.8789 \\ 280.8442 \\ 253.8962 \\ 221.8616 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 300.0000 \\ 0.0000 \\ 200.0000 \end{pmatrix}$	0.7613	0.7613	149914

Table 5-31. Solutions of (5.13) for Case III

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 78.8907 \\ 116.2039 \\ 116.2039 \\ 77.1850 \end{pmatrix}$	$\begin{pmatrix} 78.8165 \\ 192.3669 \\ 192.3669 \\ 94.0827 \end{pmatrix}$	0.7045	0.2955	58187

Table 5-32. Solutions of (5.22) for Case III

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 150.0000 \\ 200.0000 \\ 200.0000 \\ 150.0000 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 186.1648 \\ 113.8352 \\ 200.0000 \end{pmatrix}$	1.0000	0.5932	105000

Table 5-33. Solutions of (5.23) for Case III

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$
$\begin{pmatrix} 278.6934 \\ 365.4630 \\ 310.3087 \\ 297.0782 \end{pmatrix}$	$\begin{pmatrix} 100.0000 \\ 300.0000 \\ 0.0000 \\ 200.0000 \end{pmatrix}$	0.0000	1.0000	196924

Table 5-34. Solutions of (5.25) for Case III

ξ^*	x^*	$\pi(\xi^*)$	$u(x^*, \xi^*)$	${\xi^*}^T x^*$

$\begin{pmatrix} 28.2408 \\ 42.3612 \\ 42.3612 \\ 28.2408 \end{pmatrix}$	$\begin{pmatrix} 47.5771 \\ 218.2819 \\ 218.2819 \\ 47.5771 \end{pmatrix}$	0.0000	0.1076	21181

Table 5-35. The scenarios associated with the optimal solutions in Cases I and II

	(5.25)	(5.13)	(5.22)	(5.12)	(5.23)
I	$\begin{pmatrix} 53.0560 \\ 79.5840 \\ 79.5840 \\ 53.0561 \end{pmatrix}$	$\begin{pmatrix} 88.4210 \\ 128.8371 \\ 128.8371 \\ 85.3855 \end{pmatrix}$	$\begin{pmatrix} 150.0000 \\ 200.0000 \\ 200.0000 \\ 150.0000 \end{pmatrix}$	$\begin{pmatrix} 195.9639 \\ 259.0964 \\ 239.3976 \\ 202.5301 \end{pmatrix}$	$\begin{pmatrix} 248.9949 \\ 327.2792 \\ 384.8528 \\ 263.1371 \end{pmatrix}$
II	$\begin{pmatrix} 36.4944 \\ 54.7416 \\ 54.7416 \\ 36.4944 \end{pmatrix}$	$\begin{pmatrix} 81.8363 \\ 120.1086 \\ 120.1086 \\ 79.7196 \end{pmatrix}$	$\begin{pmatrix} 150.0000 \\ 200.0000 \\ 200.0000 \\ 150.0000 \end{pmatrix}$	$\begin{pmatrix} 207.1906 \\ 273.5308 \\ 249.0205 \\ 215.3607 \end{pmatrix}$	$\begin{pmatrix} 268.7939 \\ 352.7351 \\ 301.8234 \\ 285.7645 \end{pmatrix}$

Table 5-36. The scenarios associated with the optimal solutions in Cases I and III

	(5.25)	(5.13)	(5.22)	(5.12)	(5.23)
I	$\begin{pmatrix} 53.0560 \\ 79.5840 \\ 79.5840 \\ 53.0561 \end{pmatrix}$	$\begin{pmatrix} 88.4210 \\ 128.8371 \\ 128.8371 \\ 85.3855 \end{pmatrix}$	$\begin{pmatrix} 150.0000 \\ 200.0000 \\ 200.0000 \\ 150.0000 \end{pmatrix}$	$\begin{pmatrix} 195.9639 \\ 259.0964 \\ 239.3976 \\ 202.5301 \end{pmatrix}$	$\begin{pmatrix} 248.9949 \\ 327.2792 \\ 384.8528 \\ 263.1371 \end{pmatrix}$
III	$\begin{pmatrix} 28.2408 \\ 42.3612 \\ 42.3612 \\ 28.2408 \end{pmatrix}$	$\begin{pmatrix} 78.8907 \\ 116.2039 \\ 116.2039 \\ 77.1850 \end{pmatrix}$	$\begin{pmatrix} 150.0000 \\ 200.0000 \\ 200.0000 \\ 150.0000 \end{pmatrix}$	$\begin{pmatrix} 212.8789 \\ 280.8442 \\ 253.8962 \\ 221.8616 \end{pmatrix}$	$\begin{pmatrix} 278.6934 \\ 365.4630 \\ 310.3087 \\ 297.0782 \end{pmatrix}$

Table 5-37. The optimal total profit for Cases I and II

	(5.24)	(5.13)	(5.22)	(5.12)	(5.23)
I	39792	64570	105000	137831	175711
II	27371	60160	105000	145850	189853

Table 5-38. The optimal total profit for Cases I and III

	(5.25)	(5.13)	(5.22)	(5.12)	(5.23)
I	39792	64570	105000	137831	175711
III	21181	58187	105000	149914	196924

5.4.3 Results Analysis and Discussion

Let us give an explanation for numerical results shown in Section 5.4.1. Table 5-1 shows that the optimal production plan for an active manufacturer. The active production plan is (100.0000; 150.0000; 150.0000; 200.0000); the active focus point (scenario) for supporting this production plan is (191.5182; 244.4839; 244.4839; 197.4494); the relative likelihood degree of this focus point is 0.7999; when the active scenario occurs, the satisfaction level of this plan is 0.7999. In other words, the reason that an active manufacturer chooses the production plan (100.0000; 150.0000; 150.0000; 200.0000) is that the scenario (191.5182; 244.4839; 244.4839; 197.4494) is the most appropriate for this manufacturer. Table 5-2 shows that the optimal production plan for a passive manufacturer. The passive production plan is (84.9330; 180.1356; 180.1324; 124.6650); the passive focus point (scenario) for supporting this production plan is (91.9217; 133.4775; 133.4775; 88.3976); the relative likelihood degree of this scenario is 0.5945; when the passive scenario occurs, the satisfaction level of this plan is 0.4055. It means that the scenario (91.9217; 133.4775; 133.4775; 88.3976) is the most acceptable scenario amongst all unfavorable ones, a passive manufacturer chooses the production plan (84.9330; 180.1356; 180.1324; 124.6650) based on this scenario. Table 5-3 corresponds to the expected profit max manufacturer where the mean of profits per units of four products is (150; 200; 200; 150) whose relative likelihood degree is 1 and it can lead to the satisfaction level of 0.6364 for the optimal production plan. Tables 5-4 and 5-5 are for the maximax manufacturer and the maximin manufacturer, respectively. The maximax manufacturer takes into account the upper bound of profits per units of four products (250; 300; 300; 250) whereas the maximin manufacturer considers the lower bound of profits per units of four products (50; 100; 100; 50).

Tables from 5-6 to 5-10 and Tables from 5-11 to 5-15 respectively show how the solutions change with the varying of the feasible region of the realization of the random vector, that is, the varying of k and with the varying of uncertainty, that is, the varying of Σ. From these tables, we know different models provides different optimal solutions which can reflect different consideration for handling the uncertainty. However, as shown in Tables 5-16 and 5-17, in any case, the profit per unit of each product which is take into account for obtaining the optimal production plan will increase according to the order of the maximin manufacturer, the passive manufacturer, the expected profit max manufacturer, the active manufacturer and the maximax manufacturer. In addition, Tables 5-18 and 5-19 show that the profit anticipated by a maximax manufacturer is larger than the one by an active manufacturer; the profit anticipated by an active manufacturer is larger than the one by an expected profit max manufacturer; the profit anticipated by an expected profit max manufacturer is larger than the one by a passive manufacturer; the profit anticipated by a passive manufacturer is larger than the one by a maximin manufacturer. It means that the maximax manufacturer is the most optimistic, the maximin manufacturer is the most pessimistic, the expected profit max manufacturer is at the middle of the active and passive manufacturers.

When we compare the profit per unit of each product associated with the optimal production plans between Case I and Case II (shown in Table 5-16), we can find that increasing the feasible set of the realization of random variables, that is, k increasing from 2 (Case I) to 3 (Case II), the maximin manufacturer and the passive manufacturer will take a more conservative attitude so that the profit per unit of each product associated with the optimal production plan will decrease accordingly; on the contrary, the maximax manufacturer and the active manufacturer will take a more aggressive attitude so that the profit per unit of each product associated with the optimal production plan will increase accordingly; however, an expected profit max manufacturer only takes into account the mean vector of the profits per unit of products.

When we compare the profit per unit of each product associated with the optimal production plans between Case I and Case III (shown in Table 5-17), we can find that increasing the uncertainty of profits, that is, changing Σ from Σ_0 (Case I) to $1.44 * \Sigma_0$ (Case III), will cause the maximin manufacturer and the passive manufacturer more vigilant so that the profit per unit of each product associated with the optimal production plan will decrease accordingly; on the contrary, the maximax manufacturer and the active manufacturer will become more optimistic so that the profit per unit of each product associated with the optimal production plan will increase accordingly; however, the action of an expected profit max manufacturer will remain unchanged.

The obtained managerial insights can also be observed from the numerical results shown in Section 5.4.2, which are intuitively acceptable and can be used as a sort of criterion for selecting a product planning model to fit the preference of the different types of decision makers.

5.5 Concluding Remarks

We propose a new production planning model for a manufacturer that is planning to produce multiple innovative products with short life-cycles. Different from the existing production planning models, we build OSDT-based production planning models in which the optimal production quantities are obtained based on the scenarios which are the most appropriate for the manufacturer with considering the profit and the probability. We utilize the proposed reformulation methods to solve OSDT-based production planning models and use two numerical examples to show their effectiveness.

Chapter 6

Conclusions

In this dissertation, we are interested in solving a one-shot decision problem with one-shot decision theory (OSDT). Different from existing decision theories, OSDT provides a scenario-based theory of choice. According to OSDT, a decision-maker makes a one-shot decision by a two-step process in which the first step is to choose the most appropriate scenario for each alternative and the second step is to choose the best decision based on the selected scenarios. These selected scenarios are called focus points. The focus point of the best decision is a supporting scenario to carry out this decision. With different behaviors for choosing the focus points, decision-makers may make different decision.

OSDT-based decision models are bilevel programming problems with maximin or minimax lower level programs. Due to inherent mathematical difficulties, existing optimization methods are not valid to solve them directly. In this dissertation, we provide solvable single-level reformulations of these special bilevel optimization problems. The reformulated models are computationally tractable than traditional KKT-based reformulations.

As applications, we build OSDT-based newsvendor models and OSDT-based production planning models. The effectiveness of the proposed reformulation methods to these specific models is tested by preliminary numerical experiments. Applying the proposed methods to more real-world one-shot decision problems will be one of the main directions of future research.

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