A study on generic mappings under constraint conditions from the viewpoint of Singularity Theory

特異点論における、制約条件下のジェネリックな写像の研究

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2018

Acknowledgments

I am profoundly grateful to my supervisor Professor Takashi Nishimura for his encouragement and advice. I would like to thank Professors Maria Aparecida Soares Ruas and Raúl Oset Sinha for their enthusiastic discussions and helps. Also, I would like to thank Huhe Han, Xihe Liu and Yijia Gao for their encouragement.

Finally, I would like to express my sincere gratitude to my family for their understanding.

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CHAPTER 1

Introduction

In this dissertation, unless otherwise stated, all manifolds and mappings belong to class C^∞ and all manifolds are without boundary. In Chapter 1, by N (resp., P), we denote a manifold of dimension n (resp., p). Let $C^\infty(N,P)$ be the set of C^∞ mappings of N into P, and the topology on $C^\infty(N,P)$ is the Whitney C^∞ topology (for the definition of Whitney C^∞ topology, see for example [6]). For given mappings $f, g \in C^\infty(N,P)$, we say that f is $\mathcal{A}$-*equivalent* to g if there exist diffeomorphisms $\Phi : N \to N$ and $\Psi : P \to P$ such that $f = \Psi \circ g \circ \Phi^{-1}$. A mapping f is said to be *stable* if the $\mathcal{A}$-equivalence class of f is open in $C^\infty(N,P)$.

The following problem was posed by René F. Thom ([26]).

Problem 1.0.1 (Structural stability problem)**.** *Are the stable mappings of N into P dense in $C^\infty(N,P)$?*

The celebrated series by John N. Mather [15, 16, 17, 18, 19, 20] are essential for the stability of C^∞ mappings. In [20], Mather stated the following answer to Structural stability problem.

Theorem 1.0.1 ([20])**.** *Let N be a compact manifold of dimension n. Let P be a manifold of dimension p. Then, stable mappings in $C^\infty(N,P)$ are dense if and only if the pair (n,p)*

satisfies one of the following conditions.

(1) $n < \frac{6}{7}p + \frac{8}{7}$ *and* $p - n \geq 4$

(2) $n < \frac{6}{7}p + \frac{9}{7}$ *and* $3 \geq p - n \geq 0$

(3) $p < 8$ *and* $p - n = -1$

(4) $p < 6$ *and* $p - n = -2$

(5) $p < 7$ *and* $p - n \leq -3$

A dimension pair (n, p) is called a *nice dimension* if (n, p) satisfies one of the conditions (1)-(5) in Theorem 1.0.1.

After the celebrated series [15, 16, 17, 18, 19, 20], Mather also showed striking results in [21]. Let $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the set consisting of all linear mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$. We have the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$.

Theorem 1.0.2 ([21]). *Let N be a compact manifold of dimension n. Let f be an embedding of N into $\mathbb{R}^m$. If (n, ℓ) is in the nice dimensions and $m > \ell$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $\pi \circ f : N \to \mathbb{R}^\ell$ is stable.*

In Structural stability problem, the domain in which we can perturb a given mapping of N into $\mathbb{R}^\ell$ is the space $C^\infty(N, \mathbb{R}^\ell)$. On the other hand, in Theorem 1.0.2, for a given embedding $f : N \to \mathbb{R}^m$ and a linear mapping $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$, the domain in which we can perturb a mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is not $C^\infty(N, \mathbb{R}^\ell)$ but $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$. Namely, in the theorem, it is necessary to consider perturbations under a constraint condition. In this dissertation, as in Theorem 1.0.2, generic mappings under given constraint conditions are investigated.

In Chapter 2, compositions of generic linearly perturbed mappings and immersions, injections or embeddings are investigated. Let $f : N \to U$ (resp., $F : U \to \mathbb{R}^\ell$) be an immersion, an injection or an embedding (resp., a mapping), where U is an open subset of $\mathbb{R}^m$. Then, for a given linear mapping $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$, the domain in which we can perturb a mapping $(F + \pi) \circ f : N \to \mathbb{R}^\ell$ is not $C^\infty(N, \mathbb{R}^\ell)$ but $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$. Namely, it is necessary to consider perturbations under a constraint condition.

In Chapter 3 (resp., Chapter 4), we introduce the notion of a distance-squared mapping (resp., a Lorentzian distance-squared mapping), wherein each component is a distance-squared function (resp., a Lorentzian distance-squared function). In the space consisting of all distance-squared mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$ (resp., all Lorentzian distance-squared mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$), a characterization of generic distance-squared mappings (resp., generic Lorentzian distance-squared mappings) are given. Here, note that the domain in which we can perturb these quadratic mappings is $(\mathbb{R}^m)^\ell$.

In Chapter 5, we introduce the notion of a generalized distance-squared mapping. The notion is an extension of the notions of a distance-squared mapping and a Lorentzian distance-squared mapping. By applying some assertions in Chapter 2 to generalized distance-squared mappings, some properties of generic generalized distance-squared mappings are obtained.

CHAPTER 2

SOME ASSERTIONS ON GENERIC LINEAR PERTURBATIONS

2.1 Composing generic linearly perturbed mappings and immersions/injections

2.1.1 Introduction

In Section 2.1, let ℓ, m and n stand for positive integers. Let $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$, U and $F : U \to \mathbb{R}^\ell$ be a linear mapping, an open set of $\mathbb{R}^m$ and a mapping, respectively.

Set

$$F_\pi = F + \pi.$$

Note that the mapping π in $F_\pi = F + \pi$ is restricted to the open set U.

Let $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the set consisting of all linear mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$. Notice that we get the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$. An n-dimensional manifold is denoted by N. For a given mapping $f : N \to U$, a property of mappings $F_\pi \circ f : N \to \mathbb{R}^\ell$ will be said to be true for a *generic mapping* if there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ has the property. In the case of $F = 0$, by John Mather, for a given embedding $f : N \to \mathbb{R}^m$, a generic mapping $\pi \circ f : N \to \mathbb{R}^\ell$ $(m > \ell)$ is investigated in [21]. The main theorem in [21] yields a lot of applications. On the other hand, in Section 2.1, for a given immersion or a given injection $f : N \to U$, a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is investigated, where ℓ is an arbitrary positive integer which may possibly satisfy $m \leq \ell$.

The main purpose in Section 2.1 is to show two main theorems (Theorems 2.1.1 and 2.1.2 in Section 2.1.2) and to give some their applications. The first main theorem (Theorem 2.1.1) is as follows. Let $f : N \to U$ (resp., $F : U \to \mathbb{R}^\ell$) be an immersion (resp., a mapping). Then, generally, the composition $F \circ f$ does not necessarily yield a mapping transverse to a given subfiber-bundle of the jet bundle $J^1(N, \mathbb{R}^\ell)$. Nevertheless, Theorem 2.1.1 states that for any $\mathcal{A}^1$-invariant fiber, a generic mapping $F_\pi \circ f$ yields a mapping transverse to the subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the given fiber. The second main theorem (Theorem 2.1.2) is a specialized transversality theorem on crossings of a generic mapping $F_\pi \circ f$, where $f : N \to U$ is a given injection and $F : U \to \mathbb{R}^\ell$ is a given mapping.

For a given immersion (resp., injection) $f : N \to U$, we obtain the following (1)-(4) (resp., (5)) as applications of Theorem 2.1.1 (resp., Theorem 2.1.2).

(1) If $(n, \ell) = (n, 1)$, then a generic function $F_\pi \circ f : N \to \mathbb{R}$ is a Morse function.

(2) If $(n, \ell) = (n, 2n - 1)$ and $n \geq 2$, then any singular point of a generic mapping $F_\pi \circ f : N \to \mathbb{R}^{2n-1}$ is a singular point of Whitney umbrella.

(3) If $\ell \geq 2n$, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion.

(4) A generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ has corank at most k singular points (for the definition of corank at most k singular points, see Section 2.1.5), where k is the maximum integer satisfying $(n - v + k)(\ell - v + k) \leq n$ $(v = \min\{n, \ell\})$.

(5) If $\ell > 2n$, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is injective.

Furthermore, by combining the assertions (3) and (5), for a given embedding $f : N \to U$, we obtain the following assertion (6).

(6) If $\ell > 2n$ and N is compact, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an embedding.

In Section 2.1.2, some fundamental definitions are reviewed, and the two main theorems (Theorems 2.1.1 and 2.1.2) are stated. Section 2.1.3 (resp., Section 2.1.4) is devoted to the proof of Theorem 2.1.1 (resp., Theorem 2.1.2). In Section 2.1.5, the assertions (1)-(6) above are shown.

2.1.2 Preliminaries and the main results in Section 2.1

Firstly, we recall the definition of transversality. Let N and P be manifolds.

Definition 2.1.1. Let W be a submanifold of P. Let $g : N \to P$ be a mapping.

1. A mapping $g : N \to P$ is said to be *transverse* to W *at* q if $g(q) \notin W$ or in the case of $g(q) \in W$, the following holds:

$$dg_q(T_qN) + T_{g(q)}W = T_{g(q)}P.$$

2. A mapping $g : N \to P$ is said to be *transverse* to W if for any $q \in N$, the mapping g is transverse to W at q.

A mapping $g : N \to P$ is said to be $\mathcal{A}$-*equivalent* to a mapping $h : N \to P$ if there exist two diffeomorphisms $\Phi : N \to N$ and $\Psi : P \to P$ satisfying $g = \Psi \circ h \circ \Phi^{-1}$.

Let $J^r(N, P)$ be the space of r-jets of mappings of N into P. For a given mapping $g : N \to P$, the mapping $j^rg : N \to J^r(N, P)$ is defined by $q \mapsto j^rg(q)$ (for details on the space $J^r(N, P)$ or the mapping $j^rg : N \to J^r(N, P)$, see for instance, [6]).

For Theorem 2.1.1, it is sufficient to consider the case of $r = 1$ and $P = \mathbb{R}^\ell$. By $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$, we denote a coordinate neighborhood system of N. Let $\Pi : J^1(N, \mathbb{R}^\ell) \to N \times \mathbb{R}^\ell$ be the natural projection defined by $\Pi(j^1g(q)) = (q, g(q))$. Let $\Phi_\lambda : \Pi^{-1}(U_\lambda \times \mathbb{R}^\ell) \to \varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times J^1(n, \ell)$ be the homeomorphism as follows:

$$\Phi_\lambda \left(j^1g(q)\right) = \left(\varphi_\lambda(q), g(q), j^1(\psi_\lambda \circ g \circ \varphi_\lambda^{-1} \circ \widetilde{\varphi}_\lambda)(0)\right),$$

where $J^1(n,\ell)=\{j^1g(0)\mid g:(\mathbb{R}^n,0)\to(\mathbb{R}^\ell,0)\}$ and $\widetilde{\varphi}_\lambda:\mathbb{R}^n\to\mathbb{R}^n$ (resp., $\psi_\lambda:\mathbb{R}^m\to\mathbb{R}^m$) is the translation given by $\widetilde{\varphi}_\lambda(0)=\varphi_\lambda(q)$ (resp., $\psi_\lambda(g(q))=0$). Then, we see that $\{(\Pi^{-1}(U_\lambda\times\mathbb{R}^\ell),\Phi_\lambda)\}_{\lambda\in\Lambda}$ is a coordinate neighborhood system of $J^1(N,\mathbb{R}^\ell)$. We say that s subset X of $J^1(n,\ell)$ is $\mathcal{A}^1$-*invariant* if for any $j^1g(0)\in X$, and for any two germs of diffeomorphisms $H:(\mathbb{R}^\ell,0)\to(\mathbb{R}^\ell,0)$ and $h:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$, we get $j^1(H\circ g\circ h^{-1})(0)\in X$. For an $\mathcal{A}^1$-invariant submanifold X of $J^1(n,\ell)$, set

$$X(N,\mathbb{R}^\ell)=\bigcup_{\lambda\in\Lambda}\Phi_\lambda^{-1}\left(\varphi_\lambda(U_\lambda)\times\mathbb{R}^\ell\times X\right).$$

Then, the set $X(N,\mathbb{R}^\ell)$ is a subfiber-bundle of $J^1(N,\mathbb{R}^\ell)$ with the fiber X satisfying

$$\begin{aligned}\text{codim } X(N,\mathbb{R}^\ell) &= \dim J^1(N,\mathbb{R}^\ell)-\dim X(N,\mathbb{R}^\ell)\\ &= \dim J^1(n,\ell)-\dim X\\ &= \text{codim } X.\end{aligned}$$

Then, the first main theorem in Section 2.1 is the following.

Theorem 2.1.1 ([7]). *Let $f:N\to U$ be an immersion, where N is a manifold of dimension n and U is an open subset in $\mathbb{R}^m$. Let $F:U\to\mathbb{R}^\ell$ be a mapping. If X is an $\mathcal{A}^1$-invariant submanifold of $J^1(n,\ell)$, then there exists a subset $\Sigma\subset\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ with Lebesgue measure* 0 *such that for any $\pi\in\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)-\Sigma$, the mapping $j^1(F_\pi\circ f):N\to J^1(N,\mathbb{R}^\ell)$ is transverse to $X(N,\mathbb{R}^\ell)$.*

For the statement of the second main theorem (Theorem 2.1.2), we will prepare some definitions. Set $N^{(s)}=\{(q_1,q_2,\ldots,q_s)\in N^s\mid q_i\neq q_j\ (i\neq j)\}$. Note that $N^{(s)}$ is an open submanifold of N^s. For a given mapping $g:N\to P$, let $g^{(s)}:N^{(s)}\to P^s$ be the mapping defined by

$$g^{(s)}(q_1,q_2,\ldots,q_s)=(g(q_1),g(q_2),\ldots,g(q_s)).$$

Set $\Delta_s=\{(y,\ldots,y)\in P^s\mid y\in P\}$. It is not hard to see that Δ_s is a submanifold of P^s

satisfying

$$\text{codim } \Delta_s = \dim P^s - \dim \Delta_s = (s-1)\dim P.$$

Definition 2.1.2. Let $g : N \to P$ be a mapping. Then, we say that g is a *mapping with normal crossings* if for any positive integer s $(s \geq 2)$, the mapping $g^{(s)} : N^{(s)} \to P^s$ is transverse to Δ_s.

For a given injection $f : N \to \mathbb{R}^m$, set

$$s_f = \max \left\{ s \;\middle|\; \forall (q_1, q_2, \ldots, q_s) \in N^{(s)}, \dim \sum_{i=2}^{s} \mathbb{R}\overrightarrow{f(q_1)f(q_i)} = s-1 \right\}.$$

Since the mapping f is an injection, we have $2 \leq s_f$. Since $f(q_1), f(q_2), \ldots, f(q_{s_f})$ are points of $\mathbb{R}^m$, we get $s_f \leq m+1$. Hence, it follows that

$$2 \leq s_f \leq m+1.$$

Moreover, in the following, for a set X, we denote the number of its elements (or its cardinality) by $|X|$. Then, the second main theorem in Section 2.1 is the following.

Theorem 2.1.2 ([7]). *Let $f : N \to U$ be an injection, where N is a manifold of dimension n and U is an open subset in $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any s $(2 \leq s \leq s_f)$, the mapping $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to Δ_s. Furthermore, if the mapping F_π satisfies that $|F_\pi^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^\ell$, then $F_\pi \circ f : N \to \mathbb{R}^\ell$ is a mapping with normal crossings.*

The following lemma is important for the proofs of Theorems 2.1.1 and 2.1.2.

Lemma 2.1.1 ([1], [21]). *Let N, P, Z be manifolds, and let W be a submanifold of P. Let $\Gamma : N \times Z \to P$ be a mapping. If the mapping Γ is transverse to the submanifold W, then there exists a subset $\Sigma \subset Z$ with Lebesgue measure 0 such that for any $p \in Z - \Sigma$, the mapping $\Gamma_p : N \to P$ is transverse to the submanifold W, where $\Gamma_p(q) = \Gamma(q,p)$.*

Remark 2.1.1.

1. There is an advantage that the domain of the mapping F is not $\mathbb{R}^m$ but an open subset U. Suppose that $U = \mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{R}$ be the function given by

$x \mapsto |x|$. Since F is not differentiable at $x = 0$, we cannot apply Theorems 2.1.1 and 2.1.2 to the function $F : \mathbb{R} \to \mathbb{R}$.

On the other hand, if $U = \mathbb{R} - \{0\}$, then the two main theorems can be applied to $F|_U$.

2. There is a case of $s_f = 3$ as follows. If $n + 1 \leq m$, $N = S^n$ and $f : S^n \to \mathbb{R}^m$ is the inclusion $f(x) = (x, 0, \ldots, 0)$, then it is easily seen that $s_f = 3$. Indeed, suppose that there exists a point $(q_1, q_2, q_3) \in (S^n)^{(3)}$ satisfying $\dim \sum_{i=2}^{3} \mathbb{R}\overrightarrow{f(q_1)f(q_i)} = 1$. Then, since the number of the intersections of $f(S^n)$ and a straight line of $\mathbb{R}^m$ is at most two, this contradicts the assumption. Hence, we have $s_f \geq 3$. From $S^1 \times \{0\} \subset f(S^n)$, we get $s_f < 4$, where $0 = \underbrace{(0, \ldots, 0)}_{(m-2)\text{-tuple}}$. Thus, it follows that $s_f = 3$.

2.1.3 Proof of Theorem 2.1.1

Let $(\alpha_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ be a representing matrix of a linear mapping $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$. Set $F_\alpha = F_\pi$. Then, we get

$$F_\alpha(x) = \left(F_1(x) + \sum_{j=1}^{m} \alpha_{1j} x_j, F_2(x) + \sum_{j=1}^{m} \alpha_{2j} x_j, \ldots, F_\ell(x) + \sum_{j=1}^{m} \alpha_{\ell j} x_j \right), \tag{2.1.1}$$

where $F = (F_1, F_2, \ldots, F_\ell)$, $\alpha = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1m}, \ldots, \alpha_{\ell 1}, \alpha_{\ell 2}, \ldots, \alpha_{\ell m}) \in (\mathbb{R}^m)^\ell$ and $x = (x_1, x_2, \ldots, x_m)$. For a given immersion $f : N \to U$, the mapping $F_\alpha \circ f : N \to \mathbb{R}^\ell$ is given by

$$F_\alpha \circ f = \left(F_1 \circ f + \sum_{j=1}^{m} \alpha_{1j} f_j, F_2 \circ f + \sum_{j=1}^{m} \alpha_{2j} f_j, \ldots, F_\ell \circ f + \sum_{j=1}^{m} \alpha_{\ell j} f_j \right), \tag{2.1.2}$$

where $f = (f_1, f_2, \ldots, f_m)$. Since we have the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$, for the proof, it is sufficient to prove that there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $j^1(F_\alpha \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $X(N, \mathbb{R}^\ell)$.

Now, let $\Gamma : N \times (\mathbb{R}^m)^\ell \to J^1(N, \mathbb{R}^\ell)$ be the mapping given by

$$\Gamma(q, \alpha) = j^1(F_\alpha \circ f)(q).$$

If Γ is transverse to $X(N, \mathbb{R}^\ell)$, then from Lemma 2.1.1, there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $\Gamma_\alpha : N \to J^1(N, \mathbb{R}^\ell)$ ($\Gamma_\alpha = j^1(F_\alpha \circ f)$) is transverse to $X(N, \mathbb{R}^\ell)$. Thus, in order to finish the proof of Theorem 2.1.1, it is sufficient to prove that if $\Gamma(\widetilde{q}, \widetilde{\alpha}) \in X(N, \mathbb{R}^\ell)$, then the following holds:

$$d\Gamma_{(\widetilde{q},\widetilde{\alpha})}(T_{(\widetilde{q},\widetilde{\alpha})}(N \times (\mathbb{R}^m)^\ell)) + T_{\Gamma(\widetilde{q},\widetilde{\alpha})}X(N, \mathbb{R}^\ell) = T_{\Gamma(\widetilde{q},\widetilde{\alpha})}J^1(N, \mathbb{R}^\ell). \tag{2.1.3}$$

As in Section 2.1.2, let $\{(U_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ (resp., $\{(\Pi^{-1}(U_\lambda \times \mathbb{R}^\ell), \Phi_\lambda)\}_{\lambda\in\Lambda}$) be a coordinate neighborhood system of N (resp., $J^1(N, \mathbb{R}^\ell)$). Then, there exists a coordinate neighborhood $\left(U_{\widetilde{\lambda}} \times (\mathbb{R}^m)^\ell, \varphi_{\widetilde{\lambda}} \times id\right)$ containing the point $(\widetilde{q}, \widetilde{\alpha})$ of $N \times (\mathbb{R}^m)^\ell$, where $id : (\mathbb{R}^m)^\ell \to (\mathbb{R}^m)^\ell$ is the identity mapping, and the mapping $\varphi_{\widetilde{\lambda}} \times id : U_{\widetilde{\lambda}} \times (\mathbb{R}^m)^\ell \to \varphi_{\widetilde{\lambda}}(U_{\widetilde{\lambda}}) \times (\mathbb{R}^m)^\ell$ ($\subset \mathbb{R}^n \times (\mathbb{R}^m)^\ell$) is given by $\left(\varphi_{\widetilde{\lambda}} \times id\right)(q, \alpha) = \left(\varphi_{\widetilde{\lambda}}(q), id(\alpha)\right)$. There exists a coordinate neighborhood $\left(\Pi^{-1}(U_{\widetilde{\lambda}} \times \mathbb{R}^\ell), \Phi_{\widetilde{\lambda}}\right)$ containing the element $\Gamma(\widetilde{q}, \widetilde{\alpha})$ of $J^1(N, \mathbb{R}^\ell)$. Let $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$

be a local coordinate on $\varphi_{\widetilde{\lambda}}(U_{\widetilde{\lambda}})$ containing $\varphi_{\widetilde{\lambda}}(\widetilde{q})$. Then, Γ is locally given by

$$
\begin{aligned}
& (\Phi_{\widetilde{\lambda}} \circ \Gamma \circ (\varphi_{\widetilde{\lambda}} \times id)^{-1})(t, \alpha) \\
= \; & (\Phi_{\widetilde{\lambda}} \circ j^1(F_\alpha \circ f) \circ \varphi_{\widetilde{\lambda}}^{-1})(t) \\
= \; & \Big(t, (F_\alpha \circ f \circ \varphi_{\widetilde{\lambda}}^{-1})(t), \\
& \frac{\partial (F_{\alpha,1} \circ f \circ \varphi_{\widetilde{\lambda}}^{-1})}{\partial t_1}(t), \frac{\partial (F_{\alpha,1} \circ f \circ \varphi_{\widetilde{\lambda}}^{-1})}{\partial t_2}(t), \ldots, \frac{\partial (F_{\alpha,1} \circ f \circ \varphi_{\widetilde{\lambda}}^{-1})}{\partial t_n}(t), \\
& \frac{\partial (F_{\alpha,2} \circ f \circ \varphi_{\widetilde{\lambda}}^{-1})}{\partial t_1}(t), \frac{\partial (F_{\alpha,2} \circ f \circ \varphi_{\widetilde{\lambda}}^{-1})}{\partial t_2}(t), \ldots, \frac{\partial (F_{\alpha,2} \circ f \circ \varphi_{\widetilde{\lambda}}^{-1})}{\partial t_n}(t), \\
& \qquad \qquad \qquad \cdots\cdots\cdots, \\
& \frac{\partial (F_{\alpha,\ell} \circ f \circ \varphi_{\widetilde{\lambda}}^{-1})}{\partial t_1}(t), \frac{\partial (F_{\alpha,\ell} \circ f \circ \varphi_{\widetilde{\lambda}}^{-1})}{\partial t_2}(t), \ldots, \frac{\partial (F_{\alpha,\ell} \circ f \circ \varphi_{\widetilde{\lambda}}^{-1})}{\partial t_n}(t) \Big) \\
= \; & \Big(t, (F_\alpha \circ f \circ \varphi_{\widetilde{\lambda}}^{-1})(t), \\
& \frac{\partial F_1 \circ \widetilde{f}}{\partial t_1}(t) + \sum_{j=1}^m \alpha_{1j} \frac{\partial \widetilde{f}_j}{\partial t_1}(t), \frac{\partial F_1 \circ \widetilde{f}}{\partial t_2}(t) + \sum_{j=1}^m \alpha_{1j} \frac{\partial \widetilde{f}_j}{\partial t_2}(t), \ldots, \frac{\partial F_1 \circ \widetilde{f}}{\partial t_n}(t) + \sum_{j=1}^m \alpha_{1j} \frac{\partial \widetilde{f}_j}{\partial t_n}(t), \\
& \frac{\partial F_2 \circ \widetilde{f}}{\partial t_1}(t) + \sum_{j=1}^m \alpha_{2j} \frac{\partial \widetilde{f}_j}{\partial t_1}(t), \frac{\partial F_2 \circ \widetilde{f}}{\partial t_2}(t) + \sum_{j=1}^m \alpha_{2j} \frac{\partial \widetilde{f}_j}{\partial t_2}(t), \ldots, \frac{\partial F_2 \circ \widetilde{f}}{\partial t_n}(t) + \sum_{j=1}^m \alpha_{2j} \frac{\partial \widetilde{f}_j}{\partial t_n}(t), \\
& \qquad \qquad \qquad \cdots\cdots\cdots, \\
& \frac{\partial F_\ell \circ \widetilde{f}}{\partial t_1}(t) + \sum_{j=1}^m \alpha_{\ell j} \frac{\partial \widetilde{f}_j}{\partial t_1}(t), \frac{\partial F_\ell \circ \widetilde{f}}{\partial t_2}(t) + \sum_{j=1}^m \alpha_{\ell j} \frac{\partial \widetilde{f}_j}{\partial t_2}(t), \ldots, \frac{\partial F_\ell \circ \widetilde{f}}{\partial t_n}(t) + \sum_{j=1}^m \alpha_{\ell j} \frac{\partial \widetilde{f}_j}{\partial t_n}(t) \Big),
\end{aligned}
$$

where $F_\alpha = (F_{\alpha,1}, F_{\alpha,2}, \ldots, F_{\alpha,\ell})$ and $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2, \ldots, \widetilde{f}_m) = (f_1 \circ \varphi_{\widetilde{\lambda}}^{-1}, f_2 \circ \varphi_{\widetilde{\lambda}}^{-1}, \ldots, f_m \circ \varphi_{\widetilde{\lambda}}^{-1}) = f \circ \varphi_{\widetilde{\lambda}}^{-1}$. The Jacobian matrix of Γ at the point $(\widetilde{q}, \widetilde{\alpha})$ is the following:

$$J\Gamma_{(\widetilde{q},\widetilde{\alpha})} = \left(\begin{array}{c|ccccc} E_n & 0 & \cdots & & \cdots & 0 \\ \hline & * & \cdots & & \cdots & * \\ & {}^t(Jf_{\widetilde{q}}) & & 0 & & \\ * & & {}^t(Jf_{\widetilde{q}}) & & & \\ & & 0 & \ddots & & \\ & & & & & {}^t(Jf_{\widetilde{q}}) \end{array}\right)_{(t,\alpha)=(\varphi_{\widetilde{\lambda}}(\widetilde{q}),\widetilde{\alpha})},$$

where $Jf_{\widetilde{q}}$ is the Jacobian matrix of the mapping f at the point $\widetilde{q}$ and E_n is the $n \times n$ unit matrix. Notice that ${}^t(Jf_{\widetilde{q}})$ is the transpose of $Jf_{\widetilde{q}}$ and that there are ℓ copies of ${}^t(Jf_{\widetilde{q}})$ in the above description of $J\Gamma_{(\widetilde{q},\widetilde{\alpha})}$. Since the manifold $X(N, \mathbb{R}^\ell)$ is a subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the fiber X, it is clearly seen that in order to show (2.1.3), it suffices to show that the matrix M_1 given below has rank $n + \ell + n\ell$:

$$M_1 = \left(\begin{array}{c|ccccc} E_{n+\ell} & * & \cdots & & \cdots & * \\ \hline & {}^t(Jf_{\widetilde{q}}) & & 0 & & \\ 0 & & {}^t(Jf_{\widetilde{q}}) & & & \\ & & 0 & \ddots & & \\ & & & & & {}^t(Jf_{\widetilde{q}}) \end{array}\right)_{(t,\alpha)=(\varphi_{\widetilde{\lambda}}(\widetilde{q}),\widetilde{\alpha})},$$

where $E_{n+\ell}$ is the $(n+\ell) \times (n+\ell)$ unit matrix. Notice that there are ℓ copies of ${}^t(Jf_{\widetilde{q}})$ in the above description of M_1. Note that for any i $(1 \leq i \leq m\ell)$, the $(n+\ell+i)$-th column vector of M_1 coincides with the $(n+i)$-th column vector of $J\Gamma_{(\widetilde{q},\widetilde{\alpha})}$. Since f is an immersion $(n \leq m)$, it follows that the rank of M_1 is equal to $n + \ell + n\ell$. Therefore, we get (2.1.3). $\square$

2.1.4 Proof of Theorem 2.1.2

By the same method as in the proof of Theorem 2.1.1, set $F_\alpha = F_\pi$, where F_α is given by (2.1.1) in Section 2.1.3. For a given injection $f : N \to U$, the mapping $F_\alpha \circ f : N \to \mathbb{R}^\ell$ is given by the same expression as (2.1.2). Since we have the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$, in order to prove that there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any s $(2 \leq s \leq s_f)$, the mapping $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to Δ_s, it is sufficient to prove that there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, and for any s $(2 \leq s \leq s_f)$, the mapping $(F_\alpha \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to Δ_s.

Now, let s be a positive integer satisfying $2 \leq s \leq s_f$. Let $\Gamma : N^{(s)} \times (\mathbb{R}^m)^\ell \to (\mathbb{R}^\ell)^s$ be the mapping given by

$$\Gamma(q_1, q_2, \ldots, q_s, \alpha) = ((F_\alpha \circ f)(q_1), (F_\alpha \circ f)(q_2), \ldots, (F_\alpha \circ f)(q_s)) .$$

If for any positive integer s $(2 \leq s \leq s_f)$, Γ is transverse to Δ_s, then from Lemma 2.1.1, we have that for any positive integer s $(2 \leq s \leq s_f)$, there exists a subset $\Sigma_s \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma_s$, the mapping $\Gamma_\alpha : N^{(s)} \to (\mathbb{R}^\ell)^s$ $(\Gamma_\alpha = (F_\alpha \circ f)^{(s)})$ is transverse to Δ_s. Then, set $\Sigma = \bigcup_{s=2}^{s_f} \Sigma_s$. We see that Σ has Lebesgue measure 0 in $(\mathbb{R}^m)^\ell$. Hence, it follows that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, and for any s $(2 \leq s \leq s_f)$, the mapping $\Gamma_\alpha : N^{(s)} \to (\mathbb{R}^\ell)^s$ $(\Gamma_\alpha = (F_\alpha \circ f)^{(s)})$ is transverse to Δ_s.

Thus, for the proof of this theorem, it is sufficient to prove that for any positive integer s $(2 \leq s \leq s_f)$, if $\Gamma(\widetilde{q}, \widetilde{\alpha}) \in \Delta_s$ $(\widetilde{q} = (\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_s))$, then the following holds:

$$d\Gamma_{(\widetilde{q},\widetilde{\alpha})}(T_{(\widetilde{q},\widetilde{\alpha})}(N^{(s)} \times (\mathbb{R}^m)^\ell)) + T_{\Gamma(\widetilde{q},\widetilde{\alpha})}\Delta_s = T_{\Gamma(\widetilde{q},\widetilde{\alpha})}(\mathbb{R}^\ell)^s. \tag{2.1.4}$$

Let $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ be a coordinate neighborhood system of N. Then, there exists a coordinate neighborhood $(U_{\widetilde{\lambda}_1} \times U_{\widetilde{\lambda}_2} \times \cdots \times U_{\widetilde{\lambda}_s} \times (\mathbb{R}^m)^\ell, \varphi_{\widetilde{\lambda}_1} \times \varphi_{\widetilde{\lambda}_2} \times \cdots \times \varphi_{\widetilde{\lambda}_s} \times id)$ containing the point $(\widetilde{q}, \widetilde{\alpha})$ of $N^{(s)} \times (\mathbb{R}^m)^\ell$, where $id(\mathbb{R}^m)^\ell :\to (\mathbb{R}^m)^\ell$ is the identity mapping, and the mapping $\varphi_{\widetilde{\lambda}_1} \times \varphi_{\widetilde{\lambda}_2} \times \cdots \times \varphi_{\widetilde{\lambda}_s} \times id : U_{\widetilde{\lambda}_1} \times U_{\widetilde{\lambda}_2} \times \cdots \times U_{\widetilde{\lambda}_s} \times (\mathbb{R}^m)^\ell \to (\mathbb{R}^n)^s \times (\mathbb{R}^m)^\ell$ is defined by $(\varphi_{\widetilde{\lambda}_1} \times \varphi_{\widetilde{\lambda}_2} \times \cdots \times \varphi_{\widetilde{\lambda}_s} \times id)(q_1, q_2, \ldots, q_s, \alpha) = (\varphi_{\widetilde{\lambda}_1}(q_1), \varphi_{\widetilde{\lambda}_2}(q_2), \ldots, \varphi_{\widetilde{\lambda}_s}(q_s), id(\alpha))$. Let

$t_i = (t_{i1}, t_{i2}, \ldots, t_{in})$ be a local coordinate around $\varphi_{\widetilde{\lambda}_i}(\widetilde{q}_i)$ $(1 \le i \le s)$. Then, Γ is locally given by the following:

$$\begin{aligned}
&\Gamma \circ \left(\varphi_{\widetilde{\lambda}_1} \times \varphi_{\widetilde{\lambda}_2} \times \cdots \times \varphi_{\widetilde{\lambda}_s} \times id\right)^{-1}(t_1, t_2, \ldots, t_s, \alpha) \\
= &\left((F_\alpha \circ f \circ \varphi_{\widetilde{\lambda}_1}^{-1})(t_1), (F_\alpha \circ f \circ \varphi_{\widetilde{\lambda}_2}^{-1})(t_2), \ldots, (F_\alpha \circ f \circ \varphi_{\widetilde{\lambda}_s}^{-1})(t_s)\right) \\
= &\Bigg(F_1 \circ \widetilde{f}(t_1) + \sum_{j=1}^{m} \alpha_{1j} \widetilde{f}_j(t_1), F_2 \circ \widetilde{f}(t_1) + \sum_{j=1}^{m} \alpha_{2j} \widetilde{f}_j(t_1), \ldots, F_\ell \circ \widetilde{f}(t_1) + \sum_{j=1}^{m} \alpha_{\ell j} \widetilde{f}_j(t_1), \\
&F_1 \circ \widetilde{f}(t_2) + \sum_{j=1}^{m} \alpha_{1j} \widetilde{f}_j(t_2), F_2 \circ \widetilde{f}(t_2) + \sum_{j=1}^{m} \alpha_{2j} \widetilde{f}_j(t_2), \ldots, F_\ell \circ \widetilde{f}(t_2) + \sum_{j=1}^{m} \alpha_{\ell j} \widetilde{f}_j(t_2), \\
&\qquad\qquad \cdots\cdots\cdots, \\
&F_1 \circ \widetilde{f}(t_s) + \sum_{j=1}^{m} \alpha_{1j} \widetilde{f}_j(t_s), F_2 \circ \widetilde{f}(t_s) + \sum_{j=1}^{m} \alpha_{2j} \widetilde{f}_j(t_s), \ldots, F_\ell \circ \widetilde{f}(t_s) + \sum_{j=1}^{m} \alpha_{\ell j} \widetilde{f}_j(t_s) \Bigg),
\end{aligned}$$

where $\widetilde{f}(t_i) = (\widetilde{f}_1(t_i), \widetilde{f}_2(t_i), \ldots, \widetilde{f}_m(t_i)) = (f_1 \circ \varphi_{\widetilde{\lambda}_i}^{-1}(t_i), f_2 \circ \varphi_{\widetilde{\lambda}_i}^{-1}(t_i), \ldots, f_m \circ \varphi_{\widetilde{\lambda}_i}^{-1}(t_i))$ $(1 \le i \le s)$. Set $t = (t_1, t_2, \ldots, t_s)$ and $z = (\varphi_{\widetilde{\lambda}_1} \times \varphi_{\widetilde{\lambda}_2} \times \cdots \times \varphi_{\widetilde{\lambda}_s})(\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_s)$.

The Jacobian matrix of Γ at the point $(\widetilde{q}, \widetilde{\alpha})$ is the following:

$$J\Gamma_{(\widetilde{q},\widetilde{\alpha})} = \left(\begin{array}{c|c} * & B(t_1) \\ * & B(t_2) \\ \vdots & \vdots \\ * & B(t_s) \end{array}\right)_{(t,\alpha)=(z,\widetilde{\alpha})},$$

where

$$B(t_i) = \left.\begin{pmatrix} \mathbf{b}(t_i) & & & 0 \\ & \mathbf{b}(t_i) & & \\ & & \ddots & \\ 0 & & & \mathbf{b}(t_i) \end{pmatrix}\right\} \ell \text{ rows}$$

and $\mathbf{b}(t_i) = (\widetilde{f}_1(t_i), \widetilde{f}_2(t_i), \ldots, \widetilde{f}_m(t_i))$. By the construction of $T_{\Gamma(\widetilde{q},\widetilde{\alpha})}\Delta_s$, for the proof of (2.1.4), it is sufficient to prove that the rank of the following matrix M_2 is equal to ℓs:

$$M_2 = \left(\begin{array}{c|c} E_\ell & B(t_1) \\ E_\ell & B(t_2) \\ \vdots & \vdots \\ E_\ell & B(t_s) \end{array}\right)_{t=z}.$$

There exists an $\ell s \times \ell s$ regular matrix Q_1 satisfying

$$Q_1 M_2 = \left(\begin{array}{c|c} E_\ell & B(t_1) \\ 0 & B(t_2) - B(t_1) \\ \vdots & \vdots \\ 0 & B(t_s) - B(t_1) \end{array}\right)_{t=z}.$$

There exists an $(\ell + m\ell) \times (\ell + m\ell)$ regular matrix Q_2 satisfying

$$
\begin{aligned}
Q_1 M_2 Q_2 \quad &= \left(\begin{array}{c|c} E_\ell & 0 \\ & \\ 0 & B(t_2) - B(t_1) \\ & \\ \vdots & \vdots \\ & \\ 0 & B(t_s) - B(t_1) \end{array} \right)_{t=z} \\
&= \left(\begin{array}{c|cccc}
E_\ell & \multicolumn{4}{c}{0} \\ \hline
 & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} & & & 0 \\
0 & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} & & \\
 & 0 & & \ddots & \\
 & & & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_2)} \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
 & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)} & & & 0 \\
0 & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)} & & \\
 & 0 & & \ddots & \\
 & & & & \overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_s)}
\end{array} \right),
\end{aligned}
$$

where $\overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_i)} = (\widetilde{f}_1(t_i) - \widetilde{f}_1(t_1), \widetilde{f}_2(t_i) - \widetilde{f}_2(t_1), \ldots, \widetilde{f}_m(t_i) - \widetilde{f}_m(t_1))$ $(2 \leq i \leq s)$ and $t = z$. From $s - 1 \leq s_f - 1$ and the definition of s_f, we get

$$
\dim \sum_{i=2}^{s} \mathbb{R}\overrightarrow{\widetilde{f}(t_1)\widetilde{f}(t_i)} = s - 1,
$$

where $t = z$. Hence, by the construction of the matrix $Q_1 M_2 Q_2$ and $s-1 \leq m$, it follows that the rank of $Q_1 M_2 Q_2$ is equal to ℓs. Therefore, the rank of M_2 must be equal to ℓs. Thus, we get (2.1.4). Hence, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any s $(2 \leq s \leq s_f)$, the mapping $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to Δ_s.

Furthermore, suppose that the mapping F_π satisfies that $|F_\pi^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^\ell$. Since $f : N \to \mathbb{R}^m$ is an injection, we have that $|(F_\pi \circ f)^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^\ell$. Thus, it follows that for any positive integer s with $s \geq s_f + 1$, we get $(F_\pi \circ f)^{(s)}(N^{(s)}) \bigcap \Delta_s = \emptyset$. Namely, for any positive integer s with $s \geq s_f + 1$, the mapping $(F_\pi \circ f)^{(s)}$ is transverse to Δ_s. Therefore, it follows that $F_\pi \circ f : N \to \mathbb{R}^\ell$ is a mapping with normal crossings. □

2.1.5 Applications of Theorem 2.1.1

Set

$$\Sigma^k = \left\{ j^1 g(0) \in J^1(n, \ell) \mid \text{corank } Jg(0) = k \right\},$$

where corank $Jg(0) = \min\{n, \ell\} - \text{rank } Jg(0)$ and $k = 1, 2, \ldots, \min\{n, \ell\}$. Then, for any k $(k = 1, 2, \ldots, \min\{n, \ell\})$, the set Σ^k is an $\mathcal{A}^1$-invariant submanifold of $J^1(n, \ell)$. Set

$$\Sigma^k(N, \mathbb{R}^\ell) = \bigcup_{\lambda \in \Lambda} \Phi_\lambda^{-1} \left(\varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times \Sigma^k \right),$$

where Φ_λ and φ_λ are as defined in Section 2.1.2. Then, $\Sigma^k(N, \mathbb{R}^\ell)$ is a subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with the fiber Σ^k satisfying

$$\begin{aligned} \text{codim } \Sigma^k(N, \mathbb{R}^\ell) &= \dim J^1(N, \mathbb{R}^\ell) - \dim \Sigma^k(N, \mathbb{R}^\ell) \\ &= (n - v + k)(\ell - v + k), \end{aligned}$$

where $v = \min\{n, \ell\}$. (For details on Σ^k and $\Sigma^k(N, \mathbb{R}^\ell)$, see for instance [6], pp. 60–61).

As some applications of Theorem 2.1.1, we get the following Proposition 2.1.1, Corollaries 2.1.1, 2.1.2, 2.1.3 and 2.1.4.

Proposition 2.1.1. *Let $f : N \to U$ be an immersion, where N is a manifold of dimension n and U is an open subset in $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. Then, there exists a* *subset* $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ *with Lebesgue measure* 0 *such that for any* $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$*, the mapping* $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ *is transverse to* $\Sigma^k(N, \mathbb{R}^\ell)$ *for any positive integer* k *satisfying* $1 \leq k \leq v$*. Especially, in the case of* $\ell \geq 2$*, we get* $k_0 + 1 \leq v$ *and it follows that* $j^1(F_\pi \circ f)$ *satisfies that* $j^1(F_\pi \circ f)(N) \bigcap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ *for any positive integer* k *satisfying* $k_0 + 1 \leq k \leq v$*, where* k_0 *is the maximum integer satisfying* $(n - v + k_0)(\ell - v + k_0) \leq n$ $(v = \min\{n, \ell\})$.

Proof. From Theorem 2.1.1, for an arbitrary positive integer k satisfying $1 \leq k \leq v$, there exists a subset $\widetilde{\Sigma}_k \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \widetilde{\Sigma}_k$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$. Set $\Sigma = \bigcup_{k=1}^{v} \widetilde{\Sigma}_k$. Then, it is clearly seen that Σ has Lebesgue measure 0 in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$. Thus, we have that there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$ for any positive integer k satisfying $1 \leq k \leq v$.

Now, we will consider the case of $\ell \geq 2$. Firstly, we will prove that $k_0 + 1 \leq v$ in the case. Suppose that $v \leq k_0$. Then, from $(n - v + k_0)(\ell - v + k_0) \leq n$, we get $n\ell \leq n$. This contradicts the assumption $\ell \geq 2$.

Secondly, we will prove that in the case of $\ell \geq 2$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ satisfies that $j^1(F_\pi \circ f)(N) \bigcap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ for any positive integer k satisfying $k_0 + 1 \leq k \leq v$. Suppose that there exist a positive integer k $(k_0 + 1 \leq k \leq v)$ and a point $q \in N$ satisfying $j^1(F_\pi \circ f)(q) \in \Sigma^k(N, \mathbb{R}^\ell)$. Since $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$ at q, the following holds:

$$d(j^1(F_\pi \circ f))_q(T_qN) + T_{j^1(F_\pi \circ f)(q)}\Sigma^k(N, \mathbb{R}^\ell) = T_{j^1(F_\pi \circ f)(q)}J^1(N, \mathbb{R}^\ell).$$

Therefore, it follows that

$$\begin{aligned} & \dim\, d(j^1(F_\pi \circ f))_q(T_qN) \\ \geq\ & \dim\, T_{j^1(F_\pi \circ f)(q)}J^1(N, \mathbb{R}^\ell) - \dim\, T_{j^1(F_\pi \circ f)(q)}\Sigma^k(N, \mathbb{R}^\ell) \\ =\ & \operatorname{codim}\, T_{j^1(F_\pi \circ f)(q)}\Sigma^k(N, \mathbb{R}^\ell). \end{aligned}$$

Hence, we have $n \geq (n - v + k)(\ell - v + k)$. Since k_0 is the maximum integer satisfying $n \geq (n - v + k_0)(\ell - v + k_0)$, we get $k \leq k_0$. This contradicts the assumption $k_0 + 1 \leq k$. $\square$

Remark 2.1.2.

1. In Proposition 2.1.1, by $(n - v + k_0)(\ell - v + k_0) \leq n$, it is not hard to see that $k_0 \geq 0$.

2. In Proposition 2.1.1, in the case of $\ell = 1$, we get $k_0 + 1 > v$. Indeed, in the case, by $v = 1$, we have $(n - 1 + k_0)k_0 \leq n$. Thus, it follows that $k_0 = 1$.

A mapping $g : N \to \mathbb{R}$ is called a *Morse function* if all of the singularities of g are nondegenerate (for details on Morse functions, see for instance, [6], p. 63). In the case of $(n, \ell) = (n, 1)$, we get the following.

Corollary 2.1.1. *Let $f : N \to U$ be an immersion, where N is a manifold of dimension n and U is an open subset in $\mathbb{R}^m$. Let $F : U \to \mathbb{R}$ be a mapping. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}$ is a Morse function.*

Proof. From Proposition 2.1.1, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R})$ is transverse to $\Sigma^1(N, \mathbb{R})$. Therefore, if $q \in N$ is a singular point of $F_\pi \circ f$, then q is nondegenerate. $\square$

For a given mapping $g : N \to \mathbb{R}^{2n-1}$ $(n \geq 2)$, a singular point $q \in N$ is called a *singular point of Whitney umbrella* if there exist two germs of diffeomorphisms $H : (\mathbb{R}^{2n-1}, g(q)) \to (\mathbb{R}^{2n-1}, 0)$ and $h : (N, q) \to (\mathbb{R}^n, 0)$ satisfying

$$H \circ g \circ h^{-1}(x_1, x_2, \ldots, x_n) = (x_1^2, x_1 x_2, \ldots, x_1 x_n, x_2, \ldots, x_n),$$

where $(x_1, x_2, \ldots, x_n)$ is a local coordinate around $h(q) = 0 \in \mathbb{R}^n$. In the case of $(n, \ell) = (n, 2n - 1)$ $(n \geq 2)$, we get the following.

Corollary 2.1.2. *Let $f : N \to U$ be an immersion, where N is a manifold of dimension n $(n \geq 2)$ and U is an open subset in $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^{2n-1}$ be a mapping. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1})$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma$, any singular point of the mapping $F_\pi \circ f : N \to \mathbb{R}^{2n-1}$ is a singular point of Whitney* *umbrella.*

Proof. From, for instance, [6], p. 179, we see that a point $q \in N$ is a singular point of Whitney umbrella of the mapping $F_\pi \circ f$ if $j^1(F_\pi \circ f)(q) \in \Sigma^1(N, \mathbb{R}^{2n-1})$ and $j^1(F_\pi \circ f)$ is transverse to $\Sigma^1(N, \mathbb{R}^{2n-1})$ at q. Set $\ell = 2n-1$ and $v = n$ in Proposition 2.1.1. Then, it is clearly seen that we get $k_0 = 1$ in Proposition 2.1.1. Therefore, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1})$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^{2n-1}) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^{2n-1}$ is transverse to $\Sigma^k(N, \mathbb{R}^{2n-1})$ for any positive integer k satisfying $1 \leq k \leq n$, and the mapping satisfies that $j^1(F_\pi \circ f)(N) \bigcap \Sigma^k(N, \mathbb{R}^{2n-1}) = \emptyset$ for any positive integer k satisfying $2 \leq k \leq n$. Hence, if $q \in N$ is a singular point of the mapping $F_\pi \circ f$, then we have that $j^1(F_\pi \circ f)(q) \in \Sigma^1(N, \mathbb{R}^{2n-1})$ and $j^1(F_\pi \circ f)$ is transverse to $\Sigma^1(N, \mathbb{R}^{2n-1})$ at q. $\square$

In the case of $\ell \geq 2n$, the immersion property of a given mapping $f : N \to U$ is preserved by composing generic linearly perturbed mappings as follows:

Corollary 2.1.3. *Let $f : N \to U$ be an immersion, where N is a manifold of dimension n and U is an open subset in $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping $(\ell \geq 2n)$. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion.*

Proof. It is not hard to see that $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion if and only if $j^1(F_\pi \circ f)(N) \bigcap \bigcup_{k=1}^n \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$. Set $v = n$ and $\ell \geq 2n$ in Proposition 2.1.1. Then, it is not hard to see that $k_0 \leq 0$. From Remark 2.1.2, we have $k_0 = 0$. Thus, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ satisfies that $j^1(F_\pi \circ f)(N) \bigcap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ for any positive integer k $(1 \leq k \leq n)$. $\square$

A mapping $g : N \to \mathbb{R}^\ell$ *has corank at most k singular points* if

$$\sup \{\text{corank } dg_q \mid q \in N\} \leq k,$$

where corank $dg_q = \min\{n, \ell\} - \text{rank } dg_q$. From Proposition 2.1.1, we get the following.

Corollary 2.1.4. *Let $f : N \to U$ be an immersion, where N is a manifold of dimension n and U is an open subset in $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. Let k_0 be the maximum integer satisfying $(n - v + k_0)(\ell - v + k_0) \leq n$ $(v = \min\{n, \ell\})$. Then, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ has corank at most k_0 singular points.*

2.1.6 Applications of Theorem 2.1.2

Proposition 2.1.2. *Let $f : N \to U$ be an injection, where N is a manifold of dimension n and U is an open subset in $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If $(s_f - 1)\ell > ns_f$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is a mapping with normal crossings satisfying $(F_\pi \circ f)^{(s_f)}(N^{(s_f)}) \bigcap \Delta_{s_f} = \emptyset$.*

Proof. From Theorem 2.1.2, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any s $(2 \leq s \leq s_f)$, the mapping $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to Δ_s. Therefore, for the proof, it is sufficient to prove that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $(F_\pi \circ f)^{(s_f)}$ satisfies that $(F_\pi \circ f)^{(s_f)}(N^{(s_f)}) \bigcap \Delta_{s_f} = \emptyset$.

Suppose that there exists $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ such that there exists $q \in N^{(s_f)}$ satisfying $(F_\pi \circ f)^{(s_f)}(q) \in \Delta_{s_f}$. Since $(F_\pi \circ f)^{(s_f)}$ is transverse to Δ_{s_f}, we get the following:

$$d((F_\pi \circ f)^{(s_f)})_q(T_q N^{(s_f)}) + T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f} = T_{(F_\pi \circ f)^{(s_f)}(q)} (\mathbb{R}^\ell)^{s_f}.$$

Thus, it follows that

$$\begin{aligned} & \dim\, d((F_\pi \circ f)^{(s_f)})_q(T_q N^{(s_f)}) \\ \geq\ & \dim\, T_{(F_\pi \circ f)^{(s_f)}(q)} (\mathbb{R}^\ell)^{s_f} - \dim\, T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f} \\ =\ & \operatorname{codim}\, T_{(F_\pi \circ f)^{(s_f)}(q)} \Delta_{s_f}. \end{aligned}$$

Hence, we have $ns_f \geq (s_f - 1)\ell$. This contradicts the assumption $(s_f - 1)\ell > ns_f$. $\square$

In the case of $\ell > 2n$, the injection property of a given mapping $f : N \to U$ is preserved by composing generic linearly perturbed mappings as follows:

Corollary 2.1.5. *Let $f : N \to U$ be an injection, where N is a manifold of dimension n and U is an open subset in $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If $\ell > 2n$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an injection.*

Proof. Since $s_f \geq 2$ and $\ell > 2n$, it is easily seen that (n, ℓ) satisfies the assumption $(s_f - 1)\ell > ns_f$ in Proposition 2.1.2. Indeed, from $\ell > 2n$, we get $(s_f - 1)\ell > 2n(s_f - 1)$. From $s_f \geq 2$, it follows that $2n(s_f - 1) \geq ns_f$.

Thus, from Proposition 2.1.2, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $(F_\pi \circ f)^{(2)} : N^{(2)} \to (\mathbb{R}^\ell)^2$ is transverse to Δ_2. For the proof, it is sufficient to prove that the mapping $(F_\pi \circ f)^{(2)}$ satisfies that $(F_\pi \circ f)^{(2)}(N^{(2)}) \bigcap \Delta_2 = \emptyset$.

Suppose that there exists $q \in N^{(2)}$ satisfying $(F_\pi \circ f)^{(2)}(q) \in \Delta_2$. Then, we get the following:

$$d((F_\pi \circ f)^{(2)})_q(T_q N^{(2)}) + T_{(F_\pi \circ f)^{(2)}(q)}\Delta_2 = T_{(F_\pi \circ f)^{(2)}(q)}(\mathbb{R}^\ell)^2.$$

Thus, it follows that

$$\begin{aligned} & \dim\, d((F_\pi \circ f)^{(2)})_q(T_q N^{(2)}) \\ \geq\ & \dim\, T_{(F_\pi \circ f)^{(2)}(q)}(\mathbb{R}^\ell)^2 - \dim\, T_{(F_\pi \circ f)^{(2)}(q)}\Delta_2 \\ =\ & \operatorname{codim}\, T_{(F_\pi \circ f)^{(2)}(q)}\Delta_2. \end{aligned}$$

Therefore, we have $2n \geq \ell$. This contradicts the assumption $\ell > 2n$. $\square$

By combining Corollaries 2.1.3 and 2.1.5, we get the following.

Corollary 2.1.6. *Let $f : N \to U$ be an injective immersion, where N is a manifold of dimension n and U is an open subset in $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If $\ell > 2n$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in$*

$\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, *the mapping* $F_\pi \circ f : N \to \mathbb{R}^\ell$ *is an injective immersion.*

In Corollary 2.1.6, suppose that $F_\pi \circ f : N \to \mathbb{R}^\ell$ is proper. Then, an injective immersion $F_\pi \circ f$ is necessarily an embedding (see [6], p. 11). Thus, we have the following.

Corollary 2.1.7. *Let* $f : N \to U$ *be an embedding, where* N *is a compact manifold of dimension* n *and* U *is an open subset in* $\mathbb{R}^m$. *Let* $F : U \to \mathbb{R}^\ell$ *be a mapping. If* $\ell > 2n$, *then there exists a subset* $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ *with Lebesgue measure* 0 *such that for any* $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, *the mapping* $F_\pi \circ f : N \to \mathbb{R}^\ell$ *is an embedding.*

2.2 Composing generic linearly perturbed mappings and embeddings

2.2.1 Introduction

In Section 2.2, ℓ, m, n stand for positive integers. By N, we denote an n-dimensional manifold. Let $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$ be a linear mapping.

In [21], for a given embedding $f : N \to \mathbb{R}^m$, a composition $\pi \circ f : N \to \mathbb{R}^\ell$ $(m > \ell)$ is investigated, and the following assertions (M1)-(M5) are obtained for a generic mapping. All of (M1)-(M5) follow from the main result (Theorem 2.2.1 in Section 2.2.2) shown by Mather.

(M1) If $(n, \ell) = (n, 1)$, then a generic function $\pi \circ f : N \to \mathbb{R}$ is a Morse function.

(M2) If $(n, \ell) = (2, 2)$, then a generic mapping $\pi \circ f : N \to \mathbb{R}^2$ is an excellent map in the sense defined by Whitney in [27].

(M3) If $(n, \ell) = (2, 3)$, then the only singularities of the image of a generic mapping $\pi \circ f : N \to \mathbb{R}^3$ are normal crossings and pinch points.

(M4) A generic mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to the Thom-Boardman varieties (for the definition of Thom-Boardman varieties, refer to [2], [3], [22], [25]).

(M5) If (n, ℓ) is in the nice range of dimensions (for the definition of nice range of dimensions, refer to [20]), then a generic mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is locally infinitesimally stable (for the definition of local infinitesimal stability, see Section 2.2.2). Moreover, if N is compact, then a generic mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is stable (for the definition of stability, see Section 2.2.2).

Let $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the set consisting of linear mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$. For a given embedding $f : N \to \mathbb{R}^m$, a property of mappings $\pi \circ f : N \to \mathbb{R}^\ell$ will be said to be true for a *generic mapping* if there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, $\pi \circ f : N \to \mathbb{R}^\ell$ has the property.

The main purpose in Section 2.2 is to show Theorem 2.2.2 in Section 2.2.2, which is an improvement of Theorem 2.2.1 in Section 2.2.2, proved by Mather ([21]).

Let $U \subset \mathbb{R}^m$ be an open set and $F : U \to \mathbb{R}^\ell$ be a mapping. For any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$, set F_π as follows:

$$F_\pi = F + \pi.$$

For a given embedding $f : N \to U$, by Theorem 2.2.1, the assertions (I1)-(I5) hold. All of (I1)-(I5) are the properties obtained by a generic linear perturbation.

(I1) If $(n, \ell) = (n, 1)$, then a generic function $F_\pi \circ f : N \to \mathbb{R}$ is a Morse function.

(I2) If $(n, \ell) = (2, 2)$, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^2$ is an excellent map.

(I3) If $(n, \ell) = (2, 3)$, then the only singularities of the image of a generic mapping $F_\pi \circ f : N \to \mathbb{R}^3$ are normal crossings and pinch points.

(I4) A generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to the Thom-Boardman varieties.

(I5) If (n, ℓ) is in the nice range of dimensions, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is locally infinitesimally stable. Moreover, if N is compact, then a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is stable.

For a given embedding $f : N \to U$ and a given mapping $F : U \to \mathbb{R}^\ell$, a property of mappings $F_\pi \circ f : N \to \mathbb{R}^\ell$ will be said to be true for a *generic mapping* if there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ has the property. The assertion (M5) (resp., (I5)) above implies assertions (M1), (M2), and (M3) (resp., (I1), (I2) and (I3)). We get both assertions (M4) and (M5) (resp., (I4) and (I5)) from Theorem 2.2.1 (resp., Theorem 2.2.2) of Section 2.2.2. Furthermore, in the special case of $F = 0$, $U = \mathbb{R}^m$ and $m > \ell$, (I1)-(I5) are the same as (M1)-(M5), respectively. Notice that in the case of $m \leq \ell$, a generic mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is an embedding. Note also that in the same case, a generic mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is not necessarily an embedding.

2.2.2 Preliminaries and the statement of the main result in Section 2.2

Let N and P be manifolds. Let ${}_sJ^r(N,P)$ be the space consisting of the following elements

$$(j^r g(q_1), j^r g(q_2), \ldots, j^r g(q_s)) \in J^r(N,P)^s$$

satisfying $(q_1, q_2, \ldots, q_s) \in N^{(s)}$. Since $N^{(s)}$ is an open submanifold of N^s, it is clearly seen that the space ${}_sJ^r(N,P)$ is also an open submanifold of $J^r(N,P)^s$. For a given mapping $g : N \to P$, the mapping ${}_sj^r g : N^{(s)} \to {}_sJ^r(N,P)$ is given by

$$(q_1, q_2, \ldots, q_s) \mapsto (j^r g(q_1), j^r g(q_2), \ldots, j^r g(q_s)).$$

Let W be a submanifold of ${}_sJ^r(N,P)$. A mapping $g : N \to P$ will be said to be *transverse with respect to* W if ${}_sj^r g : N^{(s)} \to {}_sJ^r(N,P)$ is transverse to W.

We can partition P^s as follows. Given an arbitrary partition Π of $\{1, 2, \ldots, s\}$, let P^Π be the set of s-tuples $(y_1, y_2, \ldots, y_s) \in P^s$ such that $y_i = y_j$ if and only if i and j are in the same member of the partition Π.

By Diff N, we denote the group of diffeomorphisms of N. Then, we get the natural action of Diff N $\times$ Diff P on ${}_sJ^r(N,P)$ such that for a mapping $g : N \to P$, the equality $(h, H) \cdot {}_sj^r g(q) = {}_sj^r(H \circ g \circ h^{-1})(q')$ holds, where $q = (q_1, q_2, \ldots, q_s)$ and $q' = (h(q_1), h(q_2), \ldots, h(q_s))$. We say that a subset W of ${}_sJ^r(N,P)$ is *invariant* if it is invariant under this action.

We recall the following identification (2.2.1) from [21]. For $q = (q_1, q_2, \ldots, q_s) \in N^{(s)}$, let $g : U \to P$ be a mapping defined in a neighborhood U of $\{q_1, q_2, \ldots, q_s\}$ in N, and let $z = {}_sj^r g(q)$, $q' = (g(q_1), g(q_2), \ldots, g(q_s))$. Let ${}_sJ^r(N,P)_q$ and ${}_sJ^r(N,P)_{q,q'}$ be the fibers of ${}_sJ^r(N,P)$ over q and over (q,q') respectively. Let $J^r(N)_q$ denote the $\mathbb{R}$-algebra of r-jets at q of functions on N. Namely,

$$J^r(N)_q = {}_sJ^r(N,\mathbb{R})_q.$$

Set $g^*TP = \bigcup_{\widetilde{q}\in U} T_{g(\widetilde{q})}P$, where TP is the tangent bundle of P. By $J^r(g^*TP)_q$, we denote the $J^r(N)_q$-module of r-jets at q of sections of the bundle g^*TP. Let $\mathfrak{m}_q$ denote the ideal in $J^r(N)_q$ consisting of jets of functions which vanish at q. Namely, we have

$$\mathfrak{m}_q = \{{}_sj^r h(q) \in {}_sJ^r(N,\mathbb{R})_q \mid h(q_1) = h(q_2) = \cdots = h(q_s) = 0\}.$$

Let $\mathfrak{m}_q J^r(g^*TP)_q$ denote the set consisting of finite sums of products of an element of $\mathfrak{m}_q$ and an element of $J^r(g^*TP)_q$. Namely, we get

$$\mathfrak{m}_q J^r(g^*TP)_q = J^r(g^*TP)_q \cap \{{}_sj^r\xi(q) \in {}_sJ^r(N,TP)_q \mid \xi(q_1) = \xi(q_2) = \cdots = \xi(q_s) = 0\}.$$

Then, it is not hard to see that we get the following canonical identification of $\mathbb{R}$-vector spaces:

$$T({}_sJ^r(N,P)_{q,q'})_z = \mathfrak{m}_q J^r(g^*TP)_q. \tag{2.2.1}$$

Let W be a submanifold of ${}_sJ^r(N,P)$. Choose $q = (q_1, q_2, \ldots, q_s) \in N^{(s)}$ and $g : N \to P$. For simplicity, we set $z = {}_sj^r g(q)$ and $q' = (g(q_1), g(q_2), \ldots, g(q_s))$. Suppose that the choice is made so that $z \in W$. Set $W_{q,q'} = \widetilde{\pi}^{-1}(q,q')$, where $\widetilde{\pi} : W \to N^{(s)} \times P^s$ is given by $\widetilde{\pi}({}_sj^r\widetilde{g}(\widetilde{q})) = (\widetilde{q}, (\widetilde{g}(\widetilde{q}_1), \widetilde{g}(\widetilde{q}_2), \ldots, \widetilde{g}(\widetilde{q}_s)))$ and $\widetilde{q} = (\widetilde{q}_1, \widetilde{q}_2, \ldots, \widetilde{q}_s) \in N^{(s)}$. Suppose that $W_{q,q'}$ is a submanifold of ${}_sJ^r(N,P)$. Then, under the identification (2.2.1), $T(W_{q,q'})_z$ can be identified with a vector subspace of $\mathfrak{m}_q J^r(g^*TP)_q$. We denote this vector subspace by $E(g,q,W)$.

Definition 2.2.1. We say that a submanifold W of ${}_sJ^r(N,P)$ is *modular* if conditions (α) and (β) below are satisfied.

(α) The set W is an invariant submanifold of ${}_sJ^r(N,P)$, and lies over P^{Π} for some partition Π of $\{1,2,\ldots,s\}$.

(β) For any $q \in N^{(s)}$ and any mapping $g : N \to P$ satisfying ${}_sj^r g(q) \in W$, the subspace $E(g,q,W)$ is a $J^r(N)_q$-submodule.

Now, suppose that $P = \mathbb{R}^\ell$. The main theorem in [21] is the following.

Theorem 2.2.1 ([21]). *Let $f : N \to \mathbb{R}^m$ be an embedding, where N is a manifold of dimension n. If W is a modular submanifold of ${}_sJ^r(N,\mathbb{R}^\ell)$ and $m > \ell$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) - \Sigma$, the mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.*

The main theorem in Section 2.2 is the following. For the proof of Theorem 2.2.2, see Section 2.2.3.

Theorem 2.2.2 ([8]). *Let $f : N \to U$ be an embedding, where N is a manifold of dimension n and U is an open set in $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If W is a modular submanifold of ${}_sJ^r(N,\mathbb{R}^\ell)$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.*

Let $g : (N,S) \to (P,y)$ be a multi-germ, where S is a finite subset of N and y is a point of P. We say that $\xi : (N,S) \to (TP,\xi(S))$ is a *vector field along g* if ξ satisfies $\Pi \circ \xi = g$, where $\Pi : TP \to P$ is the canonical projection.

Let $\theta(g)_S$ be the set of vector fields along g. Set $\theta(N)_S = \theta(\mathrm{id}_N)_S$ and $\theta(P)_y = \theta(\mathrm{id}_P)_y$, where $\mathrm{id}_N : (N,S) \to (N,S)$ and $\mathrm{id}_P : (P,y) \to (P,y)$ are the identify map-germs. Then, $tg : \theta(N)_S \to \theta(g)_S$ is defined by $tg(\xi) = Tg \circ \xi$, where $Tg : TN \to TP$ is the derivative mapping of g. The mapping $\omega g : \theta(P)_y \to \theta(g)_S$ is defined by $\omega g(\eta) = \eta \circ g$. Then, we say that $g : N \to P$ is *locally infinitesimally stable* if for every $y \in P$ and every finite subset $S \subset g^{-1}(y)$, it follows that

$$tg(\theta(N)_S) + \omega g(\theta(P)_y) = \theta(g)_S.$$

By the same method as the proof of Theorem 3 of [21], we get the following as a corollary of Theorem 2.2.2.

Corollary 2.2.1. *Let $f : N \to \mathbb{R}^m$ be an embedding, where N is a manifold of dimension n and U is an open subspace of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If (n, ℓ) is in the nice dimensions, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the composition $F_\pi \circ f : N \to \mathbb{R}^\ell$ is locally infinitesimally stable.*

Remark 2.2.1.

1. In the case that $F = 0$, $U = \mathbb{R}^m$, and $m > \ell$, Theorem 2.2.2 is Theorem 2.2.1.

2. If $F_\pi \circ f : N \to \mathbb{R}^\ell$ is proper in Corollary 2.2.1, then the local infinitesimal stability of $F_\pi \circ f$ implies the stability of it (see [19]). Namely, we have the following.

   Corollary 2.2.2. *Let $f : N \to \mathbb{R}^m$ be an embedding, where N is a compact manifold of dimension n and U is an open subspace of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^\ell$ be a mapping. If (n, ℓ) is in the nice dimensions, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the composition $F_\pi \circ f : N \to \mathbb{R}^\ell$ is stable.*

3. There is an advantage that the domain of F is an open set. Suppose that $U = \mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{R}$ be the mapping given by $x \mapsto |x|$. Since F is not differentiable at $x = 0$, we can not apply Theorem 2.2.2 to the mapping $F : \mathbb{R} \to \mathbb{R}$.

   On the other hand, if $U = \mathbb{R} - \{0\}$, then Theorem 2.2.2 can be applied to $F|_U$.

2.2.3 Proof of Theorem 2.2.2

Let $(\alpha_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ be a representing matrix of a linear mapping $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$. Set $F_\alpha = F_\pi$. Then, we get

$$F_\alpha(x) = \left(F_1(x) + \sum_{j=1}^{m} \alpha_{1j} x_j, \ldots, F_\ell(x) + \sum_{j=1}^{m} \alpha_{\ell j} x_j \right),$$

where $\alpha = (\alpha_{11}, \ldots, \alpha_{1m}, \ldots, \alpha_{\ell 1}, \ldots, \alpha_{\ell m}) \in (\mathbb{R}^m)^\ell$, $F = (F_1, \ldots, F_\ell)$ and $x = (x_1, \ldots, x_m)$. For a given embedding $f : N \to U$, a mapping $F_\alpha \circ f : N \to \mathbb{R}^\ell$ is as follows:

$$F_\alpha \circ f = \left(F_1 \circ f + \sum_{j=1}^{m} \alpha_{1j} f_j, \ldots, F_\ell \circ f + \sum_{j=1}^{m} \alpha_{\ell j} f_j \right),$$

where $f=(f_1,\ldots,f_m)$. Since we have the natural identification $\mathcal{L}(\mathbb{R}^m,\mathbb{R}^\ell)=(\mathbb{R}^m)^\ell$, for the proof, it is sufficient to prove that there exists a subset $\Sigma\subset(\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $\alpha\in(\mathbb{R}^m)^\ell-\Sigma$, the mapping ${}_sj^r(F_\alpha\circ f):N^{(s)}\to {}_sJ^r(N,\mathbb{R}^\ell)$ is transverse to the given modular submanifold W.

Let $H_\Lambda:\mathbb{R}^\ell\to\mathbb{R}^\ell$ be the linear isomorphism given by

$$H_\Lambda(X_1,\ldots,X_\ell)=(X_1,\ldots,X_\ell)\Lambda,$$

where $\Lambda=(\lambda_{ij})_{1\le i\le\ell,1\le j\le\ell}$ is an $\ell\times\ell$ regular matrix. Then, we get

$$\begin{aligned}H_\Lambda\circ F_\alpha\circ f&=\left(\sum_{k=1}^{\ell}\Big(F_k\circ f+\sum_{j=1}^{m}\alpha_{kj}f_j\Big)\lambda_{k1},\ldots,\sum_{k=1}^{\ell}\Big(F_k\circ f+\sum_{j=1}^{m}\alpha_{kj}f_j\Big)\lambda_{k\ell}\right)\\&=\left(\sum_{k=1}^{\ell}\Big(F_k\circ f\Big)\lambda_{k1}+\sum_{j=1}^{m}\Big(\sum_{k=1}^{\ell}\lambda_{k1}\alpha_{kj}\Big)f_j,\ldots,\sum_{k=1}^{\ell}\Big(F_k\circ f\Big)\lambda_{k\ell}+\sum_{j=1}^{m}\Big(\sum_{k=1}^{\ell}\lambda_{k\ell}\alpha_{kj}\Big)f_j\right).\end{aligned}$$

Set $GL(\ell)=\{B\mid B:\ell\times\ell\text{ matrix},\ \det B\neq 0\}$. Let $\varphi:GL(\ell)\times(\mathbb{R}^m)^\ell\to GL(\ell)\times(\mathbb{R}^m)^\ell$ be the mapping defined by

$$\begin{aligned}&\varphi(\lambda_{11},\lambda_{12},\ldots,\lambda_{\ell\ell},\alpha_{11},\alpha_{12},\ldots,\alpha_{\ell m})\\&=\Big(\lambda_{11},\lambda_{12},\ldots,\lambda_{\ell\ell},\sum_{k=1}^{\ell}\lambda_{k1}\alpha_{k1},\sum_{k=1}^{\ell}\lambda_{k2}\alpha_{k1},\ldots,\sum_{k=1}^{\ell}\lambda_{k\ell}\alpha_{k1},\\&\sum_{k=1}^{\ell}\lambda_{k1}\alpha_{k2},\sum_{k=1}^{\ell}\lambda_{k2}\alpha_{k2},\ldots,\sum_{k=1}^{\ell}\lambda_{k\ell}\alpha_{k2},\ldots,\sum_{k=1}^{\ell}\lambda_{k1}\alpha_{km},\sum_{k=1}^{\ell}\lambda_{k2}\alpha_{km},\ldots,\sum_{k=1}^{\ell}\lambda_{k\ell}\alpha_{km}\Big).\end{aligned}$$

For the proof, it is important to show that φ is a C^∞ diffeomorphism. In order to prove that φ is a C^∞ diffeomorphism, for any point $(\Lambda',\alpha')\in GL(\ell)\times(\mathbb{R}^m)^\ell$ of the target space of φ, we will find (Λ,α) satisfying $\varphi(\Lambda,\alpha)=(\Lambda',\alpha')$, where $\Lambda=(\lambda_{11},\lambda_{12},\ldots,\lambda_{\ell\ell})$, $\Lambda'=(\lambda'_{11},\lambda'_{12},\ldots,\lambda'_{\ell\ell})$, $\alpha=(\alpha_{11},\alpha_{12},\ldots,\alpha_{\ell m})$, and $\alpha'=(\alpha'_{11},\alpha'_{12},\ldots,\alpha'_{m\ell})$. Thus, it is sufficient to find (Λ,α) satisfying

$$\begin{aligned}\lambda_{ij}&=\lambda'_{ij}\ (1\le i\le\ell,1\le j\le\ell),\\\sum_{k=1}^{\ell}\lambda_{ki}\alpha_{kj}&=\alpha'_{ji}\ (1\le i\le\ell,1\le j\le m).\end{aligned}$$

Hence, for any j $(1 \leq j \leq m)$, we have

$$\sum_{k=1}^{\ell} \lambda'_{k1}\alpha_{kj} = \alpha'_{j1}, \ \sum_{k=1}^{\ell} \lambda'_{k2}\alpha_{kj} = \alpha'_{j2}, \ldots, \ \sum_{k=1}^{\ell} \lambda'_{k\ell}\alpha_{kj} = \alpha'_{j\ell}.$$

Therefore, for any j $(1 \leq j \leq m)$, we get the following:

$$\begin{pmatrix} \lambda'_{11} & \cdots & \lambda'_{\ell 1} \\ \vdots & \ddots & \vdots \\ \lambda'_{1\ell} & \cdots & \lambda'_{\ell\ell} \end{pmatrix} \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{\ell j} \end{pmatrix} = \begin{pmatrix} \alpha'_{j1} \\ \vdots \\ \alpha'_{j\ell} \end{pmatrix}.$$

Since the matrix

$$\begin{pmatrix} \lambda'_{11} & \cdots & \lambda'_{\ell 1} \\ \vdots & \ddots & \vdots \\ \lambda'_{1\ell} & \cdots & \lambda'_{\ell\ell} \end{pmatrix}$$

is regular, for any j $(1 \leq j \leq m)$, $\alpha_{1j}, \ldots, \alpha_{\ell j}$ can be expressed by rational functions of $\lambda'_{11}, \ldots, \lambda'_{\ell\ell}$, $\alpha'_{j1}, \ldots, \alpha'_{j\ell}$. Hence, there exists the inverse mapping φ^{-1} and we see that φ^{-1} is of class C^∞. Thus, φ is a C^∞ diffeomorphism.

Now, let $\widetilde{f} : U \to \mathbb{R}^{m+\ell}$ be the mapping given by

$$\widetilde{f}(x_1, \ldots, x_m) = (F_1(x_1, \ldots, x_m), \ldots, F_\ell(x_1, \ldots, x_m), x_1, \ldots, x_m).$$

We see that $\widetilde{f}$ is an embedding. Since $f : N \to U$ is an embedding, $\widetilde{f} \circ f : N \to \mathbb{R}^{m+\ell}$ is also an embedding:

$$\widetilde{f} \circ f = (F_1 \circ f, \ldots, F_\ell \circ f, f_1, \ldots, f_m).$$

For the proof, the following lemma is important. The following lemma is the special case of Theorem 2.2.1.

Lemma 2.2.1 ([21]). *Let $\widetilde{f}\circ f : N \to \mathbb{R}^{m+\ell}$ be an embedding, where N is a manifold of dimension n. If W is a modular submanifold of ${}_sJ^r(N,\mathbb{R}^\ell)$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^{m+\ell},\mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\Pi \in \mathcal{L}(\mathbb{R}^{m+\ell},\mathbb{R}^\ell) - \Sigma$, the mapping ${}_sj^r(\Pi\circ\widetilde{f}\circ f) : N^{(s)} \to {}_sJ^r(N,\mathbb{R}^\ell)$ is transverse to W.*

From Lemma 2.2.1, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^{m+\ell},\mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\Pi \in \mathcal{L}(\mathbb{R}^{m+\ell},\mathbb{R}^\ell) - \Sigma$, the mapping ${}_sj^r(\Pi\circ(\widetilde{f}\circ f)) : N^{(s)} \to {}_sJ^r(N,\mathbb{R}^\ell)$ is transverse to W.

By the natural identification $\mathcal{L}(\mathbb{R}^{m+\ell},\mathbb{R}^\ell) = \mathbb{R}^{\ell(m+\ell)}$, we can identify the target space $GL(\ell)\times(\mathbb{R}^m)^\ell$ of φ with an open submanifold of $\mathcal{L}(\mathbb{R}^{m+\ell},\mathbb{R}^\ell)$. Since $(GL(\ell)\times(\mathbb{R}^m)^\ell)\cap\Sigma$ is a subset of $GL(\ell)\times(\mathbb{R}^m)^\ell$ with Lebesgue measure 0 and φ^{-1} is class C^∞, we have that $\varphi^{-1}((GL(\ell)\times(\mathbb{R}^m)^\ell)\cap\Sigma)$ is a subset of $GL(\ell)\times(\mathbb{R}^m)^\ell$ with Lebesgue measure 0. For any $(\Lambda,\alpha) \in GL(\ell)\times(\mathbb{R}^m)^\ell$, let $\Pi_{(\Lambda,\alpha)} : \mathbb{R}^{m+\ell} \to \mathbb{R}^\ell$ be the linear mapping given by $\varphi(\Lambda,\alpha)$ as follows:

$$
\begin{aligned}
&\Pi_{(\Lambda,\alpha)}(X_1,\ldots,X_{m+\ell}) \\
= \;& (X_1,\ldots,X_{m+\ell})
\begin{pmatrix}
\lambda_{11} & \cdots & \lambda_{1\ell} \\
\vdots & \ddots & \vdots \\
\lambda_{\ell 1} & \cdots & \lambda_{\ell\ell} \\
\sum_{k=1}^{\ell}\lambda_{k1}\alpha_{k1} & \cdots & \sum_{k=1}^{\ell}\lambda_{k\ell}\alpha_{k1} \\
\vdots & \ddots & \vdots \\
\sum_{k=1}^{\ell}\lambda_{k1}\alpha_{km} & \cdots & \sum_{k=1}^{\ell}\lambda_{k\ell}\alpha_{km}
\end{pmatrix}.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
&\Pi_{(\Lambda,\alpha)}\circ\widetilde{f}\circ f \\
= \;& \left(\sum_{k=1}^{\ell}\Big(F_k\circ f\Big)\lambda_{k1} + \sum_{j=1}^{m}\Big(\sum_{k=1}^{\ell}\lambda_{k1}\alpha_{kj}\Big)f_j, \ldots, \sum_{k=1}^{\ell}\Big(F_k\circ f\Big)\lambda_{k\ell} + \sum_{j=1}^{m}\Big(\sum_{k=1}^{\ell}\lambda_{k\ell}\alpha_{kj}\Big)f_j\right) \\
= \;& H_\Lambda\circ F_\alpha\circ f.
\end{aligned}
$$

Thus, for any $(\Lambda,\alpha)\in GL(\ell)\times(\mathbb{R}^m)^\ell-\varphi^{-1}((GL(\ell)\times(\mathbb{R}^m)^\ell)\cap\Sigma)$, we have that ${}_sj^r(\Pi_{(\Lambda,\alpha)}\circ\widetilde{f}\circ f)$ $(={}_sj^r(H_\Lambda\circ F_\alpha\circ f))$ is transverse to W. Since H_Λ is a diffeomorphism, we see that ${}_sj^r(F_\alpha\circ f)$ is transverse to W.

Let $\widetilde{\Sigma}$ be a subset consisting of $\alpha\in(\mathbb{R}^m)^\ell$ such that ${}_sj^r(F_\alpha\circ f)$ is not transverse to W. For the proof, it is sufficient to prove that $\widetilde{\Sigma}$ has Lebesgue measure 0 in $(\mathbb{R}^m)^\ell$. Suppose that $\widetilde{\Sigma}$ does not have Lebesgue measure 0 in $(\mathbb{R}^m)^\ell$. Then, $GL(\ell)\times\widetilde{\Sigma}$ does not have Lebesgue measure 0 in $GL(\ell)\times(\mathbb{R}^m)^\ell$. For any $(\Lambda,\alpha)\in GL(\ell)\times\widetilde{\Sigma}$, since ${}_sj^r(F_\alpha\circ f)$ is not transverse to the submanifold W and H_Λ is a diffeomorphism, the mapping ${}_sj^r(H_\Lambda\circ F_\alpha\circ f)$ is not transverse to W. This contradicts to the assertion that $\varphi^{-1}((GL(\ell)\times(\mathbb{R}^m)^\ell)\cap\Sigma)$ has Lebesgue measure 0 in $GL(\ell)\times(\mathbb{R}^m)^\ell$. $\square$

CHAPTER 3

Generic distance-squared mappings

3.1 Introduction

In Chapter 3, ℓ and n stand for positive integers. Let p be a given point in $\mathbb{R}^n$. The mapping $d_p : \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_p(x) = \sum_{i=1}^{n}(x_i - p_i)^2$$

is called a distance-squared function, where $x = (x_1, \ldots, x_n)$ and $p = (p_1, \ldots, p_n)$.

Definition 3.1.1. Let $p_1, \ldots, p_\ell$ be ℓ given points in $\mathbb{R}^n$. Set $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^n)^\ell$. The mapping $D_p : \mathbb{R}^n \to \mathbb{R}^\ell$ defined by

$$D_p(x) = (d_{p_1}(x), \ldots, d_{p_\ell}(x))$$

is called a *distance-squared mapping.*

Note that D_p always has a singular point if $\ell \leq n$.

We have the following motivation for investigating distance-squared mappings. Height functions and distance-squared functions have been investigated in detail so far. Moreover, they are a useful tool in the applications of singularity theory to differential geometry (see [4]). The mappings in which each component is a height function are nothing but projections. Projections as well as height functions or distance-squared functions have been investigated. For instance, in [21], the stability of projections on a given submanifold is investigated. On the other hand, the mapping in which each component is a distance-squared function is a distance-squared mapping. Hence, it is natural to investigate distance-squared mappings as well as projections.

A mapping $f : \mathbb{R}^n \to \mathbb{R}^\ell$ $(2 \leq \ell \leq n)$ is called *the normal form of definite fold mappings* if $f(x_1, \ldots, x_n) = (x_1, \ldots, x_{\ell-1}, x_\ell^2 + \cdots + x_n^2)$.

We say that ℓ points $p_1, \ldots, p_\ell \in \mathbb{R}^n$ $(1 \leq \ell \leq n+1)$ are *in general position* if $\ell = 1$ or $\overrightarrow{p_1p_2}, \ldots, \overrightarrow{p_1p_\ell}$ $(2 \leq \ell \leq n+1)$ are linearly independent.

Theorem 3.1.1 ([9]). (1) *Let ℓ and n be positive integers satisfying $2 \leq \ell \leq n$. Let $p_1, \ldots, p_\ell \in \mathbb{R}^n$ be in general position. Then, $D_p : \mathbb{R}^n \to \mathbb{R}^\ell$ is $\mathcal{A}$-equivalent to the normal form of definite fold mappings.*

(2) *Let ℓ and n be positive integers satisfying $1 \leq n < \ell$. Let $p_1, \ldots, p_{n+1} \in \mathbb{R}^n$ be in general position. Then, $D_p : \mathbb{R}^n \to \mathbb{R}^\ell$ is $\mathcal{A}$-equivalent to $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$.*

3.2 Proof of Theorem 3.1.1

3.2.1 Proof of (1) of Theorem 3.1.1

Let $H_1 : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the diffeomorphism defined by

$$\begin{aligned} & H_1(X_1, X_2, \ldots, X_\ell) \\ = & \left(\frac{1}{2}\Big(X_1 - X_2 + \sum_{j=1}^n (p_{1j} - p_{2j})^2\Big), \ldots, \frac{1}{2}\Big(X_1 - X_\ell + \sum_{j=1}^n (p_{1j} - p_{\ell j})^2\Big), X_1\right). \end{aligned}$$

Then, we have

$$
\begin{aligned}
&(H_1 \circ D_p)(x_1, x_2, \ldots, x_n) \\
= \ &\left(\sum_{j=1}^{n}(p_{2j}-p_{1j})(x_j-p_{1j}), \ldots, \sum_{j=1}^{n}(p_{\ell j}-p_{1j})(x_j-p_{1j}), \sum_{j=1}^{n}(x_j-p_{1j})^2\right).
\end{aligned}
$$

Let $H_2 : \mathbb{R}^n \to \mathbb{R}^n$ be the diffeomorphism defined by

$$H_2(x_1, x_2, \ldots, x_n) = (x_1 + p_{11}, x_2 + p_{12}, \ldots, x_n + p_{1n}).$$

The composition of $H_1 \circ D_p$ and H_2 is given by

$$
\begin{aligned}
(H_1 \circ D_p \circ H_2)(x_1, x_2, \ldots, x_n) &= \left(\sum_{j=1}^{n}(p_{2j}-p_{1j})x_j, \ldots, \sum_{j=1}^{n}(p_{\ell j}-p_{1j})x_j, \sum_{j=1}^{n} x_j^2\right) \\
&= \left((x_1, x_2, \ldots, x_n)A, \sum_{j=1}^{n} x_j^2\right),
\end{aligned}
$$

where

$$
A = \begin{pmatrix}
p_{21}-p_{11} & \cdots & p_{\ell 1}-p_{11} \\
p_{22}-p_{12} & \cdots & p_{\ell 2}-p_{12} \\
\vdots & & \vdots \\
p_{2n}-p_{1n} & \cdots & p_{\ell n}-p_{1n}
\end{pmatrix}.
$$

Since ℓ points $p_1, \ldots, p_\ell$ are in general position, the rank of A is $\ell - 1$. Therefore, there exists an $(\ell-1)\times(\ell-1)$ regular matrix B such that the set of column vectors of AB is a subset of an orthonormal basis of $\mathbb{R}^n$.

Let $H_3 : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the diffeomorphism defined by

$$H_3(X_1, X_2, \ldots, X_\ell) = ((X_1, \ldots, X_{\ell-1})B, X_\ell).$$

Then, we get

$$(H_3 \circ H_1 \circ D_p \circ H_2)(x_1, x_2, \ldots, x_n) = \left((x_1, x_2, \ldots, x_n)AB, \sum_{j=1}^{n} x_j^2 \right).$$

Set $(\mathbf{a}_1, \ldots, \mathbf{a}_\ell) = AB$. Then, there exist vectors $\mathbf{a}_{\ell-1}, \ldots, \mathbf{a}_n$ such that the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Set $C = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$. Notice that C is an $n \times n$ orthogonal matrix.

Let $H_4 : \mathbb{R}^n \to \mathbb{R}^n$ be the diffeomorphism defined by

$$H_4(x) = x^tC,$$

where $x = (x_1, \ldots, x_n)$ and ${}^tC$ is the transposed matrix of C. The composition of $H_3 \circ H_1 \circ D_p \circ H_2$ and H_4 is as follows:

$$(H_3 \circ H_1 \circ D_p \circ H_2 \circ H_4)(x) = \left(x^tCAB, \langle x^tC, x^tC \rangle\right),$$

where $\langle z, z \rangle$ $(z = (z_1, \ldots, z_n) \in \mathbb{R}^n)$ stands the inner product defined by $\langle z, z \rangle = \sum_{i=1}^{n} z_i^2$. Since $C = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is an orthogonal matrix and $AB = (\mathbf{a}_1, \ldots, \mathbf{a}_\ell)$, we have

$${}^tCAB = \begin{pmatrix} E_{\ell-1} \\ \\ O \end{pmatrix}$$

and

$$\langle x^tC, x^tC \rangle = \langle x, x \rangle,$$

where $E_{\ell-1}$ is the $(\ell-1) \times (\ell-1)$ unit matrix and O is the $(n-(\ell-1)) \times (\ell-1)$ zero matrix.

Therefore, we get

$$(H_3 \circ H_1 \circ D_p \circ H_2 \circ H_4)(x_1, \ldots, x_\ell) = \left(x_1, \ldots, x_{\ell-1}, \sum_{i=1}^{\ell} x_i^2 \right).$$

Let $H_5 : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the diffeomorphism defined by

$$H_5(X_1, X_2, \ldots, X_\ell) = \left(X_1, \ldots, X_{\ell-1}, X_\ell - \sum_{i=1}^{\ell-1} X_i^2 \right).$$

The composition of $H_3 \circ H_1 \circ D_p \circ H_2 \circ H_4$ and H_5 is as follows:

$$(H_5 \circ H_3 \circ H_1 \circ D_p \circ H_2 \circ H_4)(x_1, \ldots, x_\ell) = \left(x_1, \ldots, x_{\ell-1}, \sum_{i=\ell}^{n} x_i^2 \right).$$

□

3.2.2 Proof of (2) of Theorem 3.1.1

Since $n < \ell$ and $(n+1)$ points $p_1, \ldots, p_{n+1}$ are in general position, there exists an $(\ell - 1) \times (\ell - 1)$ regular matrix $\widetilde{B}$ satisfying

$$A\widetilde{B} = \begin{pmatrix} 1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ 0 & & 1 & 0 & \cdots & 0 \end{pmatrix},$$

where the matrix A is the same as in the proof of (1). Similarly as the proof of (1) of this theorem, we have

$$
\begin{aligned}
& (H_3 \circ H_1 \circ D_p \circ H_2)(x_1, x_2, \ldots, x_n) \\
= \ & (x_1, x_2, \ldots, x_n) \left(\begin{array}{c|c} & x_1 \\ & x_2 \\ A & \vdots \\ & \vdots \\ & x_n \end{array} \right) \left(\begin{array}{c|c} & 0 \\ \widetilde{B} & \vdots \\ & 0 \\ \hline 0 \ \cdots \ 0 & 1 \end{array} \right) \\
= \ & (x_1, x_2, \cdots, x_n) \begin{pmatrix} 1 & & 0 & 0 & \cdots & 0 & x_1 \\ & \ddots & & \vdots & & \vdots & \vdots \\ 0 & & 1 & 0 & \cdots & 0 & x_n \end{pmatrix} \\
= \ & \left(x_1, \ldots, x_n, 0, \ldots, 0, \sum_{j=1}^{n} x_j^2 \right).
\end{aligned}
$$

Let $\widetilde{H}_4 : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be the diffeomorphism given by

$$
\widetilde{H}_4(X_1, X_2, \ldots, X_\ell) = \left(X_1, \ldots, X_n, \ldots, X_{\ell-1}, X_\ell - \sum_{j=1}^{n} X_j^2 \right).
$$

Therefore, we get

$$
\begin{aligned}
& (\widetilde{H}_4 \circ H_3 \circ H_1 \circ D_p \circ H_2)(x_1, x_2, \ldots, x_n) \\
= \ & (x_1, x_2, \ldots, x_n, 0, \ldots, 0).
\end{aligned}
$$

□

CHAPTER 4

Generic Lorentzian distance-squared mappings

4.1 Introduction

In Chapter 4, by n, we denote a positive integer. For the $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$, the following quadratic form is called the *Lorentzian inner product*:

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_ny_n,$$

where $x = (x_0, x_1, \dots, x_n), y = (y_0, y_1, \dots, y_n)$ are elements of $\mathbb{R}^{n+1}$. The $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$ is called *Lorentzian* $(n+1)$*-space* and is denoted by $\mathbb{R}^{1,n}$ if the role of the Euclidean inner product $x \cdot y = \sum_{i=0}^{n} x_iy_i$ is replaced by the Lorentzian inner product. For a vector x of Lorentzian $(n+1)$-space $\mathbb{R}^{1,n}$, we say that $\sqrt{\langle x, x\rangle}$ is the *Lorentzian length* of x. We say that a non-zero vector $x \in \mathbb{R}^{1,n}$ is *space-like*, *light-like* or *time-like* if its Lorentzian length is positive, zero or pure imaginary respectively. The likeness of the vector subspace is defined as follows (see Definition 4.1.1).

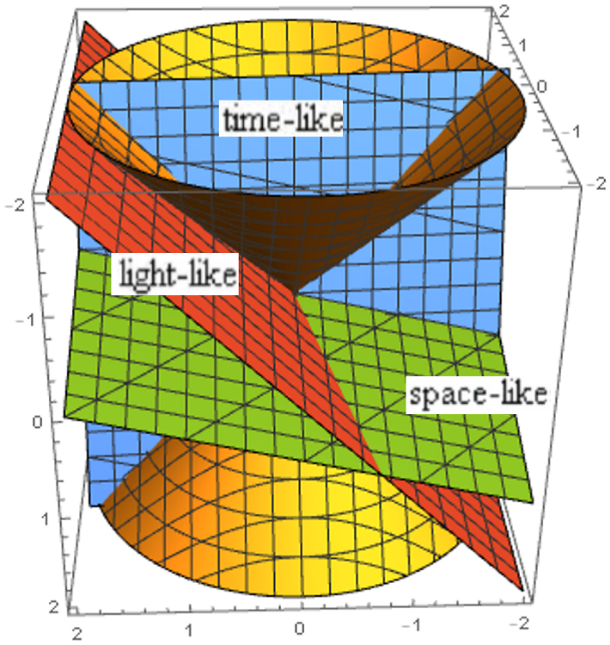

Figure 4.1: Figure of Definition 4.1.1

Definition 4.1.1 ([23]). Let V be a vector subspace of $\mathbb{R}^{1,n}$. Then, we say that V is

1. *time-like* if V has a time-like vector,
2. *space-like* if every nonzero vector in V is space-like, or
3. *light-like* otherwise.

The *light cone* of Lorentzian $(n+1)$-space $\mathbb{R}^{1,n}$, denoted by LC, is the set consisting of elements $x \in \mathbb{R}^{1,n}$ satisfying $\langle x, x\rangle = 0$.

For an arbitrary point p of $\mathbb{R}^{1,n}$, we say that $\ell_p^2 : \mathbb{R}^{1,n} \to \mathbb{R}$ is the *Lorentzian distance-squared function* if

$$\ell_p^2(x) = \langle x - p, x - p\rangle.$$

For instance, in [14], Lorentzian distance-squared functions on surfaces in Lorentzian space are investigated. They are useful for the study on Lorentzian space from the viewpoint of Singularity Theory. On the other hand, in Chapter 4, we give a different application of Singularity Theory to the study on Lorentzian space.

For $(k+1)$ points $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$ $(1 \le k)$, the *Lorentzian distance-squared mapping*, denoted by $L_p : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$, is defined as follows:

$$L_p(x) = \left(\ell^2_{p_0}(x), \ldots, \ell^2_{p_k}(x)\right),$$

where $p = (p_0, \ldots, p_k)$. The main purpose of Chapter 4 is to give a characterizations of Lorentzian distance-squared mappings (see Theorem 4.1.1).

A vector subspace $V(p_0, \ldots, p_k)$ of $\mathbb{R}^{1,n}$ is called a *recognition subspace* if

$$V(p_0, \ldots, p_k) = \sum_{i=1}^{k} \mathbb{R}\, \overrightarrow{p_0 p_i}.$$

For any two positive integers k and n satisfying $k \le n$, $\Phi_k : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is called the *normal form of definite fold mapping* if

$$\Phi_k(x_0, x_1, \ldots, x_n) = \left(x_1, \ldots, x_k, x_0^2 + \sum_{i=k+1}^{n} x_i^2\right).$$

For any two positive integers k and n satisfying $k < n$, $\Psi_k : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is called the *normal form of Lorentzian indefinite fold mapping* if

$$\Psi_k(x_0, x_1, \ldots, x_n) = \left(x_1, \ldots, x_k, -x_0^2 + \sum_{i=k+1}^{n} x_i^2\right).$$

Let j and k be two positive integers satisfying $j \le k$. Let $\tau_{(j,k)} : \mathbb{R}^{j+1} \to \mathbb{R}^{k+1}$ be the inclusion:

$$\tau_{(j,k)}(X_0, X_1, \ldots, X_j) = (X_0, X_1, \ldots, X_j, 0, \ldots, 0).$$

Theorem 4.1.1. *1. Let k and n be two positive integers. Let $(k+1)$ points $p_0, \ldots, p_k \in \mathbb{R}^{n,1}$ be the same point (i.e. $\dim V(p_0, \ldots, p_k) = 0$). Set $p = (p_0, \ldots, p_k)$. Then, $L_p : \mathbb{R}^{n,1} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to*

$$(x_0, \ldots, x_n) \mapsto \left(-x_0^2 + \sum_{i=1}^{n} x_i^2, 0, \ldots, 0\right).$$

2. *Let j, k and n be three positive integers satisfying $j < n$ and $j \leq k$. Let $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$ be $(k+1)$ points such that two recognition subspaces $V(p_0, \ldots, p_k)$ and $V(p_0, \ldots, p_j)$ have the same dimension j. Set $p = (p_0, \ldots, p_k)$. Then, the following hold:*

   (a) *The mapping $L_p : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to $\tau_{(j,k)} \circ \Phi_j$ if and only if the recognition subspace $V(p_0, \ldots, p_k)$ is time-like.*

   (b) *The mapping $L_p : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to $\tau_{(j,k)} \circ \Psi_j$ if and only if the recognition subspace $V(p_0, \ldots, p_k)$ is space-like.*

   (c) *The mapping $L_p : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to*

$$(x_0, \ldots, x_n) \mapsto \left(x_1, \ldots, x_j, x_0 x_1 + \sum_{i=j+1}^{n} x_i^2, 0, \ldots, 0 \right)$$

   if and only if the recognition subspace $V(p_0, \ldots, p_k)$ is light-like.

3. *Let k and n be two positive integers satisfying $n \leq k$. Let $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$ be $(k+1)$ points satisfying $\dim V(p_0, \ldots, p_k) = \dim V(p_0, \ldots, p_n) = n$. Set $p = (p_0, \ldots, p_k)$. Then, the following hold:*

   (a) *The mapping $L_p : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to $\tau_{(n,k)} \circ \Phi_n$ if and only if the recognition subspace $V(p_0, \ldots, p_k)$ is time-like or space-like.*

   (b) *The mapping $L_p : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to*

$$(x_0, \ldots, x_n) \mapsto (x_1, \ldots, x_n, x_0 x_1, 0 \ldots, 0)$$

   if and only if the recognition subspace $V(p_0, \ldots, p_k)$ is light-like.

4. *Let k and n be two positive integers satisfying $n < k$. Let $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$ be $(k+1)$ points satisfying $\dim V(p_0, \ldots, p_k) = \dim V(p_0, \ldots, p_{n+1}) = n+1$. Set $p = (p_0, \ldots, p_k)$. Then, $L_p : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to the inclusion*

$$(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_n, 0, \ldots, 0).$$

We say that $(k+1)$ points $p_0, \ldots, p_k$ are *in general position* if the dimension of $V(p_0, \ldots, p_k)$ is k. For $(k+1)$ points $q_0, \ldots, q_k \in \mathbb{R}^{1,n}$ in general position $(k \leq n)$, it is not hard to see that the singular set of $L_q : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is the k-dimensional affine subspace spanned by these points, where $q = (q_0, \ldots, q_k)$. Since $\tau_{(k.k)}$ is the identity mapping, we get the following.

Corollary 4.1.1. *1. Let k and n be two positive integers satisfying $k < n$. Let $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$ be $(k+1)$ points in general position. Set $p = (p_0, \ldots, p_k)$. Then, the following hold:*

(a) The mapping $L_p : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to Φ_k if and only if the recognition subspace $V(p_0, \ldots, p_k)$ is time-like.

(b) The mapping $L_p : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to Ψ_k if and only if the recognition subspace $V(p_0, \ldots, p_k)$ is space-like.

(c) The mapping $L_p : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to

$$(x_0, \ldots, x_n) \mapsto \left(x_1, \ldots, x_k, x_0 x_1 + \sum_{i=k+1}^{n} x_i^2 \right)$$

if and only if the recognition subspace $V(p_0, \ldots, p_k)$ is light-like.

2. Let n be a positive integer. Let $p_0, \ldots, p_n \in \mathbb{R}^{1,n}$ be $(n+1)$ points in general position. Set $p = (p_0, \ldots, p_k)$. Then, the following hold:

(a) The mapping $L_{(p_0, \ldots, p_n)} : \mathbb{R}^{1,n} \to \mathbb{R}^{n+1}$ is $\mathcal{A}$-equivalent to Φ_n if and only if the recognition subspace $V(p_0, \ldots, p_n)$ is time-like or space-like.

(b) The mapping $L_{(p_0, \ldots, p_n)} : \mathbb{R}^{1,n} \to \mathbb{R}^{n+1}$ is $\mathcal{A}$-equivalent to

$$(x_0, \ldots, x_n) \mapsto (x_1, \ldots, x_n, x_0 x_1)$$

if and only if the recognition subspace $V(p_0, \ldots, p_n)$ is light-like.

In Section 4.2, preliminaries for the proof of Theorem 4.1.1 are given. Section 4.3 is devoted to the proof of Theorem 4.1.1.

4.2 Preliminaries

Lemma 4.2.1. *The likeness of a vector subspace of* $\mathbb{R}^{1,n}$ *is invariant under Lorentz transformations.*

Lemma 4.2.1 clearly holds.

Lemma 4.2.2. *Set* $\mathbf{e}_i = (\underbrace{0, \ldots, 0}_{(i-1)-tuples}, 1, 0, \ldots, 0) \in \mathbb{R}^{1,n}$ $(1 \le i \le n+1)$. *Define* $\mathbf{v}_i = \alpha_i \mathbf{e}_1 + \mathbf{e}_{i+1}$ $(1 \le i \le \ell)$ *and* $\mathbf{v}_m = \mathbf{e}_{m+1}$ $(\ell + 1 \le m \le n)$. *Let* V *be the* ℓ*-dimensional vector subspace of* $\mathbb{R}^{1,n}$ *given by* $V = \sum_{i=1}^{\ell} \mathbb{R}\mathbf{v}_i$. *Let* $\widetilde{V}$ *be the* n*-dimensional vector subspace of* $\mathbb{R}^{1,n}$ *given by* $\widetilde{V} = \sum_{i=1}^{n} \mathbb{R}\mathbf{v}_i$. *Then, the following hold:*

1. *The vector subspace* $\widetilde{V}$ *is time-like if and only if* V *is time-like.*

2. *The vector subspace* $\widetilde{V}$ *is space-like if and only if* V *is space-like.*

3. *The vector subspace* $\widetilde{V}$ *is light-like if and only if* V *is light-like.*

Proof. By definition, a vector subspace V is either time-like or space-like or light-like. Hence, for the proof, it is sufficient to show only the "if parts" of 1, 2, and 3.

Suppose that V is time-like. Then, since $V \subset \widetilde{V}$, $\widetilde{V}$ is also time-like by Definition 4.1.1.

For any vector $\sum_{i=1}^{\ell} r_i \mathbf{v}_i \in V$ and $\sum_{i=1}^{n} r_i \mathbf{v}_i \in \widetilde{V}$, we get the following:

$$\left\langle \sum_{i=1}^{\ell} r_i \mathbf{v}_i, \sum_{i=1}^{\ell} r_i \mathbf{v}_i \right\rangle = -\left(\sum_{i=1}^{\ell} r_i \alpha_i \right)^2 + \sum_{i=1}^{\ell} r_i^2, \tag{4.1}$$

$$\left\langle \sum_{i=1}^{n} r_i \mathbf{v}_i, \sum_{i=1}^{n} r_i \mathbf{v}_i \right\rangle = -\left(\sum_{i=1}^{\ell} r_i \alpha_i \right)^2 + \sum_{i=1}^{\ell} r_i^2 + \sum_{i=\ell+1}^{n} r_i^2. \tag{4.2}$$

Suppose that V is space-like. By Definition 4.1.1, any nonzero vector in V is space-like. Hence, by (4.1) and (4.2), every nonzero vector in $\widetilde{V}$ is also space-like.

Suppose that V is light-like. By Definition 4.1.1, V has a nonzero light-like vector $\mathbf{v}$. The vector $\mathbf{v}$ is also in $\widetilde{V}$. Since V has no time-like vectors, by (4.1) and (4.2), We have that $\widetilde{V}$ has no time-like vectors. □

Lemma 4.2.3. *For a given element* $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, *let* V *be the* n*-dimensional vector* *subspace of* $\mathbb{R}^{1,n}$ *given by* $-x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n = 0$. *Then, the following hold:*

1. $\sum_{i=1}^{n} \alpha_i^2 > 1$ *if and only if* V *is time-like.*

2. $\sum_{i=1}^{n} \alpha_i^2 < 1$ *if and only if* V *is space-like.*

3. $\sum_{i=1}^{n} \alpha_i^2 = 1$ *if and only if* V *is light-like.*

Proof. Let H be the horizontal hyperplane $\{(1, x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$. Set $V_1 = H \cap V$. Suppose that $V_1 = \emptyset$. Then, the defining equation of V is $-x_0 = 0$. Hence, it follows that $\sum_{i=1}^{n} \alpha_i^2 = 0$ and V is space-like.

Next, suppose that $V_1 \neq \emptyset$. Then, we have $\sum_{i=1}^{n} \alpha_i^2 \neq 0$. Let q be the point $(1, 0, \ldots, 0)$. Let S_+^{n-1} be the light cone hypersurface $H \cap LC$. Then, it is clearly seen that the Euclidean distance between q and any point $x \in S_+^{n-1}$ is 1. For the proof, it is sufficient to prove that $1/\sqrt{\sum_{i=1}^{n} \alpha_i^2}$ is the Euclidean distance between q and V_1. Since $-x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ is a defining equation of V, V_1 is given by $-1 + \alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ in H. Therefore, the Euclidean distance between q and V_1 is $1/\sqrt{\sum_{i=1}^{n} \alpha_i^2}$. □

4.3 Proof of Theorem 4.1.1

4.3.1 Proof of 1 of Theorem 4.1.1

By composing L_p, the linear isomorphism of the target given by

$$(X_0, X_1, \ldots, X_k) \mapsto (X_0, X_1 - X_0, \ldots, X_k - X_0)$$

and the linear isomorphism of the source given by

$$(x_0, x_1, \ldots, x_n) \mapsto (x_0 + p_{00}, x_1 + p_{01}, \ldots, x_n + p_{0n}),$$

the desired mapping is obtained. □

4.3.2 Proof of 2 of Theorem 4.1.1

It is easily seen that any two among $\tau_{(j,k)} \circ \Phi_k$, $\tau_{(j,k)} \circ \Psi_k$ and the mapping

$$(x_0, \ldots, x_n) \mapsto (x_1, \ldots, x_j, x_0 x_1 + \sum_{i=j+1}^{n} x_i^2, 0, \ldots, 0)$$

are not $\mathcal{A}$-equivalent. Furthermore, by definition, $V(p_0, \ldots, p_k)$ is either time-like or space-like or light-like. Therefore, for the proof, it is sufficient to show only the "if parts" of 2 of Theorem 4.1.1. Set $p_i = (p_{i0}, p_{i1}, \ldots, p_{in})$ $(0 \leq i \leq k)$.

The generic case

Firstly, we will show the "if parts" of 2 of Theorem 4.1.1 in the case that $V(p_0, \ldots, p_k) \cap T = \{0\}$, where T is the time axis $\{(x_0, 0, \ldots, 0) \mid x_0 \in \mathbb{R}\}$. There are four steps.

<u>STEP 1.</u> The purpose of STEP 1 is to remove the redundant quadratic terms in k components. In order to do so, we require the affine transformation of the target space $H_1 : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ given by

$$\begin{aligned}
& H_1(X_0, X_1, \ldots, X_k) \\
= & \left(\frac{1}{2}\Big(X_0 - X_1 - (p_{00} - p_{10})^2 + \sum_{i=1}^{n} (p_{0i} - p_{1i})^2 \Big), \ldots, \right. \\
& \left. \quad \frac{1}{2}\Big(X_0 - X_k - (p_{00} - p_{k0})^2 + \sum_{i=1}^{n} (p_{0i} - p_{ki})^2 \Big), X_0 \right).
\end{aligned}$$

By composing H_1 and L_p, we get

$$\begin{aligned}
& (H_1 \circ L_p)(x_0, x_1, \ldots, x_n) \\
= & \left(-(p_{10} - p_{00})(x_0 - p_{00}) + \sum_{i=1}^{n} (p_{1i} - p_{0i})(x_i - p_{0i}), \ldots, \right. \\
& \left. \quad -(x_0 - p_{00})^2 + \sum_{i=1}^{n} (x_i - p_{0i})^2 \right).
\end{aligned}$$

<u>STEP 2.</u> The purpose of STEP 2 is to reduce the first k components to linear functions.

In order to do so, we require the affine transformation of the source space $H_2 : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ given by

$$H_2(x_0, x_1, \ldots, x_n) = (-x_0 + p_{00}, x_1 + p_{01}, \ldots, x_n + p_{0n}).$$

We define the $(n+1) \times k$ matrix A_1 as follows:

$$\begin{aligned} & (H_1 \circ L_p \circ H_2)(x_0, x_1, \ldots, x_n) \\ = & \left(\sum_{i=0}^{n} (p_{1i} - p_{0i}) x_i, \ldots, \sum_{i=0}^{n} (p_{ki} - p_{0i}) x_i, -x_0^2 + \sum_{i=1}^{n} x_i^2 \right) \\ = & \left((x_0, x_1, \ldots, x_n) A_1, -x_0^2 + \sum_{i=1}^{n} x_i^2 \right). \end{aligned}$$

Set

$$A_2 = \begin{pmatrix} p_{10} - p_{00} & \cdots & p_{j0} - p_{00} \\ p_{11} - p_{01} & \cdots & p_{j1} - p_{01} \\ \vdots & & \vdots \\ p_{1n} - p_{0n} & \cdots & p_{jn} - p_{0n} \end{pmatrix} = \begin{pmatrix} p_{10} - p_{00} \quad \cdots \quad p_{j0} - p_{00} \\ \hline \\ A_3 \\ \\ \end{pmatrix}.$$

From $\dim V(p_0, \ldots, p_j) = j$, it is clearly seen that the rank of $(n+1) \times j$ matrix A_2 is j. Furthermore, from $V(p_0, \ldots, p_j) \cap T = \{0\}$, the $n \times j$ matrix A_3 has the same rank j. Hence, there exists a $j \times j$ regular matrix B_1 such that the set of column vectors of $A_3 B_1$ is a subset of an orthonormal basis of $\mathbb{R}^n$.

STEP 3. The purpose of this step is to reduce the first j components, which are linear functions, to coordinate functions $x_1, \ldots, x_j$, preserving the Lorentzian distance-squared function $-x_0^2 + \sum_{i=1}^{n} x_i^2$ having the form $\xi(x_0, x_1, \ldots, x_j) + \sum_{i=j+1}^{n} x_i^2$. In order to do so, we construct the linear transformation of the target space $H_3 : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ below and the linear transformations of the source space $H_4, H_5 : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ below.

Let $H_3 : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ be the linear transformation of $\mathbb{R}^{k+1}$ given by

$$H_3(X_0, X_1, \dots, X_k) = ((X_0, X_1, \dots, X_{j-1})B_1, X_j, \dots, X_k).$$

We define the $k \times k$ matrix B_2 as follows:

$$H_3(X_0, X_1, \dots, X_k) = ((X_0, X_1, \dots, X_{k-1})B_2, X_k).$$

By composing H_3 and $H_1 \circ L_p \circ H_2$, we have

$$\begin{aligned} & (H_3 \circ H_1 \circ L_p \circ H_2)(x_0, x_1, \dots, x_n) \\ = & \left((x_0, x_1, \dots, x_n)A_1B_2, -x_0^2 + \sum_{i=1}^n x_i^2\right), \end{aligned}$$

Let $\mathbf{a}_i$ be the transposed matrix of the i-th column vector of A_1B_2 $(1 \le i \le j)$. From $V(p_0, \dots, p_k) = \sum_{i=1}^j \mathbb{R}\mathbf{a}_i$, there exists a $k \times k$ regular matrix B_3 satisfying

$$A_1B_2B_3 = (A_2B_1, O_{n+1,k-j}),$$

where $O_{\ell,m}$ stands for the $\ell \times m$ zero matrix. By composing H_3 and the linear isomorphism of the target defined by

$$(X_0, X_1, \dots, X_k) \mapsto ((X_0, X_1, \dots, X_{k-1})B_3, X_k)$$

if necessary, without loss of generality, from the first we may assume that

$$\begin{aligned} & (H_3 \circ H_1 \circ L_p \circ H_2)(x_0, x_1, \dots, x_n) \\ = & \left((x_0, x_1, \dots, x_n)A_2B_1, 0, \dots, 0, -x_0^2 + \sum_{i=1}^n x_i^2\right). \end{aligned}$$

Set $(\alpha_{0,i}, \alpha_{1,i}, \dots, \alpha_{n,i}) := \mathbf{a}_i$ and $\widetilde{\mathbf{a}}_i := (\alpha_{1,i}, \dots, \alpha_{n,i})$ $(1 \le i \le j)$. We can choose $\widetilde{\mathbf{a}}_{j+1}, \dots, \widetilde{\mathbf{a}}_n$ such that the set $\{\widetilde{\mathbf{a}}_1, \dots, \widetilde{\mathbf{a}}_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Note that $({}^t\widetilde{\mathbf{a}}_1, \dots, {}^t\widetilde{\mathbf{a}}_n)$ is an $n \times n$ orthogonal matrix. Let $H_4 : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ be the linear isomorphism given by

$$H_4(x_0, x_1, \ldots, x_n) = (x_0, (x_1, \ldots, x_n)({}^t\widetilde{\mathbf{a}}_1, \ldots, {}^t\widetilde{\mathbf{a}}_n)^{-1}).$$

Note that H_4 and H_4^{-1} are Lorentz transformations. We get

$$\begin{aligned}&(H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4)(x_0, x_1, \ldots, x_n)\\ = &\left(\alpha_{0,1}x_0 + x_1, \ldots, \alpha_{0,j}x_0 + x_j, 0, \ldots, 0, -x_0^2 + \sum_{i=1}^{n} x_i^2\right).\end{aligned}$$

Set $\mathbf{E}_i = (\underbrace{0, \ldots, 0}_{(i-1)-tuples}, 1, 0, \ldots, 0) \in \mathbb{R}^{k+1}$ $(1 \le i \le k+1)$. We define the linear isomorphism $H_5 : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ by

$$H_5\left(\sum_{i=0}^{k} X_i \mathbf{E}_{i+1}\right) = \sum_{i \ne j,k} X_i \mathbf{E}_{i+1} + X_k \mathbf{E}_{j+1} + X_j \mathbf{E}_{k+1}.$$

Then, we get the following:

$$\begin{aligned}&(H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4)(x_0, x_1, \ldots, x_n)\\ = &\left(\alpha_{0,1}x_0 + x_1, \ldots, \alpha_{0,j}x_0 + x_j, -x_0^2 + \sum_{i=1}^{n} x_i^2, 0, \ldots, 0\right).\end{aligned}$$

We define the $(n+1) \times (n+1)$ matrix C by

$$(x_0, \alpha_{0,1}x_0 + x_1, \ldots, \alpha_{0,j}x_0 + x_j, x_{j+1}, \ldots, x_n) = (x_0, x_1, \ldots, x_n)C.$$

For any i $(1 \le i \le n)$, let $\mathbf{c}_i$ denote the $(i+1)$-th column vector of C. Since H_4^{-1} is a Lorentz transformation, from Lemma 4.2.1, the likeness of $\sum_{i=1}^{j} \mathbb{R}{}^t\mathbf{c}_i$ is the same as that of $V(p_0, \ldots, p_j)$. Furthermore, from Lemma 4.2.2, the likeness of $\sum_{i=1}^{j} \mathbb{R}{}^t\mathbf{c}_i$ is the same as that of $\sum_{i=1}^{n} \mathbb{R}{}^t\mathbf{c}_i$. Hence, the likeness of $V(p_0, \ldots, p_j)$ is the same as that of $\sum_{i=1}^{n} \mathbb{R}{}^t\mathbf{c}_i$. Note that the n-dimensional vector subspace $\sum_{i=1}^{n} \mathbb{R}{}^t\mathbf{c}_i \subset \mathbb{R}^{1,n}$ is given by $-x_0 + \alpha_{0,1}x_1 + \cdots + \alpha_{0,j}x_j = 0$. Thus, from Lemma 4.2.3, we get the following:

Lemma 4.3.1. *1.* $\sum_{i=1}^{j} \alpha_{0,i}^2 > 1$ *if and only if* $V(p_0, \ldots, p_k)$ *is time-like.*

2. $\sum_{i=1}^{j} \alpha_{0,i}^2 < 1$ *if and only if* $V(p_0, \ldots, p_k)$ *is space-like.*

3. $\sum_{i=1}^{j} \alpha_{0,i}^2 = 1$ *if and only if* $V(p_0, \ldots, p_k)$ *is light-like.*

Since C is a regular matrix, its inverse matrix C^{-1} is well-defined. Let $H_6 : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ be the linear isomorphism given by

$$H_6(x) = xC^{-1}.$$

By expressing C^{-1} explicitly, the composition of $H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4$ and H_6 is as follows:

$$\begin{aligned}
& (H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6)(x_0, x_1, \ldots, x_n) \\
= \ & \left(x_1, x_2, \ldots, x_j, -x_0^2 + \sum_{i=1}^{j} (-\alpha_{0,i} x_0 + x_i)^2 + \sum_{i=j+1}^{n} x_i^2, 0, \ldots, 0 \right).
\end{aligned}$$

<u>STEP 4.</u> This is the last step. Firstly, the cases 2(a) and 2(b) of Theorem 4.1.1 are shown. From Lemma 4.3.1, it follows that $-1 + \sum_{i=1}^{k} \alpha_{0,i}^2 \neq 0$ in these cases. Hence, by completing the square with respect to the variable x_0, we get the following:

$$\begin{aligned}
& -x_0^2 + \sum_{i=1}^{j} (-\alpha_{0,i} x_0 + x_i)^2 + \sum_{i=j+1}^{n} x_i^2 \\
= \ & \left(-1 + \sum_{i=1}^{j} \alpha_{0,i}^2 \right) x_0^2 - 2x_0 \sum_{i=1}^{j} \alpha_{0,i} x_i + \sum_{i=1}^{n} x_i^2 \\
= \ & \left(-1 + \sum_{i=1}^{j} \alpha_{0,i}^2 \right) \left(x_0 - \frac{\sum_{i=1}^{j} \alpha_{0,i} x_i}{-1 + \sum_{i=1}^{j} \alpha_{0,i}^2} \right)^2 - \frac{\left(\sum_{i=1}^{j} \alpha_{0,i} x_i \right)^2}{-1 + \sum_{i=1}^{j} \alpha_{0,i}^2} + \sum_{i=1}^{n} x_i^2.
\end{aligned}$$

Let $H_7 : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ be the diffeomorphism given by

$$\begin{aligned}
& H_7(X_0, X_1, \ldots, X_k) \\
= \ & \left(X_0, X_1, \ldots, X_{j-1}, X_j + \frac{\left(\sum_{i=1}^{j} \alpha_{0,i} X_{i-1} \right)^2}{-1 + \sum_{i=1}^{j} \alpha_{0,i}^2} - \sum_{i=1}^{j} X_{i-1}^2, X_{j+1}, \ldots, X_{k+1} \right).
\end{aligned}$$

By composing H_7 and $H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6$, we have

$$\begin{aligned} &(H_7 \circ H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6)(x_0, x_1, \ldots, x_n) \\ = &\left(x_1, \ldots, x_j, \left(-1 + \sum_{i=1}^{j} \alpha_{0,i}^2\right) \left(x_0 - \frac{\sum_{i=1}^{j} \alpha_{0,i} x_i}{-1 + \sum_{i=1}^{j} \alpha_{0,i}^2} \right)^2 + \sum_{i=j+1}^{n} x_i^2, 0, \ldots, 0 \right). \end{aligned}$$

Let $H_8 : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ be the linear isomorphism given by

$$\begin{aligned} &H_8(x_0, x_1, \ldots, x_n) \\ = &\left(\frac{x_0}{\sqrt{\mid -1 + \sum_{i=1}^{j} \alpha_{0,i}^2 \mid}} + \frac{\sum_{i=1}^{j} \alpha_{0,i} x_i}{-1 + \sum_{i=1}^{j} \alpha_{0,i}^2}, x_1, \ldots, x_n \right). \end{aligned}$$

From Lemma 4.3.1, if $V(p_0, \ldots, p_k)$ is time-like, the composition of $H_7 \circ H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6$ and H_8 is as follows:

$$\begin{aligned} &(H_7 \circ H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6 \circ H_8)(x_0, x_1, \ldots, x_n) \\ = &\left(x_1, \ldots, x_j, x_0^2 + \sum_{i=j+1}^{n} x_i^2, 0, \ldots, 0 \right), \end{aligned}$$

and if $V(p_0, \ldots, p_k)$ is space-like, the composition of them is as follows:

$$\begin{aligned} &(H_7 \circ H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6 \circ H_8)(x_0, x_1, \ldots, x_n) \\ = &\left(x_1, \ldots, x_j, -x_0^2 + \sum_{i=j+1}^{n} x_i^2, 0, \ldots, 0 \right). \end{aligned}$$

Next, the case 2(c) of Theorem 4.1.1 is shown. From Lemma 4.3.1, we get the following:

$$\begin{aligned} &(H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6)(x_0, x_1, \ldots, x_n) \\ = &\left(x_1, \ldots, x_j, -2x_0 \sum_{i=1}^{j} \alpha_{0,i} x_i + \sum_{i=1}^{n} x_i^2, 0, \ldots, 0 \right). \end{aligned}$$

Let $H_7' : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ be the diffeomorphism given by

$$\begin{aligned} & H_7'(X_0, X_1, \dots, X_k) \\ = & \left(X_0, X_1, \dots, X_{j-1}, X_j - \sum_{i=1}^{j} X_{i-1}^2, X_{j+1}, \dots, X_k \right). \end{aligned}$$

By composing H_7' and $H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6$, we have the following.

$$\begin{aligned} & (H_7' \circ H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6)(x_0, x_1, \dots, x_n) \\ = & \left(x_1, \dots, x_j, -2x_0 \sum_{i=1}^{j} \alpha_{0,i} x_i + \sum_{i=j+1}^{n} x_i^2, 0, \dots, 0 \right). \end{aligned}$$

By $-1 + \sum_{i=1}^{j} \alpha_{0,i}^2 = 0$, there must exist an $\tilde{i}$ $(1 \le \tilde{i} \le j)$ satisfying $\alpha_{0,\tilde{i}} \neq 0$. By taking a linear transformation of the source space if necessary, without loss of generality, from the first we may assume that $\alpha_{0,1} \neq 0$. We define the $(n+1) \times (n+1)$ regular matrix D by

$$\left(x_0, -2\sum_{i=1}^{j} \alpha_{0,i} x_i, x_2, \dots, x_n \right) = (x_0, x_1, \dots, x_n) D.$$

Let $H_8' : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ be the linear isomorphism given by

$$H_8'(x) = x D^{-1},$$

where D^{-1} is the inverse matrix of D. By expressing D^{-1} explicitly, the composition of $H_7' \circ H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6$ and H_8' is as follows:

$$\begin{aligned} & (H_7' \circ H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6 \circ H_8')(x_0, x_1, \dots, x_n) \\ = & \left(-\frac{x_1}{2\alpha_{0,1}} - \frac{\sum_{i=2}^{j} \alpha_{0,i} x_i}{\alpha_{0,1}}, x_2, \dots, x_j, x_0 x_1 + \sum_{i=j+1}^{n} x_i^2, 0, \dots, 0 \right). \end{aligned}$$

Let $H_9' : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ be the linear isomorphism given by

$$H_9'(X_0, X_1, \dots, X_k) = \left(-2\left(\alpha_{0,1} X_0 + \sum_{i=1}^{j-1} \alpha_{0,i+1} X_i \right), X_1, \dots, X_k \right).$$

By composing H'_9 and $H'_7 \circ H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6 \circ H'_8$, we get the following.

$$\begin{aligned} & (H'_9 \circ H'_7 \circ H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6 \circ H'_8)(x_0, x_1, \ldots, x_n) \\ = & \left(x_1, \ldots, x_j, x_0 x_1 + \sum_{i=j+1}^{n} x_i^2, 0, \ldots, 0 \right). \end{aligned}$$

□

The case $V(p_0, \ldots, p_k) \cap T = T$

The strategy of the proof in this case is the same as the one given in Section 4.3.2. By $\dim V(p_0, \ldots, p_j) = j$, it is clearly seen that the rank of $(n+1) \times j$ matrix A_2 is j, where A_2 is the matrix given in STEP 2 of Section 4.3.2. Furthermore, from $V(p_0, \ldots, p_k) \cap T = T$, there exists a $j \times j$ regular matrix B_4 such that the set of column vectors of $A_2 B_4$ is a subset of an orthonormal basis of $\mathbb{R}^{n+1}$ and the matrix $A_2 B_4$ has the following form:

$$A_2 B_4 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \beta_{1,2} & \cdots & \beta_{1,j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \beta_{n,2} & \cdots & \beta_{n,j} \end{pmatrix}.$$

Let $\widehat{H}_3 : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ be the linear transformation of $\mathbb{R}^{k+1}$ given by

$$\widehat{H}_3(X_0, X_1, \ldots, X_k) = ((X_0, X_1, \ldots, X_{j-1})B_4, X_j, \ldots, X_k).$$

We define the $(n+1) \times k$ matrix A_4 as follows:

$$\begin{aligned} & \left(\widehat{H}_3 \circ H_1 \circ L_p \circ H_2 \right)(x_0, x_1, \ldots, x_n) \\ = & \left((x_0, x_1, \ldots, x_n) A_4, -x_0^2 + \sum_{i=1}^{n} x_i^2 \right). \end{aligned}$$

Set $\mathbf{b}_i := (0, \beta_{1,i+1}, \ldots, \beta_{n,i+1})$ $(1 \le i \le j-1)$. Since $V(p_0, \ldots, p_k) = \mathbf{e}_1 + \sum_{i=1}^{j-1} \mathbb{R}\mathbf{b}_i$, there exists a $k \times k$ regular matrix B_5 satisfying $A_4 B_5 = (A_2 B_4, O_{n+1,k-j})$. By composing $\widehat{H}_3$ and the linear isomorphism of the target given by

$$(X_0, X_1, \ldots, X_k) \mapsto ((X_0, X_1, \ldots, X_{k-1})B_5, X_k)$$

if necessary, without loss of generality, from the first we may assume

$$\begin{aligned} & \left(\widehat{H}_3 \circ H_1 \circ L_p \circ H_2\right)(x_0, x_1, \ldots, x_n) \\ = & \left((x_0, x_1, \ldots, x_n)(A_2 B_4, O_{n+1,k-j}), -x_0^2 + \sum_{i=1}^{n} x_i^2\right). \end{aligned}$$

Set $\widetilde{\mathbf{b}}_i := (\beta_{1,i+1}, \ldots, \beta_{n,i+1})$ $(1 \le i \le j-1)$. We see that $\{\widetilde{\mathbf{b}}_1, \ldots, \widetilde{\mathbf{b}}_{j-1}\}$ is a subset of an orthonormal basis of $\mathbb{R}^n$. We can choose $\widetilde{\mathbf{b}}_j, \ldots, \widetilde{\mathbf{b}}_n$ such that the set $\{\widetilde{\mathbf{b}}_1, \ldots, \widetilde{\mathbf{b}}_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Notice that $({}^t\widetilde{\mathbf{b}}_1, \ldots, {}^t\widetilde{\mathbf{b}}_n)$ is an $n \times n$ orthogonal matrix. Let $\widehat{H}_4 : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ be the linear isomorphism given by

$$\widehat{H}_4(x_0, x_1, \ldots, x_n) = (x_0, (x_1, \ldots, x_n)({}^t\widetilde{\mathbf{b}}_1, \ldots, {}^t\widetilde{\mathbf{b}}_n)^{-1}).$$

By composing $\widehat{H}_3 \circ H_1 \circ L_p \circ H_2$ and $\widehat{H}_4$, we get the following.

$$\begin{aligned} & (\widehat{H}_3 \circ H_1 \circ L_p \circ H_2 \circ \widehat{H}_4)(x_0, x_1, \ldots, x_n) \\ = & \left(x_0, x_1, \ldots, x_{j-1}, 0, \ldots, 0, -x_0^2 + \sum_{i=1}^{n} x_i^2\right). \end{aligned}$$

By composing H_5 with $\widehat{H}_3 \circ H_1 \circ L_p \circ H_2 \circ \widehat{H}_4$, we get

$$\begin{aligned} & (H_5 \circ \widehat{H}_3 \circ H_1 \circ L_p \circ H_2 \circ \widehat{H}_4)(x_0, x_1, \ldots, x_n) \\ = & \left(x_0, x_1, \ldots, x_{j-1}, -x_0^2 + \sum_{i=1}^{n} x_i^2, 0, \ldots, 0\right). \end{aligned}$$

In order to remove terms $x_0^2, \dots, x_{j-1}^2$ in the $(j+1)$-th component, we construct the diffeomorphism of the target space $\widehat{H}_5 : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ below:

$$\begin{aligned}&\widehat{H}_5(X_0, X_1, \dots, X_k)\\ = &\left(X_0, X_1, \dots, X_{j-1}, X_j + X_0^2 - \sum_{i=1}^{j-1} X_i^2, X_{j+1}, \dots, X_{k+1}\right).\end{aligned}$$

By composing $\widehat{H}_5$ and $H_5 \circ \widehat{H}_3 \circ H_1 \circ L_p \circ H_2 \circ \widehat{H}_4$, we get the following.

$$\begin{aligned}&(\widehat{H}_5 \circ H_5 \circ \widehat{H}_3 \circ H_1 \circ L_p \circ H_2 \circ \widehat{H}_4)(x_0, x_1, \dots, x_n)\\ = &\left(x_0, x_1, \dots, x_{j-1}, \sum_{i=j}^{n} x_i^2, 0, \dots, 0\right).\end{aligned}$$

Let $\widehat{H}_6 : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ be the linear isomorphism given by

$$\widehat{H}_6(x_0, x_1, \dots, x_n) = (x_1, x_2, \dots, x_j, x_0, x_{j+1}, \dots, x_n).$$

By composing $\widehat{H}_5 \circ H_5 \circ \widehat{H}_3 \circ H_1 \circ L_p \circ H_2 \circ \widehat{H}_4$ and $\widehat{H}_6$, we get the following.

$$\begin{aligned}&(\widehat{H}_5 \circ H_5 \circ \widehat{H}_3 \circ H_1 \circ L_p \circ H_2 \circ \widehat{H}_4 \circ \widehat{H}_6)(x_0, x_1, \dots, x_n)\\ = &\left(x_1, \dots, x_j, x_0^2 + \sum_{i=j+1}^{n} x_i^2, 0, \dots, 0\right).\end{aligned}$$

□

4.3.3 Proof of 3 of Theorem 4.1.1

The strategy of the proof of 3 of Theorem 4.1.1 is the same as the strategy of the proof of 2 of Theorem 4.1.1. In the case that $V(p_0, \dots, p_k)$ is space-like, compose the mapping $H_7 \circ H_5 \circ H_3 \circ H_1 \circ L_p \circ H_2 \circ H_4 \circ H_6 \circ H_8$ and the linear coordinate transformation of the target

$$(X_0, X_1, \dots, X_k) \mapsto (X_0, \dots, X_{j-1}, -X_j, X_{j+1}, \dots, X_k).$$

□

4.3.4 Proof of 4 of Theorem 4.1.1

The strategy of the proof of 4 of Theorem 4.1.1 is the same as the strategy of the proof of 2 of Theorem 4.1.1. In this case, since the rank of the $(n+1)\times k$ matrix A_1 given in STEP 2 of Section 4.3.2 is $n+1$, there exists a $k\times k$ regular matrix B_6 satisfying $A_1B_6=(E_{n+1},O_{n+1,k-(n+1)})$. By composing the target diffeomorphism

$$(X_0,X_1,\ldots,X_k)\mapsto((X_0,X_1,\ldots,X_{k-1})B_6,X_k)$$

and $H_1\circ L_p\circ H_2$ which appeared in STEP 2 of Section 4.3.2, we get

$$\begin{aligned}(x_0,x_1,\ldots,x_n) &\mapsto \left((x_0,x_1,\ldots,x_n)A_1B_6,-x_0^2+\sum_{i=1}^n x_i^2\right)\\ &= \left(x_0,x_1,\ldots,x_n,0,\ldots,0,-x_0^2+\sum_{i=1}^n x_i^2\right),\end{aligned}$$

which is clearly $\mathcal{A}$-equivalent to

$$(x_0,x_1,\ldots,x_n)\mapsto(x_0,x_1,\ldots,x_n,0,\ldots,0).$$

□

CHAPTER 5

Generic generalized distance-squared mappings

5.1 Introduction

Let $p_i = (p_{i1}, p_{i2}, \ldots, p_{im})$ $(1 \le i \le \ell)$ (resp., $A = (a_{ij})_{1\le i\le \ell, 1\le j\le m}$) be points of $\mathbb{R}^m$ (resp., an $\ell \times m$ matrix with all entries being non-zero real numbers). Set $p = (p_1, p_2, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell$. Let $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$ be the mapping defined by

$$G_{(p,A)}(x) = \left(\sum_{j=1}^{m} a_{1j}(x_j - p_{1j})^2, \sum_{j=1}^{m} a_{2j}(x_j - p_{2j})^2, \ldots, \sum_{j=1}^{m} a_{\ell j}(x_j - p_{\ell j})^2 \right),$$

where $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. The mapping $G_{(p,A)}$ is called a *generalized distance-squared mapping*, and the ℓ-tuple of points $p = (p_1, p_2, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell$ is called the *central point* of the generalized distance-squared mapping $G_{(p,A)}$. Note that a *distance-squared mapping* D_p defined in Chapter 3 (resp., a *Lorentzian distance-squared mapping* L_p defined in Chapter 4) is the mapping $G_{(p,A)}$ such that each entry of A is equal to 1 (resp., $a_{i1} = -1$ and $a_{ij} = 1$ $(j \ne 1)$).

In [13], a characterization of generic generalized distance-squared mappings of the plane into the plane is investigated. If the rank of A with all entries being non-zero real numbers is equal to two, then a generalized distance-squared mapping having a generic central point is a stable mapping of which any singular point is a fold point except one cusp point. If the rank of A with all entries being non-zero real numbers is equal to one, then a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to $(x_1, x_2) \mapsto (x_1, x_2^2)$.

In [11], a characterization of generic generalized distance-squared mappings of $\mathbb{R}^{m+1}$ into $\mathbb{R}^{2m+1}$ is investigated. If the rank of A with all entries being non-zero real numbers is equal to $m+1$, then a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to the normal form of Whitney umbrella

$$(x_1, x_2, \ldots, x_{m+1}) \mapsto (x_1^2, x_1x_2, \ldots, x_1x_{m+1}, x_2, \ldots, x_{m+1}).$$

If the rank of A with all entries being non-zero real numbers is strictly smaller than $m+1$, then a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to the inclusion

$$(x_1, x_2, \ldots, x_{m+1}) \mapsto (x_1, x_2, \ldots, x_{m+1}, 0, \ldots, 0).$$

Hence, by the results in [11] (resp., [13]), for a given 2×2 matrix A (resp., $(2m+1)\times(m+1)$ matrix A) with all entries being non-zero real numbers, there exists a subset $\Sigma_1 \subset (\mathbb{R}^2)^2$ (resp., $\Sigma_2 \subset (\mathbb{R}^{m+1})^{2m+1}$) with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^2)^2 - \Sigma_1$ (resp., $p \in (\mathbb{R}^{m+1})^{2m+1} - \Sigma_2$), the mapping $G_{(p,A)} : \mathbb{R}^2 \to \mathbb{R}^2$ (resp., $G_{(p,A)} : \mathbb{R}^{m+1} \to \mathbb{R}^{2m+1}$) is stable. On the other hand, in Chapter 5, by applying some assertions prepared in Chapter 2 to generalized distance-squared mappings, in various dimension pairs, properties of generalized distance-squared mappings having a generic central point are investigated.

5.2 Applications of Theorem 2.1.1 to $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$

Proposition 5.2.1 ([7]). *Let $f : N \to \mathbb{R}^m$ be an immersion, where N is a manifold of dimension n. Let $A = (a_{ij})_{1\le i\le \ell, 1\le j\le m}$ be an $\ell \times m$ matrix with all entries being non-*

zero real numbers. If X is an $\mathcal{A}^1$-invariant submanifold of $J^1(n,\ell)$, then there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $j^1(G_{(p,A)} \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $X(N, \mathbb{R}^\ell)$.

Proof. Let $H : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be a diffeomorphism of the target for deleting constant terms. Then, we have the following.

$$\begin{aligned} H \circ G_{(p,A)}(x) &= \left(\sum_{j=1}^m a_{1j}x_j^2 - 2\sum_{j=1}^m a_{1j}p_{1j}x_j, \sum_{j=1}^m a_{2j}x_j^2 - 2\sum_{j=1}^m a_{2j}p_{2j}x_j, \right. \\ &\quad \left. \ldots, \sum_{j=1}^m a_{\ell j}x_j^2 - 2\sum_{j=1}^m a_{\ell j}p_{\ell j}x_j \right), \end{aligned}$$

where $x = (x_1, x_2, \ldots, x_m)$.

Let $\psi : (\mathbb{R}^m)^\ell \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the mapping given by

$$\psi(p_{11}, p_{12}, \ldots, p_{\ell m}) = -2(a_{11}p_{11}, a_{12}p_{12}, \ldots, a_{\ell m}p_{\ell m}).$$

Notice that we have the natural identification $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$. Since $a_{ij} \neq 0$ for any i, j ($1 \leq i \leq \ell$, $1 \leq j \leq m$), it is not hard to see that ψ is a C^∞ diffeomorphism.

Set $F_i(x) = \sum_{j=1}^m a_{ij}x_j^2$ ($1 \leq i \leq \ell$) and $F = (F_1, F_2, \ldots, F_\ell)$. From Theorem 2.1.1, there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $X(N, \mathbb{R}^\ell)$. Since $\psi^{-1} : \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \to (\mathbb{R}^m)^\ell$ is a C^∞ mapping, $\psi^{-1}(\Sigma)$ has Lebesgue measure 0 in $(\mathbb{R}^m)^\ell$. For any $p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma)$, we get $\psi(p) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$. Thus, for any $p \in (\mathbb{R}^m)^\ell - \psi^{-1}(\Sigma)$, the mapping $j^1(H \circ G_{(p,A)} \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $X(N, \mathbb{R}^\ell)$. Then, since $H : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is a diffeomorphism, $j^1(G_{(p,A)} \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to the submanifold $X(N, \mathbb{R}^\ell)$. □

As applications of Proposition 5.2.1, we have analogies of Proposition 2.1.1, Corollaries 2.1.1, 2.1.2, 2.1.3 and 2.1.4 as follows:

Corollary 5.2.1 ([7]). *Let $f : N \to \mathbb{R}^m$ be an immersion, where N is a manifold of dimension n. Let $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ be an $\ell \times m$ matrix with all entries being non-zero* *real numbers. Then, there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $j^1(G_{(p,A)} \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$ for any integer k satisfying $1 \leq k \leq v$. Especially, in the case of $\ell \geq 2$, we get $k_0 + 1 \leq v$ and it follows that $j^1(G_{(p,A)} \circ f)$ satisfies that $j^1(G_{(p,A)} \circ f)(N) \bigcap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ for any positive integer k satisfying $k_0 + 1 \leq k \leq v$, where k_0 is the maximum integer satisfying $(n - v + k_0)(\ell - v + k_0) \leq n$ $(v = \min\{n, \ell\})$.*

Corollary 5.2.2 ([7]). *Let $f : N \to \mathbb{R}^m$ be an immersion, where N is a manifold of dimension n. Let $A = (a_{1j})_{1 \leq j \leq m}$ be a $1 \times m$ matrix with all entries being non-zero real numbers. Then, there exists a subset $\Sigma \subset \mathbb{R}^m$ with Lebesgue measure 0 such that for any $p \in \mathbb{R}^m - \Sigma$, the mapping $G_{(p,A)} \circ f : N \to \mathbb{R}$ is a Morse function.*

Corollary 5.2.3 ([7]). *Let $f : N \to \mathbb{R}^m$ be an immersion, where N is a manifold of dimension n $(n \geq 2)$. Let $A = (a_{ij})_{1 \leq i \leq 2n-1, 1 \leq j \leq m}$ be a $(2n-1) \times m$ matrix with all entries being non-zero real numbers. Then, there exists a subset $\Sigma \subset (\mathbb{R}^m)^{2n-1}$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^{2n-1} - \Sigma$, any singular point of the mapping $G_{(p,A)} \circ f : N \to \mathbb{R}^{2n-1}$ is a singular point of Whitney umbrella.*

Corollary 5.2.4 ([7]). *Let $f : N \to \mathbb{R}^m$ be an immersion, where N is a manifold of dimension n. Let $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ be an $\ell \times m$ matrix with all entries being non-zero real numbers $(\ell \geq 2n)$. Then, there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is an immersion.*

Remark 5.2.1. In the case of $\ell = m \geq 2n$, Corollary 5.2.4 is Theorem 1 of [12].

Corollary 5.2.5 ([7]). *Let $f : N \to \mathbb{R}^m$ be an immersion, where N is a manifold of dimension n. Let $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ be an $\ell \times m$ matrix with all entries being non-zero real numbers. Let k_0 be the maximum integer satisfying $(n - v + k_0)(\ell - v + k_0) \leq n$ $(v = \min\{n, \ell\})$. Then, there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ has corank at most k_0 singular points.*

5.3 Applications of Theorem 2.1.2 to $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$

From Theorem 2.1.2, we get the following proposition, which can be proved by the same argument as in the proof of Proposition 5.2.1, and we omit the proof.

Proposition 5.3.1 ([7]). *Let $f : N \to \mathbb{R}^m$ be an injection, where N is a manifold of dimension n. Let $A = (a_{ij})_{1\le i\le \ell, 1\le j\le m}$ be an $\ell \times m$ matrix with all entries being non-zero real numbers. Then, there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma$, and for any s $(2 \le s \le s_f)$, the mapping $(G_{(p,A)} \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to Δ_s. Moreover, if the mapping $G_{(p,A)}$ satisfies that $|G^{-1}_{(p,A)}(y)| \le s_f$ for any $y \in \mathbb{R}^\ell$, then $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is a mapping with normal crossings.*

As applications of Proposition 5.3.1, we have analogies of Proposition 2.1.2, Corollaries 2.1.5, 2.1.6 and 2.1.7.

Corollary 5.3.1. *Let $f : N \to \mathbb{R}^m$ be an injection, where N is a manifold of dimension n. Let $A = (a_{ij})_{1\le i\le \ell, 1\le j\le m}$ be an $\ell \times m$ matrix with all entries being non-zero real numbers. If $(s_f - 1)\ell > ns_f$, then there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is a mapping with normal crossings satisfying $(G_{(p,A)} \circ f)^{(s_f)}(N^{(s_f)}) \bigcap \Delta_{s_f} = \emptyset$.*

Corollary 5.3.2 ([7]). *Let $f : N \to \mathbb{R}^m$ be an injection, where N is a manifold of dimension n. Let $A = (a_{ij})_{1\le i\le \ell, 1\le j\le m}$ be an $\ell \times m$ matrix with all entries being non-zero real numbers. If $\ell > 2n$, then there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is an injection.*

Remark 5.3.1. In the case of $\ell = m \ge 2n + 1$, Corollary 5.3.2 is Theorem 2 of [12].

Corollary 5.3.3 ([7]). *Let $f : N \to \mathbb{R}^m$ be an injective immersion, where N is a manifold of dimension n. Let $A = (a_{ij})_{1\le i\le \ell, 1\le j\le m}$ be an $\ell \times m$ matrix with all entries being non-zero real numbers. If $\ell > 2n$, then there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is an injective immersion.*

Corollary 5.3.4 ([7]). *Let $f : N \to \mathbb{R}^m$ be an embedding, where N is a compact manifold* *of dimension n. Let $A = (a_{ij})_{1\le i\le \ell, 1\le j\le m}$ be an $\ell \times m$ matrix with all entries being non-zero real numbers. If $\ell > 2n$, then there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is an embedding.*

From Theorem 3.1.1 in Chapter 3 and Theorem 4.1.1 in Chapter 4, as the special case of the characterization of generic distance squared mappings (resp., generic Lorentzian distance-squared mappings), we get the following.

Lemma 5.3.1 ([9], [10]). *We have the following.*

1. *For any $p \in \mathbb{R}$, the mappings $D_p : \mathbb{R} \to \mathbb{R}$ and $L_p : \mathbb{R} \to \mathbb{R}$ are $\mathcal{A}$-equivalent to $x \mapsto x^2$.*

2. *For $m \ge 2$, there exists a subset Σ_D (resp., Σ_L) of $(\mathbb{R}^m)^m$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^m - \Sigma_D$ (resp., $p \in (\mathbb{R}^m)^m - \Sigma_L$), the mapping $D_p : \mathbb{R}^m \to \mathbb{R}^m$ (resp., $L_p : \mathbb{R}^m \to \mathbb{R}^m$) is $\mathcal{A}$-equivalent to $(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_{m-1}, x_m^2)$.*

3. *In the case of $1 \le m < \ell$, there exists a subset Σ_D (resp., Σ_L) of $(\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma_D$ (resp., $p \in (\mathbb{R}^m)^\ell - \Sigma_L$), the mapping $D_p : \mathbb{R}^m \to \mathbb{R}^\ell$ (resp., $L_p : \mathbb{R}^m \to \mathbb{R}^\ell$) is $\mathcal{A}$-equivalent to $(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_m, 0, \ldots, 0)$.*

Proposition 5.3.2 ([7]). *Let $f : N \to \mathbb{R}^m$ be an injection, where N is a manifold of dimension n. Then, we have the following.*

1. *For $m \ge 1$, there exists a subset Σ_D (resp., Σ_L) of $(\mathbb{R}^m)^m$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^m - \Sigma_D$ (resp., $p \in (\mathbb{R}^m)^m - \Sigma_L$), the mapping $D_p \circ f : N \to \mathbb{R}^m$ (resp., $L_p \circ f : N \to \mathbb{R}^m$) is a mapping with normal crossings.*

2. *In the case of $1 \le m < \ell$, there exists a subset Σ_D (resp., Σ_L) of $(\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma_D$ (resp., $p \in (\mathbb{R}^m)^\ell - \Sigma_L$), the mapping $D_p \circ f : N \to \mathbb{R}^\ell$ (resp., $L_p \circ f : N \to \mathbb{R}^\ell$) is injective.*

Proof. The proof for distance-squared mappings is the same as that for Lorentzian distance-squared mappings. Thus, it is sufficient to give the proof for distance-squared mappings.

Firstly, we will prove the assertion 1. From Lemma 5.3.1, there exists a subset $\Sigma_1 \subset (\mathbb{R}^m)^m$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^m - \Sigma_1$, the mapping $D_p : \mathbb{R}^m \to \mathbb{R}^m$

satisfies that $|D_p^{-1}(y)| \leq 2$ for any $y \in \mathbb{R}^m$. On the other hand, from Proposition 5.3.1, there exists a subset $\Sigma_2 \subset (\mathbb{R}^m)^m$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^m - \Sigma_2$, if the mapping D_p satisfies that $|D_p^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^m$, then the composition $D_p \circ f : N \to \mathbb{R}^m$ is a mapping with normal crossings. Set $\Sigma_D = \Sigma_1 \cup \Sigma_2$. It is not hard to see that Σ_D has Lebesgue measure 0 in $(\mathbb{R}^m)^m$. Then, for any $p \in (\mathbb{R}^m)^m - \Sigma_D$, the mapping $D_p \circ f : N \to \mathbb{R}^m$ is a mapping with normal crossings.

In the case of $m < \ell$, since from Lemma 5.3.1, there exists a subset $\Sigma_D \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^\ell - \Sigma_D$, the mapping $D_p : \mathbb{R}^m \to \mathbb{R}^\ell$ is $\mathcal{A}$-equivalent to the inclusion, we get the assertion 2. □

By combining Proposition 5.3.2 and Corollary 5.2.4, we get the following.

Corollary 5.3.5 ([7]). *Let $f : N \to \mathbb{R}^m$ be an injective immersion, where N be a manifold of dimension n $(2n \leq m)$. Then, there exists a subset Σ_D (resp., Σ_L) of $(\mathbb{R}^m)^m$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^m)^m - \Sigma_D$ (resp., $p \in (\mathbb{R}^m)^m - \Sigma_L$), the mapping $D_p \circ f : N \to \mathbb{R}^m$ (resp., $L_p \circ f : N \to \mathbb{R}^m$) is an immersion with normal crossings.*

In Corollary 5.3.5, if $m = 2n$ and the mapping $D_p \circ f : N \to \mathbb{R}^{2n}$ (resp., $L_p \circ f : N \to \mathbb{R}^{2n}$) is proper, then the immersion with normal crossings $D_p \circ f : N \to \mathbb{R}^{2n}$ (resp., $L_p \circ f : N \to \mathbb{R}^{2n}$) is necessarily stable. Hence, we have the following assertion.

Corollary 5.3.6 ([7]). *Let $f : N \to \mathbb{R}^{2n}$ be an embedding, where N is a compact manifold of dimension n. Then, there exists a subset Σ_D (resp., Σ_L) of $(\mathbb{R}^{2n})^{2n}$ with Lebesgue measure 0 such that for any $p \in (\mathbb{R}^{2n})^{2n} - \Sigma_D$ (resp., $p \in (\mathbb{R}^{2n})^{2n} - \Sigma_L$), the mapping $D_p \circ f : N \to \mathbb{R}^{2n}$ (resp., $L_p \circ f : N \to \mathbb{R}^{2n}$) is stable.*

5.4 Applications of Theorem 2.2.1 to $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$

From Theorem 2.2.2, we get the following proposition, which can be proved by the same argument as in the proof of Proposition 5.2.1, and we omit the proof.

Proposition 5.4.1 ([8]). *Let $f : N \to \mathbb{R}^m$ be an embedding, where N is a manifold of dimension n. Let $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ be an $\ell \times m$ matrix with non-zero entries. If W is a modular submanifold of ${}_sJ^r(N, \mathbb{R}^\ell)$, then there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue* *measure* 0 *such that for any* $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell - \Sigma$, $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ *is transverse with respect to the modular submanifold* W.

From Proposition 5.4.1 and the same method as that of Mather, we get the following application.

Corollary 5.4.1. *Let* $f : N \to \mathbb{R}^m$ *be an embedding, where* N *be a manifold of dimension* n*. Let* $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ *be an* $\ell \times m$ *matrix with non-zero entries. If* (n, ℓ) *is in the nice dimensions, then there exists a subset* $\Sigma \subset (\mathbb{R}^m)^\ell$ *with Lebesgue measure* 0 *such that for any* $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell - \Sigma$*, the composition* $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ *is locally infinitesimally stable.*

Remark 5.4.1.

1. Suppose that the mapping $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ is proper in Corollary 5.4.1. Then, the local infinitesimal stability of $G_{(p,A)} \circ f$ implies the stability of it (see [19]). Namely, we have the following.

   Corollary 5.4.2. *Let* $f : N \to \mathbb{R}^m$ *be an embedding, where* N *be a compact manifold of dimension* n*. Let* $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ *be an* $\ell \times m$ *matrix with non-zero entries. If* (n, ℓ) *is in the nice dimensions, then there exists a subset* $\Sigma \subset (\mathbb{R}^m)^\ell$ *with Lebesgue measure* 0 *such that for any* $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell - \Sigma$*, the composition* $G_{(p,A)} \circ f : N \to \mathbb{R}^\ell$ *is stable.*

2. Suppose that $N = \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^m$ is the identify. Let $A = (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq m}$ be an $\ell \times m$ matrix with non-zero entries. From Corollary 5.4.1, it is clearly seen that if (m, ℓ) is in the nice dimensions, then there exists a subset $\Sigma \subset (\mathbb{R}^m)^\ell$ with Lebesgue measure 0 such that for any $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell - \Sigma$, the mapping $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$ is locally infinitesimally stable.

CHAPTER 6

Future research

In Chapter 6, some conjectures on which the author is working are introduced.

6.1 An improvement of "Generic projections"

As in Chapter 2, Mather showed Theorem 2.2.1. On the other hand, in [5], an improvement of Theorem 2.2.1 is given as follows:

Theorem 6.1.1 ([5]). *Let $f : N \to \mathbb{R}^m$ be a stable mapping, where N is a manifold of dimension n. If W is a modular submanifold of ${}_sJ^r(N, \mathbb{R}^\ell)$ and $m > \ell$, then there exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure 0 such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.*

As a further improvement of Theorems 2.2.1 and 6.1.1, the following problem seems to be significant.

Problem 6.1.1. *Let N be a manifold of dimension n and let W be a modular submanifold of ${}_sJ^r(N, \mathbb{R}^\ell)$. What is the condition $(*)$ satisfying that (α) and (β) are equivalent in the case of $m > \ell$?*

(α) *A given mapping $f : N \to \mathbb{R}^m$ satisfies the condition* $(*)$.

(β) *There exists a subset $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure* 0 *such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is transverse with respect to W.*

6.2 $\mathcal{A}$-equivalence classes of generic projections

Theorem 1.0.2 is a striking result about stable mappings. As a future research, we would like to consider $\mathcal{A}$-equivalence classes of the stable mappings in Theorem 1.0.2. As a problem on which the author is working, we introduce the following.

Problem 6.2.1. *Let N be a compact manifold of dimension n. Let $f : N \to \mathbb{R}^m$ be an embedding. If (n, ℓ) is in the nice dimensions and $m > \ell$, then is the number of $\mathcal{A}$-equivalence classes of stable mappings $\pi \circ f : N \to \mathbb{R}^\ell$ $(\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell))$ finite?*

It appears difficult to investigate Problem 6.2.1. Hence, as the first step for considering Problem 6.2.1, we would like to consider the following conjecture. Namely, we consider the case that a given manifold N (resp., a given mapping $f : N \to \mathbb{R}^m$) in Problem 6.2.1 is replaced by a Nash manifold (resp., a Nash mapping). For the definitions of Nash manifolds and Nash mappings, see Section 6.3 (for the details, see for example, [24]).

Conjecture 6.2.1. *Let N be a compact Nash manifold of dimension n. Let $f : N \to \mathbb{R}^m$ be a Nash embedding. If (n, ℓ) is in the nice dimensions and $m > \ell$, then the number of $\mathcal{A}$-equivalence classes of stable mappings $\pi \circ f : N \to \mathbb{R}^\ell$ $(\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell))$ is finite.*

For Conjecture 6.2.1, the author expects that it is important to prove Conjectures 6.2.2 and 6.2.3.

Conjecture 6.2.2. *Let N be a compact Nash manifold of dimension n. Let $f : N \to \mathbb{R}^m$ be a Nash embedding. If (n, ℓ) is in the nice dimensions and $m > \ell$, then there exists a semialgebraic set $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure* 0 *such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $\pi \circ f : N \to \mathbb{R}^\ell$ is stable.*

Conjecture 6.2.3. *Let $f : N \to \mathbb{R}^m$ be an embedding, where N is a compact manifold. Let $\Phi : \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \to C^\infty(N, \mathbb{R}^\ell)$ be the mapping given by $\Phi(\pi) = \pi \circ f$. Then, Φ is continuous.*

If Conjectures 6.2.2 and 6.2.3 are true, we can prove Conjecture 6.2.1 by the following argument.

Let $\Sigma \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ be the set consisting of linear mappings $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that $\pi \circ f$ is not stable. Then, from Conjecture 6.2.2, the set Σ has Lebesgue measure 0 and is semialgebraic in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$. From Conjecture 6.2.3 and the definition of a stable mapping, it is clearly seen that $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ is open. Moreover, since Σ is semialgebraic in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$, the set $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ is also semialgebraic in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$. Hence, the number of the connected components of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ is finite. Let C be a connected component of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$. Then, it is easily seen that for any $\pi, \pi' \in C$, $\pi \circ f$ is $\mathcal{A}$-equivalent to $\pi' \circ f$. Indeed, the proof is given by the following argument. For $\pi, \pi' \in C$, we write $\pi \sim \pi'$ if $\pi \circ f$ is $\mathcal{A}$-equivalent to $\pi' \circ f$. Clearly, the relation $\sim$ on C is an equivalence relation. By $[\pi]$, we denote the equivalence class of $\pi \in C$. Since $\Phi : \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \to C^\infty(N, \mathbb{R}^\ell)$ is continuous and $C \subset \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, it is not hard to see that for any $\pi \in C$, the equivalence class $[\pi]$ is open in C. We can express $C = \cup_{\lambda \in \Lambda} [\pi_\lambda]$, where $[\pi_\lambda] \cap [\pi_{\lambda'}] = \emptyset$ for any $\lambda, \lambda' \in \Lambda$ $(\lambda \neq \lambda')$. Since the set C is connected, it is clearly seen that the set Λ has only one element. Thus, for any $\pi, \pi' \in C$, $\pi \circ f$ is $\mathcal{A}$-equivalent to $\pi' \circ f$.

Therefore, the number of $\mathcal{A}$-equivalence classes of mappings $\pi \circ f : N \to \mathbb{R}^\ell$ $(\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma)$ is equal to or less than the number of the connected components of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$. Since the number of the connected components of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ is finite, the number of $\mathcal{A}$-equivalence classes of mappings $\pi \circ f : N \to \mathbb{R}^\ell$ $(\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma)$ is also finite.

6.3 Appendix

In Section 6.3, as a appendix, we prepare the definition of Nash manifolds and Nash mappings (for the details on Nash manifolds and Nash mappings, see for example, [24]).

We say that $X \subset \mathbb{R}^n$ is a *semialgebraic subset* of $\mathbb{R}^n$ if X is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_k(x) = 0, g_1(x) > 0, \ldots, g_\ell(x) > 0\},$$

where $f_1, \ldots, f_k, g_1, \ldots, g_\ell$ are polynomial functions on $\mathbb{R}^n$. Let U and V be open semialgebraic subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. We call a mapping $f : U \to V$ a *Nash mapping* if the graph of the mapping f is semialgebraic in $\mathbb{R}^n \times \mathbb{R}^m$. A *Nash manifold* of dimension m is a manifold with a finite system of coordinate neighborhoods $\{\psi_i : U_i \to \mathbb{R}^m\}$ such that for each i and j, $\psi_i(U_i \cap U_j)$ is an open semialgebraic subset of $\mathbb{R}^m$ and the mapping

$$\psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$$

is a Nash diffeomorphism. Then, we call such coordinate neighborhoods *Nash coordinate neighborhoods*. Let N and M be Nash manifolds. We say that $f : N \to M$ is a *Nash mapping* if for every Nash coordinate neighborhoods $\psi_i : U_i \to \mathbb{R}^n$ and $\varphi_j : V_j \to \mathbb{R}^m$ of N and M, respectively, $\psi_i(f^{-1}(V_j) \cap U_i)$ is semialgebraic and open in $\mathbb{R}^n$, and the mapping

$$\varphi_j \circ f \circ \psi_i^{-1} : \psi_i(f^{-1}(V_j) \cap U_i) \to \mathbb{R}^m$$

is a Nash mapping.

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