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3-CONNECTED PLANAR GRAPHS ARE 2-DISTINGUISHABLE WITH FEW EXCEPTIONS

By

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Abstract. A graph is said to be 2-*distinguishable* if there is a subset $S \subset V(G)$ such that any automorphism $\sigma : G \to G$ with $\sigma(S) = S$ must be the identity map over G. We shall prove that every 3-connected planar graph is 2-distinguishable, except K_4, $K_{2,2,2}$, Q_3, W_4, W_5, $C_3 + \overline{K}_2$ and $C_5 + \overline{K}_2$.

Introduction

Let G be a simple graph with n vertices. An assignment $f : V(G) \to \{1, 2, \ldots, r\}$ is called an r-*distinguishing labeling* if no automorphism $\sigma : G \to G$, except the identity map id_G, preserves the labels of vertices assigned by f. That is, for an r-distinguishing labeling f, the condition that $f(\sigma(v)) = f(v)$ for all $v \in V(G)$ implies that $\sigma(v) = v$ for all $v \in V(G)$. A graph G is said to be r-*distinguishable* if G has an r-distinguishing labeling.

Since a usual vertex labeling of G with labels $1, \ldots, n$ can be regarded as an n-distinguishing labeling, there exists actually such a number r that G is r-distinguishable, as one not exceeding $n = |V(G)|$. Thus, we can consider the minimum of those numbers and call it the *distinguishing number* of G. This is denoted by $D(G)$. For example, $D(C_3) = D(C_4) = D(C_5) = 3$ and $D(C_n) = 2$ for $n \geq 6$, where C_n stands for a cycle of length n.

There have been several papers [1, 2, 5] on this topic, and they suggest that most of graphs in suitable classes seem to be 2-distinguishable as well as the above example of cycles, often called "Key Ring Problem". (Of course, there is a series of graphs whose distinguishing numbers increase arbitrarily, say $D(K_n) = n$.) It should be noticed that a graph G is 2-distinguishable if and only if there is a subset $S \subset V(G)$ such that $\sigma(S) = S$ for an automorphism σ implies that $\sigma = \mathrm{id}_G$.

To show a similar phenomenon, we would like to study the distinguishing

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numbers of graphs with an aspect of topological graph theory. That is, we shall discuss those graphs that are embedded on surfaces. In particular, we shall focus on "planar graphs", which can be embedded on the plane or on the sphere and prove the following theorem in this paper:

Theorem 1. *Every 3-connected planar graph is 2-distinguishable, except* K_4, $K_{2,2,2}$, Q_3, W_4, W_5, $C_3 + \overline{K}_2$ *and* $C_5 + \overline{K}_2$.

The exceptions K_4, $K_{2,2,2}$, Q_3 and W_n with $n = 4, 5$ are the *complete graph* over four vertices, the *complete tripartite graph* with partite sets of size 2, 2 and 2, the 3-*cube* and the *wheels* with rims of length 4 and 5 and they can be embedded on the sphere as the tetrahedron, the octahedron, the cube, and the pyramids with rectangular and pentagonal bases. The graph $C_n + \overline{K}_2$ consists of the cycle C_n of length n and two extra vertices each of which is adjacent to all vertices on C_n and can be embedded on the sphere so that C_n lies along the equator and the two extra vertices are placed at the North and South Poles. This is often called the *double wheel* with rim of length n. It is not difficult to see that $D(K_4) = 4$ and $D(G) = 3$ for the other exceptions.

More generally, we may discuss the distinguishing number of graphs G embedded on surfaces in two ways. The first way is to restrict the automorphisms within those as maps. That is, we consider only those automorphisms of G that send the boundary walks of faces to those, to evaluate its distinguishing number $D(G)$. For example, choose three vertices u, v and w so that uvw forms a corner of a face of G and $\deg v = 3$, and define $f(u) = 1$, $f(v) = 2$, $f(w) = 3$ and $f(x) = 4$ for the other vertices in G. Then f will be a distinguishing labeling of G, which implies that $D(G) \leq 4$.

In the second way, we discuss the distinguishing number of graphs just as abstract graphs, taking account of properties of their embeddings on surfaces. There is no upper bound for $D(G)$ in this case. For example, let G be a graph embeddable on a closed surface F^2 and add an independent set of n extra vertices to G so that all n vertices are adjacent to a common vertex. The resulting graph G' also is embeddable on F^2 and $D(G') \geq n$ since the extra n vertices have all distinct labels in any distinguishing labeling of G'. Thus, we need some restriction on the properties of graphs G or their embeddings to establish a theorem showing an upper bound for $D(G)$ with few exceptions.

Recently, Tucker [6] has discussed in this way, and proved that $D(G) \leq 2$ for most of graphs on surfaces as maps, classifying those graphs on surfaces with $D(G) = 3$. Our main theorem, Theorem 1, is formally included in a part of his big result since any automorphism of a 3-connected planar graph extends to a map-automorphism on the sphere by Whitney's result [7]. However, our proof is completely different from Tucker's and will suggest those arguments for graphs on surfaces in the second way rather than the first.

1. Maximal planar case

A simple graph is called a *maximal planar graph* if it is planar and if adding any new edge results in a nonplanar graph. Such a graph can be embedded on the plane or the sphere so that each face is triangular and each face can be distinguished with a triple $\{u, v, w\} \subset V(G)$ forming its three corners if it has at least four vertices. Then it is often called a *triangulation* on the sphere with a fixed embedding.

By Whitney's result [7], any 3-connected planar graph G has a unique dual, which implies that G is uniquely embeddable on the sphere and that any automorphism of G extends to an auto-homeomorphism over the sphere where G is embedded, as is pointed out in [3]. Such a graph is said to be *faithfully embedded* on the sphere. Every maximal planar graph can be faithfully embedded on the sphere since it is 3-connected.

Let G be a maximal planar graph with at least four vertices embedded on the sphere S^2 and $\sigma : G \to G$ an automorphism of G which extends to an auto-homeomorphism $h : S^2 \to S^2$. Take a face A with boundary cycle uvw. Then $\sigma(u)\sigma(v)\sigma(w)$ bounds another face of G and the face is $h(A)$. Let B be a face of G sharing the edge uv. Then $h(B)$ must be the face meeting $h(A)$ along the edge $\sigma(uv)$. Similarly, we can recongize the image of each face via h, extending a sequence of adjacent faces over the sphere. Therefore, we may say that $\sigma(u)\sigma(v)\sigma(w)$ determines the whole of σ and h. In particular, if $\sigma(u) = u$, $\sigma(v) = v$ and $\sigma(w) = w$, then σ and h must be the identity maps over G and the sphere. This fact enables us to prove the following theorem easily:

THEOREM 2. *Every maximal planar graph with at least four vertices is 2-distinguishable, except K_4, $K_{2,2,2}$, $C_3 + \overline{K_2}$, and $C_5 + \overline{K_2}$.*

Proof. Let G be a maximal planar graph and embed it on the sphere so that any automorphism of G extends an auto-homeomorphism of the sphere, which is possible by Whitney's result as mentioned in the previous. It is clear that any automorphism of G which fixes the boundary cycle of a face must be the identity map under this situation. We shall find a subset $S \subset V(G)$ so that any automorphism $\sigma : G \to G$ with $\sigma(S) = S$ must be the identity map, analyzing the following two cases separately.

CASE 1. *There is a vertex v of degree $d \geq 6$.*

Let $u_0, u_1, \ldots, u_{d-1}$ be its neighbors with subscripts taken modulo d, which form the link $u_0 u_1 \cdots u_{d-1}$ around v. Put $S_i^{\pm} = \{v, u_i, u_{i\pm1}, u_{i\pm3}\}$ for $i =$

$0, 1, \ldots, d-1$ and let $H_i^{\pm}$ be the subgraph in G with vertex set $S_i^{\pm}$ and with edges $\{vu_i, vu_{i\pm1}, vu_{i\pm3}, u_iu_{i\pm1}\}$. Then $H_i^{\pm}$ forms a triangle $vu_iu_{i\pm1}$ plus one edge joining a vertex $u_{i\pm3}$ of degree 1. These subgraphs $H_i^{\pm}$ might not be an induced subgraph in G, that is, they might have a chord, either $u_iu_{i\pm3}$ or $u_{i\pm1}u_{i\pm3}$, outside the wheel neighborhood of v in general. However, we can find an induced one among them, as follows.

Suppose that H_0^+ is not induced in G. Then either u_0u_3 or u_1u_3 exists. In the first case, the cycle vu_0u_3 separates $\{u_1, u_2\}$ and u_4, and hence u_4 is adjacent to none of u_1 and u_2 by the planarity. Thus, H_1^+ is induced. So we may assume that u_0u_3 does not exist but that H_0^+ has a chord u_1u_3. In this case, H_3^- has no chord since vu_1u_3 separates u_0 and u_2 and hence it is induced.

Without loss of generality, we assume that $H = H_0^+$ is induced in G and put $S = S_0^+$. Let $\sigma : G \to G$ be any automorphism of G with $\sigma(S) = S$. Then σ must induce an automorphism of H. It is clear that $\sigma(v) = v$, $\sigma(\{u_0, u_1\}) = \{u_0, u_1\}$ and $\sigma(u_3) = u_3$ since $\deg_H v = 3$, $\deg_H u_0 = \deg_H u_1 = 2$ and $\deg_H u_3 = 1$. If σ switched u_0 and u_1, then it would induce a reflexion over the wheel neighborhood of v with axis passing through the middle point of u_0u_1, v and u_3. This implies that $d = 5$, which is contrary to our assumption in Case 1. Thus, σ fixes all vertices in S and fixes the face vu_0u_1. This implies that σ extends to the identity map over the sphere.

CASE 2. *Every vertex has degree at most* 5.

First suppose that there is a vertex of degree 5. It is easy to see that G is isomorphic to the icosahedron if G is 5-regular and that the icosahedron is 2-distinguishable. So we may assume that there is a vertex v of degree 5 which is adjacent to a vertex u_0 of degree 3 or 4. Let $C = u_0u_1u_2u_3u_4$ be the link of v. Either if $\deg u_1 \neq \deg u_4$ or if $\deg u_2 \neq \deg u_3$, then we can take $S = \{v, u_0\}$ since any automorphism $\sigma : G \to G$ with $\sigma(S) = S$ fixes v and C.

Thus, we may assume that there is an automorphism $\sigma' : G \to G$ which flips C, fixing v and u_0 and that $\deg u_1 = \deg u_4$ and $\deg u_2 = \deg u_3$. If $\deg u_i \neq 5$ for some $i \neq 0$ in addition, then we may assume that $\deg u_{i+1} = \deg u_{i-1}$ and $\deg u_{i+2} = \deg u_{i-2}$, too by the same argument as in the previous paragraph. These two assumptions imply that all u_0, u_1, u_2, u_3, u_4 have the same degree and its value should be 4 since two vertices of degree 3 are not adjacent to each other in any triangulation except K_4. It is easy to see that G is isomorphic to $C_5 + \overline{K}_2$ in this case. This is one of the exceptions.

If $\deg u_i = 5$ for all $i \neq 0$, then we can take $S = \{v, u_0, u_2\}$. If there were an edge u_0u_2, then there would be an edge $\sigma'(u_0u_2) = u_0u_3$, which implies that $\deg u_0 = 5$, contrary to our assumption. Thus, u_0 is not adjacent to u_2 while u_0 is adjacent to v. Since no automorphism can swap v and u_2, we have $\sigma(v) = v$

for any automorphism $\sigma : G \to G$ with $\sigma(S) = S$. This implies that σ fixes C and extends to the identity map.

The remaining case is when every vertex has degree 3 or 4. In this case, G is isomorphic to $C_i + \overline{K}_2$ ($i = 3, 4$) or K_4. They are listed as three of the exceptions; $C_4 + \overline{K}_2$ is isomorphic to $K_{2,2,2}$. ■

2. General case

It should be noticed that each face of a 3-connected planar graph is bounded by a cycle in any embedding on the shpere. The same arguments as given before Theorem 2 works for 3-connected planar graphs, too. That is, any automorphism of a 3-connected planar graph extends to an auto-homeomorphism over the sphere where the graph is embedded. Now we shall prove Theorem 1, dividing the cases with respect to the face size that a given 3-connected planar graph admits. We shall find a subset $S \subset V(G)$, in each case, so that any automorphism $\sigma : G \to G$ with $\sigma(S) = S$ must be the identity map over G:

Proof of Theorem 1. Let G be a 3-connected planar graph embedded on the sphere. We shall discuss the distinguishability of G, depending on its maximum face size.

Case 1. *There is a face A of G bounded by a cycle of length at least* 6.

Let $C = u_0u_1 \cdots u_{d-1}$ be the cycle bounding A with $d \geq 6$. Put $S = \{u_0, u_2, u_3\}$ and suppose that an authomorphism $\sigma : G \to G$ fixes S as a set, that is, $\sigma(S) = S$.

Since σ extends to an auto-homeomorphism $h : S^2 \to S^2$ over the sphere, $\sigma(C)$ bounds a face $h(A)$. If $h(A) = A$, then it is clear that σ fixes all vertices along $C = h(C)$ and hence it must be the identity map over the whole of G. On the other hand, if $h(A) \neq A$, then their boundary cycles $\sigma(C)$ and C meet each other at S. In this case, a subset of S forms a 2-cut of G, which is contrary to G being 3-connected. Thus, this is not the case.

Case 2. *There is a pentagonal face A of G.*

Let $C = u_0u_1u_2u_3u_4$ be its boundary cycle and put $S = \{u_0, u_2, u_3\}$ as well as in the previous case. However, this S does not work well since we can consider the reflexion over the sphere which fixes u_0 and swaps u_2 and u_3. We would like to find a subset $S' \subset V(G)$ such that any automorphism $\sigma : G \to G$ with $\sigma(S') = S'$ must be the identity map. Let B be the face of G sharing the edge u_3u_4 with A and $C_B = u_4u_3w_1w_2 \cdots$ its boundary cycle. Put $S' = S \cup \{w_1\}$ and let σ be any automorphism of G with $\sigma(S') = S'$. Then σ extends to an auto-homeomorphism $h : S^2 \to S^2$ over the sphere.

First assume that $h(A) \neq A$. Then $|\sigma(S) \cap S| \geq 2$ since $\sigma(S') = S'$ and $S' - S = \{w_1\}$. If $\sigma(S) \cap S$ contains a pair of non-adjacent vertices, then the pair would form a 2-cut of G, contrary to G being 3-connected. Otherwise, we have $\sigma(S) \cap S = \{u_2, u_3\}$ and $\sigma(S) = \{u_2, u_3, w_1\}$. These three vertices forms a path $u_2u_3w_1$ of length 2. However, at most one of two pairs $\{u_2, u_3\}$ and $\{u_3, w_1\}$ can be joined by an edge lying on $\sigma(C)$ and the remaining non-adjacent pair would form a 2-cut of G, contrary to our assumption again.

Therefore, we can assume that $h(A) = A$ and $\sigma(C) = C$. Then we have $\sigma(u_0) = u_0$ and $\sigma(\{u_2, u_3\}) = \{u_2, u_3\}$. If σ does not swap u_2 and u_3, then h fixes A pointwise and σ must be the identity map. Otherwise, h can be assumed to be the reflextion $\rho : S^2 \to S^2$ over the sphere whose fixed point set forms a simple closed curve ℓ passing through u_0 and the middle point of u_2u_3. In this case, $h(B)$ must be the face meeting A along u_1u_2. and with boundary cycle $u_1u_2\sigma(w_1)\sigma(w_2)\cdots$. Since w_1 belongs to S', we have $\sigma(w_1) = w_1$ and hence B and $h(B)$ meet each other at $w_1 = \sigma(w_1)$.

Let B_i be the face of G meeting $u_{i+1}u_{i+2}$ for $i \equiv 0, 1, 2, 3, 4 \pmod 5$. By similar arguments, we can conclude that if we cannot find the desired subset $S' \subset V(G)$, then B_i and B_{i+2} meet at a vertex for all i. The assumption of G being 3-connected forces all B_i to be triangular and to meet at a common vertex, which corresponds to $w_1 = \sigma(w_1)$ in the previous paragraph. That is, G must be a wheel with rim of length 5. This is one of the exceptions.

Case 3. *There is a quadrilateral face A of G and there is no face with boundary of length more than* 4.

We shall show that if G is not 2-distinguishable, then all of vertices lying on the boundary cycle of any quadrilateral face have degree 3. After showing it, it is easy to see that G is isomorphic to one of Q_3, W_4 and $C_3 \times K_2$. The first two are exceptions for the theorem while the last one, the triangular prism, is not and $D(C_3 \times K_2) = 2$.

Let $C = u_0u_1u_2u_3$ be the boundary cycle of A. If there is an edge joining u_0 and u_2, then it runs outside A and we can find a simple closed which separates u_1 and u_3. This implies that $\{u_0, u_2\}$ forms a 2-cut of G, contrary to G being 3-connected. Therefore, u_0 and u_2 are not adjacent and also u_1 and u_3 are not adjacent similarly.

First suppose that u_0u_1 is incident to a triangular face u_0u_1w. Put $S = \{u_0, u_1, u_3, w\}$ and let H be the subgraph induced by S. Let $\sigma : G \to G$ be any automorphism of G with $\sigma(S) = S$ which is not the identity map and $h : S^2 \to S^2$ its extention. If w and u_3 are not adjacent, then the subgraph H induced by S consists of a triangle u_0u_1w and one edge joining u_0 and u_3. This implies that

$\sigma(u_0) = u_0$ and $\sigma(u_3) = u_3$. If σ fixes u_1 and w, then h fixes u_0u_1w and hence it must be the identitiy map. This is however contrary to our assumption on σ. Thus, σ swaps u_1 and w, and h is a reflexion whose axis runs through u_0u_3 and the middle point of u_1w. In this case, wu_0u_3 forms a corner of $h(A)$. This implies that u_0 has degree 3 as we expect.

Suppose that w and u_3 are adjacent. Then H consists of two triangles wu_0u_1 and wu_0u_3 which meet along wu_0. If σ swaps these triangles, then wu_0u_3 also bounds a face and u_0 must have degree 3. Otherwise, σ fixes each of u_1 and u_3, swaps u_0 and w since σ is not the identity map. In this case, $h(A)$ and A would meet each other only at $\{u_1, u_3\}$, which forms a 2-cut, contrary to G being 3-connected. Thus, only the former case happens and hence u_0 has degree 3.

Suppose that at least one of four faces adjacent to A is triangular, say u_0u_1w. Then we have $\deg w = 3$ as shown in the previous. If u_0u_3 is incident to a quadrilateral face u_0u_3xw in addition, then we can show that G would be 2-distinguishable, taking $S = \{u_0, u_3, w\}$. Thus, there must be a triangular face u_0u_3w. Repeating this argument, we can conclude that all faces incident to A are triangular and that G is isomorphic to W_4. Therefore, we may assume that all faces sharing edges with A are quadrilateral.

Let B_i be such a face bounded by a cycle $C_{B_i} = u_iu_{i+1}x_iy_i$ for $i \equiv 0, 1, 2, 3 \pmod 4$. Look at B_0 and put $S = \{u_0, u_1, u_2, u_3, y_0\}$. Let $\sigma : G \to G$ be any automorphism of G with $\sigma(S) = S$ which is not the identity map and has an extention $h : S^2 \to S^2$. Since $\sigma(S - \{y_0\})$ must contain either $\{u_0, u_2\}$ or $\{u_1, u_3\}$, if $h(A) \neq A$, then $h(A) \cup A$ includes a simple closed curve which passes through one of $\{u_0, u_2\}$ and $\{u_1, u_3\}$ and separates the other, which is contrary to G being 3-connected. Thus, we have $h(A) = A$.

If $h(B_0) = B_0$, then h would fix $A \cup B_0$ pointwise and σ would be the identity map, a contradiction. Thus, we have $h(B_0) \neq B_0$ and $\sigma(C_{B_0})$ contains $\sigma(y_0) = y_0$. If $\sigma(C_{B_0})$ contained u_1, then $\{u_1, y_0\}$ would form a 2-cut. Thus, $h(B_0)$ is either B_2 or B_3. In either case, $\sigma(y_0)$ must be one of x_3, y_3, x_2 and y_2 and it must be equal to y_0. If $y_0 = x_3$, then u_0y_0 coincides with u_0x_3 to exclude the multiple edges between u_0 and x_3 and hence u_0 has degree 3 in this case as we want.

Look at B_3 and put $S' = \{u_0, u_1, u_2, u_3, x_3\}$. By similar arguments with an automorphism $\sigma' : G \to G$ with $\sigma'(S') = S'$, we can conclude that one of y_0, x_0, y_1 and x_1 must be x_3. By the planarity, most of these cases are not compatible to the cases in the previous paragraph. The unique compatible case is when $y_0 = x_3$ and u_0 has degree 3 in this case. Now we have shown that $\deg u_0 = 3$ in all cases.

Case 4. *Every face of G is triangular.*

That is, G is a maximal planar graph. By Theorem 2, G is 2-distinguishable unless G is one of the exceptions. Since our graphs are all simple, this is the final case for the proof. ■

Theorem 1 is best possible with respect to the connectivity. That is, if we omit the assumption of G being 3-connected in Theorem 1, then there is no upper bound for $D(G)$. For example, the complete bipartite graph $K_{2,n}$ $(n \geq 3)$ is 2-connected planar graph but is not 2-distinguishable and $D(K_{2,n}) = n$.

An *outer planar graph* is one that can be embedded on the plane so that the boundary of its unique unbounded face contains all vertices. If an outer planar graph is 2-connected, then the boundary of its unbounded face is its hamilton cycle. It is very easy to prove the following theorem on outer planar graphs and we cannot omit the assumption of being 2-connected. For example, the star $K_{1,n}$ $(n \geq 2)$ is outer planar but is not 2-connected and $D(K_{1,n}) = n$.

THEOREM 3. *Every 2-connected outer planar graph is 2-distinguishable, except* C_3, C_4 *and* C_5.

Proof. Let G be a 2-connected outer planar graph embedded on the plane and C a hamilton cycle of G, which contains all vertices and bounds the outer face. It is easy to see that there is no other hamilton cycle of G than C and hence $\sigma(C) = C$ for any automorphism $\sigma : G \to G$. Then C contains a vertex of degree 2. If G is not 2-regular, that is, if it is not a cycle, then there are two consecutive vertices u and v along C such that $\deg u = 2$ and $\deg v > 2$.

Put $S = \{u, v\}$ and let $\sigma : G \to G$ be any automorphism of G with $\sigma(S) = S$. Then we have $\sigma(u) = u$ and $\sigma(v) = v$ since u and v have different degrees and hence σ fixes C and must be the identity map. ■

3. On other surfaces

The point in our previous arguments for 3-connected planar graphs is the fact that their all automorphisms extend to auto-homeomorphisms over the surface where the graph is embedded. Also the planarity excludes complicated local structures and allows us to carry out the same argument around vertices or faces as for "Key Ring Problem". Thus, we will be able to establish similar theorems for those graphs embedded on surfaces such that these can be assumed for them.

A *triangulation* on a closed surface F^2 is a simple graph G embedded on F^2 so that every face is bounded by a cycle of length 3 and any two faces share at most one edge. In particular, if every cycle of length 3 in G bounds a face, then G is said to be *clean*.

It is easy to see that the neighbors of every vertex v in a triangulation G form a cycle surrounding v on F^2. Such a cycle is called the *link* of v and is often denoted by $\mathrm{lk}(v)$. If G is clean, then the link of every vertex v in G must be an induced cycle in G, that is, it has no chord. This implies that $\mathrm{lk}(v)$ is the unique cycle consisting of the neighbors of v. This fact excludes complicated local sturctures in G and guarantees the following:

Lemma 4. *Every automorphism of a clean triangulation on a closed surface extends to an auto-homeomorphism over the surface.*

Proof. Let G be a clean triangulation on a closed surface F^2 and $\sigma : G \to G$ its automorphism as an abstract graph. Let uvw be the boundary cycle of any face A of G. Then $\sigma(u)\sigma(v)\sigma(w)$ is a cycle of length 3 in G. Since G is clean, this cycle $\sigma(u)\sigma(v)\sigma(w)$ must bound a face A' and hence σ extends so that it maps A to A'. Therefore, σ extends to an auto-homeomorphism over F^2. ∎

Theorem 5. *Every clean triangulation on a closed surface is* 2*-distinguishable, except* K_4*,* $K_{2,2,2}$ *and* $C_5 + \overline{K}_2$ *on the sphere.*

Proof. Let G be a clean triangulation on a closed surface F^2 and let V_i denote the number of vertices in G of degree i. Then we have the following well-known formula:

$$3V_3 + 2V_4 + V_5 = 6\chi(F^2) + \sum_{i \geq 7}(i-6)V_i,$$

where $\chi(F^2)$ stands for the Euler characteristic of F^2.

If there is no vertex of degree at least 6, then this formula is reduced to:

$$3V_3 + 2V_4 + V_5 = 6\chi(F^2)$$

This implies that $\chi(F^2)$ must be positive and that F^2 is homeomorphic to either the sphere or the projective plane. We have already discussed the first case as Theorem 2 and have known those triangulations that are not 2-distinguishable and all of them except $C_3 + \overline{K}_2$ are clean. They are the exceptions for the theorem, too. On the other hand, if F^2 is the projective plane, then we have $V_3 + V_4 + V_5 \leq 6$ and that G must be isomorphic to K_6 since K_6 is the smallest triangulation on the projective plane. However, K_6 is not clean and hence there is no exception for the theorem in this case.

Therefore, we may assume that G has a vertex v of degree at least 6. Let $\mathrm{lk}(v) = u_0u_1 \cdots u_{d-1}$ be the link around v with $d \geq 6$. Put $S = \{v, u_0, u_1, u_3\}$. Then the subgraph H induced by S consists of the triangle vu_0u_1 and one edge vu_3 joining v and the unique vertex u_3 of degree 1 since G is clean. Thus, for any automorphism $\sigma : G \to G$ with $\sigma(S) = S$, we have $\sigma(v) = v$, $\sigma(u_3) = u_3$ and

$\sigma(\{u_0, u_1\}) = \{u_0, u_1\}$. By Lemma 4, σ extends to an auto-homeomorphism over F^2 and hence σ preserves $\mathrm{lk}(v)$. Since $d \geq 6$, σ must be the identity map over G as well as in Case 1 in our proof of Theorem 2. Therefore, G is 2-distinguishable. ■

Let G be a graph embedded on a closed surface F^2, except the sphere. Then G is said to be *r-representative* on F^2 if any non-contractible simple closed curve on F^2 meets G in at least r points and the minimum of such a number r is called the *representativity* of G on F^2. It has been known that if a 3-connected graph G has sufficiently large representativity on a closed surface F^2, then G is uniquely and faithfully embeddable on F^2 (see [4] for example) and hence there is no gap between automorphisms of G as an abstract graph and those as a map on F^2. Thus the theorem for those graphs will be included in Tucker's result [6].

On the other hand, we cannot establish such a theorem for graphs with low representativity. That is, a closed surface admits infinitely many graphs that are not 2-distinguishable, as follows:

Theorem 6. *There exists no upper bound for the distinguishing number of 2-connected graphs embedded on any closed surface.*

Proof. Let G be any 2-connected graph embedded on a closed surface F^2. Take one edge uv and replace it with $K_{2,n}$ $(n \geq 3)$ so that the partite set of size 2 in $K_{2,n}$ coincides with $\{u, v\}$. Let G' be the resulting graph embedded on F^2. Then the labels of vertices in the partite set of size n in $K_{2,n}$ are all distinct for any distinguishing labeling of G'. Therefore, $D(G') \geq n$. ■

Theorem 7. *There exist infinitely many 4-connected graphs G embedded on the orientable closed surface of genus $g \geq 1$ such that $D(G) \geq 2g$.*

Proof. Prepare the sphere where a 4-connected graph is embedded. Choose two faces A and B and let $\{a_1, a_2\}$ and $\{b_1, b_2\}$ be two disjoint sets of vertices lying on the boundary cycles of A and B, respectively. Join A and B with g annuli and put $2g$ extra vertices $x_1, \ldots, x_{2g}$ on the annuli so that each annulus contains two of them. Add four edges between x_i and $\{a_1, a_2, b_1, b_2\}$ to the annulus where x_i lies for $i = 1, \ldots, 2g$. It is easy to see that the resulting graph G' is 4-connected and that $x_1, \ldots, x_{2g}$ must get different labels in any distinguishing labeling of G'. Therefore, $D(G') \geq 2g$. ■

A similar argument works for non-orientable closed surfaces and we can show the following:

Corollary 8. *Any closed surface F^2 with $\chi(F^2) \leq -2$ admits infinitely* *many 4-connected graphs that are not 2-distinguishable.*

Proof. Because we can take at least two annuli to place $x_1, x_2, x_3, \ldots$ as in the proo of Theorem 7. ■

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References

[1] M.O. Albertson and K.L. Collings, Symmetry breaking in graphs, *Electron. J. Combin.* **3** (1996)

[2] B. Bogstad, Bill and L.J. Cowen, The distinguishing number of the hypercube, *Discrete Math.* **283** (2004)

[3] S. Negami, Uniqueness and faithfulness of embedding of toroidal graphs, *Discrete Math.* **44** (1983), 161–180.

[4] N. Robertson and R. Vitray, Representativity of surface embeddings, In: *Paths, flows, and VLSI layout*, B. Korte, L. Lovász, H.J. Prömel, and A. Schrijver, eds., Springer-Verlag, Berlin, Heidelberg, 1990.

[5] A. Russell and R. Sundaram, A note on the asymptotics and computational complexity of graph distinguishability, *Electron. J. Combin.* **5** (1998)

[6] T.W. Tucker, Distinguishability of maps, preprint 2005.

[7] H. Whitney, Congruent graphs and the connectivity of graphs, *Amer. J. Math.* **54** (1932), 150–168.

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