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THE CONVERGENCE RATE ESTIMATES OF MANN ITERATIVE SEQUENCES INVOLVING ϕ-STRONG PSEUDOCONTRACTIONS IN BANACH SPACES

By

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Abstract. Let C be a bounded closed and convex subset of an arbitrary Banach space, whose diameter is $M > 0$, and let $T : C \to C$ be a Lipschitz continuous and ϕ-strong pseudocontraction with a unique fixed point p. In this paper, for any $\beta \in (0, 1)$ we determine a non-negative integer sequence $\{n(K)\}$, and with respect to $\{n(K)\}$ we show to construct Mann iterative sequence $\{x_n\}$ which has the following rate estimate: For any K,

$$\|x_n - p\| \leq M\beta^K \quad \text{for all} \quad n \geq n(K).$$

We also give estimates of $n(K)$ from above.

1. Introduction

Let X be an arbitrary Banach space and let $T : X \to X$ be a nonlinear mapping such that the set $F(T)$ of fixed points of T is nonempty. Let J denote the normalized duality mapping from X into 2^{X^*} given by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where X^* denotes the dual space of X, $\langle \cdot, \cdot\rangle$ denotes the duality pairing and $\|\cdot\|$ denotes the norm on X and X^* while there are no confusion.

A mapping $T : D(T) \to R(T)$ with domain $D(T)$ and range $R(T)$ in X is called a strong pseudocontraction if there exists $c > 1$ such that for all $x, y \in D(T)$, there is $j(x - y) \in J(x - y)$ satisfying

$$\langle Tx - Ty, j(x - y)\rangle \leq \frac{1}{c} \|x - y\|^2 .$$

It is well-known that if $T : X \to X$ is a continuous and strong pseudocontraction, then T has a unique fixed point p (see [8]). There are many results of strong

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convergence theorems to p; see, for instance, [1, 5, 6, 7, 9]. For the T, let us consider the following Mann iterative sequence with a coefficient sequence $\{t_n\} \subset [0,1]$:

$$\text{(1.1)} \qquad \begin{cases} x_0 \in X, \\ x_{n+1} = t_n T x_n + (1-t_n) x_n \qquad \text{for all } \ n \in N \cup \{0\}, \end{cases}$$

where N is the set of positive integers. In particular, when $D(T) = C$ is a nonempty closed convex and bounded subset of X and $T : C \to C$ is additionally Lipschitz continuous with a Lipschitz constant L, Liu [6] gave a convergence rate estimate of the $\{x_n\}$ as follows:

$$\text{(1.2)} \qquad \|x_{n+1} - p\| \leq \rho^n \|x_0 - p\| \quad \text{for all } n \in N \cup \{0\},$$

where

$$t_n = \frac{k}{2(3+3L+L^2)}, \quad k = 1 - \frac{1}{c} \qquad \text{and} \quad \rho = 1 - \frac{k^2}{4(3+3L+L^2)}.$$

Sastry and Babu [9] also showed, without condition of boundedness of C, that the $\{x_n\}$ has the same convergence rate estimate as (1.2) for the following constant ρ :

$$\rho = 1 - \frac{k^2}{4(L+1)(L+2-k)+2k}.$$

In this paper, we give convergence rate estimate of $\{x_n\}$ involving the following mapping T. Let $\alpha : [0,\infty) \to [0,\infty)$ be a function for which $\alpha(0) = 0$ and the $\liminf_{r \to r_0} \alpha(r) > 0$ for every $r_0 > 0$. For a function α, a mapping $T : D(T) \to R(T)$ is called an α-strong pseudocontraction with α if for all $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \alpha(\|x-y\|) \|x-y\|.$$

Kirk and Morales [3, 4] showed that a continuous and α-strong pseudocontraction T on a closed convex subset $C = D(T) = R(T)$ has a unique fixed point p. The mapping T according to a strictly increasing function ϕ instead of α is called a ϕ-strong pseudocontraction. The case of ϕ-strong pseudocontractions has been studied extensively, and it is well-known that the class of strong pseudocontractions is a proper subset of the class of ϕ-strong pseudocontractions (see [1, 3, 5, 7, 11]). We give convergence rate estimates of Mann iterative sequence involving a Lipschitz continuous and ϕ-strong pseudocontraction T on an arbitrary Banach space. Precisely, for this mapping T and any $\beta \in (0,1)$ we determine a non-negative integer sequence $\{n(K)\}_{K\in N\cup\{0\}}$ and a coefficient sequence $\{t_n\}$ in a suitable way and we construct Mann iterative sequence with the $\{t_n\}$ as (1. 1) which satisfies the following rate estimate: For any non-negative integer K,

$$\|x_n - p\| \le M\beta^K \quad \text{for all } n \ge n(K), \tag{1.3}$$

where $M > 0$ is a diameter of a bounded subset C. We also give estimate of $n(K)$ from above.

2. Lemmas

In this section, we shall prove two lemmas which are crucial in the proofs of main results. Let $N_0 = N \cup \{0\}$ and let Φ be the set of all strictly increasing functions $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$. We can prove the first lemma by using Kato's lemma (see [2]).

Lemma 1. *Let C be a subset of a Banach space X and let $T : C \to C$ be a ϕ-strong pseudocontraction with $\phi \in \Phi$. Then for any $x, y \in C$ with $x \ne y$, the following inequality holds:*

$$\|x - y + t\{(I - T - \gamma_{xy}I)x - (I - T - \gamma_{xy}I)y\}\| \ge \|x - y\| \quad \textit{for all} \quad t > 0,$$

where $\gamma_{xy} = \frac{\phi(\|x-y\|)}{\|x-y\|}$ and I is an identity mapping.

Proof. From the definition of a ϕ-strong pseudocontraction T, we have that for any $x, y \in C$ with $x \ne y$,

$$\begin{aligned} \langle Tx - Ty, j(x-y)\rangle &\le \|x-y\|^2 - \phi(\|x-y\|)\,\|x-y\| \\ &= (1 - \frac{\phi(\|x-y\|)}{\|x-y\|})\,\|x-y\|^2 \\ &= (1 - \frac{\phi(\|x-y\|)}{\|x-y\|})\,\langle x-y, j(x-y)\rangle . \end{aligned}$$

So, we have

$$\langle (I - T - \gamma_{xy}I)x - (I - T - \gamma_{xy}I)y, j(x-y)\rangle \ge 0.$$

From Kato's lemma [2] this inequality is equivalent to

$$\|x - y + t\{(I - T - \gamma_{xy}I)x - (I - T - \gamma_{xy}I)y\}\| \ge \|x - y\| \quad \text{for all} \quad t > 0.$$

□

Let C be a bounded closed and convex subset of a Banach space X and let M be $\delta(C) > 0$, the diameter of C. Consider $\phi \in \Phi$ such that the function ψ defined by

$$\psi(t) = \begin{cases} \frac{\phi(t)}{t} & \text{if} \quad t \in (0, M], \\ 0 & \text{if} \quad t = 0 \end{cases}$$

is increasing, $\lim_{t \to 0+} \psi(t) = 0$ and $0 \leq \psi(t) \leq 1$ for all $t \in [0, M]$. Such a ϕ is said to be a P-function with ψ on $[0, M]$.

For example,

$$\phi(t) = \left(\frac{t}{M} \right)^2 \quad \text{for all} \quad t \in [0, M],$$

and

$$\phi(t) = \frac{t}{1 + \log M - \log t} \quad \text{for all} \quad t \in (0, M], \text{ and } \phi(0) = 0$$

are P-functions on $[0, M]$. By virtue of Lemma 1, we obtain the following important lemma.

Lemma 2. *Let C be a bounded closed and convex subset of a Banach space X. Suppose $M = \delta(C) > 0$, and let ϕ be a P-function with ψ on $[0, M]$. Let $T : C \to C$ be ϕ-strongly pseudocontractive and Lipschitz continuous with a Lipschitz constant L. Then, for any coefficient sequence $\{t_n\} \subset (0, 1)$, Mann iterative sequence $\{x_n\}$ defined by (1.1) has the following estimate: For a unique fixed point p,*

$$\|x_{n+1} - p\| \leq (1 - \gamma_n t_n + \tilde{L} t_n^2) \|x_n - p\| \quad \textit{for any} \quad n \in N_0,$$

where $\gamma_n = \psi(\|x_{n+1} - p\|)$ and $\tilde{L} = 3 + 3L + L^2$.

Proof. It is enough to consider the case of $x_{n+1} \neq p$. Let γ be any real number. From $x_{n+1} = t_n T x_n + (1 - t_n) x_n$, as in [6] we get

$$\begin{aligned} x_n &= x_{n+1} - t_n T x_n + t_n x_n \\ &= (1 + t_n) x_{n+1} + t_n (I - T - \gamma I) x_{n+1} - t_n (2 - \gamma) x_{n+1} \\ &\qquad + t_n (T x_{n+1} - T x_n) + t_n x_n \\ &= (1 + t_n) x_{n+1} + t_n (I - T - \gamma I) x_{n+1} - t_n (2 - \gamma)(t_n T x_n + (1 - t_n) x_n) \\ &\qquad + t_n (T x_{n+1} - T x_n) + t_n x_n \\ &= (1 + t_n) x_{n+1} + t_n (I - T - \gamma I) x_{n+1} - t_n^2 (2 - \gamma)(T x_n - x_n) \\ &\qquad - t_n (1 - \gamma) x_n + t_n (T x_{n+1} - T x_n). \end{aligned}$$

For p, we also have

$$\begin{aligned} p &= p - t_nTp + t_np \\ &= (1+t_n)p + t_n(I - T - \gamma I)p - t_n(1-\gamma)p. \end{aligned}$$

Thus we have

(2. 1)

$$\begin{aligned} x_n - p = {} & (1+t_n)(x_{n+1} - p) + t_n\{(I - T - \gamma I)x_{n+1} - (I - T - \gamma I)p\} \\ & - t_n(1-\gamma)(x_n - p) + t_n(Tx_{n+1} - Tx_n) - t_n^2(2-\gamma)(Tx_n - x_n) \\ = {} & (1+t_n)\{x_{n+1} - p + \frac{t_n}{1+t_n}((I - T - \gamma I)x_{n+1} - (I - T - \gamma I)p)\} \\ & - t_n(1-\gamma)(x_n - p) + t_n(Tx_{n+1} - Tx_n) - t_n^2(2-\gamma)(Tx_n - x_n) \end{aligned}$$

and hence

$$\begin{aligned} \|x_n - p\| \geq {} & (1+t_n)\left\| x_{n+1} - p + \frac{t_n}{1+t_n}\{(I - T - \gamma I)x_{n+1} - (I - T - \gamma I)p\}\right\| \\ & - t_n \|(1-\gamma)(x_n - p)\| - t_n \|Tx_{n+1} - Tx_n\| - t_n^2 \|(2-\gamma)(x_n - Tx_n)\|. \end{aligned}$$

Here, since we have from Lemma 1 for γ_n

$$\left\| x_{n+1} - p + \frac{t_n}{1+t_n}\{(I - T - \gamma_n I)x_{n+1} - (I - T - \gamma_n I)p\}\right\| \geq \|x_{n+1} - p\|,$$

we obtain from (2. 1), using γ_n instead of γ,

$$\begin{aligned} \|x_n - p\| \geq {} & (1+t_n)\|x_{n+1} - p\| - t_n(1-\gamma_n)\|x_n - p\| - t_n\|Tx_{n+1} - Tx_n\| \\ & - t_n^2(2-\gamma_n)\|x_n - Tx_n\| \end{aligned}$$

and this inequality implies

(2. 2)

$$\begin{aligned} & \{1 + t_n(1-\gamma_n)\}\|x_n - p\| + t_n\|Tx_{n+1} - Tx_n\| + t_n^2(2-\gamma_n)\|x_n - Tx_n\| \\ & \quad \geq (1+t_n)\|x_{n+1} - p\|. \end{aligned}$$

Since T is Lipschitz continuous with a Lipschitz constant L, we have

$$\begin{aligned} t_n\|Tx_{n+1} - Tx_n\| + t_n^2(2-\gamma_n)\|x_n - Tx_n\| & \\ \leq {} & t_nL\|\{t_nTx_n + (1-t_n)x_n\} - x_n\| \\ & \qquad + t_n^2(2-\gamma_n)\|x_n - Tx_n\| \\ = {} & t_n^2L\|Tx_n - x_n\| + t_n^2(2-\gamma_n)\|Tx_n - x_n\| \\ = {} & t_n^2\{L + (2-\gamma_n)\}\|Tx_n - p + p - x_n\| \\ \leq {} & t_n^2\{L + (2-\gamma_n)\}(L+1)\|x_n - p\|. \end{aligned}$$

Thus we obtain from (2. 2)

$$\{1+(1-\gamma_n)t_n\}\|x_n-p\|+t_n^2\{L+(2-\gamma_n)\}(L+1)\|x_n-p\| \\ \geq (1+t_n)\|x_{n+1}-p\|.$$

This implies that

$$(2.3)\qquad \|x_{n+1}-p\| \\ \leq \frac{\{1+(1-\gamma_n)t_n\}}{(1+t_n)}\|x_n-p\|+\frac{t_n^2\{L+(2-\gamma_n)\}(L+1)}{(1+t_n)}\|x_n-p\|.$$

Moreover, since we have

$$\begin{aligned}\frac{1+(1-\gamma_n)t_n}{1+t_n} &\leq (1+(1-\gamma_n)t_n)(1-t_n+t_n^2) \\ &= 1+(1-\gamma_n)t_n+(-t_n+t_n^2)+(1-\gamma_n)t_n(-t_n+t_n^2) \\ &= 1-\gamma_n t_n+t_n^2-(1-\gamma_n)(1-t_n)t_n^2 \\ &\leq 1-\gamma_n t_n+t_n^2,\end{aligned}$$

we obtain from (2. 3)

$$\begin{aligned}\|x_{n+1}-p\| &\leq \frac{1+(1-\gamma_n)t_n}{1+t_n}\|x_n-p\|+\frac{t_n^2\{L+(2-\gamma_n)\}(L+1)}{1+t_n}\|x_n-p\| \\ &\leq (1-\gamma_n t_n+t_n^2)\|x_n-p\|+t_n^2\{L+(2-\gamma_n)\}(L+1)\|x_n-p\| \\ &= (1-\gamma_n t_n)\|x_n-p\| \\ &\qquad +t_n^2\{1+L(L+1)+(2-\gamma_n)(L+1)\}\|x_n-p\| \\ &\leq \{1-\gamma_n t_n+t_n^2(3+3L+L^2)\}\|x_n-p\|.\end{aligned}$$

Setting $\tilde{L}=3+3L+L^2$, we obtain that

$$\|x_{n+1}-p\|\leq (1-\gamma_n t_n+\tilde{L}t_n^2)\|x_n-p\| \quad \text{for all} \quad n\in N_0.$$

□

3. Main Results

Let C be a bounded closed and convex subset of a Banach space X, and suppose $M=\delta(C)>0$. Let ϕ be a P-function with ψ on $[0,M]$, and let $T:C\to C$ be ϕ-strongly pseudocontractive and Lipschitz continuous with a

Lipschitz constant L. Set $\tilde{L} = (3 + 3L + L^2)$ and define a function $C_T : (0, 1] \to (0, 1)$ as follows:

$$C_T(\alpha) = 1 - \frac{1}{4\tilde{L}}(\psi(M\alpha))^2 \quad \text{for all} \quad \alpha \in (0, 1].$$

Then $C_T(\alpha_1) \leq C_T(\alpha_2)$ for $\alpha_1 \geq \alpha_2$ since $\psi(t)$ is increasing. Fix $\beta \in (0, 1)$. For a sufficiently large $K \in N_0$, we have

$$\beta < C_T(\beta^K) = 1 - \frac{1}{4\tilde{L}}(\psi(M\beta^K))^2 < 1.$$

For any $K \in N_0$, define m_K depending on β as follows:

$$m_K = \min\{m \in N : \ (C_T(\beta^K))^m \leq \beta \ \}. \tag{3.1}$$

Then $m_K \leq m_{K+1}$ and $(C_T(\beta^K))^n \leq \beta$ for $n \geq m_K$. Moreover, define $n_\beta(0) = 0$ and

$$n_\beta(K) = n_\beta(K - 1) + m_K, \quad K \in N.$$

Then we have

$$n_\beta(K) = n_\beta(0) + \sum_{j=1}^{K} m_j = \sum_{j=1}^{K} m_j, \quad K \in N. \tag{3.2}$$

Denoting $n_\beta(K)$ by $n(K)$ for simplification, we have that $0 = n(0) < n(1) < \cdots < n(K) < \cdots$. For each $n \in N_0$ we can find some $K \in N$ such that $n(K - 1) \leq n < n(K)$. So, using such a K, define t_n by

$$t_n = \frac{1}{2\tilde{L}}\psi(M\beta^K). \tag{3.3}$$

With respect to this coefficient sequence $\{t_n\}$, let us construct Mann iterative sequence $\{x_n\} \subset C$ as (1.1):

$$\begin{cases} x_0 \in C, \\ x_{n+1} = t_n T x_n + (1 - t_n)x_n, \quad n \in N_0. \end{cases}$$

Then, we obtain the following theorem concerning convergence rate.

Theorem 1. *Let C be a bounded closed convex subset of a Banach space X. Let ϕ be a P-function with ψ on $[0, M]$, and let $T : C \to C$ be ϕ-strongly pseudcontractive and Lipschitz continuous with a Lipschitz constant L. For $\beta \in (0, 1)$, define $\{n(K)\}_{K \in N_0}$ by (3.2) and suppose the coefficient sequence $\{t_n\}$ is determined by the $\{n(K)\}_{K \in N_0}$ as (3.3). Then Mann iterative sequence $\{x_n\}$ constructed by $\{t_n\}$ as (1.1) has the following convergence rate estimate: For any $K \in N_0$,*

$$\|x_n - p\| \leq M\beta^K \quad \textit{for all} \quad n \geq n(K).$$

Proof. Suppose $\beta \in (0,1)$ is fixed. For this $\beta \in (0,1)$, consider $\{n(K)\}_{K\in N_0}$ and the Mann iterative sequence $\{x_n\}$ with the coefficient sequence $\{t_n\}$ defined by (3.3). We first prove that $\left\|x_{n(K)} - p\right\| \le M\beta^K$ for all $K \in N_0$ by induction. Because $M = \delta(C)$, we have $\left\|x_{n(0)} - p\right\| \le M = M\beta^0$. Assume that $\left\|x_{n(K-1)} - p\right\| \le M\beta^{K-1}$ for some $K \in N$. Then we shall show that $\left\|x_{n(K)} - p\right\| \le M\beta^K$. Note first that $n(K) = n(K-1) + m_K$. From Lemma 2 we have

$$\begin{aligned}\left\|x_{n(K)} - p\right\| &\le (1 - \gamma_{n(K)-1}t_{n(K)-1} + \tilde{L}t^2_{n(K)-1})\left\|x_{n(K)-1} - p\right\| \\ &= \{1 - \psi(\left\|x_{n(K)} - p\right\|)\frac{1}{2\tilde{L}}\psi(M\beta^K) \\ &\quad + \frac{1}{4\tilde{L}}(\psi(M\beta^K))^2\}\left\|x_{n(K)-1} - p\right\|.\end{aligned}$$

If $\left\|x_{n(K)} - p\right\| > M\beta^K$, then we have $\psi(\left\|x_{n(K)} - p\right\|) \ge \psi(M\beta^K)$ from the increasing property of ψ. So we have

$$\begin{aligned}\left\|x_{n(K)} - p\right\| &\le \{1 - \psi(\left\|x_{n(K)} - p\right\|)\frac{1}{2\tilde{L}}\psi(M\beta^K) \\ &\quad + \frac{1}{4\tilde{L}}(\psi(M\beta^K))^2\}\left\|x_{n(K)-1} - p\right\| \\ &\le \{1 - \psi(M\beta^K)\frac{1}{2\tilde{L}}\psi(M\beta^K) + \frac{1}{4\tilde{L}}(\psi(M\beta^K))^2\}\left\|x_{n(K)-1} - p\right\| \\ &= \{1 - \frac{1}{4\tilde{L}}(\psi(M\beta^K))^2\}\left\|x_{n(K)-1} - p\right\| \\ &= C_T(\beta^K)\left\|x_{n(K)-1} - p\right\|.\end{aligned}$$

Thus we obtain

$$\left\|x_{n(K)} - p\right\| \le \max\{M\beta^K,\ C_T(\beta^K)\left\|x_{n(K)-1} - p\right\|\}.$$

Similarly, since $\left\|x_{n(K)-1} - p\right\| \le \max\{M\beta^K, C_T(\beta^K)\left\|x_{n(K)-2} - p\right\|\}$, we have

$$\left\|x_{n(K)} - p\right\| \le \max\{M\beta^K,\ C_T(\beta^K)M\beta^K, (C_T(\beta^K))^2\left\|x_{n(K)-2} - p\right\|\}.$$

Since $0 < C_T(\beta^K) < 1$, we have

$$\left\|x_{n(K)} - p\right\| \le \max\{M\beta^K,\ (C_T(\beta^K))^2\left\|x_{n(K)-2} - p\right\|\}. \tag{3.4}$$

From m_K-times repeating such a way and by the inductive assumption, we have

$$\begin{aligned}\left\|x_{n(K)} - p\right\| &\le \max\{M\beta^K,\ (C_T(\beta^K))^{m_K}\left\|x_{n(K-1)} - p\right\|\} \\ &\le \max\{M\beta^K, \beta\left\|x_{n(K-1)} - p\right\|\} \\ &\le \max\{M\beta^K, \beta M\beta^{K-1}\} \\ &= M\beta^K.\end{aligned}$$

Hence we obtain by induction

$$\left\|x_{n(K)} - p\right\| \leq M\beta^K \quad \text{for all} \quad K \in N_0.$$

Next, we shall prove that for all $K \in N_0$ and $m \in N$ with $0 < m < m_{K+1}$,

$$\left\|x_{n(K)+m} - p\right\| \leq M\beta^K.$$

Similarly as the above, if $\left\|x_{n(K)+m} - p\right\| > M\beta^K$, then we get that

$$\begin{aligned}
\left\|x_{n(K)+m} - p\right\| &\leq \{1 - \gamma_{n(K)+m-1}t_{n(K)+m-1} \\
&\qquad + \tilde{L}t^2_{n(K)+m-1}\} \left\|x_{n(K)+m-1} - p\right\| \\
&= \{1 - \psi(\left\|x_{n(K)+m} - p\right\|)\frac{1}{2\tilde{L}}\psi(M\beta^{K+1}) \\
&\qquad + \frac{1}{4\tilde{L}}(\psi(M\beta^{K+1}))^2\} \left\|x_{n(K)+m-1} - p\right\| \\
&\leq \{1 - \psi(M\beta^K)\frac{1}{2\tilde{L}}\psi(M\beta^{K+1}) \\
&+ \frac{1}{4\tilde{L}}(\psi(M\beta^{K+1}))^2\} \left\|x_{n(K)+m-1} - p\right\| \\
&\leq \{1 - \psi(M\beta^{K+1})\frac{1}{2\tilde{L}}\psi(M\beta^{K+1}) \\
&\qquad + \frac{1}{4\tilde{L}}(\psi(M\beta^{K+1}))^2\} \left\|x_{n(K)+m-1} - p\right\| \\
&= \{1 - \frac{1}{4\tilde{L}}(\psi(M\beta^{K+1}))^2\} \left\|x_{n(K)+m-1} - p\right\| \\
&= C_T(\beta^{K+1}) \left\|x_{n(K)+m-1} - p\right\|.
\end{aligned}$$

Thus we obtain

$$\left\|x_{n(K)+m} - p\right\| \leq \max\{M\beta^K, C_T(\beta^{K+1}) \left\|x_{n(K)+m-1} - p\right\|\}.$$

Repeating similarly, we have

$$\begin{aligned}
\left\|x_{n(K)+m} - p\right\| &\leq \max\{M\beta^K, (C_T(\beta^{K+1}))^m \left\|x_{n(K)} - p\right\|\} \\
&\leq \max\{M\beta^K, C_T(\beta^{K+1})^m M\beta^K\} \\
&\leq M\beta^K.
\end{aligned}$$

So, we obtain that for all $K \in N_0$ and $m \in N$ with $0 < m < m_{K+1}$,

$$\left\|x_{n(K)+m} - p\right\| \leq M\beta^K.$$

As a consequence, noting $\left\|x_{n(K+1)} - p\right\| \leq M\beta^{K+1} \leq M\beta^K$, we have that for any $K \in N_0$,

$$\|x_n - p\| \leq M\beta^K \quad \text{for all} \quad n \geq n(K).$$

□

Next, for a given β and for a P-function ϕ, we shall give estimates of $\{n(K)\}_K$ defined by (3.2). For the sake of simplification, we assume that $M = 1$.

THEOREM 2. *Under the assumption of the previous theorem, the following holds:*

$$n(K) \leq (1 + \log\beta)K - 8\tilde{L}(\log\beta)\sum_{j=1}^{K}\frac{1}{(\psi(\beta^j))^2}, \quad K \in N.$$

Moreover, in the case of $\beta = \frac{1}{2}$, the following hold:

(1) *If $\phi(t) = t^2$, i.e., $\psi(t) = t$, then*

$$n(K) \leq (1 - \log 2)K + \frac{32\tilde{L}}{3}4^K \log 2, \quad K \in N;$$

(2) *if $\phi(0) = 0$ and $\phi(t) = \frac{t}{1-\log t}$, $t \in (0, 1]$, i.e., $\psi(0) = 0$ and*

$$\psi(t) = \frac{1}{1 - \log t}, \quad t \in (0, 1],$$

then for any $K \in N$,

$$\begin{aligned} n(K) \leq \{1 + (8\tilde{L} - 1)(\log 2)\}K + 8\tilde{L}(\log 2)^2 K(K + 1) \\ + \frac{4}{3}\tilde{L}(\log 2)^3 K(K + 1)(2K + 1). \end{aligned}$$

Proof. Let $\{m_j\}_{j\in N}$ be the sequence defined by (3.1). For any $x \in (0, \infty)$, define

$$[x] = \max\{\ n \in N_0 : n \leq x\}.$$

Then, we have, for any $j \in N$,

$$\begin{aligned} m_j &= \min\{\ m \in N : m \geq \frac{\log\beta}{\log C_T(\beta^j)}\ \} \\ &\leq \left[\frac{\log\beta}{\log C_T(\beta^j)}\right] + 1. \end{aligned}$$

The Taylor expansion of $\log(1 + s)$ gives the following equation:

$$\begin{aligned} \frac{(-1)}{\log C_T(\beta^j)} &= \frac{(-1)}{\log(1 - \frac{1}{4\tilde{L}}(\psi(\beta^j))^2)} \\ &= \frac{1}{\sum_{r=1}^{\infty}\frac{1}{r}(\frac{1}{4\tilde{L}}(\psi(\beta^j))^2)^r}. \end{aligned}$$

Since $(\frac{1}{r}) \geq (\frac{1}{2})^r$ for all $r \in N$, we have that

$$\begin{aligned}\frac{(-1)}{\log C_T(\beta^j)} &\leq \frac{1}{\sum_{r=1}^{\infty}(\frac{1}{8\tilde{L}}(\psi(\beta^j))^2)^r} \\ &= \frac{1-(8\tilde{L})^{-1}(\psi(\beta^j))^2}{(8\tilde{L})^{-1}(\psi(\beta^j))^2} \\ &= \frac{8\tilde{L}}{(\psi(\beta^j))^2} - 1.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}n(K) = \sum_{j=1}^{K} m_j &\leq \sum_{j=1}^{K}\{\left[\frac{\log\beta}{\log C_T(\beta^j)}\right] + 1\} \\ &\leq \sum_{j=1}^{K}\{(-\log\beta)(\frac{8\tilde{L}}{(\psi(\beta^j))^2} - 1) + 1\} \\ &= K(1+\log\beta) - 8\tilde{L}(\log\beta)\sum_{j=1}^{K}\frac{1}{(\psi(\beta^j))^2}.\end{aligned}$$

(1) From $\beta = \frac{1}{2}$ and $\psi(t) = t$, we have

$$\begin{aligned}n(K) &\leq K(1-\log 2) + 8\tilde{L}(\log 2)\sum_{j=1}^{K} 4^j \\ &= K(1-\log 2) + \frac{32}{3}\tilde{L}(\log 2)4^K.\end{aligned}$$

(2) From $\beta = \frac{1}{2}$, $\psi(0) = 0$ and $\psi(t) = \frac{1}{\log\frac{e}{t}}$, $t \in (0,1]$, i.e., $\psi(0) = 0$ and

$$\frac{1}{\psi(t)} = \log(\frac{e}{t}), \quad t \in (0,1],$$

we have

$$\begin{aligned}n(K) &\leq K(1-\log 2) + 8\tilde{L}(\log 2)\sum_{j=1}^{K}(\log\frac{e}{2^{-j}})^2 \\ &\leq K(1-\log 2) + 8\tilde{L}(\log 2)\sum_{j=1}^{K}(1+j\log 2)^2 \\ &\leq K(1-\log 2) \\ &\quad + 8\tilde{L}(\log 2)\{K + (\log 2)K(K+1) + \frac{(\log 2)^2}{6}K(K+1)(2K+1)\}.\end{aligned}$$

□

These results seem to show that the convergence rate of the iterative sequence $\{x_n\}$ with $\phi(t) = \frac{t}{1-\log t}$ is better than that with $\phi(t) = t^2$.

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