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VISCOSITY APPROXIMATION METHODS FOR EQUILIBRIUM PROBLEM AND NONEXPANSIVE MAPPING IN HILBERT SPACES

By

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Abstract. In this paper, we introduce an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in real Hilbert space. Then, we prove a strong convergence theorem which is connected with Combettes and Hirstoaga's result [2], Wittmann's result [18] and Suzuki [12]. Using our theorem, we obtain two corollaries and consider the problem of finding a minimizer of a convex function which is one of applicable example for the equilibrium problem.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let T be a bifunction from $C \times C$ to $\boldsymbol{R}$, where $\boldsymbol{R}$ is the set of all real numbers. The equilibrium problem for a bifunction T is to find $x \in C$ such that

$$T(x, y) \geq 0 \text{ for all } y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(T)$. Given a proper lower semicontinuous convex function g from C to $\boldsymbol{R}$, let $T(x, y) = g(y) - g(x)$ for all $x, y \in C$. Then, $z \in EP(T)$ if and only if $g(y) \geq g(x)$ for all $y \in C$, i.e., z is a solution of the convex minimization problem which is connected to the convex feasibility problem. Numerous problems in physics, optimization, and economics reduce to find a solution of (1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [1], [3] and [8]. Recently, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when $EP(T)$ is nonempty and proved a strong convergence theorem.

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A mapping S is called nonexpansive of C into H if

$$\|Sx - Sy\| \leq \|x - y\| \text{ for all } x, y \in C.$$

We denote by $F(S)$ the set of all fixed points of S. If $C \subset H$ is bounded, closed and convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty; for instance, see [15]. There are some methods for approximation of fixed points of a nonexpansive mapping. Wittmann [18] introduced the following iterative method for approximation of fixed points of a nonexpansive S of C into itself: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S x_n \text{ for all } n = 1, 2, 3, \cdots, \tag{2}$$

where $\alpha_n \in [0, 1]$. Then by the additional assumption, the sequence $\{x_n\}$ converges strongly to z; see originally Halpern [4]. In 2007, Suzuki [12] proved for an averaged mapping U, that is, there exists a nonexpansive mapping S of C into itself and $\lambda \in (0, 1)$ such that $U = \lambda I + (1 - \lambda) S$ that the sequence $\{x_n\}$ generated by (2) converges strongly to $z \in F(T)$.

In 2000, as one of another methods for approximation of fixed points, Moudafi [7] proved the following strong convergence theorem.

Theorem 1.1 (Moudafi [7]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let S be a nonexpansive mapping of C into itself such that $F(S)$ is nonempty. Let f be a contraction of C into itself and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \frac{\varepsilon_n}{1 + \varepsilon_n} f(x_n) + \frac{1}{1 + \varepsilon_n} S x_n \text{ for all } n \in \boldsymbol{N},$$

where $\{\varepsilon_n\} \subset (0, 1)$ satisfies

$$\lim_{n \to \infty} \varepsilon_n = 0, \ \sum_{n=1}^{\infty} \varepsilon_n = \infty \text{ and } \lim_{n \to \infty} \left| \frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n} \right| = 0.$$

Then the sequence $\{x_n\}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Such a method for approximation of fixed points is called the *viscosity approximation method.*

On the other hand, using Halpern's method, Tada and Takahashi [13] introduced the iterative method for finding a common element of the set of the solutions of (1) and the set of fixed points of a nonexpansive mapping of C into H and proved the strong convergence theorem. More recently, using the viscosity method, Takahashi and Takahashi [14] proved the strong convergence theorem. In this paper, motivated Suzuki [12], Tada and Takahashi [13] and Takahashi and Takahashi [14], we introduce an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of (1) and the set of fixed points of a nonexpansive mapping in a real Hilbert space H. Then, we prove a strong convergence theorem which is connected with Combettes and Hirstoaga's result [2], Wittmann's result [18] and Suzuki [12]. Using our main theorem, we consider the problem of finding a minimizer of a convex function from C to $\boldsymbol{R}$ which is one of applicable example for the equilibrium problem.

2. Preliminaries

Throughout this paper, we denote the set of all natural numbers by $\boldsymbol{N}$ and the set of all real numbers by $\boldsymbol{R}$. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. When $\{x_n\}$ is a sequence in H, we denote strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda \|x\|^2 + (1-\lambda) \|y\|^2 - \lambda(1-\lambda) \|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

For any $x \in H$, there exists a unique element in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. Such a mapping P_C is called the *metric projection* of H onto C. $P_C(x)$ is characterized as follows:

$$y = P_C(x) \Leftrightarrow \langle x-y, y-z \rangle \geq 0 \text{ for all } z \in C.$$

It is well-known that P_C is nonexpansive; see [15] for more details.

For solving the equilibrium problem, let us assume that a bifunction T satisfies the following conditions:

(A1) $T(x,x) = 0$ for all $x \in C$;
(A2) T is monotone, i.e., $T(x,y) + T(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t \to 0} T(tz + (1-t)x, y) \leq T(x,y);$$

(A4) $T(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The properties of $T : C \times C \to \boldsymbol{R}$ which satisfies (A1)–(A4) are proved in [1] and [2].

Lemma 2.1 (Blum-Oettli [1]). *Let C be a nonempty closed convex subset of H, let T be a bifunction from $C \times C$ to $\boldsymbol{R}$ satisfying* (A1)–(A4) *and let $r > 0$*

and $x \in H$. Then there exists $z \in C$ such that

$$T(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \text{ for all } y \in C.$$

Lemma 2.2 (Combettes-Hirstoaga [2]). *For $r > 0$, $x \in H$, define a mapping T_r of H into C as follows:*

$$T_r(x) = \left\{ z \in C : T(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ ^{\forall} y \in C \right\}.$$

Then the following hold:

(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive, i.e. for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle ;$$

(3) *$F(T_r) = EP(T)$;*
(4) *$EP(T)$ is closed and convex.*

The following lemma is proved in [11].

Lemma 2.3 (Suzuki [11]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in H and let $\{\beta_n\} \subset [0, 1]$ be a sequence with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1 - \beta_n) x_n$ for all $n \in \boldsymbol{N}$ and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

The following lemma is well known.

Lemma 2.4 (Xu [19]). *Let $\{c_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ and let $\{\beta_n\}$ be a sequence of $\boldsymbol{R}$. Suppose that*

$$c_{n+1} \leq (1 - \alpha_n) c_n + \alpha_n \beta_n \text{ for all } n \in \boldsymbol{N},$$
$$\sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \limsup_{n\to\infty} \beta_n \leq 0.$$

Then $\lim_{n\to\infty} c_n = 0$.

3. main theorem

In this section, we show the strong convergence theorem by the viscosity approximation method for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a nonexpansive in a real Hilbert space H.

Theorem 3.1. *Let C be a nonempty closed convex subset of H. Let T be a bifunction from $C \times C$ to $\boldsymbol{R}$ satisfying* (A1)–(A4) *and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(T) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and*

$$\begin{cases} u_n \in C \text{ such that } T(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \text{ for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) (\beta_n S u_n + (1 - \beta_n) x_n) \text{ for all } n \in \boldsymbol{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (a, \infty)$ for some $a > 0$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$$
$$\text{and } \lim_{n\to\infty} |r_{n+1} - r_n| = 0.$$

Then $\{x_n\}$ converges strongly to $z \in F(S) \cap EP(T)$, where $z = P_{F(S)\cap EP(T)} f(z)$.

Proof. We first show that $P_{F(S)\cap EP(T)} f$ is a contraction of H into itself. Since f is a contraction of H into itself, there exists $r \in [0, 1)$ such that $\|f(x) - f(y)\| \leq r \|x - y\|$ for all $x, y \in H$. So, we have that

$$\left\|P_{F(S)\cap EP(T)} f(x) - P_{F(S)\cap EP(T)} f(y)\right\| \leq \|f(x) - f(y)\| \leq r \|x - y\|$$

for all $x, y \in H$ and hence $P_{F(S)\cap EP(T)} f$ is a contraction of H into itself. From Banach's contraction principle, there exists a unique element $z \in H$ such that $z = P_{F(S)\cap EP(T)} f(z)$. Such a $z \in H$ is an element of C.

By Lemma 2.1, it follows that $\{u_n\}$ and $\{x_n\}$ are well defined. Let $v \in F(S) \cap EP(T)$ and $u_n = T_{r_n} x_n$, where T_{r_n} is as in Lemma 2.2. Then we have that

$$\|u_n - v\| = \|T_{r_n} x_n - T_{r_n} v\| \leq \|x_n - v\| \tag{3}$$

and that

$$\|S u_n - v\| \leq \|u_n - v\| \leq \|x_n - v\| \tag{4}$$

for all $n \in \boldsymbol{N}$. Put $M = \max\{\|x_1 - v\|, \frac{1}{1-r} \|f(v) - v\|\}$ and

$$y_n = \frac{\alpha_n f(x_n) + (1 - \alpha_n) \beta_n S u_n}{\alpha_n + (1 - \alpha_n) \beta_n} \tag{5}$$

for each $n \in \boldsymbol{N}$. Then we note that

$$\begin{aligned} x_{n+1} &= (\alpha_n + (1-\alpha_n)\beta_n)\, y_n + (1-\alpha_n)(1-\beta_n)x_n \\ &= (1-(1-\alpha_n)(1-\beta_n))\, y_n + (1-\alpha_n)(1-\beta_n)x_n \end{aligned}$$

for all $n \in \boldsymbol{N}$ and that

$$0 < \liminf_{n\to\infty}(1-\alpha_n)(1-\beta_n) \leq \limsup_{n\to\infty}(1-\alpha_n)(1-\beta_n) < 1 \tag{6}$$

because of $\lim_{n\to\infty} \alpha_n = 0$ and the assumption of $\{\beta_n\}$. It is obvious that $\|x_1 - v\| \leq M$. Suppose that $\|x_n - v\| \leq M$ for some $n \in \boldsymbol{N}$. From (3) and (4), we have

$$\begin{aligned} \|x_{n+1} - v\| &= \|\alpha_n f(x_n) + (1-\alpha_n)(\beta_n S u_n + (1-\beta_n)x_n) - v\| \\ &\leq \alpha_n \|f(x_n) - v\| + (1-\alpha_n)\beta_n \|Su_n - v\| \\ &\quad + (1-\alpha_n)(1-\beta_n)\|x_n - v\| \\ &\leq \alpha_n(\|f(x_n) - f(v)\| + \|f(v) - v\|) + (1-\alpha_n)\beta_n \|x_n - v\| \\ &\quad + (1-\alpha_n)(1-\beta_n)\|x_n - v\| \\ &\leq \alpha_n(r\|x_n - v\| + \|f(v) - v\|) + (1-\alpha_n)\beta_n \|x_n - v\| \\ &\quad + (1-\alpha_n)(1-\beta_n)\|x_n - v\| \\ &= (1-\alpha_n(1-r))\|x_n - v\| + \alpha_n(1-r)\frac{1}{1-r}\|f(v) - v\| \\ &\leq (1-\alpha_n(1-r))\, M + \alpha_n(1-r)M = M. \end{aligned}$$

So, we have that $\|x_n - v\| \leq M$ for any $n \in \boldsymbol{N}$ and hence $\{x_n\}$ is bounded. We also have that $\{u_n\}, \{Su_n\}$ and $\{f(x_n)\}$ are bounded. Further, we have that

$$\begin{aligned} \|y_n - v\| &= \left\| \frac{\alpha_n f(x_n) + (1-\alpha_n)\beta_n S u_n}{\alpha_n + (1-\alpha_n)\beta_n} - v \right\| \\ &\leq \left\| \frac{\alpha_n (f(x_n) - v) + (1-\alpha_n)\beta_n (S u_n - v)}{\alpha_n + (1-\alpha_n)\beta_n} \right\| \\ &\leq \frac{\alpha_n \|f(x_n) - v\| + (1-\alpha_n)\beta_n \|Su_n - v\|}{\alpha_n + (1-\alpha_n)\beta_n} \\ &\leq \frac{\alpha_n \|f(x_n) - f(v)\| + \alpha_n \|f(v) - v\| + (1-\alpha_n)\beta_n \|Su_n - v\|}{\alpha_n + (1-\alpha_n)\beta_n} \\ &\leq \frac{\alpha_n r \|x_n - v\| + \alpha_n \|f(v) - v\| + (1-\alpha_n)\beta_n \|x_n - v\|}{\alpha_n + (1-\alpha_n)\beta_n} \\ &\leq \frac{\alpha_n r M + \alpha_n (1-r)M + (1-\alpha_n)\beta_n M}{\alpha_n + (1-\alpha_n)\beta_n} \end{aligned}$$

$$\leq \frac{\alpha_n M + (1-\alpha_n)\beta_n M}{\alpha_n + (1-\alpha_n)\beta_n} = M$$

for all $n \in \boldsymbol{N}$. Therefore $\{y_n\}$ is also bounded.

Since $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we have

$$T(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \quad \text{for all } y \in C \tag{7}$$

and

$$T(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0 \quad \text{for all } y \in C. \tag{8}$$

Putting $y = u_{n+1}$ in (7) and $y = u_n$ in (8), we have

$$T(u_n, u_{n+1}) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0 \tag{9}$$

and

$$T(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0 \tag{10}$$

for all $n \in \boldsymbol{N}$. Therefore, from (9) and (10), we obtain

$$T(u_n, u_{n+1}) + T(u_{n+1}, u_n) + \left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0$$

for all $n \in \boldsymbol{N}$. So, from (A2), we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0.$$

This implies that

$$\begin{aligned}
\|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \\
&= \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\
&\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}
\end{aligned}$$

$$\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{a} \|u_{n+1} - x_{n+1}\| \right\}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{a} \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{a} N, \end{aligned} \tag{11}$$

where $N = \sup\{\|u_n - x_n\| : n \in \boldsymbol{N}\}$. Next we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Putting $K = \max\{\sup \|f(x_n)\|, \sup \|Su_n\|\}$, from (5) and (11) we have that

$$\begin{aligned}
&\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \\
&= \limsup_{n\to\infty} \Bigg(\left\| \frac{\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1})\beta_{n+1} S u_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} - \frac{\alpha_n f(x_n) + (1-\alpha_n)\beta_n S u_n}{\alpha_n + (1-\alpha_n)\beta_n} \right\| \\
&\qquad - \|x_{n+1} - x_n\| \Bigg) \\
&\leq \limsup_{n\to\infty} \Bigg(\frac{\alpha_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} \|f(x_{n+1}) - f(x_n))\| \\
&\qquad + \left| \frac{\alpha_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} - \frac{\alpha_n}{\alpha_n + (1-\alpha_n)\beta_n} \right| \|f(x_n)\| \\
&\qquad + \frac{(1-\alpha_{n+1})\beta_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} \|Su_{n+1} - Su_n\| \\
&\qquad + \left| \frac{(1-\alpha_{n+1})\beta_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} - \frac{(1-\alpha_n)\beta_n}{\alpha_n + (1-\alpha_n)\beta_n} \right| \|Su_n\| \\
&\qquad - \|x_{n+1} - x_n\| \Bigg) \\
&\leq \limsup_{n\to\infty} \Bigg\{ \frac{\alpha_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} r \|x_{n+1} - x_n\| \\
&\qquad + \left| \frac{\alpha_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} - \frac{\alpha_n}{\alpha_n + (1-\alpha_n)\beta_n} \right| K \\
&\qquad + \frac{(1-\alpha_{n+1})\beta_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} \left(\|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{a} N \right) \\
&\qquad + \left| \frac{(1-\alpha_{n+1})\beta_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} - \frac{(1-\alpha_n)\beta_n}{\alpha_n + (1-\alpha_n)\beta_n} \right| K \\
&\qquad - \|x_{n+1} - x_n\| \Bigg\} \\
&\leq \limsup_{n\to\infty} \Bigg\{ \left(\frac{\alpha_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} + \frac{(1-\alpha_{n+1})\beta_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} \right) \|x_{n+1} - x_n\| \\
&\qquad + \frac{(1-\alpha_{n+1})\beta_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} \frac{|r_{n+1} - r_n|}{a} N
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\left| \frac{\alpha_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} - \frac{\alpha_n}{\alpha_n + (1-\alpha_n)\beta_n} \right| \right. \\
&\qquad \left. + \left| \frac{(1-\alpha_n)\beta_n}{\alpha_n + (1-\alpha_n)\beta_n} - \frac{(1-\alpha_{n+1})\beta_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} \right| \right) K \\
&\quad \left. - \|x_{n+1} - x_n\| \right\} \\
&\le \limsup_{n\to\infty} \left\{ \frac{(1-\alpha_{n+1})\beta_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} \frac{|r_{n+1} - r_n|}{a} N \right. \\
&\quad + \left(\left| \frac{\alpha_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} - \frac{\alpha_n}{\alpha_n + (1-\alpha_n)\beta_n} \right| \right. \\
&\qquad \left. \left. + \left| \frac{(1-\alpha_n)\beta_n}{\alpha_n + (1-\alpha_n)\beta_n} - \frac{(1-\alpha_{n+1})\beta_{n+1}}{\alpha_{n+1} + (1-\alpha_{n+1})\beta_{n+1}} \right| \right) K \right\}.
\end{aligned}
$$

From $\lim_{n\to\infty} \alpha_n = 0$, the assumption of $\{\beta_n\}$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, we obtain

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

By Lemma 2.3, we get $\lim_{n\to\infty} \|y_n - x_n\| = 0$. Therefore, from (6) we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - (1-\alpha_n)(1-\beta_n)) \|y_n - x_n\| = 0. \tag{12}$$

From (11), (12) and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

We also have

$$
\begin{aligned}
\|x_{n+1} - Su_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)(\beta_n S u_n + (1-\beta_n)x_n) - Su_n\| \\
&\le \alpha_n \|f(x_n) - Su_n\| + (1-\alpha_n)(1-\beta_n) \|x_n - Su_n\| \\
&\le \alpha_n \|f(x_n) - Su_n\| + (1-\alpha_n)(1-\beta_n) \|x_n - x_{n+1}\| \\
&\quad + (1-\alpha_n)(1-\beta_n) \|x_{n+1} - Su_n\|.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
(1 - (1-\alpha_n)(1-\beta_n)) \|x_{n+1} - Su_n\| &\le \alpha_n \|f(x_n) - Su_n\| \\
&\quad + (1-\alpha_n)(1-\beta_n) \|x_n - x_{n+1}\|.
\end{aligned}
$$

From $\lim_{n\to\infty} \alpha_n = 0$, the assumption of $\{\beta_n\}$ and (12), we have

$$\lim_{n\to\infty} \|x_{n+1} - Su_n\| = 0.$$

Since $\|x_n - Su_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - Su_n\|$, we obtain

$$\lim_{n\to\infty} \|x_n - Su_n\| = 0. \tag{13}$$

From Lemma 2.2, we have

$$\begin{aligned}
\|u_n - v\|^2 &= \|T_{r_n}x_n - T_{r_n}v\|^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}v, x_n - v\rangle \\
&= \langle u_n - v, x_n - v\rangle \\
&= \frac{1}{2}\left\{\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2\right\}
\end{aligned}$$

and hence

$$\|u_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - u_n\|^2 .$$

Using this inequality, we have

$$\begin{aligned}
&\|x_{n+1} - v\|^2 \\
&\le \alpha_n \|f(x_n) - v\|^2 + (1-\alpha_n)\beta_n \|Su_n - v\|^2 + (1-\alpha_n)(1-\beta_n)\|x_n - v\|^2 \\
&\le \alpha_n \|f(x_n) - v\|^2 + (1-\alpha_n)\beta_n \|u_n - v\|^2 + (1-\alpha_n)(1-\beta_n)\|x_n - v\|^2 \\
&\le \alpha_n \|f(x_n) - v\|^2 + (1-\alpha_n)\beta_n(\|x_n - v\|^2 - \|x_n - u_n\|^2) \\
&\quad + (1-\alpha_n)(1-\beta_n)\|x_n - v\|^2 \\
&= \alpha_n \|f(x_n) - v\|^2 + (1-\alpha_n)\|x_n - v\|^2 - (1-\alpha_n)\beta_n \|x_n - u_n\|^2 \\
&\le \alpha_n \|f(x_n) - v\|^2 + \|x_n - v\|^2 - (1-\alpha_n)\beta_n \|x_n - u_n\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
(1-\alpha_n)\beta_n \|x_n - u_n\|^2 &\le \alpha_n \|f(x_n) - v\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\
&= \alpha_n \|f(x_n) - v\|^2 \\
&\quad + (\|x_n - v\| + \|x_{n+1} - v\|)(\|x_n - v\| - \|x_{n+1} - v\|) \\
&\le \alpha_n \|f(x_n) - v\|^2 + \|x_n - x_{n+1}\| (\|x_n - v\| + \|x_{n+1} - v\|).
\end{aligned}$$

From $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n > 0$ and (12), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{14}$$

Since $\|Su_n - u_n\| \le \|Su_n - x_n\| + \|x_n - u_n\|$, from (13) and (14) we also have

$$\lim_{n\to\infty} \|Su_n - u_n\| = 0. \tag{15}$$

Next we show that

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle \le 0, \tag{16}$$

where $z = P_{F(S)\cap EP(T)}f(z)$. It is sufficient to show that

$$\limsup_{n\to\infty} \langle f(z) - z, u_n - z\rangle \le 0$$

because $x_n - u_n \to 0$ as $n \to \infty$. To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\to\infty} \langle f(z) - z, u_{n_i} - z\rangle = \limsup_{n\to\infty} \langle f(z) - z, u_n - z\rangle .$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to $w \in C$. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w \in C$.

First we show $w \in EP(T)$. By $u_n = T_{r_n}x_n$, we have

$$T(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0$$

for all $y \in C$. From (A2), we obtain

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge T(y, u_n)$$

and hence

$$\frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \ge T(y, u_{n_i})$$

for all $y \in C$. Since the sequence $\{r_n\} \in (a, \infty)$ for some $a > 0$, we have

$$\begin{aligned} T(y, u_{n_i}) &\le \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \\ &\le \frac{1}{r_{n_i}}\|y - u_{n_i}\|\,\|u_{n_i} - x_{n_i}\| \\ &\le \frac{1}{a}\|y - u_{n_i}\|\,\|u_{n_i} - x_{n_i}\| . \end{aligned}$$

So, from $u_{n_i} \rightharpoonup w$ and (14) we have $T(y, w) \le 0$ for all $y \in C$. For $t \in (0, 1]$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$ and hence $T(y_t, w) \le 0$. So, we have

$$0 = T(y_t, y_t)$$

$$\begin{aligned}&\leq tT(y_t, y) + (1-t)T(y_t, w)\\ &\leq tT(y_t, y)\end{aligned}$$

for all $y \in C$. Dividing by t, we get $T(y_t, y) \geq 0$ for all $y \in C$. Letting $t \to 0$, from (A3), we obtain

$$T(w, y) \geq 0 \text{ for all } y \in C.$$

Therefore we obtain $w \in EP(T)$.

We shall show $w \in F(S)$. Assume that $w \notin F(S)$. Then from $u_{n_i} \rightharpoonup w$ and Opial's condition [9] (see also [15–17]), we have

$$\begin{aligned}\liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - Sw\|\\ &\leq \liminf_{i\to\infty} \{\|u_{n_i} - Su_{n_i}\| + \|Su_{n_i} - Sw\|\}\\ &\leq \liminf_{i\to\infty} \|u_{n_i} - w\|.\end{aligned}$$

This is a contradiction. So, we get $w \in F(S)$. Therefore we obtain $w \in F(S) \cap EP(T)$. Since $z = P_{F(S)\cap EP(T)} f(z)$, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle f(z) - z, u_n - z\rangle &= \lim_{i\to\infty} \langle f(z) - z, u_{n_i} - z\rangle\\ &= \langle f(z) - z, w - z\rangle \leq 0.\end{aligned}$$

So, the inequality (16) holds. It follows from (4) that

$$\begin{aligned}\|\beta_n Su_n + (1-\beta_n)x_n - z\| &\leq \beta_n \|Su_n - z\| + (1-\beta_n)\|x_n - z\|\\ &\leq \beta_n \|x_n - z\| + (1-\beta_n)\|x_n - z\|\\ &= \|x_n - z\|.\end{aligned}$$

Since $x_{n+1} - z = \alpha_n(f(x_n) - z) + (1-\alpha_n)(\beta_n Su_n + (1-\beta_n)x_n - z)$, we have

$$\begin{aligned}(1-\alpha_n)^2 \|x_n - z\|^2 &\geq (1-\alpha_n)^2 \|\beta_n Su_n + (1-\beta_n)x_n - z\|^2\\ &= \|(x_{n+1} - z) - \alpha_n(f(x_n) - z)\|^2\\ &\geq \|x_{n+1} - z\|^2 - 2\alpha_n \langle f(x_n) - z, x_{n+1} - z\rangle\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n \langle f(x_n) - z, x_{n+1} - z\rangle\\ &= (1-\alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n \langle f(x_n) - f(z), x_{n+1} - z\rangle\\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n r \|x_n - z\| \|x_{n+1} - z\| \\
&\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle \\
&\leq (1-\alpha_n)^2 \|x_n - z\|^2 + \alpha_n r (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z \rangle .
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \frac{(1-\alpha_n)^2 + \alpha_n r}{1 - \alpha_n r} \|x_n - z\|^2 + \frac{2\alpha_n}{1 - \alpha_n r} \langle f(z) - z, x_{n+1} - z \rangle \\
&= \frac{1 - 2\alpha_n + \alpha_n r}{1 - \alpha_n r} \|x_n - z\|^2 + \frac{\alpha_n^2}{1 - \alpha_n r} \|x_n - z\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n r} \langle f(z) - z, x_{n+1} - z \rangle \\
&\leq \left(1 - \frac{2(1-r)\alpha_n}{1 - \alpha_n r}\right) \|x_n - z\|^2 \\
&\quad + \frac{2(1-r)\alpha_n}{1 - \alpha_n r} \left(\frac{\alpha_n}{2(1-r)} L + \frac{1}{1-r} \langle f(z) - z, x_{n+1} - z \rangle \right),
\end{aligned}
$$

where $L = \sup\{\|x_n - z\|^2 : n \in \boldsymbol{N}\}$. Put $\beta_n = \frac{2(1-r)\alpha_n}{1-\alpha_n r}$. Then we have that $\lim_{n\to\infty} \beta_n = 0$. From $\alpha_n \in [0,1]$ and $r \in [0,1)$, we have

$$
1 \leq \frac{1-r}{1 - \alpha_n r}
$$

and hence

$$
2\alpha_n \leq \frac{2(1-r)\alpha_n}{1 - \alpha_n r}.
$$

From $\sum_{n=1}^{\infty} \alpha_n = \infty$, we have $\sum_{n=1}^{\infty} \beta_n = \infty$. As in the proof of Takahashi and Takahashi [14], we can conclude that the sequence $\{x_n\}$ converges strongly to z. $\square$

As direct consequences of Theorem 3.1, we obtain the following corollaries.

Corollary 3.1. *Let C be a nonempty closed convex subset of H and let S be a nonexpansive mapping of C into H such that $F(S) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$ and*

$$
x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)(\beta_n S P_C x_n + (1-\beta_n) x_n) \text{ for all } n \in \boldsymbol{N},
$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ satisfy

$$
\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.
$$

Then $\{x_n\}$ converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$.

Proof. Put $T(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \boldsymbol{N}$ in Theorem 3.1. Then, we have $u_n = P_C x_n$. So, from Theorem 3.1, the sequence $\{x_n\}$ generated in Corollary 3.1 converges strongly to $z \in F(S)$, where $z = P_{F(S)} f(z)$. $\square$

Corollary 3.2. *Let C be a nonempty closed convex subset of H. Let T be a bifunction from $C \times C$ to $\boldsymbol{R}$ satisfying (A1)–(A4) such that $EP(T) \neq \emptyset$ and let f be a contraction of H into itself. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and*

$$\begin{cases} u_n \in C \text{ such that } T(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \text{ for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)(\beta_n u_n + (1 - \beta_n) x_n) \text{ for all } n \in \boldsymbol{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (a, \infty)$ for some $a > 0$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$$
$$\text{and } \lim_{n\to\infty} |r_{n+1} - r_n| = 0.$$

Then $\{x_n\}$ converges strongly to $z \in EP(T)$, where $z = P_{EP(T)} f(z)$.

Proof. Let S be a identity mapping of C into itself in Theorem 3.1. Then, from Theorem 3.1 the sequence $\{x_n\}$ generated in Corollary 3.2 converges strongly to $z \in EP(T)$, where $z = P_{EP(T)} f(z)$. $\square$

4. Applications

Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let g be a proper lower semicontinuous convex function from C to $\boldsymbol{R}$. We define a bifunction T from $C \times C$ to $\boldsymbol{R}$ by

$$T(x,y) = g(y) - g(x) \tag{17}$$

for all $x, y \in C$. Then $z \in EP(T)$, that is, $T(z,y) \geq 0$ for all $y \in C$ if and only if

$$g(y) \geq g(z) \text{ for all } y \in C. \tag{18}$$

This means that $z \in EP(T)$ is the minimizer of g. So, the convex minimization problem is one of applicable examples for the equilibrium problem.

In this section, using our main theorem, we consider to find a solution z of (18). For each g, we can define the subdifferential of g as follows:

$$\partial g(x) = \{z \in H : g(y) \geq g(x) + \langle z, y - x \rangle \ \text{ for all } y \in C\} \tag{19}$$

for all $x \in C$. If $g(z) = \infty$, then we define $\partial g(z) = \emptyset$; see [15–17] for more details.

First, we show the following lemma.

Lemma 4.1. *Let C be a nonempty closed convex subset of H. Let g be a proper lower semicontinuous convex function from C to $\boldsymbol{R}$ with $(\partial g)^{-1}0 \neq \emptyset$. If we define a bifunction T from $C \times C$ to $\boldsymbol{R}$ by (17), then $EP(T) = (\partial g)^{-1}0$.*

Proof. Since $(\partial g)^{-1}0 \neq \emptyset$, let $z \in (\partial g)^{-1}0$. Then we have

$$\begin{aligned} z \in (\partial g)^{-1}0 \neq \emptyset &\Leftrightarrow \partial g(z) = 0 \\ &\Leftrightarrow g(y) \geq g(z) + \langle y - z, 0 \rangle \text{ for all } y \in C \\ &\Leftrightarrow g(y) - g(z) \geq 0 \text{ for all } y \in C \\ &\Leftrightarrow T(z, y) \geq 0 \text{ for all } y \in C \\ &\Leftrightarrow z \in EP(T). \end{aligned}$$

So, we get the conclusion. □

Using Theorem 3.1 and Lemma 4.1, we can prove the following theorem.

Theorem 4.1. *Let C be a nonempty closed convex subset of H. Let g be a proper lower semicontinuous convex function from C to $\boldsymbol{R}$ with $(\partial g)^{-1}0 \neq \emptyset$, let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and*

$$\begin{cases} u_n \in C \text{ such that } g(y) - g(u_n) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \text{ for all } y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\left(\beta_n u_n + (1 - \beta_n) x_n\right) \text{ for all } n \in \boldsymbol{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset (a, \infty)$ for some $a > 0$ satisfy

$$\begin{aligned} &\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty,\ 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1 \\ &\text{and } \lim_{n\to\infty} |r_{n+1} - r_n| = 0. \end{aligned}$$

Then $\{x_n\}$ converges strongly to $z \in (\partial g)^{-1}0$, where $z = P_{(\partial g)^{-1}0} f(z)$.

Proof. Let S be a identity mapping of C into itself in Theorem 3.1 and let T be a bifunction from $C \times C$ to $\boldsymbol{R}$ defined by (17). It is easy to show that T satisfies (A1)–(A4) because T is upper semicontinuous in its first co-ordinate. Then, from Theorem 3.1 and Lemma 4.1, the sequence $\{x_n\}$ generated in Theorem 4.1 converges strongly to $z \in (\partial g)^{-1}0$, where $z = P_{(\partial g)^{-1}0} f(z)$. □

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