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STUDY ON IGUSA'S PROBLEM FOR THE $\mathfrak{p}$-ADIC STATIONARY PHASE FORMULA

By

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Abstract. Let $Z(t)$ be Igusa local zeta function of a linear combination $\alpha f(x)+\beta g(y)$ of two strongly nondegenerate forms $f(x)$ and $g(y)$ with $\mathfrak{p}$-adic integers coefficients α and β. We show that the successive applications of the $\mathfrak{p}$-adic stationary phase formula to $Z(t)$ teminate by periodically, hence the explicit formula of $Z(t)$ is obtained.

Introduction

We have some examples of ecplicit formulae of Igusa Local zeta functions and some methods of compyting them ([2], [3], [4], [7]). In [5], Chap.10, § 10.2, J. Igusa states the $\mathfrak{p}$-adic stationary phase formula for Igusa local zeta functions and , applying this formula successibly, he gives the explicit formulae of Igusa local zeta functions of two polynomials $x_1^2+x_2^3$ and $x_1^2+x_2^3+x_3^5$. Moreover J. Igusa points out that, in these examples, it is crusial that the successive applications of the $\mathfrak{p}$-adic stationary phase formula terminate by periodically and states the problem; Does this fact hold in general case? This is *Igusa's problem* for the $\mathfrak{p}$-adic stationary formula in the title.

We shall state the subjects of this paper. We denote by K a $\mathfrak{p}$-adic number field and O_K its subring of ht $mathfrakp$-adic integers. We fix a prime element π_K of K once and for all. Thus we observe that $\pi_K O_K$ is the unique maximal ideal of O_K and the residue field $O_K/\pi_K O_K$ is finite. We put $\mathbb{F}_q = O_K/\pi_K O_K$. For a polynomial $f(x) = f(x_1, \ldots, x_n)$ in n-letters $x_1, \ldots, x_n$ with its coefficients in O_K, we put $\overline{f}(x) = f(x) \bmod \pi_K$. We denotes by $\overline{S}_{\overline{f}}$ the set of $\mathbb{F}_q$-rarional singular points of the hypersurface defined by $\overline{f}(x) = 0$. We say that $f(x)$ is *strongly nondegenrate* if $\overline{S}_{\overline{f}}$ consists of only the origin of $_mathbbF_1^n$, and if $f(x)$ is also homogeneous we call it *strongly nondegenerate form.* Now let $f(x)$ and $g(y)$ by two strongly nondegenerate forms (with some assumptions) and $\alpha f(x)+\beta g(y)$ a linear combination of $f(x)$ a nd $g(y)$ with $\mathfrak{p}$-adic integers coefficients α and β, then we shall consider the subjects; *Which is the answer to Igusa's problem for*

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the $\mathfrak{p}$-adic stationary phase formula in th case of $\alpha f(x) + \beta g(y)$, yes or no?, and if the answer is yes, then, give the explicit formula of Igusa local zeta function of $\alpha f(x) + \beta g(y)$. We find the answer to the former subject is yes, and the latter subject is also solvable, these are the main results of this paper.

Contents.

In § 1 we shall recall the $\mathfrak{p}$-adic stationary phase formula and give some examples of its applications. In § 2 we shall give some recurrent formulae related to the $\mathfrak{p}$-adic stationary phase formula and we shall show that our subjects can be reduced to some simple cases — Case I and Case II — by these formulae. In § 3 we shall give some sequences related to the successive applications of the $\mathfrak{p}$-adic stationary phase formula. Using these sequences we shall give our main results in § 4 and § 5.

1. $\mathfrak{p}$-adic stationary phase formula

In this section, we shall recall the $\mathfrak{p}$-adic stationary phase formula (§ 1.1) and, applying this formula, we shall geve the explicit formulae of Igusa local zeta functions of some polynomials (§ 1.2).

1.1 $\mathfrak{p}$-adic stationary phase formula

Let $K, O_K, \pi_K, \ldots$ be as in the introduction. We denote by $O_K^\times$ the multiplicative group of all unit in O_K. We can uniquely express every element α in the multiplicative group $K^\times$ of K as $\alpha = \pi_K^{\mathrm{ord}_K(\alpha)}\,\mathrm{ac}_K(\alpha)$ is a rational integer and $\mathrm{ac}_K(\alpha)$ is in $O_K^\times$. We call $\mathrm{ord}_K(\alpha)$ the order of α and $\mathrm{ac}_K(\alpha)$ the argument component of α. We denote the $\mathfrak{p}$-adic absolute value $|\alpha|_K$ of α in K by $|\alpha|_K = q^{-\mathrm{ord}_K(\alpha)}$ $(\alpha \in K^\times)$ and $|0|_K = 0$.

For a positive integer n, we put K^n the n-dimensional vector space over K with the canonical basis. We observe that K^n is a locally compact additive group. We denote by $|dx|_K$ the Haar measure on K^n such that the measure of an open compact subset O_K^n is 1. For a polynomial $f(x) = f(x_1, \ldots, x_n)$ in n-letters $x_1, \ldots, x_n$ with its coefficients in K, we define a $\mathfrak{p}$-adic integral associated to $f(x)$ as follows

$$Z(s) = \int_{x\in O_K^n} |f(x)|_K^s |dx|_K \quad (s \in \mathbb{C},\ \mathrm{Re}(s) > 0).$$

We immediately observe that the integral in the right hand side is absolutely convergent for $\mathrm{Re}(s) > 0$. Moreover we find that it has an analytic continuation to a rational function of $T = q^{-s}$ on the whole complex plane $\mathbb{C}$ (see [5], Chap.8, § 8.2, Theorem 8.2.1 or [6], II Theorem 1). We call this rational function *Igusa local zeta function* of $f(x)$ after J.P. Serre. In this paper, we denote it by $Z(T)$ instead by $Z(s)$.

We shall recall the $\mathfrak{p}$-adic stationary phase formula for Igusa local zeta function $Z(T)$ of a polynomial $f(x)$. After multiplying a suitable power of π_K to $f(x)$, we may assume that the coefficients of $f(x)$ are in O_K but not all in $\pi_K O_K$. Hence we observe that $\overline{f}(x) = f(x) \bmod \pi_K$ is a nonzero polynomial with its coefficients in $\mathbb{F}_q$. As in the introduction, we denote by $\overline{S}_{\overline{f}}$ the set of $\mathbb{F}_q$-rational singular points of the hypersurface defined by $\overline{f}(x) = 0$, namely,

$$\overline{S}_{\overline{f}} = \left\{ x \in \mathbb{F}_q^n : \overline{f}(x) = 0 \ \left(\mathrm{grad}_x \overline{f}\right)(x) = (0, 0, \dots, 0) \right\},$$

in which we put

$$\left(\mathrm{grad}_x \overline{f}\right)(x) = \left(\frac{\partial \overline{f}}{\partial x_1}(x), \frac{\partial \overline{f}}{\partial x_2}(x), \cdots, \frac{\partial \overline{f}}{\partial x_n}(x) \right),$$

and by $\sharp \overline{S}_{\overline{f}}$ the cardinarity of $\overline{S}_{\overline{f}}$. Moreover we denote by $S_{\overline{f}}$ the preimage of $\overline{S}_{\overline{f}}$ under the canonical homomorphism $O_K^n \longrightarrow \left(O_K/\pi_K O_K\right)^n = (\mathbb{F}_q)^n : x \to \overline{x} = x \bmod \pi_K$ and by $N_{\overline{f}}$ the number of zeros of $\overline{f}(x)$ in $\mathbb{F}_q^n$. Then we have

$$Z(T) = \left(1 - q^{-n} N_{\overline{f}}\right) + q^{-n}\left(N_{\overline{f}} - \sharp \overline{S}_{\overline{f}}\right) \frac{(1-q^{-1})T}{1-q^{-1}T} + \int_{x \in S_{\overline{f}}} |f(x)|_K^s \, |dx|_K.$$

This is *the* $\mathfrak{p}$*-adic stationary phase formula for* $Z(T)$ *of* $f(x)$ (see [5], Chap.10, § 10.2, Theorem 10.2.1).

1.2 Examples

As the first example, we start from Igusa local zeta functions of the polynomials $f_0(x) = x_1^2 + x_2^3$, $f_1(x) = x_1^2 + \pi_K x_2^3$, $f_2(x) = \pi_K x_1^2 + x_2^3$ and $f_3(x) = x_1^2 + \pi_1^2 x_2^3$. We denote by $Z_{(i)}(T)$ Igusa local zeta functions for $f_i(x)$ $(i = 0, 1, 2, 3)$. In this example, we assume that 2 and 3 are in $O_K^\times$. Since the hypersurface defined by $\overline{f_0}$ parametrized by $x_1 = u^3$, $x_2 = -u^2$ and

$$\left(\mathrm{grad}_x \overline{f_0}\right)(x) = \left(\frac{\partial \overline{f_0}}{\partial x_1}(x),\ \frac{\partial \overline{f_0}}{\partial x_2}(x) \right) = (2x_1,\ 3x_2^3),$$

we have $N_{\overline{f_0}} = q$, $\overline{S}_{\overline{f_0}} = \{(0,0)\}$ and $S_{\overline{f_0}} = \pi_K O_K \times \pi_K O_K$. Similarly, we have $N_{\overline{f_1}} = q$, $\overline{S}_{\overline{f_1}} = \{(0,y) : y \in \mathbb{F}_q\}$ and $S_{\overline{f_1}} = \pi_K O_K \times O_K$; $N_{\overline{f_2}} = q$, $\overline{S}_{\overline{f_2}} = \{(x,0) : x \in \mathbb{F}_q\}$ and $S_{\overline{f_2}} = O_K \times \pi_K O_K$; $N_{\overline{f_3}} = q$, $\overline{S}_{\overline{f_3}} = \{(0,y) : y \in \mathbb{F}_q\}$ and $S_{\overline{f_1}} = \pi_K O_K \times O_K$. Hence, applying the $\mathfrak{p}$-adic stationary phase formula successively, we obtain the following foumulae

$$Z_{(0)}(T) = (1-q^{-1}) + \frac{(1-q^{-1})^2 q^{-1} T}{1-q^{-1}T} + q^{-2}T^2 Z_{(1)}(T),$$

$$Z_{(i)}(T) = (1-q^{-1}) + q^{-1} T Z_{(i+1)}(T) \quad (i = 1, 2),$$

$$Z_{(3)}(T) = (1-q^{-1}) + q^{-1} T^2 Z_{(0)}(T).$$

Therefore we can compute all $Z_{(i)}(T)$ $(i = 0, 1, 2, 3)$;

$$Z_{(0)}(T) = \frac{(1-q^{-1})\big(1 - q^{-2}T(1-T) - q^{-5}T^5\big)}{(1-q^{-1}T)(1-q^{-5}T^6)},$$

$$Z_{(1)}(T) = \frac{(1-q^{-1})\big(1 - q^{-3}T^3(1-T) - q^{-5}T^5\big)}{(1-q^{-1}T)(1-q^{-5}T^6)},$$

$$Z_{(2)}(T) = \frac{(1-q^{-1})\big(1 - q^{-2}T^2(1-T)(1+T^2) - q^{-5}T^6\big)}{(1-q^{-1}T)(1-q^{-5}T^6)},$$

$$Z_{(3)}(T) = \frac{(1-q^{-1})\big(1 - q^{-1}T(1-T)(1+T^2) - q^{-5}T^6\big)}{(1-q^{-1}T)(1-q^{-5}T^6)}.$$

Next we consider Igusa local zeta functions of strongly nondegenerate forms. Let $f(x)$ be a strongly nondegenerate and d its homogeneous degree, then we have that $\overline{S}_{\overline{f}}$ consists of only the origin of $\mathbb{F}_q$, $S_{\overline{f}} = \pi_K O_K$ and $f(\pi_K x) = \pi_K^d f(x)$. Hence, by the $\mathfrak{p}$-stationary phase formula, we have

$$Z = \big(1 - q^{-n} N_{\overline{f}}\big) + q^{-n}\big(N_{\overline{f}} - 1\big) \frac{(1-q^{-1})T}{1-q^{-1}T} + q^{-n} T^d Z(T).$$

Therefore we have

$$Z(t) = \frac{(1-q^{-1})(1-q^{-n})T + \big(1 - q^{-n} N_{\overline{f}}\big)(1-T)}{(1-q^{-1}T)(1-q^{-n}T^d)}.$$

This formula is given by J. Igusa (see [5], Chap.10, § 10.2, Proposition 10.2.1). On the other hand, J.R. Goldman gives the explicit formula of the Póincare series for strongly nondegenerate forms by using Hensel's lemma (see [1], § 3, Theorem (B)). By the relation between Igusa local zeta functions and Póincare series (see [5], Chap.8, § 8.2, Theorem 8.2.2), we can immediately derive the above formula from Goldman's fomula.

Notation. For a strongly nondegenerate form $f(x)$, we denote by $D_{\overline{f}}(T)$ the polynomial in the denominator of $Z(T)$ of $f(T)$;

$$D_{\overline{f}}(T) = (1-q^{-1})(1-q^{-n})T + \bigl(1-q^{-n}N_{\overline{f}}\bigr)(1-T),$$

because it repeatedlly appears our results.

2. Recurrent formulae

In this section, we shall give some recurrent formulae of the Igusa local zeta function of $\alpha f + \beta g$ (§ 2.1) and we shall show that our subjects can be reduced to some simple cases by these formulae (§ 2.2).

2.1 Recurrent formula

Let $\alpha f(x) + \beta g(y)$ be as in the introduction. We denote by d_f, d_g the homogeneous degree of $f(x)$, $g(y)$, respectively and assume that

$$(A_1) \qquad d_f,\ d_g \in O_K^{\times}.$$

We put $d = \mathrm{ord}_K(\alpha)$, $e = \mathrm{ord}_K(\beta)$ and denote by $Z_{(d,e)}(T)$ Igusa local zeta function of $\alpha f(x) + \beta g(y)$. We shall give some recurrent formulae of $Z_{(d,e)}(T)$ in the following Lemma.

Lemma 1. (1) *We put $d_1 = \min(d_f, d_g)$, then we have*

$$Z_{(0,0)}(T) = \frac{D_{\overline{\alpha f+\beta g}}(T)}{1-q^{-1}T} + q^{-(n+m)}T^{d_1} Z_{(d_f-d_1, d_g-d_1)}(T).$$

(2) *For a positive integer e, we have*

$$Z_{(0,e)}(T) = \begin{cases} \dfrac{D_{\overline{f}}(T)}{1-q^{-1}T} + q^{-n}T^{e} Z_{(d_f-e,0)}(T) & (0 < e < d_f) \\[2ex] \dfrac{D_{\overline{f}}(T)}{1-q^{-1}T} + q^{-n}T^{d_f} Z_{(d_f-e,0)}(T) & (e \geqq d_f). \end{cases}$$

(3) *For a positive integer d, we have*

$$Z_{(d,0)}(T) = \begin{cases} \dfrac{D_{\overline{g}}(T)}{1-q^{-1}T} + q^{-m}T^{d} Z_{(0,d_g-d)}(T) & (0 < d < d_g) \\[2ex] \dfrac{D_{\overline{g}}(T)}{1-q^{-1}T} + q^{-m}T^{d_g} Z_{(d-d_g,0)}(T) & (d \geqq d_g). \end{cases}$$

Proof. (1) At first we shall show that $\alpha f + \beta g$ is strongly nondegenerate. Let $\overline{C}_{\overline{f}}$ be the set of $\mathbb{F}_q$-rational singular points of $\overline{f}(x)$;

$$\overline{C}_{\overline{f}} = \left\{ x \in \mathbb{F}_q^n : \left(\mathrm{grad}_x \overline{f}\right)(x) = (0,0,\dots,0) \right\}.$$

Since $\overline{f}(x)$ is a homogeneous polynomial of degree d_f, for any point x in $\overline{C}_{\overline{f}}$, we have

$$d_f \overline{f}(x) = x_1 \frac{\partial \overline{f}}{\partial x_1}(x) + x_2 \frac{\partial \overline{f}}{\partial x_2}(x) + \cdots + x_n \frac{\partial \overline{f}}{\partial x_n}(x) = 0.$$

and, by the assumption (A_1), $\overline{f}(x) = 0$. Hence we observe that $\overline{C}_{\overline{f}}$ coincides with $\overline{S}_{\overline{f}}$. Similarly we observe that the set $\overline{C}_{\overline{g}}$ of $\mathbb{F}_q$-rational singular points of $\overline{g}(x)$ coincides with $\overline{S}_{\overline{g}}$. Since f and g are strongly nondegenerate, we have

$$\overline{C}_{\overline{f}} = \left\{(0,0,\dots,0)\right\}, \quad \overline{C}_{\overline{g}} = \left\{(0,0,\dots,0)\right\}.$$

On the other hand, we have

$$\left(\mathrm{grad}_{(x,y)} \overline{\alpha f + \beta g}\right)(x,y) = \left(\left(\mathrm{grad}_x \overline{f}\right)(x), \left(\mathrm{grad}_y \overline{g}\right)(y)\right),$$

since, for $i = 1,2,\dots,n$ and $j = 1,2,\dots,m$, we have

$$\frac{\partial \overline{\alpha f + \beta g}}{\partial x_i}(x,y) = \frac{\partial \overline{f}}{\partial x_i}(x), \quad \frac{\partial \overline{\alpha f + \beta g}}{\partial y_j}(x,y) = \frac{\partial \overline{f}}{\partial y_j}(y).$$

Therefore we have

$$\overline{S}_{\overline{\alpha f+\beta g}} = \left\{ (x,y) \in \mathbb{F}_q^n \times \mathbb{F}_q^m : \overline{\alpha f(x) + \beta g(y)} = 0 \right\} \cap \left(\overline{S}_{\overline{f}} \times \overline{S}_{\overline{g}} \right),$$

hence it consists of only the origin of $\mathbb{F}_q^n \times \mathbb{F}_q^m$ and $S_{\overline{\alpha f+\beta g}} = \pi_K O_K^n \times \pi_K O_K^m$. We apply the $\mathfrak{p}$-stationary phase formula to $Z_{(0,0)}(T)$, then we have

$$\begin{aligned} Z_{\alpha f+\beta g}(T) &= \left(1 - q^{-(n+m)} N_{\overline{\alpha f+\beta g}}\right) + q^{-(n+m)} \left(N_{\overline{\alpha f+\beta g}} - 1\right) \frac{(1-q^{-1})T}{1-q^{-1}T} \\ &\quad + \int_{(x,y)\in O_K^n \times O_K^m} |\alpha f(\pi_K x) + \beta g(\pi_K y)|_K^s \, |dxdy|_K \\ &= \frac{D_{\overline{\alpha f+\beta g}}(T)}{1-q^{-1}T} + q^{-n+m} T^{d_1} Z_{(d_f-d_1, d_g-d_1)}(T). \end{aligned}$$

For the cases (2) and (3), the substantial idea has already appeared in the first example in § 2.1. Here we shall show the case (2). Since $d = \mathrm{ord}_K(\alpha) = 0$ and $e = \mathrm{ord}_K(\beta) > 0$, we have $\overline{\alpha f + \beta g} = \overline{\alpha}\overline{f}$ and hence $N_{\overline{\alpha f+\beta g}} = N_{\overline{f}}$, $\overline{S}_{\overline{\alpha f+\beta g}} = \overline{S}_{\overline{f}} \times \mathbb{F}_q^m$ and $S_{\overline{\alpha f+\beta g}} = \pi_K O_K^n \times O_K^m$. We apply the $\mathfrak{p}$-stationary phase formula to $Z_{(0,e)}(T)$, then we have

$$Z_{(0,e)}(T) = \left(1 - q^{-(n+m)} q^m N_{\overline{f}}\right) + q^{-(n+m)} \left(q^m N_{\overline{f}} - q^m\right) \frac{(1-q^{-1})T}{1-q^{-1}T}$$

$$+ \int_{(x,y)\in O_K^n \times O_K^m} |\alpha f(\pi_K x) + \beta g(y)|_K^s \ |dxdy|_K$$

$$= \frac{D_{\overline{f}}(T)}{1-q^{-1}T} + q^{-n} \int_{(x,y)\in O_K^n \times O_K^m} |\alpha \pi_K^e f(x) + \beta g(y)|_K^s \ |dxdy|_K.$$

Here the last integral in the above formula is given by

$$T^e Z_{(d_f - e, 0)}(T) \ (0 < e < d_f), \quad T^{d_f} Z_{(0, e-d_f)}(T) \ (e \geqq d_f).$$

Thus we have our formula. Q.E.D.

2.2 Reduction of subjects

At first we may add the following assumptions.

(A_2) $$d_f \leqq d_g$$

and at least one of α and β is in $O_K^\times$, namely

(A_3) $$\min(d, e) = 0.$$

For nonnegatime integer e, we denote $Z_{(0,e)}$ by $Z_e(T)$. Then we shall show that our $Z(T)$ can be described with $D_{\overline{f}}(T)/(1-q^{-1}T)$, $D_{\overline{g}}(T)/(1-q^{-1}T)$ and $Z_e(T)$ $(e = 0, 1, \ldots, d_f - 1)$, hence our subjects can be reduced to the case of $Z_e(T)$ $(e = 0, 1, \ldots, d_f - 1)$.

(1) $d = 0$, $e \geqq d_f$. By successibly applying the $\mathfrak{p}$-adic stationary phase fourmula to $Z(T)$, we have, from Lemma 1., (2),

$$\begin{aligned} Z(T) &= Z_e(T) \\ &= \frac{D_{\overline{f}}(T)}{1-q^{-1}T} \cdot \frac{1 - q^{-\lceil e/d_f \rceil n} T^{\lceil e/d_f \rceil d_f}}{1 - q^n T^{d_f}} \\ &\quad + q^{-\lceil e/d_f \rceil n} T^{\lceil e/d_f \rceil d_f} \int_{(x,y)\in O_K^n \times O_K^m} \left| \alpha f(x) + \pi^{-\lceil e/d_f \rceil d_f} \beta g(y) \right|_K^s |dxdy|_K. \end{aligned}$$

Here we have

$$\pi_K^{-\lceil e/d_f\rceil d_f}\beta = \pi_K^{e-\lceil e/d_f\rceil d_f}\mathrm{ac}_K(\beta) \quad 0 \leqq e - \left\lceil \frac{e}{d_f}\right\rceil d_f < d_f.$$

(2) $d \geqq 1$, $e = 0$. By the same way of the case (1), we have

$$\begin{aligned} Z(T) &= Z_{(d,0)}(T) \\ &= \frac{D_{\overline{g}}\,(T)}{1-q^{-1}T}\cdot\frac{1-q^{-\lceil d/d_g\rceil m}T^{\lceil d/d_g\rceil d_g}}{1-q^mT^{d_g}} \\ &\quad + q^{-\lceil d/d_g\rceil m}T^{\lceil d/d_g\rceil d_g}\int_{(x,y)\in O_K^n\times O_K^m}\left|\pi_K^{-\lceil d/d_g\rceil d_g}\alpha f(x)+\beta g(y)\right|_K^s |dxdy|_K. \end{aligned}$$

Here we have

$$\pi_K^{-\lceil d/d_g\rceil d_g}\beta = \pi_K^{d-\lceil d/d_g\rceil d_g}\mathrm{ac}_K(\alpha) \quad 0 \leqq d - \left\lceil \frac{d}{d_g}\right\rceil d_g < d_g.$$

We put $r_1 = d - [d/d_g]d_g$. Our statement holds in the case of $r_1 = 0$. Hence we may consider the case of $0 < r_1 < d_g$. Here we apply the $\mathfrak{p}$-adic stationary phase formula once more to the last integral of the above formula, then we have, from Lemma 1., (3),

$$\begin{aligned} &\int_{(x,y)\in O_K^n\times O_K^m}\left|\pi_K^{-[d/d_g]d_g}\alpha f(x)+\beta g(y)\right|_K^s |dxdy|_K \\ &\qquad = \frac{D_{\overline{g}}(T)}{1-q^{-1}T} + q^mT^{r_1}\int_{(x,y)\in O_K^n\times O_K^m}\left|\alpha f(x)+\pi_K^{d_g-r_1}\beta g(y)\right|_K^s |dxdy|_K. \end{aligned}$$

Thus we obtain

$$\begin{aligned} Z(T) &= \frac{D_{\overline{g}}\,(T)}{1-q^{-1}T}\left(\frac{1-q^{-\lceil d/d_g\rceil m}T^{\lceil d/d_g\rceil d_g}}{1-q^mT^{d_g}} + q^{-\lceil d/d_g\rceil m}T^{\lceil d/d_g\rceil d_g}\right) \\ &\quad + q^{-(\lceil d/d_g\rceil+1)m}T^d\int_{(x,y)\in O_K^n\times O_K^m}\left|\alpha f(x)+\pi_K^{d_g-r_1}\beta g(y)\right|_K^s |dxdy|_K. \end{aligned}$$

with $\alpha, \beta \in O_K^\times$ and $e = 0, 1, \ldots, d_f - 1$ under the assumptions $(A1)$, $(A2)$ and $(A3)$. We divide our subjects into two cases — Case I : $d_f \mid d_g$ and Case II : $d_f \nmid d_g$. We shall consider Case I in § 4 and Case II in § 5.

Notation. We define some notations as follows,

$$d_{(f,g)} = (d_f, d_g) : \text{the greatest common divisors of } d_f \text{ and } d_g,$$

$$\ell = \frac{d_f d_g}{d_{(f,g)}} : \text{the least common multiplier of } d_f \text{ and } d_g,$$

$$e_f = \frac{d_f}{d_{(f,g)}}, \quad e_g = \frac{d_g}{d_{(f,g)}},$$

$$d_{g/f} = \left[\frac{e_g}{e_f}\right] : \text{the quotation of the division of } e_g \text{ by } e_f,$$

$$r_{g/f} = e_g - d_{(f,g)} e_f : \text{the remainder of the division of } e_g \text{ by } e_f,$$

then we observe that the e_f and r are mutually relatively prime and the quotient and the remainder of the division of d_g by d_f are $d_{g/f}$ and $d_{(f,g)} r_{g/f}$ respectively.

3. Some sequences

In this section we define some sequences which are related to the successive applications the $\mathfrak{p}$-adic stationary phase formula to $Z_e(T)$ $(e = 0, 1, \ldots, d_f - 1)$;

$$\left\{(\widetilde{\alpha}_k, \widetilde{\beta}_k)\right\}_k, \left\{(\alpha_k, \beta_k)\right\} : \text{sequences in } O_K \times O_K,$$

$$\left\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\right\}_k, \left\{(\nu_k, \mu_k)\right\} : \text{sequences in } \mathbb{N} \times \mathbb{N},$$

$$\{\rho_k\}_k : \text{sequence in } \mathbb{N}.$$

3.1 Definition

1. (1) $e = 0$. We put

$$(\widetilde{\alpha}_1, \widetilde{\beta}_1) = (\alpha \pi_K^{d_f}, \beta \pi_K^{d_g}), \quad (\widetilde{\nu}_1, \widetilde{\mu}_1) = (d_f, d_g), \quad \rho_1 = d_f$$

and

$$(\alpha_1, \beta_1) = (\widetilde{\alpha}_1 \pi_K^{-\rho_1}, \widetilde{\beta}_1 \pi_K^{-\rho_1}) = (\alpha, \beta \pi_K^{d_g - d_f}), \quad (\nu_1, \mu_1) = (0, d_g - d_f).$$

(2) $e = 1, \ldots, d_f - 1$. We put

$$(\widetilde{\alpha}_1, \widetilde{\beta}_1) = (\alpha \pi_K^{d_f}, \beta \pi_K^{d_g}), \quad (\widetilde{\nu}_1, \widetilde{\mu}_1) = (d_f, e), \quad \rho_1 = e$$

and

$$(\alpha_1, \beta_1) = (\widetilde{\alpha}_1 \pi_K^{-\rho_1}, \widetilde{\beta}_1 \pi_K^{-\rho_1}) = (\alpha \pi_K^{d_f - e}, \beta), \quad (\nu_1, \mu_1) = (d_f - e, 0).$$

2. For $k \geqq 1$, if (α_k, β_k) is defined and satisfies the assumption

$$(A) \qquad \text{At least one of } \alpha_k \text{ and } \beta_k \text{ is in } O_K^\times,$$

then we define the terms $(\widetilde{\alpha}_{k+1}, \widetilde{\beta}_{k+1})$, $(\widetilde{\nu}_{k+1}, \widetilde{\mu}_{k+1})$, ρ_{k+1}, $(\alpha_{k+1}, \beta_{k+1})$ and (ν_{k+1}, μ_{k+1}) as follows

(1) $\alpha_k, \beta_k \in O_K^\times$.

$$(\widetilde{\alpha}_{k+1}, \widetilde{\beta}_{k+1}) = (\alpha\pi_K^{d_f}, \beta\pi_K^{d_g}),$$
$$(\widetilde{\nu}_{k+1}, \widetilde{\mu}_{k+1}) = \big(\mathrm{ord}_K(\widetilde{\alpha}_{k+1}), \mathrm{ord}_K(\widetilde{\beta}_{k+1})\big) = (d_f, d_g), \quad \rho_{k+1} = d_f$$

and

$$(\alpha_{k+1}, \beta_{k+1}) = (\widetilde{\alpha}_1\pi_K^{-\rho_{k+1}}, \widetilde{\beta}_1\pi_K^{-\rho_{k+1}}) = (\alpha_k, \beta_k\pi_K^{d_g - d_f}),$$
$$(\nu_{k+1}, \mu_{k+1}) = (0, d_g - d_f).$$

(2) $\alpha_k \in O_K^\times,\ \beta_k \in \pi_K O_K$ or $\alpha_k \in \pi_K O_K,\ \beta_k \in O_K^\times$.

$$(\widetilde{\alpha}_{k+1}, \widetilde{\beta}_{k+1}) = \begin{cases} (\alpha_k\pi_K^{d_f},\ \beta_k) & (\alpha_k \in O_K^\times,\ \beta_k \in \pi_K O_K) \\ (\alpha_k,\ \beta_k\pi_K^{d_g}) & (\alpha_k \in \pi_k O_K,\ \beta_k \in O_K^\times) \end{cases}$$
$$(\widetilde{\nu}_{k+1}, \widetilde{\mu}_{k+1}) = \big(\mathrm{ord}_K(\widetilde{\alpha}_{k+1}),\ \mathrm{ord}(\widetilde{\beta}_{k+1})\big), \quad \rho_{k+1} = \min(\widetilde{\nu}_{k+1},\ \widetilde{\mu}_{k+1})$$

and

$$(\alpha_{k+1},\ \beta_{k+1}) = (\widetilde{\alpha}_1\pi_K^{-\rho_{k+1}},\ \widetilde{\beta}_1\pi_K^{-\rho_{k+1}})$$
$$(\nu_{k+1},\ \mu_{k+1}) = \big(\mathrm{ord}_K(\widetilde{\alpha}_{k+1}),\ \mathrm{ord}_K(\widetilde{\beta}_{k+1})\big).$$

Here we should give some remarks.

(i) In the above definition, we have added the restrictive assumption (A). However, actually (α_k, β_k) satisfies (A) for any k, this can be proved by the induction on k.

(ii) We observe that $\mathrm{ac}_K(\alpha_k) = \mathrm{ac}_K(\alpha)$, $\mathrm{ac}_K(\beta_k) = \mathrm{ac}_K(\beta)$.

3.2 Relations with the $\mathfrak{p}$-adic stationary phase formula

For comprehension of relations between the sequences $\{(\widetilde{\alpha}_k, \widetilde{\beta}_k)\}_k, \{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k$, $\{\rho_k\}_k$, $\{(\alpha_k, \beta_k)\}_k$, $\{(\nu_k, \mu_k)\}_k$ and the successive applications fo the $\mathfrak{p}$-adic stationary phase formula to $Z_e(T)$, we shall recall the case of $x_1^2 + x_2^3$, whici is considered in the first example of § 1.2. Applying of the $\mathfrak{p}$-adic stationary phase formula successively, we obtain

$$Z_{(0)}(T) = \frac{D_{\overline{x_1^2+x_2^3}}(T)}{1-q^{-1}T} + q^{-1}T^2 Z_{(1)}(T), \quad Z_{(1)}(T) = \frac{D_{\overline{x_1^2}}(T)}{1-q^{-1}T} + q^{-1}T Z_{(2)}(T),$$

$$Z_{(2)}(T) = \frac{D_{\overline{x_2^3}}(T)}{1-q^{-1}T} + q^{-1}T Z_{(3)}(T), \qquad Z_{(3)}(T) = \frac{D_{\overline{x_1^2}}(T)}{1-q^{-1}T} + q^{-1}T^2 Z_{(0)}(T).$$

Here we observe that two sequences appear in the right hand side of the above formulae. One is the sequence of $D(T)/(1-q^{-1}T)$'s ;

$$(S_1) \qquad \frac{D_{\overline{x_1^2+x_2^3}}(T)}{1-q^{-1}T}, \quad \frac{D_{\overline{x_1^2}}(T)}{1-q^{-1}T}, \quad \frac{D_{\overline{x_2^3}}(T)}{1-q^{-1}T}, \quad \frac{D_{\overline{x_1^2}}(T)}{1-q^{-1}T},$$

and the othe is the sequence of the coefficients of $Z_{(i)}(T)$'s ;

$$(S_2) \qquad q^{-2}T^2, \quad q^{-1}T, \quad q^{-1}T, \quad q^{-1}T^2.$$

We should notice that (S_1) and (S_2) lead to the explicit formula of $Z_{(0)}$.

On the other hand, we observe that the sequences $\{(\widetilde{\alpha}_k, \widetilde{\beta}_k)\}_k$, $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k$, $\{\rho_k\}_k$, $\{(\alpha_k, \beta_k)\}_k$, $\{(\nu_k, \mu_k)\}_k$ are given as follows.

k	1	2	3	4	5	$\cdots$
$(\widetilde{\alpha}_k,\ \widetilde{\beta}_k)$	(π_K^2, π_K^3)	(π_K^2, π_K)	(π_K, π_K^3)	(π_K^2, π_K^2)	(π_K^2, π_K^3)	$\cdots$
$(\widetilde{\nu}_k,\ \widetilde{\mu}_k)$	$(2,3)$	$(2,1)$	$(1,3)$	$(2,2)$	$(2,3)$	$\cdots$
ρ_k	2	1	1	2	2	$\cdots$
$(\alpha_k,\ \beta_k)$	$(1, \pi_K)$	$(\pi_K, 1)$	$(1, \pi_K^2)$	$(1,1)$	$(1, \pi_K)$	$\cdots$
$(\nu_k,\ \mu_k)$	$(0,1)$	$(1,0)$	$(0,2)$	$(0,0)$	$(0,1)$	$\cdots$

Table 1 the sequences $\{(\widetilde{\alpha}_k, \widetilde{\beta}_k)\}_k$, $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k, \ldots$ for $x_1^2 + x_3^3$

Comparing (S_1), (S_2) and Table 1, we find that the successive applications of the $\mathfrak{p}$-adic stationary phase formula to $Z_{(0)}(T)$ are completely descrived with the sequences $\{(\widetilde{\alpha}_k, \widetilde{\beta}_k)\}_k$, $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k, \ldots$.

We observe that this fact generally holds for $Z_e(T)$ in our case, In particular, if the sequences $\{(\widetilde{\alpha}_k, \widetilde{\beta}_k)\}_k$, $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k, \ldots$ are periodical, then the successive applications of the $\mathfrak{p}$-adic stationary phase formula to $Z_e(T)$ terminates by periodecally, and the periods of them concide. These facts are crucial in the proof of the theorems in the coming two sections.

4. Case I

In this section we shall consider Case I : $d_f \mid d_g$.

Theorem 1. *The successive appications of the* $\mathfrak{p}$*-adic stationary phase formula to* $Z_e(T)$ *in Case I teminates by periodically and its period is*

$$d_{g/f}(e=0), \quad d_{g/f}+1 \ (0<e<d_f).$$

As a consequence of this the explicit formula of $Z_e(T)$ *is given as follows*

$$Z_0(T) = \frac{D_{\overline{\alpha f+\beta g}}(T) + D_{\overline{f}}(T)\,P_0(T)}{(1-q^{-1}T)(1-q^{-(e_g n+e_f m)}T^{\ell})},$$

$$Z_e(T) = \frac{D_{\overline{f}}(T)(1+P_0(T)) + D_{\overline{g}}(T)q^{-n}T^{e}}{(1-q^{-1}T)(1-q^{-(e_g n+e_f m)}T^{\ell})} \quad (0<e<d_f),$$

in which we put

$$P_0(T) = q^{-(n+m)}T^{d_f} + q^{-(2n+m)}T^{2d_f} + \cdots + q^{-\left((d_{g/f}-1)n+m\right)}T^{d_g-d_f}.$$

Proof. At first we remark that, in this case, $d_{(f,g)} = d_f$, $\ell = d_g$, $e_f = 1$, $e_g = d_g/d_f$, $d_{g/f} = e_g = d_g/d_f$ and $r_{g/f} = 0$.

Case of $e=0$. We successibily apply the $\mathfrak{p}$-adic stationary phase formula to $Z_0(T)$, then, by Lemma 1., (1), (2) in § 2.1, we have

$$Z_0(T) = \frac{D_{\overline{\alpha f+\beta g}}(T)}{1-q^{-1}T} + q^{-(n+m)}T^{d_f}Z_{(0,d_g-d_f)}(T)$$

$$Z_{(0,d_g-d_f)}(T) = \frac{D_{\overline{f}}(T)}{1-q^{-1}T}\left(1+q^{-n}T^{d_f}+\cdots+q^{-\left((d_{g/f}-2)n+m\right)}T^{(d_{g/f}-2)d_f}\right) + q^{-\left((d_{g/f}-1)n\right)}T^{(d_{g/f}-1)d_f}Z_0(T).$$

We can obtain these formulae from the corresponding sequences $\{(\widetilde{\alpha}_k,\widetilde{\beta}_k)\}_k$, $\{(\widetilde{\nu}_k,\widetilde{\mu}_k)\}_k,\ldots$ which are given as follows.

Thus we find that the successive applications of the $\mathfrak{p}$-adic stationary phase formula to $Z_0(T)$ terminates periodically and its period is $d_{g/f}$, and we obtain

$$Z_0(T) = \frac{D_{\overline{\alpha f+\beta g}}(T)}{1-q^{-1}T} + \frac{D_{\overline{f}}(T)}{1-q^{-1}T}P(T) + q^{-(d_{g/f}n+m)}T^{d_g}Z_0(T)$$

and hence the formula stated in the theorem.

k	1	2	$\cdots$	$d_{g/f}$
$(\widetilde{\alpha}_k,\ \widetilde{\beta}_k)$	$(\pi_K^{d_f}\alpha, \pi_K^{d_g}\beta)$	$(\pi_K^{d_f}\alpha, \pi_K^{d_g-d_f}\beta)$	$\cdots$	$(\pi_K^{d_f}\alpha, \pi_K^{d_f}\beta)$
$(\widetilde{\nu}_k,\ \widetilde{\mu}_k)$	(d_f, d_g)	$(d_f, d_g - d_f)$	$\cdots$	(d_f, d_f)
ρ_k	d_f	d_f	$\cdots$	d_f
$(\alpha_k,\ \beta_k)$	$(\alpha, \pi_K^{d_g-d_f}\beta)$	$(\alpha, \pi_K^{d_g-2d_f}\beta)$	$\cdots$	(α, β)
$(\nu_k,\ \mu_k)$	$(0, d_g - d_f)$	$(0, d_g - 2d_f)$	$\cdots$	$(0, 0)$

Table 2 The sequence $\{(\widetilde{\alpha}_k,\ \widetilde{\beta}_k)\}_k$, $\{(\widetilde{\nu}_k,\ \widetilde{\mu}_k)\}_k$, ... for $Z_{\alpha f+\beta g}(T)$ in Case I

k	1	2	$\cdots$	$d_{g/f}$
$(\widetilde{\nu}_k,\ \widetilde{\mu}_k)$	(d_f, d_g)	$(d_f, d_g - d_f)$	$\cdots$	(d_f, d_f)
ρ_k	d_f	d_f	$\cdots$	d_f
$(\nu_k,\ \mu_k)$	$(0, d_g - d_f)$	$(0, d_g - 2d_f)$	$\cdots$	$(0, 0)$

Table 3 $\{(\widetilde{\nu}_k,\ \widetilde{\mu}_k)\}_k$, $\{\rho_k\}_k$, $\{(\nu_k,\ \mu_k)\}_k$ for $Z_{\alpha f+\beta g}(T)$ in Case I

Here we remark that it is enough to consider the sequences $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k$, $\{\rho_k\}_k$ and $\{(\nu_k, \mu_k)\}_k$ for the explicit formula to Igusa local zeta function. Hence we may consider the following Table 3 instead of the above Table 2.

Case of $0 < e < d_f$. In this case we again successibly apply the $\mathfrak{p}$-adic stationary phase formula to $Z_e(T)$, then the corresponding sequences $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k$, $\{\rho_k\}_k$ and $\{(\nu_k, \mu_k)\}_k$ are as follows.

k	1	2	3	$\cdots$	$d_{g/f}+1$
$(\widetilde{\nu}_k,\ \widetilde{\mu}_k)$	(d_f, e)	$(d_f - e, d_g)$	$(d_f, d_g - d_f + e)$	$\cdots$	$(d_f, d_f + e)$
ρ_k	e	$d_f - e$	d_f	$\cdots$	d_f
$(\nu_k,\ \mu_k)$	$(d_f - e, 0)$	$(0, d_g - d_f + e)$	$(0, d_g - 2d_f + e)$	$\cdots$	$(0, e)$

Table 4 $\{(\widetilde{\nu}_k,\ \widetilde{\mu}_k)\}_k$, $\{\rho_k\}_k$, $\{(\nu_k,\ \mu_k)\}_k$ for $Z_{\alpha f+\pi^e\beta g}(T)$ $(0 < e < d_f)$ in Case I

Therefore we have

$$Z_e(T) = \frac{D_{\overline{f}}(T)}{1-q^{-1}T} + q^{-n}T^{e}\left\{\frac{D_{\overline{g}}(T)}{1-q^{-1}T} + q^{-m}T^{d_f-e}\left(\frac{D_{\overline{g}}(T)}{1-q^{-1}T}(1+P_0(T)) + q^{-\left((d_{g/f}-1)n+m\right)}T^{(d_{g/f}-1)d_f}Z_e(T)\right)\right\}.$$

From this formula we obtain the formula stated in the theorem. Q.E.D.

5. Case II

In this section we consider Case II : $d_f \nmid d_g$. At first we give some lemmas (§ 5.1), and then, using these lemmas, we give our results (§ 5.2).

5.1 Lemmas

We denote $\mathbb{Z}/e_f\mathbb{Z}$ the set of congruence classes modulo e_f and we take

$$\Lambda = \{0, 1, \dots, e_f - 1\}$$

as a complete set of residues modulo e_f. For a multiple $\lambda r_{g/f}$ $(\lambda \in \Lambda)$, we take a representative point $\overline{\lambda r_{g/f}}$ of the congruence class $\lambda r_{g/f}$ mod e_f from Λ and we put

$$\underline{\lambda r_{g/f}} = r_{g/f} + \overline{\lambda r_{g/f}}.$$

We define two subsets Λ_0 and Λ_1 of Λ as follows

$$\Lambda_0 = \left\{\lambda \in \Lambda : (\lambda - 1)r_{g/f} < he_f < \lambda r_{g/f} \text{ for some } h \geqq 1\right\}, \quad \Lambda_1 = \Lambda - \Lambda_0.$$

Lemma 2.

$$\overline{\lambda r_{g/f}} = \begin{cases} \underline{(\lambda - 1)r_{g/f}} - e_f & (\lambda \in \Lambda_0) \\ \underline{(\lambda - 1)r_{g/f}} & (\lambda \in \Lambda_1). \end{cases}$$

Proof. For each λ in Λ_0, there exists an integer h such that $(\lambda - 1)r_{g/f} < he_f < \lambda r_{g/f}$. Then we observe that the quotient and the remainder of the division of $(\lambda - 1)r_{g/f}$ by e_f are $h - 1$ and $\overline{(\lambda - 1)r_{g/f}}$ respectively. Hence we have

$$(\lambda - 1)r_{g/f} = (h - 1)e_f + \overline{(\lambda - 1)r_{g/f}},$$

and

$$\begin{aligned} \underline{(\lambda - 1)r_{g/f}} &= r_{g/f} + \overline{(\lambda - 1)r_{g/f}} \\ &= r + (\lambda - 1)r_{g/f} - (h - 1)e_f = e_f + (\lambda r_{g/f} - he_f). \end{aligned}$$

From the definitions of λ and h, we have $\lambda r_{g/f} - he_f > 0$ and

$$e_f - (\lambda r_{g/f} - he_f) = (e_f - r_{g/f}) + (he_f - (\lambda - 1)e_f) > 0.$$

Therefore we have

$$\overline{(\lambda-1)r_{g/f}} = \lambda r_{g/f} - he_f \in \Lambda \quad \text{and} \quad \underline{(\lambda-1)r_{g/f}} = e_f + \overline{\lambda r_{g/f}}.$$

For a $\lambda \in \Lambda_1$, we denote by h' the quotient of the division $(\lambda-1)r_{g/f}$ by e_f then we have

$$(\lambda-1)r = h'a + \overline{(\lambda-1)r_{g/f}}$$

and hence

$$\underline{(\lambda-1)r_{g/f}} = r_{g/f} + (\lambda-1)r_{g/f} - h'e_f = \lambda r_{g/f} - h'e_f.$$

Thus we observe that $\underline{(\lambda-1)r_{g/f}}$ is congruent to $\lambda r_{g/f}$ modulo e_f. Hence we need to show $0 \leqq \underline{(\lambda-1)r_{g/f}} < e_f$. It is plain that $\underline{(\lambda-1)r_{g/f}} \geqq 0$. If we assume $\underline{(\lambda-1)r_{g/f}} = \lambda r_{g/f} - h'e_f \geqq e_f$, then we have $\lambda r_{g/f} \geqq (h'+1)e_f$. Here we put $h = h'+1$, then we have $h \geqq 1$ and have $(\lambda-1)r_{g/f} < h'e_f + e_f = he_f$. On the other hand, since e_f and $r_{g/f}$ are mutually relatively prime, we have $\lambda r_{g/f} \neq he_f$, and hence $he_f < \lambda r_{g/f}$. Thus we have $(\lambda-1)r_{g/f} < he_f < \lambda r_{g/f}$. It is contragradient to the face $\lambda \in \Lambda_1$, hence we observe that $\underline{(\lambda-1)r_{g/f}} < e_f$. Thus we have $\underline{(\lambda-1)r_{g/f}} = \overline{\lambda r_{g/f}}$. Q.E.D.

Lemma 3. *The cardinarity* $\sharp\Lambda_0$ *of* Λ_0 *is* $r_{g/f} - 1$.

Proof. We consider the interval $I = [0,\, e_f r_{g/f}] = \{x \in \mathbb{R} : 0 \leqq x \leqq e_f r_{g/f}\}$. We decompose I into subintervals $I_i = [e_f(i-1),\, e_f i]$ $(i = 1, 2, \ldots, r_{g/f})$.

We take a non-zero element λ_0 of Λ, then we observe that $\lambda r_{g/f}$ is contained in one and only one of subintervals I_1 $(i = 1, 2, \ldots, r_{g/f})$, but it is not the end of the subinterval, since e_f and $r_{g/f}$ are mutually relatively prime.

On the other hand, we observe that the distance of two numbers $\lambda r_{g/f}$ and $(\lambda+1)r_{g/f}$ is $r_{g/f}$ and it is less than the length e_f of each subinterval I_i. Hence we observe that each subinterval I_i contains at least one of $\lambda r_{g/f}$ $(\lambda \in \Lambda, \neq 0)$ as its interior point.

We notice the above facts and take the smallest number of the form $\lambda r_{g/f}$ in I_i $(i = 1, 2, \ldots, r_{g/f})$, then we find that Λ_0 consists of these numbers, and hence $\sharp\Lambda_0 = r_{g/f} - 1$. Q.E.D.

Lemma 4. $\Lambda = \left\{ \overline{\lambda r_{g/f}} : \lambda \in \Lambda \right\}$.

Proof. We consider the additive group of $\mathbb{Z}/e_f\mathbb{Z}$, which is a cyclic group of order e_f. Since e_f and $r_{g/f}$ are mutually relatively prime, we have the congruence class $r_{g/f}$ mod e_f is a generator of this cyclic group ;

$$\left\{ \lambda r_{g/f} \bmod e_f : \lambda \in \Lambda \right\} = \mathbb{Z}/e_f\mathbb{Z}.$$

Since we have taken $\overline{\lambda r_{g/f}}$ as a representative point of the congruence class $r_{g/f}$ mod e_f, we have $\Lambda = \left\{ \overline{\lambda r_{g/f}} : \lambda \in \Lambda \right\}$. Q.E.D.

5.2 Results in Case II

Here we shall give our results in Case II.

Proposition 1. *For each $e = 0, 1, \ldots, d_{(f,g)} - 1$, the sequences $\{(\widetilde{\alpha}_k, \widetilde{\beta}_k)\}_k$, $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k, \ldots$ are periodical and the period c_e of them is given by*

$$c_e = e_f + e_g - \epsilon_e$$

in which we put $\epsilon_e = 1\,(e = 0),\ 0\,(e = 1, \ldots, d_{(f,g)} - 1)$.

Proof. From the definitions, we may consider the sequences $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k$, $\{\rho_k\}_k$ and $\{(\nu_k, \mu_k)\}_k$. We can represent the tables of those sequences as in Table 5 $(e = 0)$ and Table 6 $(e = 1, \ldots, d_{(f,g)} - 1)$ cited in the end of this paper. Here we should give some remarks on Table 5 and Table 6

(i) We decompose the terms of each row into 2 type classes ; the former class has 2 terms, except the first row in the case $e = 0$, it has only one term, and the latter class has $d_{g/f} - 1$ terms or $d_{g/f}$ terms.

(ii) The number of terms consisting in a second class depends on its number of row. Namely, the number of terms consisting in the second class of the λ-th row is $d_{g/f}$ if $\lambda \in \Lambda_0$ or $\lambda = e_f$ and $d_{g/f} - 1$ if $\lambda \in \Lambda_1$ and $\lambda \neq e_f$. In Table 5 and 6, we have assumed that $3, \ldots \in \Lambda_0$ and $2, 4, \ldots \in \Lambda_1$.

(iii) The end term (ν, μ) of the λ-th row is given as follows

$$(\nu, \mu) = \begin{cases} \Big(0, d_{(f,g)}(\underline{\lambda r_{g/f}} - e_f) + e\epsilon_e\Big) & (\lambda \in \Lambda_0) \\ \Big(0, d_{(f,g)}(\underline{\lambda r_{g/f}} + e\epsilon_e\Big). & (\lambda \in \Lambda_1) \end{cases}$$

Hence, from Lemma 2, we have $(\nu, \mu) = \Big(0, d_{(f,g)}\big(\overline{\lambda + 1) r_{g/f}}\big) + e\epsilon_e\Big)$.

Let c_e be the number of the end term of the e_f-th row in Table 5 or Table 6, then we observe that the end term (ν_{c_e}, μ_{c_e}) of the e_f-th row in Table 5 or Table 6 is $(0, e)$, hence the first term $(\widetilde{\nu}_{c_e+1}, \widetilde{\nu}_{c_e+1})$ of the next row is (d_f, d_g) $(e = 0)$ or $(d_f, 0)$ $(e = 1, \ldots, d_f - 1)$. Therefore we observe that the sequences $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k$, $\{\rho_k\}_k$ and $\{(\nu_k, \mu_k)\}_k$ are periodical, and the period of them is c_e. On the other hand, from Lemma 3 and the above remarks, we have

$$\begin{aligned} c_0 &= 1 + 2(e_f - 1) + d_{g/f}(\sharp\Lambda_0 + 1) + (d_{g/f} - 1)(\sharp\Lambda_1 - 1) \\ &= e_f + d_{g/f} + r_{g/f} - 1 = e_f + e_g - 1. \end{aligned}$$

Similarly we have the number c_e $(e = 1, \dots, d_f - 1)$ for Table 6 is

$$c_e = 2(e_f - 1) + d_{g/f}(\sharp\Lambda_0 + 1) + (d_{g/f} - 1)(\sharp\Lambda_1 - 1) = e_f + e_g.$$

Thus we complete the proof of our theorem. Q.E.D.

Corollary. *Let $S_e = \rho_1 + \rho_2 + \cdots + \rho_{c_e}$ then we have $S_e = \ell$.*

Proof. From Table 5 and 6, we have $S_e = d_f e_g = d_f d_g / d_{(f,g)} = \ell$. Q.E.D.

We observe that Proposition 1 guarantees that the successive applications of the $\mathfrak{p}$-adic stationary phase formula leads the explicit formula of $Z_e(T)$ $(e = 0, 1, \dots, d_f - 1)$. On the bass of this observation we shall settle teh explicit formula of $Z_e(T)$.

At first, we define two polynomials $P(T)$ and $Q(T)$ associated to Table 5. For each λ $(\lambda = 0, 1, \dots, e_f)$, we put

$$\delta_\lambda = 1 \ (\lambda \ in \Lambda_0), \quad 0 \ (\lambda \in \Lambda_1) \quad \text{and} \quad \delta_{e_f} = 1,$$

and

$$\Delta_\lambda = \delta_0 + \cdots + \delta_\lambda, \quad \alpha_\lambda = d_{g/f}\lambda + \Delta_\lambda.$$

We define

$$p_\lambda(T) = q^{-(\alpha_{\lambda-1} + \lambda m)} T^{\alpha_{\lambda-1} d_f} \sum_{k=1}^{d_{g/f} + \delta_\lambda} q^{-kn} T^{k d_f} \qquad (\lambda = 1, \dots, e_f)$$

$$q_\lambda(T) = q^{-(\alpha_{\lambda-1} + \lambda m)} T^{\lambda d_g} \qquad (\lambda = 1, \dots, e_f - 1)$$

and

$$P(T) = p_1(T) + \cdots + p_{e_f}(T), \quad Q(T) = q_1(T) + \cdots + q_{e_f - 1}(T).$$

For comprehension of the relation between Table 5 and the polynomials $P(T)$ and $Q(T)$, we observe an example ; the case of $d_f = 3$, $d_f = 8$. We have two tables ; one is Table 5 of this case and the other is the table of monomials of the form $q^{-a}T^{-b}$ associated to the formaer table.

1	$(3,8)$ 3 $(0,5)$		$(3,5)$ 3 $(0,2)$	
2	$(3,2)$ 2 $(1,0)$	$(1,8)$ 1 $(0,7)$	$(3,7)$ 3 $(0,4)$	$(3,4)$ 3 $(0,1)$
3	$(3,1)$ 1 $(2,0)$	$(2,8)$ 2 $(0,6)$	$(3,6)$ 3 $(0,3)$	$(3,3)$ 3 $(0,0)$

1	$q^{-n+m}T^3$		$q^{-n}T^3$	
2	$q^{-n}T^2$	$q^{-m}T$	$q^{-n}T^3$	$q^{-n}T^3$
3	$q^{-n}T$	$q^{-m}T^2$	$q^{-n}T^3$	$q^{-n}T^3$

Observing the letter table, we obtain that

$$\begin{aligned} P(T) &= q^{-(n+m)}T^3(1+q^{-n}T^3(1+q^{-n}T^2(0+q^{-m}T \\ &\qquad \times (1+q^{-n}T^3(1+q^{-n}T^3(1+q^{-n}T(0+q^{-m}T^2(1+q^{-n}T^3)\cdots) \\ &= q^{-(n+m)}T^3(1+q^{-(2n+m)}T^6+q^{-(3n+2m)}T^9+q^{-(4n+2m)}T^{12} \\ &\qquad + q^{-(5n+2m)}T^{15}+q^{-(6n+3m)}T^{18}+q^{-(7n+3m)}T^{21}, \end{aligned}$$

$$\begin{aligned} Q(T) &= q^{-(n+m)}T^3(0+q^{-n}T^3(0+q^{-n}T^2(1+q^{-m}T \\ &\qquad \times (0+q^{-3}T^3(0+q^{-n}T^3(0+q^{-n}T(1+q^{-m}T^2(0+q^{-n}T^3)\cdots) \\ &= q^{-(3n+m)}T^8+q^{-(6n+2m)}T^{16}. \end{aligned}$$

Here we should give some remarks on the above formulae.

(i) The corresponding coefficients in $P(T)$ and $Q(T)$, which are 1 or 0, are opposite to each other. For example, we see $P(T) = q^{-(n+m)}T^3(1+\cdots$; $Q(T) = q^{-(n+m)}T^3(0+\cdots$.

(ii) The coefficient 1 in $P(T)$ corresponds to $D_{\overline{f}}/(1-q^{-1}T)$ in teh successive applications of the $\mathfrak{p}$-adic stationary phase formula to $Z_0(T)$. On the other hand, the coefficients 1 in $Q(T)$ corresponds to $D_{\overline{g}}(T)/(1-q^{-1}T)$. We should emphasize the above facts generally hold in our case.

Lemma 5. $P(T) = q^{-\left((e_g-1)n+e_f m\right)}T^{\ell-d_f}$ + (terms of lower degree).

Proof. The term of the highest degree in $P(T)$ is

$$\begin{aligned} &q^{-(\alpha_{e_f-1}n+e_f m)}T^{\alpha_{e_f-1}d_f}\cdot q^{-(d_{g/f}+\delta_{e_f})n}T^{(d_{g/f}+\delta_{e_f})}d_f \\ &\qquad = q^{-\left((\alpha_{e_f-1}+d_{g/f}+\delta_{e_f}n+e_f m\right)}T^{(\alpha_{e_f-1}+d_{g/f}+\delta_{e_f})d_f}. \end{aligned}$$

Here we have

$$\begin{aligned} \alpha_{e_f-1}+d_{g/f}+\delta_{e_f} &= d_{g/f}(e_f-1)+\Delta_{e_f-1}+d_{g/f}+\delta_{e_f} \\ &= d_{g/f}e_f+\Delta_{e_f} = d_{g/f}e_f+(r-1) = e_g-1, \end{aligned}$$

since $\Delta_{e_f} = \sharp\Lambda = r-1$ from Lemma 3., and $(e_g-1)d_f = \ell - d_f$. Thus we have the formula in the lemma. Q.E.D.

Theorem 2. *The successive applications of the $\mathfrak{p}$-adic stationary phase formula to $Z_e(T)$ in Case I terminates by periodically and its period is*

$$e_f + e_g - \delta_{d_{(f,g)},e},$$

in which we put $\delta_{d_{(f,g)},e} = 1(d_{(f,g)} \mid e)$, $0(d_{(f,g)} \nmid e)$. As a consequence of this, the explicit formula of $Z_e(T)$ is given as follows

(i) $e = 0$.

$$Z_0(T) = \frac{D_{\overline{\alpha f+\beta g}}(T) + D_{\overline{f}}(T)\,P(T) + D_{\overline{g}}(T)\,Q(T)}{(1-q^{-1}T)(1-q^{-(e_g n+e_f m)}T^{\ell}}$$

(ii) $e = 1, \dots, d_f - 1$

(ii)$_a$ $d_{g/f} \mid e$. *For such a e, there is a unique λ in Λ satisfying $e = d\big(\overline{\lambda r_{g/f}}\big)$ from* Lemma *4. Here we define two polynomials $P_e(T)$ and $Q_e(T)$ as follows,*

$$\begin{aligned} P_e(T) = {} & \big(1 - q^{-(e_g n+e_f m)}T^{\ell}\big) \\ & + \big(p_1(T) + \cdots + p_\lambda(T)\big) q^{-\big((e_g-\alpha_\lambda)n+(e_f-\lambda)m\big)} T^{\ell-\alpha_\lambda d_f} \\ & + \big(p_{\lambda+1}(T) + \cdots + p_{e_f}(T)\big)\cdot\Big(q^{-(\alpha_\lambda n+\lambda m)}T^{\alpha_\lambda d_f}\Big)^{-1}, \end{aligned}$$

$$\begin{aligned} Q_e(T) = {} & \big(q_1(T) + \cdots + q_{\lambda-1}(T)\big) q^{-\big((e_g-\alpha_\lambda)n+(e_f-\lambda)m\big)} T^{\ell-\alpha_\lambda d_f} \\ & + \big(q_\lambda(T) + \cdots + p_{e_f-1}(T)\big)\cdot\Big(q^{-(\alpha_\lambda n+\lambda m)}T^{\alpha_\lambda d_f}\Big)^{-1}. \end{aligned}$$

Then we have

$$Z_e(T) = \frac{D_{\overline{\alpha f+\beta g}}(T) q^{-(\alpha_\lambda n+\lambda m)}T^{\alpha_\lambda d_f} + D_{\overline{f}}(T)\,P_e(T) + D_{\overline{g}}(T)\,Q_e(T)}{(1-q^{-1}T)\big(1-q^{-(e_g n+e_f m)}T^{\ell}\big)}.$$

(ii)$_b$ $d_{g/f} \nmid e$. *We denote by $\overline{\lambda r_{g/f}}$ and r_1 the divisor and the remainder of the division of e by $d_{g/f}$ respectively ;*

$$e = d\big(\overline{\lambda r_{g/f}}\big) + r_1, \quad 1 \leqq r_1 < d_{g/f}.$$

Here we define two polynomials $\widetilde{P}_e(T)$ and $\widetilde{Q}_e(T)$ as follows ;

$\lambda = 0$.

$$\widetilde{P}_e(T) = 1 + P(T), \quad \widetilde{Q}_e(T) = q^{-nr_1}\big(1 + Q(T)\big).$$

$\lambda \neq 0$.

$$\widetilde{P}_e(T) = q^{\big((e_g-\alpha_\lambda)n+(e_f-\lambda)m\big)} T^{\ell-\alpha_\lambda d_f} + P_{d\big(\overline{\lambda r_{g/f}}\big)}(T),$$

$$\widetilde{Q}_e(T) = q^{-n}T^{r_1}\left(q^{-\big((e_g-\alpha_\lambda)n+(e_f-\lambda)m\big)} T^{\ell-\alpha_\lambda d_f} + Q_{d\big(\overline{\lambda r_{g/f}}\big)}(T)\right).$$

Then we have

$$Z_e(T) = \frac{D_{\overline{f}}(T)\,\widetilde{P}_e(T) + D_{\overline{g}}(T)\,\widetilde{Q}_e(T)}{(1-q^{-1}T)\big(1-q^{-(e_g n + e_f m)}T^{\ell}\big)}.$$

Proof. We start with the first statement. We shall notice the remark given in the end of § 3, then we find that, to prove the first statement, it is sufficient to show the sequences $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k$, $\{\rho_k\}_k$ and $\{(\nu_k, \mu_k)\}_k$ for $Z_e(T)$ is periodical and its period is as in the theorem. When $e = 0, 1, \ldots, d_{(f,g)} - 1$, then we imediately obtain our result from PROPOSITION 1. When e is as in the case $(\mathrm{ii})_a$. We take λ as in the theorem, then we find that the end term (ν, μ) of the λ-th row of Table 5, then we obtain the table of the sequences $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k$, $\{\rho_k\}_k$ and $\{(\nu_k, \mu_k)\}_k$ for $Z_e(T)$, and we observe that this table is periodical and its period coincides with the period of Table 5 ; $e_f + e_g - 1$. When e is as in the case $(\mathrm{ii})_b$. We take λ and r_1 as in the theorem, then we find that the end term (ν, μ) of the λ-th row of Table 6 is $\big(0, d\big(\overline{\lambda r_{g/f}}\,\big) + r_1\big)$. Therefore, in a similar way of the above case, we observe that the table of the sequences $\{(\widetilde{\nu}_k, \widetilde{\mu}_k)\}_k$, $\{\rho_k\}_k$ and $\{(\nu_k, \mu_k)\}_k$ for $Z_e(T)$ is periodical and its period coincides with the period of Table 6 ; $e_f + e_g$.

Now we consider the explicit formula of $Z_e(T)$ $(e = 0, 1, \ldots, d_f - 1)$.

(i) Observing the relation between Table 5 and the polynomials $P(T)$ and $Q(T)$, we have

$$\begin{aligned} Z_0(T) = \frac{D_{\overline{\alpha f + \beta g}}(T)}{1-q^{-1}T} + \frac{D_{\overline{f}}(T)}{1-q^{-1}T}\,P(T) + \frac{D_{\overline{g}}(T)}{1-q^{-1}T}\,Q(T) \\ + q^{-(e_g n + e_f m)}T^{\ell} Z_0(T). \end{aligned}$$

The reason that the coefficient monomimal of $Z_e(T)$ in the right hand side is $q^{-(e_g n + e_f m)}T^{\ell}$ is as follows. From LEMMA 5., we observe that the last term of $P(T)$ is $q^{-\left((e_g-1)n + e_f m\right)}T^{\ell - d_f}$ and this term corresponds to the $e_f + e_g - 1$-th application of the $\mathfrak{p}$-adic stationary phase formula. Applying the $\mathfrak{p}$-adic stationary phase formula once more, we return to $Z_0(T)$ and observe that its coefficient monomial is

$$q^{-\left((e_g-1)n + e_f m\right)}T^{\ell - d_f} \cdot q^{-n}T^{d_f} = q^{-(e_g n + e_f m)}T^{\ell}$$

(see the end term of e_f-th row of Table 5). Thus we obtain the formula in the case (i).

$(\mathrm{ii})_a$ Let λ be as in the theorem, then, as stated in the above, we observe that the end term (ν, μ) of the λ-th row is $\big(0, d\big(\overline{\lambda r_{g/f}}\,\big)\big)$ and this term corresponds to $Z_e(T)$. Therefore we have

$$Z_0(T) = \frac{D_{\overline{\alpha f+\beta g}}(T)}{1-q^{-1}T} + \frac{D_{\overline{f}}(T)}{1-q^{-1}T}\Big(p_1(T)+\cdots+p_\lambda(T) - q^{-(\alpha_\lambda n+\lambda m)}T^{\alpha_\lambda d_f}\Big) + \frac{D_{\overline{g}}(T)}{1-q^{-1}T}\big(q_1(T)+\cdots+q_{\lambda-1}(T)\big) + q^{-(\alpha_\lambda n+\lambda m)}T^{\alpha_\lambda d_f}\, Z_e(T).$$

Combing this formula and the formula in the case (i), we obtain the formula in the case $(\mathrm{ii})_a$.

$(\mathrm{ii})_b$. At first we consider the case of $e = 1, \dots, d_f - 1$; in this case $\lambda = 0$ and $e = r_1$. Observing Table 6, we have

$$Z_e(T) = \frac{D_{\overline{f}}(T)}{1-q^{-1}T}\big(1+P(T)\big) + \frac{D_{\overline{g}}(T)}{1-q^{-1}T}\big(q^{-n}T^e + q^{-n}T^e Q(T)\big) + q^{-(e_g n+e_f m)}T^\ell Z_e(T).$$

Hence we obtain

$$Z_e(T) = \frac{D_{\overline{f}}(T)\big(1+P_e(T)\big) + D_{\overline{g}}(T)q^{-n}T^e\big(1+Q_e(T)\big)}{(1-q^{-1}T)\big(1-q^{-(e_g n+e_f m)}T^\ell\big)},$$

which is the formula stated in the theorem.

Next we consider the otherwise case. We take λ and r_1 as in the theorem, then, as stated in the above, we observe that the end term (ν, μ) of the λ-th row of Table 6 is $\big(0, d\big(\overline{\lambda r_{g/f}}\big) + r_1\big)$, and this term corresponds to $Z_e(T)$. Therefore we have

$$Z_{r_1}(T) = \frac{D_{\overline{f}}(T)}{1-q^{-1}T}\Big(1+p_1(T)+\cdots+p_\lambda(T) - q^{-(\alpha_\lambda n+\lambda m)}T^{\alpha_\lambda d_f}\Big) + \frac{D_{\overline{g}}(T)}{1-q^{-1}T}\big(q_1(T)+\cdots+q_{\lambda-1}(T)\big) + q^{-(\alpha_\lambda n+\lambda m)}T^{\alpha_\lambda d_f}\, Z_e(T).$$

Combing this formula and the formula in the above, we obtain the formula in the case $(\mathrm{ii})_b$. Q.E.D.

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	1	2	1	2	$\cdots$	$d_{g/f}-1$	$d_{g/f}$
1	(d_f, d_g) d_f $(0, d_g - d_f)$		$(d_f, d_g - d_f)$ d_f $(0, d_g - 2d_f)$	$(d_f, d_g - 2d_f)$ d_f $(0, d_g - 3d_f)$	$\cdots$	$(d_f, d_f + d_{(f,g)}0\underline{r_{g/f}})$ d_f $(0, d_{(f,g)}\overline{r_{g/f}})$	skip
2	$(d_f, d_{(f,g)}\overline{r_{r/g}})$ $d_{(f,g)}\overline{r_{g/f}}$ $(d_g - d_f + d_{(f,g)}\overline{r_{r/g}}, 0)$	$(d_f - d_{(f,g)}\overline{r_{g/f}}, d_g)$ $d_f - d_{(f,g)}\overline{r_{g/f}}$ $(0, d_g - d_f + d_{(f,g)}\overline{r_{g/f}})$	$(d_f, d_g - d_f + d_{(f,g)}\overline{r_{g/f}})$ d_f $(0, d_g - 2d_f + d_{(f,g)}\overline{r_{g/f}})$	$(d_f, d_g - 2d_f + d_{(f,g)}\overline{r_{g/f}})$ d_f $(0, d_g - 3d_f + d_{(f,g)}\overline{r_{g/f}})$	$\cdots$	$(d_f, d_f + d_{(f,g)}\underline{r_{g/f}})$ d_f $(0, d_{(f,g)}\overline{2r_{g/f}})$	skip
3	$(d_f, d_{(f,g)}\overline{2r_{r/g}})$ $d_{(f,g)}\overline{2r_{g/f}}$ $(d_g - d_f + d_{(f,g)}\overline{2r_{r/g}}, 0)$	$(d_f - d_{(f,g)}\overline{2r_{g/f}}, d_g)$ $d_f - d_{(f,g)}\overline{2r_{g/f}}$ $(0, d_g - d_f + d_{(f,g)}\overline{2r_{g/f}})$	$(d_f, d_g - d_f + d_{(f,g)}\overline{2r_{g/f}})$ d_f $(0, d_g - 2d_f + d_{(f,g)}\overline{2r_{g/f}})$	$(d_f, d_g - 2d_f + d_{(f,g)}\overline{2r_{g/f}})$ d_f $(0, d_g - 3d_f + d_{(f,g)}\overline{2r_{g/f}})$	$\cdots$	$(d_f, d_f + d_{(f,g)}2\underline{r_{g/f}})$ d_f $(0, d_{(f,g)}2\underline{r_{g/f}})$	$(d_f, d_{(f,g)}2\underline{r_{g/f}})$ d_f $(0, d_{(f,g)}\overline{3r_{g/f}})$
4	$(d_f, d_{(f,g)}\overline{3r_{r/g}})$ $d_{(f,g)}\overline{3r_{g/f}}$ $(d_g - d_f + d_{(f,g)}\overline{3r_{r/g}}, 0)$	$(d_f - d_{(f,g)}\overline{3r_{g/f}}, d_g)$ $d_f - d_{(f,g)}\overline{3r_{g/f}}$ $(0, d_g - d_f + d_{(f,g)}\overline{3r_{g/f}})$	$(d_f, d_g - d_f + d_{(f,g)}\overline{3r_{g/f}})$ d_f $(0, d_g - 2d_f + d_{(f,g)}\overline{3r_{g/f}})$	$(d_f, d_g - 2d_f + d_{(f,g)}\overline{3r_{g/f}})$ d_f $(d_f, d_g - 3d_f + d_{(f,g)}\overline{3r_{g/f}})$	$\cdots$	$(d_f, d_f + d_{(f,g)}3\underline{r_{g/f}})$ d_f $(0, d_{(f,g)}\overline{4r_{g/f}})$	skip
$\vdots$			$\vdots$				
$l+1$	$(d_f, d_{(f,g)}\overline{lr_{r/g}})$ $d_{(f,g)}\overline{lr_{g/f}}$ $(d_g - d_f + d_{(f,g)}\overline{lr_{r/g}}, 0)$	$(d_f - d_{(f,g)}\overline{lr_{g/f}}, d_g)$ $d_f - d_{(f,g)}\overline{lr_{g/f}}$ $(0, d_g - d_f + d_{(f,g)}\overline{lr_{g/f}})$	$(d_f, d_g - d_f + d_{(f,g)}\overline{lr_{g/f}})$ d_f $(0, d_g - 2d_f + d_{(f,g)}\overline{lr_{g/f}})$	$(d_f, d_g - 2d_f + d_{(f,g)}\overline{lr_{g/f}})$ d_f $(0, d_g - 3d_f + d_{(f,g)}\overline{lr_{g/f}})$	$\cdots$	$(d_f, d_f + d_{(f,g)}l\underline{r_{g/f}})$ d_f $(0, d_{(f,g)}l\underline{r_{g/f}})$	$(d_f, d_{(f,g)}l\underline{r_{g/f}})$ d_f $(0, d_{(f,g)}\overline{(l+1)r_{g/f}})$
$\vdots$			$\vdots$				
e_f	$(d_f, d_{(f,g)}\overline{(e_f-1)r_{r/g}})$ $d_f - d_{(f,g)}r_{g/f}$ $(d_f - d_{(f,g)}\overline{(e_f-1)r_{r/g}}, 0)$	$(d_f - d_{(f,g)}\overline{(e_f-1)r_{g/f}}, d_g)$ $d_{(f,g)}r_{g/f}$ $(0, d_{(f,g)}d_f)$	$(d_f, d_{(f,g)}d_f)$ d_f $(0, (d_{(f,g)}-1)d_f)$	$(d_f, (d_{(f,g)}-1)d_f$ d_f $(0, (d_{(f,g)}-2)d_f)$	$\cdots$	$(d_f, 2d_f)$ d_f $(0, d_f)$	(d_f, d_f) d_f $(0, 0)$

[illegible] I

	1	2	1	2	$\cdots$	$d_{g/f}-1$	$d_{g/f}$
1	(d_f,e) e $(d_f-e,0)$	(d_f-e,d_g) d_f-e $(0,d_g-d_f+e)$	(d_f,d_g-d_f+e) d_f $(0,d_g-2d_f+e)$	(d_f,d_g-2d_f+e) d_f $(0,d_g-3d_f+e)$	$\cdots$	$(d_f,d_f+d_{(f,g)}0\underline{r_{g/f}}+e)$ d_f $(0,d_{(f,g)}\overline{r_{g/f}}+e)$	skip
2	$(d_f,d_{(f,g)}\overline{r_{r/g}}+e)$ $d_{(f,g)}\overline{r_{g/f}}+e$ $(d_g-d_f+d_{(f,g)}\overline{r_{r/g}}-e,0)$	$(d_f-d_{(f,g)}\overline{r_{g/f}}-e,d_g)$ $d_f-d_{(f,g)}\overline{r_{g/f}}-e$ $(0,d_g-d_f+d_{(f,g)}\overline{r_{g/f}}+e)$	$(d_f,d_g-d_f+d_{(f,g)}\overline{r_{g/f}}+e)$ d_f $(0,d_g-2d_f+d_{(f,g)}\overline{r_{g/f}}+e)$	$(d_f,d_g-2d_f+d_{(f,g)}\overline{r_{g/f}}+e)$ d_f $(0,d_g-3d_f+d_{(f,g)}\overline{r_{g/f}}+e)$	$\cdots$	$(d_f,d_f+d_{(f,g)}\underline{r_{g/f}}+e)$ d_f $(0,d_{(f,g)}\overline{2r_{g/f}}+e)$	skip
3	$(d_f,d_{(f,g)}\overline{2r_{r/g}}+e)$ $d_{(f,g)}\overline{2r_{g/f}}+e$ $(d_g-d_f+d_{(f,g)}\overline{2r_{r/g}}-e,0)$	$(d_f-d_{(f,g)}\overline{2r_{g/f}}-e,d_g)$ $d_f-d_{(f,g)}\overline{2r_{g/f}}-e$ $(0,d_g-d_f+d_{(f,g)}\overline{2r_{g/f}}+e)$	$(d_f,d_g-d_f+d_{(f,g)}\overline{2r_{g/f}}+e)$ d_f $(0,d_g-2d_f+d_{(f,g)}\overline{2r_{g/f}}+e)$	$(d_f,d_g-2d_f+d_{(f,g)}\overline{2r_{g/f}}+e)$ d_f $(0,d_g-3d_f+d_{(f,g)}\overline{2r_{g/f}}+e)$	$\cdots$	$(d_f,d_f+d_{(f,g)}\underline{2r_{g/f}}+e)$ d_f $(0,d_{(f,g)}\underline{2r_{g/f}}+e)$	$(d_f,d_{(f,g)}\underline{2r_{g/f}}+e)$ d_f $(0,d_{(f,g)}\overline{3r_{g/f}}+e)$
4	$(d_f,d_{(f,g)}\overline{3r_{r/g}}+e)$ $d_{(f,g)}\overline{3r_{g/f}}+e$ $(d_g-d_f+d_{(f,g)}\overline{3r_{r/g}}-e,0)$	$(d_f-d_{(f,g)}\overline{3r_{g/f}}-e,d_g)$ $d_f-d_{(f,g)}\overline{3r_{g/f}}-e$ $(0,d_g-d_f+d_{(f,g)}\overline{3r_{g/f}}+e)$	$(d_f,d_g-d_f+d_{(f,g)}\overline{3r_{g/f}}+e)$ d_f $(0,d_g-2d_f+d_{(f,g)}\overline{3r_{g/f}}+e)$	$(d_f,d_g-2d_f+d_{(f,g)}\overline{3r_{g/f}}+e)$ d_f $(d_f,d_g-3d_f+d_{(f,g)}\overline{3r_{g/f}}+e)$	$\cdots$	$(d_f,d_f+d_{(f,g)}\underline{3r_{g/f}}+e)$ d_f $(0,d_{(f,g)}\overline{4r_{g/f}}+e)$	skip
$\vdots$	$\vdots$						
$l+1$	$(d_f,d_{(f,g)}\overline{lr_{r/g}}+e)$ $d_{(f,g)}\overline{lr_{g/f}}+e$ $(d_g-d_f+d_{(f,g)}\overline{lr_{r/g}}-e,0)$	$(d_f-d_{(f,g)}\overline{lr_{g/f}}-e,d_g)$ $d_f-d_{(f,g)}\overline{lr_{g/f}}-e$ $(0,d_g-d_f+d_{(f,g)}\overline{lr_{g/f}}+e)$	$(d_f,d_g-d_f+d_{(f,g)}\overline{lr_{g/f}}+e)$ d_f $(0,d_g-2d_f+d_{(f,g)}\overline{lr_{g/f}}+e)$	$(d_f,d_g-2d_f+d_{(f,g)}\overline{lr_{g/f}}+e)$ d_f $(0,d_g-3d_f+d_{(f,g)}\overline{lr_{g/f}}+e)$	$\cdots$	$(d_f,d_f+d_{(f,g)}\underline{lr_{g/f}}+e)$ d_f $(0,d_{(f,g)}\underline{lr_{g/f}}+e)$	$(d_f,d_{(f,g)}\underline{lr_{g/f}}+e)$ d_f $(0,d_{(f,g)}\overline{(l+1)r_{g/f}}+e)$
$\vdots$	$\vdots$						
e_f	$(d_f,d_{(f,g)}\overline{(e_f-1)r_{r/g}}+e)$ $d_f-d_{(f,g)}r_{g/f}+e$ $(d_f-d_{(f,g)}\overline{(e_f-1)r_{r/g}}-e,0)$	$(d_f-d_{(f,g)}\overline{(e_f-1)r_{g/f}}-e,d_g)$ $d_{(f,g)}r_{g/f}-e$ $(0,d_{(f,g)}d_f+e)$	$(d_f,d_{(f,g)}d_f+e)$ d_f $(0,(d_{(f,g)}-1)d_f+e)$	$(d_f,(d_{(f,g)}-1)d_f+e)$ d_f $(0,(d_{(f,g)}-2)d_f+e$	$\cdots$	$(d_f,2d_f+e)$ d_f $(0,d_f+e)$	(d_f,d_f+e) d_f $(0,e)$

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