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TEXTILE SYSTEMS ON LAMBDA-GRAPH SYSTEMS

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Abstract. The notions of symbolic matrix system and λ-graph system for a subshift are generalizations of symbolic matrix and λ-graph (= finite symbolic matrix) for a sofic shift respectively ([Doc. Math. 4(1999), 285-340]). M. Nasu introduced the notion of textile system for a pair of graph homomorphisms to study automorphisms and endomorphisms of topological Markov shifts ([Mem. Amer. Math. Soc. 546,114(1995)]). In this paper, we formulate textile systems on λ-graph systems and study automorphisms on subshifts. We will prove that for a forward automorphism ϕ of a subshift (Λ, σ), the automorphisms $\phi^k\sigma^n, k \geq 0, n \geq 1$ can be explicitly realized as a subshift defined by certain symbolic matrix systems coming from both the strong shift equivalence representing ϕ and the subshift (Λ, σ). As an application of this result, if an automorphism ϕ of a subshift Λ is a simple automorphism, the dynamical system $(\Lambda, \phi \circ \sigma)$ is topologically conjugate to the subshift (Λ, σ).

1. Introduction

Let Σ be a finite set with its discrete topology, that is called an alphabet. Let $\Sigma^{\mathbb{Z}}$ be the compact Hausdorff space of all bi-infinite sequences of Σ. One has the homeomorphism σ defined by the left-shift that sends a point $(\alpha_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ into the point $(\alpha_{i+1})_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$. A subshift (Λ, σ) is the topological dynamical system that is obtained by restricting the shift to a closed shift-invariant subset Λ of $\Sigma^{\mathbb{Z}}$. The space $\Lambda \subset \Sigma^{\mathbb{Z}}$ is uniquely determined by a set of forbidden words, such as a sequence $(x_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ of Σ belongs to Λ if and only if any word in the forbidden words can not apper as a subward of $(x_i)_{i\in\mathbb{Z}}$. If a subshift is obtained by a finite set of its forbidden words, it is said to be a shift of finite type. It is well-known that the class of shifts of finite type coincides with the class of topological Markov shifts, that are defined by finite square nonnegative matrices. For an introduction to the theory of topological Markov shifts see [8] or [13]. R. F. Williams [23] proved that two shifts of finite type are topologically conjugate if and only if their defining nonnegative matrices are strong shift equivalent. This result also says a structure of automorphisms

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of topological Markov shifts. That is, an automorphism is given by a strong shift equivalence from the defining matrix to itself, and conversely a strong shift equivalence from the defining matrix to itself gives rise to an automorphism of the shift of finite type. M. Nasu [18] formulated strong shift equivalence between finite symbolic matrices and generalized the above Williams's result to sofic shifts. He proved that two sofic shifts are topologically conjugate if and only if their canonical symbolic matrices are strong shift equivalent ([18]).

For a subshift (Λ, σ), a homeomorphism φ on Λ satisfying $\varphi \circ \sigma = \sigma \circ \varphi$ is called an automorphism of (Λ, σ). It is also well-known that if an automorphism φ of a subshift (Λ, σ) is expansive, it is topologically conjugate to a subshift [5]. The problem studied in this paper is to *subshift-identify* the dynamical system (Λ, φ). Namely, for an expansive automorphism φ of a subshift (Λ, σ), the problem is to find, in an explicit way, a subshift $(\Lambda_\varphi, \sigma)$ that is topologically conjugate to (Λ, φ). This problem has been studied in several situations for the case of topological Markov shifts and sofic shifts. Boyle and Krieger [1] proved that for an automorphism φ of topological Markov shift (Λ_A, σ_A) defined by a nonnegative matrix A and for all integers n greater than a coding bound for φ and φ^{-1}, the dynamical system $(\Lambda_A, \varphi\sigma_A^n)$ is topologically conjugate to a topological Markov shift, and specified its dimension triple. M. Nasu [19] has introduced the notion of textile system, which is very useful to analyze the automorphisms and endomorphisms of topological Markov shifts. A textile system is defined by an ordered pair of graph homomorphisms p and q of a directed finite graph Γ into a directed finite graph G, that is written as $T = (p, q : \Gamma \to G)$. Nasu also generaized the fiomulation of the textile systems to textile systems on finite labeled graphs. Among other things, he proved that if φ is a forward automorphism of a sofic shift $(\Lambda_{\mathcal{A}}, \sigma_{\mathcal{A}})$ defined by a finite symbolic matrix $\mathcal{A}$ and is given by a strong shift equivalence

$$\mathcal{A} \overset{\kappa_0}{\simeq} \mathcal{P}_1\mathcal{Q}_1, \quad \mathcal{Q}_1\mathcal{P}_1 \overset{\kappa_1}{\simeq} \mathcal{P}_2\mathcal{Q}_2, \quad \cdots, \quad \mathcal{Q}_{N-1}\mathcal{P}_{N-1} \overset{\kappa_{N-1}}{\simeq} \mathcal{P}_N\mathcal{Q}_N, \quad \mathcal{Q}_N\mathcal{P}_N \overset{\kappa_N}{\simeq} \mathcal{A},$$

then the dynamical system $(\Lambda_{\mathcal{A}}, \varphi^k\sigma_{\mathcal{A}}^n)$ is topologically conjugate to the sofic shift defined by the symbolic matrix $\mathcal{P}^k\mathcal{A}^n$ for all $k \geq 0$ and $n \geq 1$, where $\mathcal{P} = \mathcal{P}_1 \cdots \mathcal{P}_N$. If in particular φ is expansive, the dynamical system $(\Lambda_{\mathcal{A}}, \varphi)$ is topologically conjugate to a sofic shift.

In [15], the author has introduced the notion of symbolic matrix system and λ-graph system from an idea of C^*-algebras (cf. [14], [16]). The symbolic matrix system is a generalization of symbolic matrix, and λ-graph system is a generalization of λ-graph (= finite labeled graph). We henceforth denote by $\mathbb{Z}_+$ the set of all nonnegative integers and by $\mathbb{N}$ the set of all positive integers. A symbolic matrix system over alphabet Σ consists of two sequences of rectangular matrices $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{Z}_+$. The matrices $\mathcal{M}_{l,l+1}, l \in \mathbb{Z}_+$ have their entries

in formal sums of Σ and the matrices $I_{l,l+1}, l \in \mathbb{Z}_+$ have their entries in $\{0,1\}$. They satisfy the commutation relations:

$$I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{Z}_+.$$

We further assume that each row of $I_{l,l+1}$ has at least one 1 and each column of $I_{l,l+1}$ has exactly one 1. We denote $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{Z}_+$ by $(\mathcal{M}, I)$ or $(\mathcal{M}, I^{\mathcal{M}})$. A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ consists of a vertex set $V = V_0 \cup V_1 \cup V_2 \cup \cdots$, an edge set $E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \cdots$, a labeling $\lambda : E \to \Sigma$ and a surjective map $\iota(= \iota_{l,l+1}) : V_{l+1} \to V_l$ for each $l \in \mathbb{Z}_+$. It naturally arises from a symbolic matrix system $(\mathcal{M}, I)$. The edges from a vertex $v_i^l \in V_l$ to a vertex $v_j^{l+1} \in V_{l+1}$ are given by the (i,j)-component $\mathcal{M}_{l,l+1}(i,j)$ of the matrix $\mathcal{M}_{l,l+1}$. The matrix $I_{l,l+1}$ defines a surjection $\iota_{l,l+1}$ from V_{l+1} to V_l for each $l \in \mathbb{Z}_+$. The symbolic matrix systems and the λ-graph systems are the same objects and give rise to subshifts by gathering all the label sequences appearing in the labeled Bratteli diagram. A canonical method to construct a symbolic matrix system and a λ-graph system from an arbitrary subshift has been introduced in [15]. The obtained symbolic matrix system and the λ-graph system are said to be canonical for the subshift. The notion of strong shift equivalence for nonnegative matrices and symbolic matrices has been generalized to symbolic matrix systems as properly strong shift equivalence. Two symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *properly strong shift equivalent in 1-step* if there exist alphabets C, D and specifications $\kappa : \Sigma \to CD$, $\kappa' : \Sigma' \to DC$ and increasing sequences $n(l), n'(l)$ on $l \in \mathbb{Z}_+$ such that for each $l \in \mathbb{Z}_+$, there exist an $n(l) \times n'(l+1)$ matrix $\mathcal{P}_l$ over C, an $n'(l) \times n(l+1)$ matrix $\mathcal{Q}_l$ over D, an $n(l) \times n(l+1)$ matrix X_l over $\{0,1\}$ and an $n'(l) \times n'(l+1)$ matrix Y_l over $\{0,1\}$ satisfying the following equations:

$$\begin{aligned} \mathcal{M}_{l,l+1} &\overset{\kappa}{\simeq} \mathcal{P}_{2l}\mathcal{Q}_{2l+1}, & \mathcal{M}'_{l,l+1} &\overset{\kappa'}{\simeq} \mathcal{Q}_{2l}\mathcal{P}_{2l+1}, \\ I_{l,l+1} &= X_{2l}X_{2l+1}, & I'_{l,l+1} &= Y_{2l}Y_{2l+1} \end{aligned}$$

and

$$X_l\mathcal{P}_{l+1} = \mathcal{P}_lY_{l+1}, \qquad Y_l\mathcal{Q}_{l+1} = \mathcal{Q}_lX_{l+1}.$$

This situation is written as $(\mathcal{P}, \mathcal{Q}, X, Y) : (\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}', I')$. A finite chain of properly strong shift equivalences in 1-step with length N is called a properly strong shift equivalence (in N-step). Then the previously mentioned Williams's result and Nasu's result have been generalized to topological conjugacy between subshifts. That is, if two symbolic matrix systems are properly strong shift equivalent, then their presented subshifts are topologically conjugate. Furthermore, two subshifts are topologically conjugate if and only if their canonical symbolic

matrix systems are properly strong shift equivalent ([15]). Hence, in particular, a properly strong shift equivalence from a symbolic matrix system to itself gives rise to an automorphism of the presented subshift. And an automorphism of a subshift exactly corresponds to a properly strong shift equivalence from the canonical symbolic matrix system of the subshift to itself.

In this paper, we will generalize the Nasu's textile systems for graph homomorphisms between finite directed (labeled) graphs to graph homomorphisms between λ-graph systems and generalize the Nasu's formalism for topological Markov shifts and sofic shifts to general subshifts. Namely we will formulate textile systems for graph homomorphisms between λ-graph systems and study automorphisms of general subshifts by using the generalized textile systems. Let $(\mathcal{M}, I^{\mathcal{M}}), (\mathcal{K}, I^{\mathcal{K}}), (\mathcal{N}, I^{\mathcal{N}})$ be symbolic matrix systems and $\mathfrak{L}^{\mathcal{K}}, \mathfrak{L}^{\mathcal{M}}, \mathfrak{L}^{\mathcal{N}}$ their respect λ-graph systems. Assume that the vertex sets $V_l^{\mathcal{M}}$ of $\mathfrak{L}^{\mathcal{M}}$ and the vertex sets $V_l^{\mathcal{N}}$ of $\mathfrak{L}^{\mathcal{N}}$ coincide and that the condition $I^{\mathcal{M}}_{l,l+1} = I^{\mathcal{N}}_{l,l+1}$ hold for all $l \in \mathbb{Z}_+$. We further assume that the vertex set $V_l^{\mathcal{K}}$ of $\mathfrak{L}^{\mathcal{K}}$ is identified with the edge set $E^{\mathcal{N}}_{l,l+1}$ of $\mathfrak{L}^{\mathcal{N}}$ for $l \in \mathbb{Z}_+$. A label preserving graph homomorphism $p : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ compatible to ι is called a λ-graph system homomorphism if $p(V_l^{\mathcal{K}}) = V_l^{\mathcal{M}}, l \in \mathbb{Z}_+$. A label preserving graph homomorphism $q : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ compatible to ι is called a one-shift λ-graph system homomorphism if $q(V_l^{\mathcal{K}}) = V_{l+1}^{\mathcal{M}}, l \in \mathbb{Z}_+$. Hence the source map $s^{\mathcal{K}} : E^{\mathcal{K}}_{l,l+1} \longrightarrow V_l^{\mathcal{K}} = E^{\mathcal{N}}_{l,l+l}$ and the terminal map $t^{\mathcal{K}} : E^{\mathcal{K}}_{l,l+1} \longrightarrow V_{l+1}^{\mathcal{K}} = E^{\mathcal{N}}_{l+1,l+2}$ of $\mathfrak{L}^{\mathcal{K}}$ yield a λ-graph system homomorphism and a one-shift λ-graph system homomorphism respectively. Then for a λ-graph system homomorphism $p : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ and a one-shift λ-graph system homomorphism $q : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$, the diagram

$$\begin{array}{ccccc} & & \mathfrak{L}^{\mathcal{M}} & & \\ & & \uparrow{\scriptstyle p} & & \\ \mathfrak{L}^{\mathcal{N}} & \xleftarrow{s^{\mathcal{K}}} & \mathfrak{L}^{\mathcal{K}} & \xrightarrow{t^{\mathcal{K}}} & \mathfrak{L}^{\mathcal{N}} \\ & & \downarrow{\scriptstyle q} & & \\ & & \mathfrak{L}^{\mathcal{M}} & & \end{array}$$

is called a *textile system on λ-graph systems* if some further conditions are satisfied. It is written as $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$. This formulation is a generalization of Nasu's sofic textile systems [19]. We will follow and generalize Nasu's machinery of [19] so that the dual of $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ can be defined and we may consider LR textile systems on λ-graph systems. We will prove that for a forward automorphism ϕ of a subshift (Λ, σ), the automorphisms $\phi^k\sigma^n, k \geq 0, n \geq 1$ can be explicitly realized as a subshift defined by certain symbolic matrix systems coming from both the strong shift equivalence representing ϕ and the subshift (Λ, σ). Suppose that Λ is equipped with a metric for which σ has 1 as its expansive constant. If in

particular, ϕ is expansive with $\frac{1}{m}$ as its expansive constant for some $m \in \mathbb{N}$, the dynamical system (Λ, ϕ) can be realized as a subshift defined by certain symbolic matrix system coming from the strong shift equivalence representing ϕ and the subshift (Λ, σ) (Theorem 7.6).

We will prove the following

THEOREM 1.1 (Theorem 7.8). *Let $(\Lambda, \sigma)(= (\Lambda_{\mathcal{M}}, \sigma_{\mathcal{M}}))$ be a subshift presented by a symbolic matrix system $(\mathcal{M}, I)$. Let ϕ be a forward automorphism on (Λ, σ) defined by a properly strong shift equivalence*

$$(\mathcal{P}^{(j)}, \mathcal{Q}^{(j)}, X^{(j)}, Y^{(j)}) : (\mathcal{M}^{(j-1)}, I^{(j-1)}) \underset{1-pr}{\approx} (\mathcal{M}^{(j)}, I^{(j)}), \quad j = 1, 2, \dots, N$$

in N-step where $(\mathcal{M}^{(0)}, I^{(0)}) = (\mathcal{M}^{(N)}, I^{(N)}) = (\mathcal{M}, I)$. Then the dynamical system $(\Lambda, \phi^k\sigma^n)$ is topologically conjugate to the subshift $(\Lambda_{\mathcal{P}^k\mathcal{M}^n}, \sigma_{\mathcal{P}^k\mathcal{M}^n})$ presented by the symbolic matrix system $(\mathcal{P}^k\mathcal{M}^n, I^{kN+n})$ for $k \geq 0, n \geq 1$ defined by

$$\begin{aligned}
&(\mathcal{P}^k\mathcal{M}^n)_{l,l+1} \\
=&\mathcal{P}_{l(kN+n),l(kN+n)+N}\mathcal{P}_{l(kN+n)+N,l(kN+n)+2N}\cdots\mathcal{P}_{l(kN+n)+(k-1)N,l(kN+n)+kN}\cdot \\
&\cdot\mathcal{M}_{l(kN+n)+kN,l(kN+n)+kN+1}\mathcal{M}_{l(kN+n)+kN+1,l(kN+n)+kN+2} \\
&\cdots\mathcal{M}_{(l+1)(kN+n)-1,(l+1)(kN+n)},\ I^{kN+n}_{l,l+1} \\
=&I_{l(kN+n),l(kN+n)+1}I_{l(kN+n)+1,l(kN+n)+2}\cdots I_{(l+1)(kN+n)-1,(l+1)(kN+n)},\ l \in \mathbb{Z}_+
\end{aligned}$$

where $\mathcal{P}_{l,l+1} = \mathcal{P}^{(1)}_{2l}Y^{(1)}_{2l+1}\mathcal{P}^{(2)}_{2l+2}Y^{(2)}_{2l+3}\cdots\mathcal{P}^{(N)}_{2l+2N-2}Y^{(N)}_{2l+2N-1}$ and $\mathcal{P}^{(i)}_{2l+2(i-1)}$, $Y^{(i)}_{2l+2i-1}$, $i = 1, \dots, N$ are matrices appearing in the above properly strong shift equivalence.

Namely these automorphisms $\phi^k\sigma^n, k \geq 0, n \geq 1$ are subshift-identified. As an application of this result, if an automorphism ϕ of a subshift (Λ, σ) is a simple automorphism, that is conjugate to a symbolic automorphism fixing vertices of a λ-graph system, the dynamical system $(\Lambda, \phi \circ \sigma^n)$ is topologically conjugate to the n-th power (Λ, σ^n) of the subshift (Λ, σ) for $n \in \mathbb{Z}, n \neq 0$ (Theorem 8.2).

This paper is organized as in the following way.

1. Introduction
2. Symbolic matrix systems and λ-graph systems
3. Textile systems on λ-graph systems
4. Textile shifts on λ-graph systems
5. LR textile systems on λ-graph systems
6. LR textile systems and properly strong shift equivalences

7. Subshift-identifications of automorphisms of subshifts
8. An application

2. Symbolic matrix systems and λ-graph systems

We call each element of a finite set Σ a symbol or a label. The transformation σ on the infinite product space $\Sigma^{\mathbb{Z}}$ given by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$ is called the (full) shift. Let Λ be a shift-invariant closed subset of $\Sigma^{\mathbb{Z}}$ i.e. $\sigma(\Lambda) = \Lambda$. We write the subshift (Λ, σ) as Λ for short. We denote by $\Lambda^+ (\subset \Sigma^{\mathbb{N}})$ the set of all right-infinite sequences $(x_i)_{i\in\mathbb{N}}$ that $(x_i)_{i\in\mathbb{Z}}$ belongs to Λ. A finite sequence $\mu = (\mu_1, ..., \mu_k)$ of elements $\mu_j \in \Sigma$ is called a block or a word of length k. We write the empty symbol $\emptyset$ in Σ as 0. We denote by $\mathfrak{S}_\Sigma$ the set of all finite formal sums of elements of Σ. By a symbolic matrix $\mathcal{A}$ over Σ we mean a finite matrix with entries in $\mathfrak{S}_\Sigma$. A square symbolic matrix $\mathcal{A}$ naturally gives rise to a labeled directed graph, called a λ-graph, which we denote by $G_{\mathcal{A}}$. The labeled directed graph defines a subshift over Σ which consist of all infinite labeled sequences following the labeled edges in $G_{\mathcal{A}}$. Such a subshift is called a sofic shift presented by $G_{\mathcal{A}}$ (cf. [4], [9], [10], [22], [8], [13]). If, in particular, different edges have different labels, the sofic shift is called a topological Markov shift.

Let $\mathcal{A}$ and $\mathcal{A}'$ be symbolic matrices over Σ and Σ' respectively such that the size of $\mathcal{A}$ is the same as that of $\mathcal{A}'$. Let κ be a bijection from a subset of Σ onto a subset of Σ'. Following M. Nasu in [18],[19], we say that $\mathcal{A}$ and $\mathcal{A}'$ are specified equivalence under specification κ if $\mathcal{A}'$ can be obtained from $\mathcal{A}$ by replacing every symbol a appearing in the components of $\mathcal{A}$ by $\kappa(a)$. We write it as $\mathcal{A} \overset{\kappa}{\simeq} \mathcal{A}'$.

Two symbolic matrix systems $(\mathcal{M}, I)$ over Σ and $(\mathcal{M}', I')$ over Σ' are said to be isomorphic if for $l \in \mathbb{Z}_+$ the size of $\mathcal{M}_{l,l+1}$ coincides with that of $\mathcal{M}'_{l,l+1}$ and there exist a specification κ from Σ to Σ' and an $m(l)\times m(l)$-square permutation matrix S_l for each $l \in \mathbb{Z}_+$ such that

$$S_l\mathcal{M}_{l,l+1} \overset{\kappa}{\simeq} \mathcal{M}'_{l,l+1}S_{l+1}, \qquad S_lI_{l,l+1} = I'_{l,l+1}S_{l+1}.$$

Recall that a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ is a directed Bratteli diagram with vertex set

$$V = \cup_{l\in\mathbb{Z}_+} V_l$$

and edge set

$$E = \cup_{l\in\mathbb{Z}_+} E_{l,l+1}$$

that is labeled by a map $\lambda(= \lambda_{l,l+1}) : E_{l,l+1} \to \Sigma$ with symbols in Σ for $l \in \mathbb{Z}_+$, and that is supplied with a sequence of surjective maps

$$\iota(= \iota_{l,l+1}) : V_{l+1} \to V_l \qquad \text{for} \quad l \in \mathbb{Z}_+.$$

Here each vertex set V_l and each edge set $E_{l,l+1}$ are both finite sets. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l and its terminal vertex $t(e)$ in V_{l+1}. Every vertex in V has a successor and every vertex in V, except the verteces in V_0 at level 0, has a predecessor. It is then required that there exists an edge in $E_{l,l+1}$ with label α and its terminal is $v \in V_{l+1}$ if and only if there exists an edge in $E_{l-1,l}$ with label α and its terminal is $\iota(v) \in V_l$. A λ-graph system is said to be essential if there is no distinct edges that have the same source vertices, the same terminal vertices and the same labels. Throughout this paper, we will treat essential λ-graph systems. For $u \in V_{l-1}$ and $v \in V_{l+1}$, we put

$$E^{\iota}_{l,l+1}(u,v) = \{e \in E_{l,l+1} \mid t(e) = v, \iota(s(e)) = u\},$$
$$E^{l-1,l}_{\iota}(u,v) = \{e \in E_{l-1,l} \mid s(e) = u, t(e) = \iota(v)\}.$$

Then there exists a bijective map $\varphi^{\mathfrak{L}}_{(u,v)}$ from $E^{\iota}_{l,l+1}(u,v)$ to $E^{l-1,l}_{\iota}(u,v)$ such that

$$\lambda(\varphi^{\mathfrak{L}}_{(u,v)}(e)) = \lambda(e) \qquad \text{for } e \in E^{\iota}_{l,l+1}(u,v).$$

Hence two sets $E^{\iota}_{l,l+1}(u,v)$ and $E^{l-1,l}_{\iota}(u,v)$ bijectively correspond in preserving labels for all pairs $(u,v) \in V_{l-1} \times V_{l+1}$. We call this property the local property of the λ-graph system. We immediately see

Lemma 2.1. *For a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ, there exists a surjection*

$$\varphi^{\mathfrak{L}}_{l} : E_{l,l+1} \longrightarrow E_{l-1,l}$$

for each $l \in \mathbb{N}$ such that

$$\varphi^{\mathfrak{L}}_{l}|_{E^{\iota}_{l,l+1}(u,v)} = \varphi^{\mathfrak{L}}_{(u,v)} \qquad \textit{for } u \in V_{l-1}, v \in V_{l+1}$$

and

$$\iota_{l-1,l}(s(e)) = s(\varphi^{\mathfrak{L}}_{l}(e)), \quad \iota_{l,l+1}(t(e)) = t(\varphi^{\mathfrak{L}}_{l}(e)) \qquad \textit{for } e \in E_{l,l+1}.$$

We call an edge $e \in E_{l,l+1}$ a λ-edge and a connecting finite sequence of λ-edges a λ-path. For $u \in V_l$ and $v \in V_{l+1}$, if $\iota(v) = u$, we say that there exists an ι-edge from u to v. Similarly we use the term ι-path.

Two λ-graph systems (V, E, λ, ι) over Σ and $(V', E', \lambda', \iota')$ over Σ' are said to be isomorphic if there exist bijections $\Phi_V : V_l \to V'_l$, $\Phi_E : E_{l,l+1} \to E'_{l,l+1}$ for $l \in \mathbb{Z}_+$, and a specification $\kappa : \Sigma \to \Sigma'$ such that $\Phi_V(s(e)) = s(\Phi_E(e))$, $\Phi_V(t(e)) = t(\Phi_E(e))$ and $\lambda'(\Phi_E(e)) = \kappa(\lambda(e))$ for $e \in E$, and $\iota'(\Phi_V(v)) = \Phi_V(\iota(v))$ for $v \in V$. There exists a bijective correspondence between the set of all isomorphism classes of symbolic matrix systems and the set of all isomorphism

classes of λ-graph systems. We identify isomorphic symbolic matrix systems, and similarly isomorphic λ-graph systems.

A symbolic matrix system $(\mathcal{M}, I)$ is denoted by $(\mathcal{M}, I^{\mathcal{M}})$ although the matrices $I_{l,l+1}$ are not determined by the symbolic matrices $\mathcal{M}_{l,l+1}, l \in \mathbb{Z}_+$. We denote its λ-graph system by $\mathfrak{L}^{\mathcal{M}} = (V^{\mathcal{M}}, E^{\mathcal{M}}, \lambda^{\mathcal{M}}, \iota^{\mathcal{M}})$. The surjections $\varphi_l^{\mathfrak{L}^{\mathcal{M}}} : E_{l,l+1}^{\mathcal{M}} \longrightarrow E_{l-1,l}^{\mathcal{M}}, l \in \mathbb{Z}_+$ defined in Lemma 2.1 are denoted by $\varphi_l^{\mathcal{M}}, l \in \mathbb{Z}_+$.

A λ-graph (= a finite labeled graph) defines a λ-graph system as in the following way. Let $\mathcal{G} = (G, \lambda)$ be a λ-graph with underlying finite directed graph G and its labeling λ. Let V^G be the vertex set of G. Put $V_l = V^G$ for all $l \in \mathbb{Z}_+$ and $\iota = \mathrm{id}$. Write labeled edges from V_l to V_{l+1} for $l \in \mathbb{Z}_+$ following the directed graph G with labeling λ. It is clear to see that the resulting labeled Bratteli diagram with $\iota(= \mathrm{id})$ becomes a λ-graph system. A λ-graph and also a λ-graph system are said to be left-resolving if different edges with the same label must have different terminals. By the construction if a λ-graph $\mathcal{G}$ is left-resolving, so is the above defined λ-graph system by $\mathcal{G}$. In what follows, we assume that a λ-graph system is left-resolving.

For a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ and a natural number $N \geq 2$, the N-higher block $\mathfrak{L}^{[N]}$ of $\mathfrak{L}$ is defined to be a λ-graph system $(V^{[N]}, E^{[N]}, \lambda^{[N]}, \iota^{[N]})$ over $\Sigma^{[N]} = \underbrace{\Sigma \cdots \Sigma}_{\text{N-times}}$ as follows ([15]):

$$\begin{aligned} V_l^{[N]} &= \{(e_1, e_2, \ldots, e_{N-1}) \in E_{l,l+1} \times E_{l+1,l+2} \times \cdots \times E_{l+N-2,l+N-1} \mid \\ &\qquad t(e_i) = s(e_{i+1}) \text{ for } i = 1, 2, \ldots, N-2\}, \\ E_{l,l+1}^{[N]} &= \{((e_1, \ldots, e_{N-1}),(f_1, \ldots, f_{N-1})) \in V_l^{[N]} \times V_{l+1}^{[N]} \mid \\ &\qquad t(e_{N-1}) = s(f_{N-1}), e_{i+1} = f_i \text{ for } i = 1, 2, \ldots, N-2\}. \end{aligned}$$

The maps

$$s^{[N]} : E_{l,l+1}^{[N]} \to V_l^{[N]}, \qquad t^{[N]} : E_{l,l+1}^{[N]} \to V_{l+1}^{[N]}$$

are defined by

$$\begin{aligned} s^{[N]}((e_1, \ldots, e_{N-1}), (f_1, \ldots, f_{N-1})) &= (e_1, \ldots, e_{N-1}), \\ t^{[N]}((e_1, \ldots, e_{N-1}), (f_1, \ldots, f_{N-1})) &= (f_1, \ldots, f_{N-1}). \end{aligned}$$

Set $V^{[N]} = \cup_{l \in \mathbb{Z}_+} V_l^{[N]}$ and $E^{[N]} = \cup_{l \in \mathbb{Z}_+} E_{l,l+1}^{[N]}$. Hence $(V^{[N]}, E^{[N]}, s^{[N]}, t^{[N]})$ is a Bratteli diagram. A labeling $\lambda^{[N]}$ on $(V^{[N]}, E^{[N]})$ is defined by

$$\lambda^{[N]}((e_1, \ldots, e_{N-1}), (f_1, \ldots, f_{N-1})) = \lambda(e_1)\lambda(e_2)\ldots\lambda(e_{N-1})\lambda(f_{N-1}) \in \Sigma^{[N]}$$

for $((e_1, \ldots, e_{N-1}), (f_1, \ldots, f_{N-1})) \in E^{[N]}$. A sequence of surjections $\iota^{[N]} : V_{l+1}^{[N]} \to V_l^{[N]}, l \in \mathbb{Z}_+$ is defined as follows. As the λ-graph system (V, E, λ, ι) is left-resolving, for $(e_1, \ldots, e_{N-1}) \in V_{l+1}^{[N]}$, there uniquely exist $e_i' \in E_{l+i-1,l+i}$ for

$i = 1, 2, \ldots, N-2$ such that

$$\iota(s(e_i)) = s(e'_i), \quad \iota(t(e_i)) = t(e'_i), \quad \lambda(e_i) = \lambda(e'_i).$$

As $(e'_1, \ldots, e'_{N-1}) \in V_l^{[N]}$, by setting $\iota^{[N]}(e_1, \ldots, e_{N-1}) = (e'_1, \ldots, e'_{N-1})$, we get a λ-graph system $(V^{[N]}, E^{[N]}, \lambda^{[N]}, \iota^{[N]})$ over $\Sigma^{[N]}$. We set $\mathfrak{L}^{[1]} = \mathfrak{L}$. The N-higher block $(\mathcal{M}^{[N]}, I^{[N]})$ of a symbolic matrix system $(\mathcal{M}, I)$ is defined to be the symbolic matrix system for the N-higher block $\mathfrak{L}^{[N]}$ of the λ-graph system $\mathfrak{L}$ for $(\mathcal{M}, I)$.

3. Textile systems on λ-graph systems

In what follows, let $(\mathcal{K}, I^{\mathcal{K}})$, $(\mathcal{M}, I^{\mathcal{M}})$ and $(\mathcal{N}, I^{\mathcal{N}})$ be symbolic matrix systems over alphabets $\Sigma^{\mathcal{K}}$, $\Sigma^{\mathcal{M}}$ and $\Sigma^{\mathcal{N}}$ respectively. Let us consider their respect λ-graph systems $\mathfrak{L}^{\mathcal{K}} = (V^{\mathcal{K}}, E^{\mathcal{K}}, \lambda^{\mathcal{K}}, \iota^{\mathcal{K}})$, $\mathfrak{L}^{\mathcal{M}} = (V^{\mathcal{M}}, E^{\mathcal{M}}, \lambda^{\mathcal{M}}, \iota^{\mathcal{M}})$ and $\mathfrak{L}^{\mathcal{N}} = (V^{\mathcal{N}}, E^{\mathcal{N}}, \lambda^{\mathcal{N}}, \iota^{\mathcal{N}})$. We denote by $(s^{\mathcal{K}}, t^{\mathcal{K}}), (s^{\mathcal{M}}, t^{\mathcal{M}})$ and $(s^{\mathcal{N}}, t^{\mathcal{N}})$ their source maps and terminal maps in the λ-graph systems respectively.

A *λ-graph system homomorphism* $p = (p^V, p^E, p^\Sigma) : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ consists of sequences of maps

$$p^V (= p_l^V) : V_l^{\mathcal{K}} \longrightarrow V_l^{\mathcal{M}}, \qquad p^E (= p_{l,l+1}^E) : E_{l,l+1}^{\mathcal{K}} \longrightarrow E_{l,l+1}^{\mathcal{M}}, \qquad l \in \mathbb{Z}_+$$

together with a map $p^\Sigma : \Sigma^{\mathcal{K}} \longrightarrow \Sigma^{\mathcal{M}}$ such that

(1) $p_l^V(s^{\mathcal{K}}(e)) = s^{\mathcal{M}}(p_{l,l+1}^E(e)), \quad p_{l+1}^V(t^{\mathcal{K}}(e)) = t^{\mathcal{M}}(p_{l,l+1}^E(e))$ for $e \in E_{l,l+1}^{\mathcal{K}}$,
(2) $p_l^V(\iota_{l,l+1}^{\mathcal{K}}(v)) = \iota_{l,l+1}^{\mathcal{M}}(p_{l+1}^V(v))$ for $v \in V_{l+1}^{\mathcal{K}}$,
(3) $p^\Sigma(\lambda^{\mathcal{K}}(e)) = \lambda^{\mathcal{M}}(p_{l,l+1}^E(e))$ for $e \in E_{l,l+1}^{\mathcal{K}}$.

We call it a homomorphism and write it as $p : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ for short.

A *one-shift λ-graph system homomorphism* $q = (q^V, q^E, q^\Sigma) : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ consists of sequences of maps

$$q^V (= q_l^V) : V_l^{\mathcal{K}} \longrightarrow V_{l+1}^{\mathcal{M}}, \qquad q^E (= q_{l,l+1}^E) : E_{l,l+1}^{\mathcal{K}} \longrightarrow E_{l+1,l+2}^{\mathcal{M}}, \qquad l \in \mathbb{Z}_+$$

together with a map $q^\Sigma : \Sigma^{\mathcal{K}} \longrightarrow \Sigma^{\mathcal{M}}$ such that

(1) $q_l^V(s^{\mathcal{K}}(e)) = s^{\mathcal{M}}(q_{l,l+1}^E(e)), \quad q_{l+1}^V(t^{\mathcal{K}}(e)) = t^{\mathcal{M}}(q_{l,l+1}^E(e))$ for $e \in E_{l,l+1}^{\mathcal{K}}$,
(2) $q_l^V(\iota_{l,l+1}^{\mathcal{K}}(v)) = \iota_{l+1,l+2}^{\mathcal{M}}(q_{l+1}^V(v))$ for $v \in V_{l+1}^{\mathcal{K}}$,
(3) $q^\Sigma(\lambda^{\mathcal{K}}(e)) = \lambda^{\mathcal{M}}(q_{l,l+1}^E(e))$ for $e \in E_{l,l+1}^{\mathcal{K}}$.

We call it a one-shift homomorphism and write it as $q : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ for short. For a one-shift homomorphism $q = (q^V, q^E, q^\Sigma) : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$, put $p_{q\ l}^V = \iota_{l,l+1}^{\mathcal{M}} \circ q_l^V : V_l^{\mathcal{K}} \longrightarrow V_l^{\mathcal{M}}, p_{q\ l,l+1}^E = \varphi_{l+1,l+2}^{\mathcal{M}} \circ q_{l,l+1}^E : E_{l,l+1}^{\mathcal{K}} \longrightarrow E_{l,l+1}^{\mathcal{M}}, l \in \mathbb{Z}_+$, and $p_q^\Sigma = q^\Sigma : \Sigma^{\mathcal{K}} \longrightarrow \Sigma^{\mathcal{M}}$. Then $p_q = (p_q^V, p_q^E, p_q^\Sigma) : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ is a homomorphism.

Lemma 3.1.

(i) *For a homomorphism* $p : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$, *we have* $p^E \circ \varphi_l^{\mathcal{K}} = \varphi_l^{\mathcal{M}} \circ p^E$.
(ii) *For a one-shift homomorphism* $q : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$, *we have* $q^E \circ \varphi_l^{\mathcal{K}} = \varphi_l^{\mathcal{M}} \circ q^E$.

Proof. Let $p : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ be a homomorphism. For $u \in V_{l-1}^{\mathcal{K}}, v \in V_{l+1}^{\mathcal{K}}$ and $e \in E^{\mathcal{K}}{}_{l,l+1}^{\iota^{\mathcal{K}}}(u, v)$, it is direct to see that $s^{\mathcal{M}}(p^E(\varphi_l^{\mathcal{K}}(e))) = s^{\mathcal{M}}(\varphi_l^{\mathcal{M}}(p^E(e)))$, $t^{\mathcal{M}}(p^E(\varphi_l^{\mathcal{K}}(e))) = t^{\mathcal{M}}(\varphi_l^{\mathcal{M}}(p^E(e)))$ and $\lambda^{\mathcal{M}}(p^E(\varphi_l^{\mathcal{K}}(e))) = \lambda^{\mathcal{M}}(\varphi_l^{\mathcal{M}}(p^E(e)))$. As $\mathfrak{L}^{\mathcal{M}}$ is essential, one sees that $p^E(\varphi_l^{\mathcal{K}}(e)) = \varphi_l^{\mathcal{M}}(p^E(e))$. Hence (i) holds. The assertion for (ii) is similarly shown to (i). □

We say that $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ *form squares* if

$$V_l^{\mathcal{M}} = V_l^{\mathcal{N}}, \qquad I_{l,l+1}^{\mathcal{M}} = I_{l,l+1}^{\mathcal{N}}, \qquad l \in \mathbb{Z}_+. \tag{3.1}$$

In this case, one may see a square as in the following figure:

$$\begin{array}{ccc} V_l^{\mathcal{N}} = V_l^{\mathcal{M}} & \xrightarrow{E_{l,l+1}^{\mathcal{M}}} & V_{l+1}^{\mathcal{M}} = V_{l+1}^{\mathcal{N}} \\ \Big\downarrow{\scriptstyle E_{l,l+1}^{\mathcal{N}}} & & \Big\downarrow{\scriptstyle E_{l+1,l+2}^{\mathcal{N}}} \qquad \text{for} \quad l \in \mathbb{Z}_+. \\ V_{l+1}^{\mathcal{N}} = V_{l+1}^{\mathcal{M}} & \xrightarrow{E_{l+1,l+2}^{\mathcal{M}}} & V_{l+2}^{\mathcal{M}} = V_{l+2}^{\mathcal{N}} \end{array}$$

We will formulate textile system on λ-graph systems as in the following way.

Definition (Textile system on λ-graph systems). For λ-graph systems $\mathfrak{L}^{\mathcal{M}}$, $\mathfrak{L}^{\mathcal{N}}$ and $\mathfrak{L}^{\mathcal{K}}$ with a homomorphism $p : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ and a one-shift homomorphism $q : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$, the diagram

$$\begin{array}{ccccc} & & \mathfrak{L}^{\mathcal{M}} & & \\ & & \Big\uparrow{\scriptstyle p} & & \\ \mathfrak{L}^{\mathcal{N}} & \xleftarrow{s^{\mathcal{K}}} & \mathfrak{L}^{\mathcal{K}} & \xrightarrow{t^{\mathcal{K}}} & \mathfrak{L}^{\mathcal{N}} \\ & & \Big\downarrow{\scriptstyle q} & & \\ & & \mathfrak{L}^{\mathcal{M}} & & \end{array}$$

is called a *textile system on λ-graph systems* if the following six conditions are satisfied:

(1) $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ form squares.
(2) $V_l^{\mathcal{K}} = E_{l,l+1}^{\mathcal{N}}, \qquad l \in \mathbb{Z}_+$.
(3) Under the equality (2),

$$(\iota_{l,l+1}^{\mathcal{K}} : V_{l+1}^{\mathcal{K}} \to V_l^{\mathcal{K}}) = (\varphi_{l+1}^{\mathcal{N}} : E_{l+1,l+2}^{\mathcal{N}} \to E_{l,l+1}^{\mathcal{N}}), \quad l \in \mathbb{Z}_+.$$

(4) Under the equalities (3.1) and (2),

$$(p^V : V_l^{\mathcal{K}} \to V_l^{\mathcal{M}}) = (s^{\mathcal{N}} : E_{l,l+1}^{\mathcal{N}} \to V_l^{\mathcal{N}}),$$
$$(q^V : V_l^{\mathcal{K}} \to V_{l+1}^{\mathcal{M}}) = (t^{\mathcal{N}} : E_{l,l+1}^{\mathcal{N}} \to V_{l+1}^{\mathcal{N}}), \qquad l \in \mathbb{Z}_+.$$

(5) The quadruple:

$$(s^{\mathcal{K}}(e), t^{\mathcal{K}}(e), p^E(e), q^E(e)) \in V_l^{\mathcal{K}} \times V_{l+l}^{\mathcal{K}} \times E_{l,l+1}^{\mathcal{M}} \times E_{l+1,l+2}^{\mathcal{M}}$$

determines $e \in E_{l,l+l}^{\mathcal{K}}$.

(6) Under the equality (2), there exists a specified equivalence between $\Sigma^{\mathcal{N}} \times \Sigma^{\mathcal{N}} \times \Sigma^{\mathcal{M}} \times \Sigma^{\mathcal{M}}$ and $\Sigma^{\mathcal{K}}$ by the correspondence between the symbols:

$$(\lambda^{\mathcal{N}}(s^{\mathcal{K}}(e)), \lambda^{\mathcal{N}}(t^{\mathcal{K}}(e)), \lambda^{\mathcal{M}}(p^E(e)), \lambda^{\mathcal{M}}(q^E(e))) \in \Sigma^{\mathcal{N}} \times \Sigma^{\mathcal{N}} \times \Sigma^{\mathcal{M}} \times \Sigma^{\mathcal{M}}$$

and $\lambda^{\mathcal{K}}(e) \in \Sigma^{\mathcal{K}}$.

A textile system on λ-graph systems is called a *textile λ-graph system* for short. We write the textile λ-graph system as $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}} = (p, q : \mathfrak{L}^{\mathcal{K}} \to \mathfrak{L}^{\mathcal{M}})$, or simply as $\mathcal{T}$. In viewing the textile λ-graph system, one uses the following square

$$\begin{array}{ccc} \cdot & \xrightarrow{\lambda^{\mathcal{M}}(p^E(e))} & \cdot \\ {\scriptstyle \lambda^{\mathcal{N}}(s^{\mathcal{K}}(e))}\downarrow & & \downarrow{\scriptstyle \lambda^{\mathcal{N}}(t^{\mathcal{K}}(e))} \\ \cdot & \xrightarrow{\lambda^{\mathcal{M}}(q^E(e))} & \cdot \end{array} \qquad \text{for} \quad e \in E^{\mathcal{K}}.$$

Proposition 3.2. *For a textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}} = (p, q : \mathfrak{L}^{\mathcal{K}} \to \mathfrak{L}^{\mathcal{M}})$, there exists a λ-graph system $\mathfrak{L}^{\mathcal{K}^*}$ and a textile λ-graph system $\mathcal{T}_{\mathcal{K}^{*\mathcal{N}}_{\mathcal{M}}} = (s^{\mathcal{K}}, t^{\mathcal{K}} : \mathfrak{L}^{\mathcal{K}^*} \to \mathfrak{L}^{\mathcal{N}})$ defined by the diagram:*

$$\begin{array}{ccccc} \mathcal{T}_{\mathcal{K}^{*\mathcal{N}}_{\mathcal{M}}} & & \mathfrak{L}^{\mathcal{N}} & & \\ & & \uparrow{\scriptstyle s^{\mathcal{K}}} & & \\ \mathfrak{L}^{\mathcal{M}} & \xleftarrow{p} & \mathfrak{L}^{\mathcal{K}^*} & \xrightarrow{q} & \mathfrak{L}^{\mathcal{M}} \\ & & \downarrow{\scriptstyle t^{\mathcal{K}}} & & \\ & & \mathfrak{L}^{\mathcal{N}} & & \end{array}$$

Proof. We define a λ-graph system $\mathfrak{L}^{\mathcal{K}^*} = (V^{\mathcal{K}^*}, E^{\mathcal{K}^*}, \lambda^{\mathcal{K}^*}, \iota^{\mathcal{K}^*})$ over $\Sigma^{\mathcal{K}^*}$ by setting

$$V_l^{\mathcal{K}^*} = E_{l,l+1}^{\mathcal{M}}, \quad E_{l,l+1}^{\mathcal{K}^*} = E_{l,l+1}^{\mathcal{K}}, \quad \iota_{l,l+1}^{\mathcal{K}^*} = \varphi_{l+1}^{\mathcal{M}}, \quad \Sigma^{\mathcal{K}^*} = \Sigma^{\mathcal{K}} \qquad \text{for } l \in \mathbb{Z}_+$$

and for $e \in E^{\mathcal{K}^*}_{l,l+1} = E^{\mathcal{K}}_{l,l+1}$

$$s^{\mathcal{K}^*}(e)=p^E(e) \in E^{\mathcal{M}}_{l,l+1}=V^{\mathcal{K}^*}_l,\ t^{\mathcal{K}^*}(e)=q^E(e) \in E^{\mathcal{M}}_{l+1,l+2}=V^{\mathcal{K}^*}_{l+1},\ \lambda^{\mathcal{K}^*}(e)=\lambda^{\mathcal{K}}(e).$$

For $u \in V^{\mathcal{K}^*}_{l-1}, v \in V^{\mathcal{K}^*}_{l+1}$ put $w = \iota^{\mathcal{K}^*}_{l,l+1}(v)$. It then follows that

$$\begin{aligned} E^{l-1,l}_{\iota^{\mathcal{K}^*}}(u,v) &= \{e \in E^{\mathcal{K}^*}_{l-1,l} \mid s^{\mathcal{K}^*}(e) = u,\, t^{\mathcal{K}^*}(e) = w\} \\ &= \{e \in E^{\mathcal{K}}_{l-1,l} \mid p^E(e) = u,\, q^E(e) = w\}, \\ E^{\iota^{\mathcal{K}^*}}_{l,l+1}(u,v) &= \{f \in E^{\iota^{\mathcal{K}^*}}_{l,l+1} \mid \iota^{\mathcal{K}^*}_{l-1,l}(s^{\mathcal{K}^*}(f)) = u,\, t^{\mathcal{K}^*}(f) = v\} \\ &= \{f \in E^{\mathcal{K}}_{l,l+1} \mid \varphi^{\mathcal{M}}_l(p^E(f)) = u,\, q^E(f) = v\}. \end{aligned}$$

For $e \in E^{l-1,l}_{\iota^{\mathcal{K}^*}}(u,v)$, one sees $t^{\mathcal{K}}(e) \in E^{\mathcal{N}}_{l,l+1}$. As $\mathfrak{L}^{\mathcal{N}}$ is left-resolving, there uniquely exists $v' \in E^{\mathcal{N}}_{l+1,l+2} = V^{\mathcal{K}}_{l+1}$ such that

$$\lambda^{\mathcal{N}}(v') = \lambda^{\mathcal{N}}(t^{\mathcal{K}}(e)), \qquad t^{\mathcal{N}}(v') = t^{\mathcal{M}}(v), \qquad \varphi^{\mathcal{N}}_{l+1}(v') = t^{\mathcal{K}}(e).$$

For the two vertices $s^{\mathcal{K}}(e) \in V^{\mathcal{K}}_{l-1}, v' \in V^{\mathcal{K}}_{l+1}$ with $\iota^{\mathcal{K}}(v') = t(e)$, by the local property of $\mathfrak{L}^{\mathcal{K}}$, there uniquely exists $f \in E^{\mathcal{K}}_{l,l+1}$ such that

$$\iota^{\mathcal{K}}_{l-1,l}(s^{\mathcal{K}}(f)) = s^{\mathcal{K}}(e), \qquad t^{\mathcal{K}}(f) = v', \qquad \lambda^{\mathcal{K}}(f) = \lambda^{\mathcal{K}}(e).$$

$$\begin{array}{ccc} \cdot & \xrightarrow{p^E(e)=u\in E^{\mathcal{M}}_{l-1,l}} & \cdot \\ {\scriptstyle s^{\mathcal{K}}(e)\in E^{\mathcal{N}}_{l-1,l}}\downarrow & & \downarrow{\scriptstyle t^{\mathcal{K}}(e)\in E^{\mathcal{N}}_{l,l+1}} \\ \cdot & \xrightarrow{q^E(e)=w\in E^{\mathcal{M}}_{l,l+1}} & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \xrightarrow{p^E(f)=u\in E^{\mathcal{M}}_{l,l+1}} & \cdot \\ {\scriptstyle s^{\mathcal{K}}(f)\in E^{\mathcal{N}}_{l,l+1}}\downarrow & & \downarrow{\scriptstyle t^{\mathcal{K}}(f)\in E^{\mathcal{N}}_{l+1,l+2}} \\ \cdot & \xrightarrow{q^E(f)=v\in E^{\mathcal{M}}_{l+1,l+2}} & \cdot \end{array}$$

Hence one has

$$\begin{aligned} &(\lambda^{\mathcal{N}}(s^{\mathcal{K}}(f)), \lambda^{\mathcal{N}}(t^{\mathcal{K}}(f)), \lambda^{\mathcal{M}}(p^E(f)), \lambda^{\mathcal{M}}(q^E(f)) \\ =&(\lambda^{\mathcal{N}}(s^{\mathcal{K}}(e)), \lambda^{\mathcal{N}}(t^{\mathcal{K}}(e)), \lambda^{\mathcal{M}}(p^E(e)), \lambda^{\mathcal{M}}(q^E(e)). \end{aligned}$$

Since $\mathfrak{L}^{\mathcal{M}}$ is left-resolving, the edge $p^E(f)$, whose label is $\lambda^{\mathcal{M}}(p^E(e))$ and terminal is the source of $v' \in E^{\mathcal{N}}_{l+1,l+2}$, is unique, and also the edge $q^E(f)$, whose label is $\lambda^{\mathcal{M}}(q^E(e))$ and terminal is the terminal of $v' \in E^{\mathcal{N}}_{l+1,l+2}$, is unique, Since $\mathfrak{L}^{\mathcal{N}}$ is left-resolving, the edge $s^{\mathcal{K}}(f)$, whose label is $\lambda^{\mathcal{N}}(s^{\mathcal{K}}(e))$ and terminal is

the source of $v \in E^{\mathcal{M}}_{l+1,l+2}$, is unique, and also the edge $t^{\mathcal{K}}(f)$, whose label is $\lambda^{\mathcal{N}}(t^{\mathcal{K}}(e))$ and terminal is the terminal of $v \in E^{\mathcal{M}}_{l+1,l+2}$, is unique, Hence the square

$$\begin{array}{ccc} \cdot & \xrightarrow{p^E(f)=u\in E^{\mathcal{M}}_{l,l+1}} & \cdot \\ {\scriptstyle s^{\mathcal{K}}(f)\in E^{\mathcal{N}}_{l,l+1}}\big\downarrow & & \big\downarrow{\scriptstyle t^{\mathcal{K}}(f)\in E^{\mathcal{N}}_{l+1,l+2}} \\ \cdot & \xrightarrow[q^E(f)=v\in E^{\mathcal{M}}_{l+1,l+2}]{} & \cdot \end{array}$$

is uniquely determined by $e \in E^{l-1,l}_{\iota^{\mathcal{K}^*}}(u,v)$ so that $f \in E^{\iota^{\mathcal{K}^*}}_{l,l+1}(u,v)$ is uniquely determined and hence $\varphi^{\mathcal{K}}_l(f) = e$. Conversely, for an edge $f \in E^{\iota^{\mathcal{K}^*}}_{l,l+1}$ there uniquely exists $e \in E^{l-1,l}_{\iota^{\mathcal{K}^*}}(u,v)$ such that $\varphi^{\mathcal{K}}_l(f) = e$. Hence $(V^{\mathcal{K}^*}, E^{\mathcal{K}^*}, \lambda^{\mathcal{K}^*}, \iota^{\mathcal{K}^*})$ satisfies the local property so that it yields a λ-graph system over $\Sigma^{\mathcal{K}}$, that is written as $\mathfrak{L}^{\mathcal{K}^*}$.

We define a homomorphism $p^* : \mathfrak{L}^{\mathcal{K}^*} \to \mathfrak{L}^{\mathcal{N}}$ by setting

$$p^*(e) = s^{\mathcal{K}}(e) \in E^{\mathcal{N}}_{l,l+1} \qquad \text{for} \quad e \in E^{\mathcal{K}^*}_{l,l+1}$$

and a one-shift homomorphism $q^* : \mathfrak{L}^{\mathcal{K}^*} \to \mathfrak{L}^{\mathcal{N}}$ by setting

$$q^*(e) = t^{\mathcal{K}}(e) \in E^{\mathcal{N}}_{l+1,l+2} \qquad \text{for} \quad e \in E^{\mathcal{K}^*}_{l,l+1}.$$

Then the diagram below

$$\begin{array}{ccccc} & & \mathfrak{L}^{\mathcal{N}} & & \\ & & \big\uparrow{\scriptstyle p^*=s^{\mathcal{K}}} & & \\ \mathfrak{L}^{\mathcal{M}} & \xleftarrow{s^{\mathcal{K}^*}=p} & \mathfrak{L}^{\mathcal{K}^*} & \xrightarrow{t^{\mathcal{K}^*}=q} & \mathfrak{L}^{\mathcal{M}} \\ & & \big\downarrow{\scriptstyle q^*=t^{\mathcal{K}}} & & \\ & & \mathfrak{L}^{\mathcal{N}} & & \end{array}$$

yields a textile λ-graph system $\mathcal{T}_{\mathcal{K}^*{}^{\mathcal{N}}_{\mathcal{M}}} = (s^{\mathcal{K}}, t^{\mathcal{K}} : \mathfrak{L}^{\mathcal{K}^*} \to \mathfrak{L}^{\mathcal{N}})$. □

We call the textile λ-graph system $\mathcal{T}_{\mathcal{K}^*{}^{\mathcal{N}}_{\mathcal{M}}} = (s^{\mathcal{K}}, t^{\mathcal{K}} : \mathfrak{L}^{\mathcal{K}^*} \to \mathfrak{L}^{\mathcal{N}})$ the *dual* of $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}} = (p, q : \mathfrak{L}^{\mathcal{K}} \to \mathfrak{L}^{\mathcal{M}})$. It is written as $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}{}^*$. It is clear that $(\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}{}^*)^* = \mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$.

For $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}} = (p, q : \mathfrak{L}^{\mathcal{K}} \to \mathfrak{L}^{\mathcal{M}})$ and $N \in \mathbb{N}$, we will define the N-higher block $\mathcal{T}^{[N]}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ of $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ as in the following way. Let $\mathfrak{L}^{\mathcal{K}^{[N]}}$ and $\mathfrak{L}^{\mathcal{M}^{[N]}}$ be the N-higher blocks of $\mathfrak{L}^{\mathcal{K}}$ and $\mathfrak{L}^{\mathcal{M}}$ respectively. For $N \geq 2$, we will define the λ-graph system

$$\mathfrak{L}^{\mathcal{N}^{[N]}_{\mathcal{T}}} = (V^{\mathcal{N}^{[N]}_{\mathcal{T}}}, E^{\mathcal{N}^{[N]}_{\mathcal{T}}}, \lambda^{\mathcal{N}^{[N]}_{\mathcal{T}}}, \iota^{\mathcal{N}^{[N]}_{\mathcal{T}}})$$

over $\Sigma^{\mathcal{N}_{\mathcal{T}}^{[N]}} = \underbrace{\Sigma^{\mathcal{K}} \times \cdots \times \Sigma^{\mathcal{K}}}_{N-1 \text{ times}}$ by setting

$$\begin{aligned}
V_l^{\mathcal{N}_{\mathcal{T}}^{[N]}} &= V_l^{\mathcal{M}^{[N]}}, \\
E_{l,l+1}^{\mathcal{N}_{\mathcal{T}}^{[N]}} &= V_l^{\mathcal{K}^{[N]}} \\
&\quad (= \{(e_1, e_2, \ldots, e_{N-1}) \in E_{l,l+1}^{\mathcal{K}} \times E_{l+1,l+2}^{\mathcal{K}} \times \cdots \times E_{l+N-2,l+N-1}^{\mathcal{K}} \mid \\
&\qquad t^{\mathcal{K}}(e_i) = s^{\mathcal{K}}(e_{i+1}), i = 1, 2, \ldots, N-2\}), \\
\lambda^{\mathcal{N}_{\mathcal{T}}^{[N]}} &= \underbrace{\lambda^{\mathcal{K}} \times \lambda^{\mathcal{K}} \times \cdots \times \lambda^{\mathcal{K}}}_{N-1 \text{ times}}, \\
\iota^{\mathcal{N}_{\mathcal{T}}^{[N]}} &= \iota^{\mathcal{M}^{[N]}},
\end{aligned}$$

and

$$\begin{aligned}
s^{\mathcal{N}_{\mathcal{T}}^{[N]}}(e_1, e_2, \ldots, e_{N-1}) &= p^E(e_1)p^E(e_2)\cdots p^E(e_{N-1}), \\
t^{\mathcal{N}_{\mathcal{T}}^{[N]}}(e_1, e_2, \ldots, e_{N-1}) &= q^E(e_1)q^E(e_2)\cdots q^E(e_{N-1}),
\end{aligned}$$

for $(e_1, e_2, \ldots, e_{N-1}) \in E_{l,l+1}^{\mathcal{N}_{\mathcal{T}}^{[N]}} = V_l^{\mathcal{K}^{[N]}}$. The λ-graph system $\mathfrak{L}^{\mathcal{N}_{\mathcal{T}}^{[N]}}$ is called the *N-higher block of $\mathfrak{L}^{\mathcal{N}}$ relative to $\mathcal{T}_{\mathcal{K},\mathcal{N}}$*. For $N = 1$, we put $\mathfrak{L}^{\mathcal{N}_{\mathcal{T}}^{[1]}} = \mathfrak{L}^{\mathcal{N}}$. The homomorphism $p^{[N]} : \mathfrak{L}^{\mathcal{K}^{[N]}} \to \mathfrak{L}^{\mathcal{M}^{[N]}}$ and the one-shift homomorphism $q^{[N]} : \mathfrak{L}^{\mathcal{K}^{[N]}} \to \mathfrak{L}^{\mathcal{M}^{[N]}}$ are defined by

$$\begin{aligned}
p^{[N]}(e_1, e_2, \ldots, e_N) &= p^E(e_1)p^E(e_2)\cdots p^E(e_N), \\
q^{[N]}(e_1, e_2, \ldots, e_N) &= q^E(e_1)q^E(e_2)\cdots q^E(e_N)
\end{aligned}$$

for $(e_1, e_2, \ldots, e_N) \in E_{l,l+1}^{\mathcal{K}} \times E_{l+1,l+2}^{\mathcal{K}} \times \cdots \times E_{l+N-1,l+N}^{\mathcal{K}}$. Then we have

Proposition 3.3. *The diagram*

$$\begin{array}{ccccc}
 & & \mathfrak{L}^{\mathcal{M}^{[N]}} & & \\
 & & \uparrow{\scriptstyle p^{[N]}} & & \\
\mathfrak{L}^{\mathcal{N}_{\mathcal{T}}^{[N]}} & \xleftarrow{s^{\mathcal{K}^{[N]}}} & \mathfrak{L}^{\mathcal{K}^{[N]}} & \xrightarrow{t^{\mathcal{K}^{[N]}}} & \mathfrak{L}^{\mathcal{N}_{\mathcal{T}}^{[N]}} \\
 & & \downarrow{\scriptstyle q^{[N]}} & & \\
 & & \mathfrak{L}^{\mathcal{M}^{[N]}} & &
\end{array}$$

defines a textile λ-graph system.

We write the above textile λ-graph system $\mathcal{T}_{\mathcal{K}^{[N]}\mathcal{N}_{\mathcal{T}}^{[N]}}^{\mathcal{M}^{[N]}} = (p^{[N]}, q^{[N]} : \mathfrak{L}^{\mathcal{K}^{[N]}} \to \mathfrak{L}^{\mathcal{M}^{[N]}})$ as $\mathcal{T}_{\mathcal{K}\mathcal{N}}^{\mathcal{M}[N]}$ and call it the *N-higher block* of $\mathcal{T}_{\mathcal{K}\mathcal{N}}^{\mathcal{M}}$.

4. Textile shifts on λ-graph systems

For a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$, we set

$$X_{\mathfrak{L}} = \{(z_l)_{l=0}^{\infty} \in \prod_{l=0}^{\infty} E_{l,l+1} \mid z_l \in E_{l,l+1}, t(z_l) = s(z_{l+1}), l = 0, 1, \dots\},$$

$$X_{\mathfrak{L}_0} = \{(z_l)_{l=1}^{\infty} \in \prod_{l=1}^{\infty} E_{l,l+1} \mid z_l \in E_{l,l+1}, t(z_l) = s(z_{l+1}), l = 1, 2, \dots\}.$$

We define $S : X_{\mathfrak{L}} \longrightarrow X_{\mathfrak{L}_0}$ by setting

$$S((z_l)_{l=0}^{\infty}) = (z_l)_{l=1}^{\infty}, \qquad (z_l)_{l=0}^{\infty} \in X_{\mathfrak{L}}.$$

For a textile λ-graph system $\mathcal{T}_{\mathcal{K}\mathcal{N}}^{\mathcal{M}} = (p, q : \mathfrak{L}^{\mathcal{K}} \to \mathfrak{L}^{\mathcal{M}})$, there exist maps $p_X : X_{\mathfrak{L}^{\mathcal{K}}} \longrightarrow X_{\mathfrak{L}^{\mathcal{M}}}$ and $q_X : X_{\mathfrak{L}^{\mathcal{K}}} \longrightarrow X_{\mathfrak{L}_0^{\mathcal{M}}}$ defined by $p_X((z_l)_{l=0}^{\infty}) = (p^E(z_l))_{l=0}^{\infty}$ and $q_X((z_l)_{l=0}^{\infty}) = (q^E(z_{l-1}))_{l=1}^{\infty}$ respectively.

Following Nasu's notation, we say that a textile λ-graph system $\mathcal{T}_{\mathcal{K}\mathcal{N}}^{\mathcal{M}}$ is *non-degenerate* if both factor maps $p_X : X_{\mathfrak{L}^{\mathcal{K}}} \to X_{\mathfrak{L}^{\mathcal{M}}}$ and $q_X : X_{\mathfrak{L}^{\mathcal{K}}} \to X_{\mathfrak{L}_0^{\mathcal{M}}}$ are surjective. We henceforth assume that textile λ-graph systems $\mathcal{T}_{\mathcal{K}\mathcal{N}}^{\mathcal{M}}$ and ${\mathcal{T}_{\mathcal{K}\mathcal{N}}^{\mathcal{M}}}^*$ are both nondegenerate.

Let $\triangle$ be the lattice of the lower right half plane: $\triangle = \{(i, j) \in \mathbb{Z}^2 \mid i+j \geq 0\}$, where the vertical coordinate is reversed. A *textile edge weaved by* $\mathcal{T}_{\mathcal{K}\mathcal{N}}^{\mathcal{M}}$ is a configuration

$$(e_{i,j})_{(i,j)\in\triangle}$$

such that

(1) $e_{i,j} \in E_{i+j,i+j+1}^{\mathcal{K}}$ for $(i, j) \in \triangle$,
(2) $(e_{i,-i+l})_{l\in\mathbb{Z}_+} \in X_{\mathfrak{L}^{\mathcal{K}}}$ for each $i \in \mathbb{Z}_+$,
(3) $p^E(e_{i,j}) = q^E(e_{i-1,j})$ for $i, j \in \mathbb{Z}$ with $i + j \geq 1$.

That is a sequence

$$(e_i)_{i\in\mathbb{Z}}$$

such that

(1) $e_i = (e_{i,-i+l})_{l\in\mathbb{Z}_+} \in X_{\mathfrak{L}^{\mathcal{K}}}$ for $i \in \mathbb{Z}$,
(2) $S \circ p_X(e_i) = q_X(e_{i-1})$ for $i \in \mathbb{Z}$.

A textile edge weaved by $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ is regarded as a configuration of concatenated edges of $\mathfrak{L}^{\mathcal{K}}$ on the lattice $\triangle$ of the lower right half plane as in the following way.

$$\begin{array}{ccccccccccc}
 & & & & & & & & \cdots & & \\
 & & & & & & e_{-l,l} & & \cdots & e_{-l} & \\
 & & & & & e_{-(l-1),l-1} & e_{-(l-1),l} & & \cdots & e_{-(l-1)} & \\
 & & & & \cdots & \cdots & \cdots & & \cdots & & \\
 & & & e_{-1,1} & e_{-1,2} & e_{-1,3} & e_{-1,4} & & \cdots & e_{-1} & \\
 & & e_{0,0} & e_{0,1} & e_{0,2} & e_{0,3} & e_{0,4} & & \cdots & e_{0} & \\
 & e_{1,-1} & e_{1,0} & e_{1,1} & e_{1,2} & e_{1,3} & e_{1,4} & & \cdots & e_{1} & \\
e_{2,-2} & e_{2,-1} & e_{2,0} & e_{2,1} & e_{2,2} & e_{2,3} & e_{2,4} & & \cdots & e_{2} & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & &
\end{array}$$

It is easy to see that $(e_{i,j})_{(i,j)\in\triangle}$ is a textile edge weaved by $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ if and only if $(e_{j,i})_{(i,j)\in\triangle}$ is a textile edge weaved by ${\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}}^*$.

Consider the set $X(\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}})$ of all textile edges weaved by $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$

$$X(\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}) = \{(e_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} X_{\mathfrak{L}^{\mathcal{K}}} \mid S\circ p_X(e_i) = q_X(e_{i-1}),\, i\in\mathbb{Z}\}.$$

For $e \in E^{\mathcal{K}}_{l,l+1}, l \in \mathbb{Z}_+$, there exists a specified equivalence between

$$(\lambda^{\mathcal{N}}(s(e)), \lambda^{\mathcal{N}}(t(e)), \lambda^{\mathcal{M}}(p^E(e)), \lambda^{\mathcal{M}}(q^E(e))) \in \Sigma^{\mathcal{N}}\times\Sigma^{\mathcal{N}}\times\Sigma^{\mathcal{M}}\times\Sigma^{\mathcal{M}}$$

and $\lambda^{\mathcal{K}}(e) \in \Sigma^{\mathcal{K}}$. We may identify them, and assume that

$$\Sigma^{\mathcal{K}} = \{\lambda^{\mathcal{K}}(e) \mid e \in E^{\mathcal{K}}_{l,l+1}, l \in \mathbb{Z}_+\}.$$

We define

$$\begin{aligned}
J_{upper} &: \Sigma^{\mathcal{K}} \to \Sigma^{\mathcal{M}} \quad \text{by} \quad J_{upper}(\lambda^{\mathcal{K}}(e)) = \lambda^{\mathcal{M}}(p^E(e)),\\
J_{lower} &: \Sigma^{\mathcal{K}} \to \Sigma^{\mathcal{M}} \quad \text{by} \quad J_{lower}(\lambda^{\mathcal{K}}(e)) = \lambda^{\mathcal{M}}(q^E(e)),\\
J_{right} &: \Sigma^{\mathcal{K}} \to \Sigma^{\mathcal{N}} \quad \text{by} \quad J_{right}(\lambda^{\mathcal{K}}(e)) = \lambda^{\mathcal{N}}(t^{\mathcal{K}}(e)),\\
J_{left} &: \Sigma^{\mathcal{K}} \to \Sigma^{\mathcal{N}} \quad \text{by} \quad J_{left}(\lambda^{\mathcal{K}}(e)) = \lambda^{\mathcal{N}}(s^{\mathcal{K}}(e)).
\end{aligned}$$

Let $\Lambda_{\mathcal{M}}, \Lambda_{\mathcal{K}}, \Lambda_{\mathcal{N}}$ be the two-sided subshifts presented by $\mathfrak{L}^{\mathcal{M}}, \mathfrak{L}^{\mathcal{K}}, \mathfrak{L}^{\mathcal{N}}$ respectively. The above one-block maps $J_{left}, J_{right}, J_{upper}, J_{lower}$ give rise to sliding block codes between the subshifts:

$$\begin{aligned}
\xi &= (J_{upper})_\infty : \Lambda_{\mathcal{K}} \to \Lambda_{\mathcal{M}},\\
\eta &= (J_{lower})_\infty : \Lambda_{\mathcal{K}} \to \Lambda_{\mathcal{M}},\\
\xi^* &= (J_{right})_\infty : \Lambda_{\mathcal{K}^*} \to \Lambda_{\mathcal{N}},\\
\eta^* &= (J_{left})_\infty : \Lambda_{\mathcal{K}^*} \to \Lambda_{\mathcal{N}}
\end{aligned}$$

respectively. We say that $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ is 1-1 if the factor codes $\xi : \Lambda_{\mathcal{K}} \to \Lambda_{\mathcal{M}}$ and $\eta : \Lambda_{\mathcal{K}} \to \Lambda_{\mathcal{M}}$ are both one-to-one. Since $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ is nondegenerate, the codes $\xi : \Lambda_{\mathcal{K}} \to \Lambda_{\mathcal{M}}$ and $\eta : \Lambda_{\mathcal{K}} \to \Lambda_{\mathcal{M}}$ are both surjective. Hence, in this case, we have an automorphism $\varphi_{\mathcal{T}} = \eta \circ \xi^{-1}$ on the subshift $\Lambda_{\mathcal{M}}$.

Let $\mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$ be the set of configurations of labels of $\mathfrak{L}^{\mathcal{K}}$ on $X(\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}})$:

$$\mathcal{U}_{\mathfrak{L}^{\mathcal{K}}} = \{(\lambda^{\mathcal{K}}(e_{i,j}))_{(i,j)\in\triangle} \in \prod_{(i,j)\in\triangle} \Sigma^{\mathcal{K}} \mid (e_{i,j})_{(i,j)\in\triangle} \in X(\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}})\}.$$

Consider the natural product topology on $\prod_{(i,j)\in\triangle} \Sigma^{\mathcal{K}}$ and restrict it to $\mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$ so that $\mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$ is compact. For a connected subset Ω of $\triangle$, we set

$$\begin{aligned} X(\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}; \Omega) = \{(e_{i,j})_{(i,j)\in\Omega} \in \prod_{(i,j)\in\Omega} E^{\mathcal{K}}_{i+j,i+j+1} \mid \\ p^E(e_{i,j}) = q^E(e_{i-1,j}) \text{ for } (i,j),(i-1,j) \in \Omega, \\ q^E(e_{i,j}) = p^E(e_{i+1,j}) \text{ for } (i,j),(i+1,j) \in \Omega, \\ s^{\mathcal{K}}(e_{i,j}) = t^{\mathcal{K}}(e_{i,j-1}) \text{ for } (i,j),(i,j-1) \in \Omega, \\ t^{\mathcal{K}}(e_{i,j}) = s^{\mathcal{K}}(e_{i,j+1}) \text{ for } (i,j),(i,j+1) \in \Omega\} \end{aligned}$$

and

$$\mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}(\Omega) = \{\lambda^{\mathcal{K}}(e_{i,j})_{(i,j)\in\Omega} \in \prod_{(i,j)\in\Omega} \Sigma^{\mathcal{K}} \mid (e_{i,j})_{(i,j)\in\Omega} \in X(\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}; \Omega)\}.$$

Hence $\mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}(\triangle) = \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$. Put

$$\triangle_{(n,k)} = \{(i,j) \in \triangle \mid i \le n,\, j \le k\} \qquad \text{for } (n,k) \in \triangle.$$

By noticing the assumption that $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ and $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}{}^*$ are both nondegenerate, we have

Lemma 4.1. *For $(n,k) \in \triangle$ and $(a_{i,j})_{(i,j)\in\triangle_{(n,k)}} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}(\triangle_{(n,k)})$, there exists $(b_{i,j})_{(i,j)\in\triangle_{(n+1,k+1)}} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}(\triangle_{(n+1,k+1)})$ such that*

$$b_{i,j} = a_{i,j} \qquad \textit{for all } (i,j) \in \triangle_{(n,k)}.$$

Proof. For $(a_{i,j})_{(i,j)\in\triangle_{(n,k)}} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}(\triangle_{(n,k)})$, take $(e_{i,j})_{(i,j)\in\triangle_{(n,k)}} \in X(\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}; \triangle_{(n,k)})$ such that $a_{i,j} = \lambda^{\mathcal{K}}(e_{i,j})$ for all $(i,j) \in \triangle_{(n,k)}$. Since $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}{}^*$ is nondegenerate, there exist $e_{i,k+1} \in E^{\mathcal{K}}_{i,k+1}$ for $i = -k-1, -k, \ldots, n-1, n$ such that

$$p^E(e_{i,k+1}) = q^E(e_{i-1,k+1}) \quad \text{and} \quad s^{\mathcal{K}}(e_{i,k+1}) = t^{\mathcal{K}}(e_{i,k})$$

for $i = -k, -k+1, \dots, n$. Hence we have $(e_{i,j})_{(i,j)\in\triangle_{(n,k+1)}} \in X(\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}; \triangle_{(n,k+1)})$. Similarly by the condition that $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ is nondegenerate, there exist $e_{n+1,j} \in E^{\mathcal{K}}_{n+1,j}$ for $j = -n-1, -n, \dots, k, k+1$ such that

$$p^E(e_{n+1,j}) = q^E(e_{n,j}) \quad \text{and} \quad s^{\mathcal{K}}(e_{n+1,j}) = t^{\mathcal{K}}(e_{n+1,j-1})$$

for $j = -n, -n+1, \dots, k, k+1$. This implies that $(e_{i,j})_{(i,j)\in\triangle_{(n+1,k+1)}} \in X(\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}; \triangle_{(n+1,k+1)})$ so that by putting $b_{i,j} = \lambda^{\mathcal{K}}(e_{i,j})$ for $(e_{i,j})_{(i,j)\in\triangle_{(n+1,k+1)}}$ we get the assertion. □

Hence one has

Corollary 4.2. *For $(n,k) \in \triangle$ and $(a_{i,j})_{(i,j)\in\triangle_{(n,k)}} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}(\triangle_{(n,k)})$, there exists $(\alpha_{i,j})_{(i,j)\in\triangle} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$ such that*

$$\alpha_{i,j} = a_{i,j} \qquad \text{for all } (i,j) \in \triangle_{(n,k)}.$$

Proposition 4.3. *Let $(\alpha_{i,j})_{(i,j)\in\triangle}$ be a family $\alpha_{i,j} \in \Sigma^{\mathcal{K}}$ of symbols indexed by $(i,j) \in \triangle$. Then $(\alpha_{i,j})_{(i,j)\in\triangle} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$ if and only if $(\alpha_{i,j})_{(i,j)\in\triangle_{(n,k)}} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}(\triangle_{(n,k)})$ for all $(n,k) \in \triangle$.*

Proof. The only if part is clear. Suppose that $(\alpha_{i,j})_{(i,j)\in\triangle_{(n,k)}} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}(\triangle_{(n,k)})$ for all $(n,k) \in \triangle$. By Corollary 4.2, for $(n,k) \in \triangle$ there exists $\alpha^{(n,k)} = (\alpha^{(n,k)}_{i,j})_{(i,j)\in\triangle} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$ such that

$$\alpha^{(n,k)}_{i,j} = \alpha_{i,j} \qquad \text{for all } (i,j) \in \triangle_{(n,k)}.$$

As $\mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$ is compact, there exists $\bar{\alpha} = (\bar{\alpha}_{i,j})_{(i,j)\in\triangle} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$ such that $\bar{\alpha} = \lim_{n,k\to\infty} \alpha^{(n,k)}$. Since $\bar{\alpha}_{i,j} = \alpha^{(n,k)}_{i,j} = \alpha_{i,j}$ for all $(i,j) \in \triangle_{(n,k)}$, one has $\bar{\alpha}_{i,j} = \alpha_{i,j}$ for all $(i,j) \in \triangle$ and hence $(\alpha_{i,j})_{(i,j)\in\triangle} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$. □

Lemma 4.4. *For $\alpha = (\alpha_{i,j})_{(i,j)\in\triangle} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$, put*

$$S_R(\alpha)_{i,j} = \alpha_{i,j+1}, \quad S_D(\alpha)_{i,j} = \alpha_{i+1,j} \qquad \text{for } (i,j) \in \triangle.$$

Then we have $S_R(\alpha), S_D(\alpha) \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$.

Proof. For $\alpha = (\alpha_{i,j})_{(i,j)\in\triangle} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$, take $(e_{i,j})_{(i,j)\in\triangle} \in X(\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}})$ such that $\alpha_{i,j} = \lambda^{\mathcal{K}}(e_{i,j})$, where $e_{i,j} \in E^{\mathcal{K}}_{i+j,i+j+1}$. By the map $\varphi^{\mathcal{K}}_{i+j,i+j+1} : E^{\mathcal{K}}_{i+j,i+j+1} \to E^{\mathcal{K}}_{i+j-1,i+j}$ in Lemma 3,1, one has $\varphi^{\mathcal{K}}(e_{i,j+1})_{(i,j)\in\triangle} \in X(\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}})$. As $\lambda^{\mathcal{K}}(\varphi^{\mathcal{K}}(e_{i,j+1})) = \lambda^{\mathcal{K}}(e_{i,j+1}) = \alpha_{i,j+1}$, one sees that $S_R(\alpha) \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$. One may symmetrically prove that $S_D(\alpha) \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$ by considering $\varphi^{\mathcal{K}^*}$. □

The assertions above mean that $\mathcal{U}_{\mathfrak{L}^\kappa}$ can be shifted to both left and upper. We note that $S_R \circ S_D = S_D \circ S_R$ on $\mathcal{U}_{\mathfrak{L}^\kappa}$ to set

$$\mathcal{U}_{\mathfrak{L}^\kappa}^{\infty} = \bigcap_{n,m=0}^{\infty} S_R^n \circ S_D^m(\mathcal{U}_{\mathfrak{L}^\kappa}).$$

Hence one has

$$S_R(\mathcal{U}_{\mathfrak{L}^\kappa}^{\infty}) = \mathcal{U}_{\mathfrak{L}^\kappa}^{\infty} = S_D(\mathcal{U}_{\mathfrak{L}^\kappa}^{\infty}).$$

A *textile label weaved by* $\mathcal{T} = \mathcal{T}_{\mathcal{N}}^{\mathcal{M}}$ is a two-dimensional configuration $(\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2}$ of $\Sigma^{\mathcal{K}}$ such that

$$(\alpha_{i-k,j})_{(i,j)\in\triangle} \in \mathcal{U}_{\mathfrak{L}^\kappa}^{\infty} \qquad \text{for all} \quad k \in \mathbb{Z}.$$

The condition is equivalent to the condition

$$(\alpha_{i,j-k})_{(i,j)\in\triangle} \in \mathcal{U}_{\mathfrak{L}^\kappa}^{\infty} \qquad \text{for all} \quad k \in \mathbb{Z}.$$

Let $\mathcal{U}_{\mathcal{T}}$ be the set of all textile labels weaved by $\mathcal{T}$. We note

Lemma 4.5. *For* $(\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \mathcal{U}_{\mathcal{T}}$ *one has* $(\alpha_{i,j})_{j\in\mathbb{Z}} \in \Lambda_{\mathcal{K}}$ *for all* $i \in \mathbb{Z}$, *and* $(\alpha_{i,j})_{i\in\mathbb{Z}} \in \Lambda_{\mathcal{K}^*}$ *for all* $j \in \mathbb{Z}$.

Proof. For $(\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \mathcal{U}_{\mathcal{T}}$, one sees that $(\alpha_{i,j-k})_{(i,j)\in\triangle} \in \mathcal{U}_{\mathfrak{L}^\kappa}$ for $k \in \mathbb{Z}_+$. Hence there exists $(e_{i,j-k})_{(i,j)\in\triangle} \in X(\mathcal{T}_{\mathcal{N}}^{\mathcal{M}})$ such that $\alpha_{i,j-k} = \lambda^{\mathcal{K}}(e_{i,j-k})$, so that $(\alpha_{i,j-k})_{j\in\mathbb{Z},(i,j)\in\triangle} \in \Lambda_{\mathcal{K}}^+$ for all $i \in \mathbb{Z}$ and $k \in \mathbb{Z}_+$. We then have $(\alpha_{i,j})_{j\in\mathbb{Z}} \in \Lambda_{\mathcal{K}}$ for all $i \in \mathbb{Z}$. We similarly have $(\alpha_{i,j})_{i\in\mathbb{Z}} \in \Lambda_{\mathcal{K}^*}$ for all $j \in \mathbb{Z}$. □

We define a metric $\delta_{\mathcal{M}}$ on $\Lambda_{\mathcal{M}}$ by setting

$$\delta(\alpha,\alpha') = \begin{cases} 0 & \text{if} \quad \alpha = \alpha', \\ \frac{1}{k+1} & \text{if} \quad \alpha \neq \alpha' \end{cases}$$

for $\alpha = (\alpha_i)_{i\in\mathbb{Z}}, \alpha' = (\alpha'_i)_{i\in\mathbb{Z}}$ in $\Lambda_{\mathcal{M}}$, where $k = \min\{|i| \mid i \in \mathbb{Z}, \alpha_i \neq \alpha'_i\}$. Similarly we define a metric $\delta_{\mathcal{N}}$ on $\Lambda_{\mathcal{N}}$. We next define a metric $\delta_{\mathcal{T}}$ on $\mathcal{U}_{\mathcal{T}}$ by setting

$$\delta_{\mathcal{T}}(u,u') = \begin{cases} 0 & \text{if } u = u', \\ \frac{1}{k+1} & \text{if } u \neq u' \end{cases}$$

for $u = (\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2}, u' = (\alpha'_{i,j})_{(i,j)\in\mathbb{Z}^2}$ in $\mathcal{U}_{\mathcal{T}}$, where $k = \min\{|i| + |j| \mid i,j \in \mathbb{Z}, \alpha_{i,j} \neq \alpha'_{i,j}\}$.

Lemma 4.6. $\mathcal{U}_{\mathcal{T}}$ *is compact.*

Proof. We first note that the set $X(\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}})$ of all textile edges is a compact set in a natural topology of the edge set so that the label sets $\mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$ and $\mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}^{\infty}$ are both compact. Let $\prod_{(i,j)\in\mathbb{Z}^2}\Sigma^{\mathcal{K}}$ be the set $\alpha_{i,j}\in\Sigma^{\mathcal{K}}$, $(i,j)\in\mathbb{Z}^2$ of all two-dimensional configurations of $\Sigma^{\mathcal{K}}$, that is endowed with the topology similarly defined by the above $\delta_{\mathcal{T}}$. Consider the sequence of the following continuous maps

$$\zeta_k : (\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \prod_{(i,j)\in\mathbb{Z}^2}\Sigma^{\mathcal{K}} \longrightarrow (\alpha_{i-k,j})_{(i,j)\in\triangle} \in \prod_{(i,j)\in\triangle}\Sigma^{\mathcal{K}}, \qquad k\in\mathbb{Z}.$$

Since we have

$$\mathcal{U}_{\mathcal{T}} = \bigcap_{k\in\mathbb{Z}} \zeta_k^{-1}(\mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}),$$

the set $\mathcal{U}_{\mathcal{T}}$ is compact. □

Define a one-block code

$$\Phi_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{K}}$$

by setting

$$\Phi_{\mathcal{T}}((\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2}) = (\alpha_{0,j})_{j\in\mathbb{Z}}, \qquad (\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2}\in\mathcal{U}_{\mathcal{T}}.$$

We say that the textile λ-graph system $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ is *surjective* if the map $\Phi_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{K}}$ is surjective. Define the one-block codes

$$\Theta_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{M}}, \qquad \Theta_{\mathcal{T}}^{*} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{N}}$$

by setting

$$\Theta_{\mathcal{T}}((\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2}) = (J_{lower}(\alpha_{0,j}))_{j\in\mathbb{Z}}, \qquad \Theta_{\mathcal{T}}^{*}((\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2}) = (J_{right}(\alpha_{i,0}))_{i\in\mathbb{Z}}.$$

They are continuous in the topology defined by the metric $\delta_{\mathcal{K}}$ on $\mathcal{U}_{\mathcal{T}}$. Since $\eta : \Lambda_{\mathcal{K}} \to \Lambda_{\mathcal{M}}$ is always surjective and $\Theta_{\mathcal{T}} = \eta\circ\Phi_{\mathcal{T}}$, if $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ is surjective, the map $\Theta_{\mathcal{T}}$ is surjective. For $k,n\in\mathbb{Z}$, the homeomorphism

$$\sigma_{\mathcal{T}}^{(k,n)} : \mathcal{U}_{\mathcal{T}} \longrightarrow \mathcal{U}_{\mathcal{T}}$$

is defined by

$$\sigma_{\mathcal{T}}^{(k,n)}((\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2}) = (\alpha_{i+k,j+n})_{(i,j)\in\mathbb{Z}^2} \qquad \text{for} \quad (\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2}\in\mathcal{U}_{\mathcal{T}}.$$

The dynamical system

$$(\mathcal{U}_{\mathcal{T}}, \sigma_{\mathcal{T}}^{(k,n)})$$

is called the (k,n)-textile shift on λ-graph systems.

Lemma 4.7.

(i) *If* $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ *is 1-1 and surjective, then* $\Theta_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{M}}$ *is a homeomorphism such that* $\Theta_{\mathcal{T}} \circ \sigma_{\mathcal{T}}^{(0,n)} = \sigma_{\mathcal{M}}^{n} \circ \Theta_{\mathcal{T}}$.

(ii) *If* $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}{}^{*}$ *is 1-1 and surjective, then* $\Theta_{\mathcal{T}}^{*} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{N}}$ *is a homeomorphism such that* $\Theta_{\mathcal{T}}^{*} \circ \sigma_{\mathcal{T}}^{(k,0)} = \sigma_{\mathcal{N}}^{k} \circ \Theta_{\mathcal{T}}^{*}$.

Proof. (i) If $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ is 1-1 and surjective, then $\Theta_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{M}}$ is one-to-one and surjective so that it is a homeomorphism. (ii) The assertion is symmetric to (i). □

Proposition 4.8. *If* $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ *is 1-1 and surjective, then* $(\Lambda_{\mathcal{M}}, \varphi_{\mathcal{T}}^{k}\sigma_{\mathcal{M}}^{n})$ *is conjugate to* $(\mathcal{U}_{\mathcal{T}}, \sigma_{\mathcal{T}}^{(k,n)})$ *for all* $k, n \in \mathbb{Z}$.

Proof. We note that $\varphi_{\mathcal{T}}^{k}((J_{lower}(\alpha_{0,j}))_{j\in\mathbb{Z}}) = (J_{lower}(\alpha_{k,j}))_{j\in\mathbb{Z}})$ for $(\alpha_{k,j})_{(k,j)\in\mathbb{Z}^2}$ $\in \mathcal{U}_{\mathcal{T}}$. Since $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ is 1-1 and surjective, the map $\Theta_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{M}}$ is a homeomorphism that gives rise to a conjugacy between $(\Lambda_{\mathcal{M}}, \varphi_{\mathcal{T}}^{k}\sigma_{\mathcal{M}}^{n})$ and $(\mathcal{U}_{\mathcal{T}}, \sigma_{\mathcal{T}}^{(k,n)})$. □

Now we reach the following theorem.

Theorem 4.9. *Suppose that* $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ *and* $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}{}^{*}$ *are both 1-1 and surjective. Then there exists a homeomorphism*

$$\chi_{\mathcal{T}} : \Lambda_{\mathcal{M}} \to \Lambda_{\mathcal{N}}$$

such that the diagrams

$$\begin{array}{ccc} \Lambda_{\mathcal{M}} & \xrightarrow{\varphi_{\mathcal{T}}} & \Lambda_{\mathcal{M}} \\ \downarrow{\scriptstyle \chi_{\mathcal{T}}} & & \downarrow{\scriptstyle \chi_{\mathcal{T}}} \\ \Lambda_{\mathcal{N}} & \xrightarrow{\sigma_{\mathcal{N}}} & \Lambda_{\mathcal{N}}, \end{array} \qquad \begin{array}{ccc} \Lambda_{\mathcal{M}} & \xrightarrow{\sigma_{\mathcal{M}}} & \Lambda_{\mathcal{M}} \\ \downarrow{\scriptstyle \chi_{\mathcal{T}}} & & \downarrow{\scriptstyle \chi_{\mathcal{T}}} \\ \Lambda_{\mathcal{N}} & \xrightarrow{\varphi_{\mathcal{T}^*}} & \Lambda_{\mathcal{N}} \end{array}$$

are both commutative.

Proof. We set

$$\chi_{\mathcal{T}} = \Theta_{\mathcal{T}}^{*} \circ {\Theta_{\mathcal{T}}}^{-1} : \Lambda_{\mathcal{M}} \to \Lambda_{\mathcal{N}}.$$

It satisfies

$$\chi_{\mathcal{T}} \circ \varphi_{\mathcal{T}} = \sigma_{\mathcal{N}} \circ \chi_{\mathcal{T}}, \qquad \chi_{\mathcal{T}} \circ \sigma_{\mathcal{M}} = \varphi_{\mathcal{T}^*} \circ \chi_{\mathcal{T}}.$$

□

There are various metrics on $\Lambda_{\mathcal{M}}$ by which the product topology on $\Lambda_{\mathcal{M}}$ is given. Any such metric makes the homeomorphism $\sigma_{\mathcal{M}}$ on $\Lambda_{\mathcal{M}}$ expansive. We may fix the previously defined metric on $\Lambda_{\mathcal{M}}$. By the metric, $\sigma_{\mathcal{M}}$ has 1 as its expansive constant. Theorem 4.9 is generalized as follows:

THEOREM 4.10 (cf. [19; Theorem 4.1]). *Assume that $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ is 1-1 and surjective.*

(i) *If $\varphi_{\mathcal{T}}$ is expansive and its expansive constant c satisfies $c \geq \frac{1}{k}$ for some $k \in \mathbb{N}$, then ${\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}}^{[2k]^*}$ is 1-1. Hence if there is no $n \in \mathbb{N}$ such that ${\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}}^{[n]^*}$ is 1-1, then $\varphi_{\mathcal{T}} : \Lambda_{\mathcal{M}} \to \Lambda_{\mathcal{M}}$ is not expansive.*

(ii) *If there is $n \in \mathbb{N}$ such that $\left({\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}}^{[n]}\right)^*$ is 1-1 and surjective, then one has topological conjugacies:*

$$(\Lambda_{\mathcal{M}}, \varphi_{\mathcal{T}}) \simeq (\Lambda_{\mathcal{N}_{\mathcal{T}}^{[n]}}, \sigma_{\mathcal{N}_{\mathcal{T}}^{[n]}}), \qquad (\Lambda_{\mathcal{N}_{\mathcal{T}}^{[n]}}, \varphi_{\mathcal{T}^{[n]^*}}) \simeq (\Lambda_{\mathcal{M}}, \sigma_{\mathcal{M}}),$$

where $\mathcal{N}_{\mathcal{T}}^{[n]}$ is the n-higher block of $\mathfrak{L}^{\mathcal{N}}$ relative to $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$. Hence the topological dynamical system $(\Lambda_{\mathcal{M}}, \varphi_{\mathcal{T}})$ is realized as the subshift $(\Lambda_{\mathcal{N}_{\mathcal{T}}^{[n]}}, \sigma_{\mathcal{N}_{\mathcal{T}}^{[n]}})$. If in particular $\varphi_{\mathcal{T}}$ is expansive and its expansive constant c satisfies $c \geq \frac{1}{k}$ for some $k \in \mathbb{N}$ and ${\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}}^{[2k]^*}$ is surjective, then the topological dynamical system $(\Lambda_{\mathcal{M}}, \varphi_{\mathcal{T}})$ is topologically conjugate to the subshift $(\Lambda_{\mathcal{N}_{\mathcal{T}}^{[2k]}}, \sigma_{\mathcal{N}_{\mathcal{T}}^{[2k]}})$ presented by the λ-graph system $\mathfrak{L}^{\mathcal{N}_{\mathcal{T}}^{[2k]}}$.

Proof. The proofs below are essentially similar to the proofs of [19, Theorem 4.1]. We will give the proofs for the sake of completeness. (i) Assume that $\varphi_{\mathcal{T}}$ is expansive and its expansive constant c satisfies $c \geq \frac{1}{k}$ for some $k \in \mathbb{N}$. Suppose that ${\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}}^{[2k]^*}$ is not 1-1. There are distinct textile labels $s = (\beta_{i,j})_{(i,j)\in\mathbb{Z}^2}$ and $s' = (\beta'_{i,j})_{(i,j)\in\mathbb{Z}^2}$ in $\mathcal{U}_{\mathcal{T}}$ such that $\beta_{i,j} = \beta'_{i,j}$ for $i \in \mathbb{Z}, -(k-1) \leq j \leq k-1$. Now $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ is 1-1, by putting $y = (y_j)_{j\in\mathbb{Z}} = \Theta_{\mathcal{T}}(s), y' = (y'_j)_{j\in\mathbb{Z}} = \Theta_{\mathcal{T}}(s')$, we have $y \neq y' \in \Lambda_{\mathcal{M}}$. Since one has $y_j = y'_j$ for $-(k-1) \leq j \leq k-1$, one sees that $\varphi_{\mathcal{T}}^i(y)_j = \varphi_{\mathcal{T}}^i(y')_j$ for $i \in \mathbb{Z}$ and $-(k-1) \leq j \leq k-1$. Hence we have

$$d(\varphi_{\mathcal{T}}^i(y), \varphi_{\mathcal{T}}^i(y')) < \frac{1}{k}$$

for all $i \in \mathbb{Z}$, a contradiction.

(ii) Since $(\Lambda_{\mathcal{M}^{[n]}}, \varphi_{\mathcal{T}^{[n]}})$ is topologically conjugate to $(\Lambda_{\mathcal{M}}, \varphi_{\mathcal{T}})$ and $(\Lambda_{\mathcal{M}^{[n]}}, \sigma_{\mathcal{M}^{[n]}})$ is topologically conjugate to $(\Lambda_{\mathcal{M}}, \sigma_{\mathcal{M}})$, The assetion holds from Theorem 4.9. □

Following Nasu's consideration as in [19, Section 2], we will define bias shifts on textile λ-graph systems. For a symbolic matrix system $(\mathcal{M}, I)$, we set for

$k \in \mathbb{N}$

$$\mathcal{M}_{l,l+k} = \mathcal{M}_{l,l+1} \cdots \mathcal{M}_{l,l+k}, \qquad I_{l,l+k} = I_{l,l+1} \cdots I_{l,l+k}, \qquad l \in \mathbb{Z}_+.$$

Let $(\mathcal{M}, I^{\mathcal{M}})$ and $(\mathcal{N}, I^{\mathcal{N}})$ form squares. Then for $k, n \in \mathbb{Z}_+$, we set

$$(\mathcal{N}^k\mathcal{M}^n)_{l,l+1} = \mathcal{N}_{l(k+n),(l+1)k+ln}\mathcal{M}_{(l+1)k+ln,(l+1)(k+n)},$$
$$(I^{\mathcal{N}^k\mathcal{M}^n})_{l,l+1} = I^{\mathcal{N}}_{l(k+n),(l+1)k+ln}I^{\mathcal{M}}_{(l+1)k+ln,(l+1)(k+n)}, \qquad l \in \mathbb{Z}_+.$$

As $I^{\mathcal{N}}_{l,l+1} = I^{\mathcal{M}}_{l,l+1}$, one sees that $(\mathcal{N}^k\mathcal{M}^n, I^{\mathcal{N}^k\mathcal{M}^n})$ becomes a symbolic matrix system over $(\Sigma^{\mathcal{N}})^k(\Sigma^{\mathcal{M}})^n$. Similarly we have a symbolic matrix system $(\mathcal{M}^n\mathcal{N}^k, I^{\mathcal{M}^n\mathcal{N}^k})$ over $(\Sigma^{\mathcal{M}})^n(\Sigma^{\mathcal{N}})^k$. For $\alpha = (\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \mathcal{U}_{\mathcal{T}}$, we set

$$(\check{c}^{(k,n)}(\alpha)_{(ik,in)}) = b_{ik,in}b_{ik+1,in}\cdots b_{ik+k-1,in}a_{ik+k,in}a_{ik+k,in+1}\cdots a_{ik+k,in+n-1}$$
$$(\hat{c}^{(k,n)}(\alpha)_{(ik,in)}) = a_{ik,in}a_{ik,in+1}\cdots a_{ik,in+n-1}b_{ik,ik+in}b_{ik+1,ik+in}\cdots b_{ik+k-1,ik+n}$$

where $a_{i,j} = J_{upper}(\alpha_{i,j}), b_{i,j} = J_{left}(\alpha_{i,j})$. Define

$$\check{\Theta}^{(k,n)}_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{N}^k\mathcal{M}^n}, \qquad \hat{\Theta}^{(k,n)}_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{M}^n\mathcal{N}^k}$$

by setting

$$\check{\Theta}^{(k,n)}_{\mathcal{T}}(\alpha) = (\check{c}^{(k,n)}(\alpha)_{(ik,in)})_{i\in\mathbb{Z}} \in \Lambda_{\mathcal{N}^k\mathcal{M}^n},$$
$$\hat{\Theta}^{(k,n)}_{\mathcal{T}}(\alpha) = (\hat{c}^{(k,n)}(\alpha)_{(ik,in)})_{i\in\mathbb{Z}} \in \Lambda_{\mathcal{M}^n\mathcal{N}^k}.$$

We set

$$\check{\mathcal{U}}^{(k,n)}_{\mathcal{T}} = \check{\Theta}^{(k,n)}_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}}), \qquad \hat{\mathcal{U}}^{(k,n)}_{\mathcal{T}} = \hat{\Theta}^{(k,n)}_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}}),$$
$$\check{\sigma}^{(k,n)}_{\mathcal{T}}((\check{c}^{(k,n)}(\alpha)_{(ik,in)})_{i\in\mathbb{Z}}) = (\check{c}^{(k,n)}(\alpha)_{((i+1)k,(i+1)n)})_{i\in\mathbb{Z}},$$
$$\hat{\sigma}^{(k,n)}_{\mathcal{T}}((\hat{c}^{(k,n)}(\alpha)_{(ik,in)})_{i\in\mathbb{Z}}) = (\hat{c}^{(k,n)}(\alpha)_{((i+1)k,(i+1)n)})_{i\in\mathbb{Z}}.$$

We have subshifts

$$(\check{\mathcal{U}}^{(k,n)}_{\mathcal{T}}, \check{\sigma}^{(k,n)}_{\mathcal{T}}) \quad \text{and} \quad (\hat{\mathcal{U}}^{(k,n)}_{\mathcal{T}}, \hat{\sigma}^{(k,n)}_{\mathcal{T}})$$

over $(\Sigma^{\mathcal{N}})^k(\Sigma^{\mathcal{M}})^n$ and over $(\Sigma^{\mathcal{M}})^n(\Sigma^{\mathcal{N}})^k$ respectively.

Lemma 4.11. *$(\check{\mathcal{U}}^{(k,n)}_{\mathcal{T}}, \check{\sigma}^{(k,n)}_{\mathcal{T}})$ is topologically conjugate to $(\hat{\mathcal{U}}^{(k,n)}_{\mathcal{T}}, \hat{\sigma}^{(k,n)}_{\mathcal{T}})$.*

Proof. Define $\psi : \hat{\mathcal{U}}^{(k,n)}_{\mathcal{T}} \to \check{\mathcal{U}}^{(k,n)}_{\mathcal{T}}$ by setting

$$\psi(\hat{\Theta}^{(k,n)}_{\mathcal{T}}(\alpha)) = \check{\Theta}^{(k,n)}_{\mathcal{T}}(\sigma^{(0,n)}_{\mathcal{T}}(\alpha))$$

for $\alpha \in \mathcal{U}_{\mathcal{T}}$. It is direct to see that ψ is a topological conjugacy between $(\check{\mathcal{U}}^{(k,n)}_{\mathcal{T}}, \check{\sigma}^{(k,n)}_{\mathcal{T}})$ and $(\hat{\mathcal{U}}^{(k,n)}_{\mathcal{T}}, \hat{\sigma}^{(k,n)}_{\mathcal{T}})$. □

We call the subshift $(\check{\mathcal{U}}_{\mathcal{T}}^{(k,n)}, \check{\sigma}_{\mathcal{T}}^{(k,n)})$ the (k,n)-*bias shift* defined by $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$.

5. LR textile systems on λ-graph systems

In this section, we formulate LR textile λ-graph systems, that are generalization of sofic LR textile systems defined by Nasu [19].

Proposition 5.1. *Assume that λ-graph systems $\mathfrak{L}^{\mathcal{M}}$ and $\mathfrak{L}^{\mathcal{N}}$ form squares. If there exists a specification κ between $\Sigma^{\mathcal{M}}\Sigma^{\mathcal{N}}$ and $\Sigma^{\mathcal{N}}\Sigma^{\mathcal{M}}$ that gives specified equivalences*

$$\mathcal{M}_{l,l+1}\mathcal{N}_{l+1,l+2} \overset{\kappa}{\simeq} \mathcal{N}_{l,l+1}\mathcal{M}_{l+1,l+2}, \qquad l \in \mathbb{Z}_+, \tag{5.1}$$

then there exists a λ-graph system $\mathfrak{L}^{\mathcal{K}}$ and a textile λ-graph system $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}} = (p, q : \mathfrak{L}^{\mathcal{K}} \to \mathfrak{L}^{\mathcal{M}})$.

Proof. We identify the vertex sets $V_l^{\mathcal{M}}$ and $V_l^{\mathcal{N}}$ for $l \in \mathbb{Z}_+$. Put

$$\begin{aligned} V_l^{\mathcal{K}} =& E_{l,l+1}^{\mathcal{N}}, \\ E_{l,l+1}^{\mathcal{K}} =& \{(f', f, e, e') \in E_{l,l+1}^{\mathcal{N}} \times E_{l+1,l+2}^{\mathcal{N}} \times E_{l,l+1}^{\mathcal{M}} \times E_{l+1,l+2}^{\mathcal{M}} \mid \\ & s^{\mathcal{M}}(e) = s^{\mathcal{N}}(f'), t^{\mathcal{M}}(e) = s^{\mathcal{N}}(f), t^{\mathcal{N}}(f') = s^{\mathcal{M}}(e'), \\ & t^{\mathcal{M}}(e') = t^{\mathcal{N}}(f), \kappa(\lambda^{\mathcal{M}}(e)\lambda^{\mathcal{N}}(f)) = \lambda^{\mathcal{N}}(f')\lambda^{\mathcal{M}}(e')\}, \qquad l \in \mathbb{Z}_+ \end{aligned}$$

and

$$V^{\mathcal{K}} = \cup_{l\in\mathbb{Z}_+} V_l^{\mathcal{K}}, \qquad E^{\mathcal{K}} = \cup_{l\in\mathbb{Z}_+} E_{l,l+1}^{\mathcal{K}}.$$

Each element $(f', f, e, e') \in E_{l,l+1}^{\mathcal{K}}$ is visualized as a square:

$$\begin{array}{ccc} \cdot & \xrightarrow{e} & \cdot \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \cdot & \xrightarrow{e'} & \cdot \end{array} \quad .$$

We define $s^{\mathcal{K}} : E_{l,l+1}^{\mathcal{K}} \to V_l^{\mathcal{K}}, t^{\mathcal{K}} : E_{l,l+1}^{\mathcal{K}} \to V_{l+1}^{\mathcal{K}}$ by setting

$$s^{\mathcal{K}}(f', f, e, e') = f', \qquad t^{\mathcal{K}}(f', f, e, e') = f \qquad \text{for } (f', f, e, e') \in E_{l,l+1}^{\mathcal{K}},$$

and $\iota_{l,l+1}^{\mathcal{K}} : V_{l+1}^{\mathcal{K}} \to V_l^{\mathcal{K}}$ by setting $\iota_{l,l+1}^{\mathcal{K}} = \varphi_{l+1}^{\mathcal{N}}$. We put

$$\begin{aligned} \Sigma^{\mathcal{K}} = \{(\lambda^{\mathcal{N}}(f'), \lambda^{\mathcal{N}}(f), \lambda^{\mathcal{M}}(e), \lambda^{\mathcal{M}}(e')) \in \Sigma^{\mathcal{N}} \times \Sigma^{\mathcal{N}} \times \Sigma^{\mathcal{M}} \times \Sigma^{\mathcal{M}} \mid \\ (f', f, e, e') \in E_{l,l+1}^{\mathcal{K}}, l \in \mathbb{Z}_+\} \end{aligned}$$

and

$$\lambda^{\mathcal{K}} : E^{\mathcal{K}} \ni (f', f, e, e') \longrightarrow \Sigma^{\mathcal{K}} \ni (\lambda^{\mathcal{N}}(f'), \lambda^{\mathcal{N}}(f), \lambda^{\mathcal{M}}(e), \lambda^{\mathcal{M}}(e')).$$

Then we will show that $\mathfrak{L}^{\mathcal{K}} = (V^{\mathcal{K}}, E^{\mathcal{K}}, \lambda^{\mathcal{K}}, \iota^{\mathcal{K}})$ is a λ-graph system over $\Sigma^{\mathcal{K}}$. For $u \in V^{\mathcal{K}}_{l-1}, v \in V^{\mathcal{K}}_{l+1}$, put $w = \iota^{\mathcal{K}}_{l,l+1}(v)$. One sees

$$(E^{\mathcal{K}})^{l-1,l}_{\iota^{\mathcal{K}}}(u, v) = \{(u, w, e, e') \in E^{\mathcal{K}}_{l-1,l} \mid e \in E^{\mathcal{M}}_{l-1,l}, e' \in E^{\mathcal{M}}_{l,l+1}\}.$$

As $w \in V^{\mathcal{K}}_l = E^{\mathcal{N}}_{l,l+1}$ is fixed, if we choose $e \in E^{\mathcal{M}}_{l-1,l}$ such that $(u, w, e, e') \in (E^{\mathcal{K}})^{l-1,l}_{\iota^{\mathcal{K}}}(u, v)$ for some e', the label $\lambda^{\mathcal{M}}(e)\lambda^{\mathcal{N}}(w) \in \Sigma^{\mathcal{M}}\Sigma^{\mathcal{N}}$ determines the labels $\lambda^{\mathcal{N}}(u)$ and $\lambda^{\mathcal{M}}(e')$ of u and e' through the specification κ. Since the label $\lambda^{\mathcal{M}}(e')$ and the terminal $t^{\mathcal{M}}(e') = t^{\mathcal{N}}(w)$ are determined, the edge e' is uniquely determined because $\mathfrak{L}^{\mathcal{M}}$ is left-resolving. Hence under fixing both $u \in V^{\mathcal{K}}_{l-1}$ and $v \in V^{\mathcal{K}}_{l+1}$, the edge $e' \in E^{\mathcal{M}}_{l,l+1}$ is uniquely determined by $e \in E^{\mathcal{M}}_{l-1,l}$, so that $(E^{\mathcal{K}})^{l-1,l}_{\iota^{\mathcal{K}}}(u, v)$ is identified with $\{e \in E^{\mathcal{M}}_{l-1,l} \mid s^{\mathcal{M}}(e) = s^{\mathcal{N}}(u), t^{\mathcal{M}}(e) = s^{\mathcal{N}}(w)\}$. Now $\iota^{\mathcal{N}} = \iota^{\mathcal{M}}$ so that $s^{\mathcal{N}}(w) = \iota^{\mathcal{N}}_{l,l+1}(s^{\mathcal{N}}(v)) = \iota^{\mathcal{M}}_{l,l+1}(s^{\mathcal{N}}(v))$. Hence $(E^{\mathcal{K}})^{l-1,l}_{\iota^{\mathcal{K}}}(u, v)$ is identified with $(E^{\mathcal{M}})^{l-1,l}_{\iota^{\mathcal{M}}}(s^{\mathcal{N}}(u), s^{\mathcal{N}}(v))$. On the other hand, one sees

$$(E^{\mathcal{K}})^{\iota^{\mathcal{K}}}_{l,l+1}(u, v) = \{(w', v, g, g') \in E^{\mathcal{K}}_{l,l+1} \mid \iota^{\mathcal{K}}_{l-1,l}(w') = u\}.$$

Similarly to the discussion of $(E^{\mathcal{K}})^{l-1,l}_{\iota^{\mathcal{K}}}(u, v)$, if we choose $g \in E^{\mathcal{M}}_{l,l+1}$ such that $(w', v, g, g') \in (E^{\mathcal{K}})^{\iota^{\mathcal{K}}}_{l,l+l}(u, v)$ for some g', the label $\lambda^{\mathcal{M}}(g)\lambda^{\mathcal{N}}(v) \in \Sigma^{\mathcal{M}}\Sigma^{\mathcal{N}}$ determines the labels $\lambda^{\mathcal{N}}(w')$ and $\lambda^{\mathcal{M}}(g')$ of w' and g' through the specification κ. Since the label $\lambda^{\mathcal{M}}(g')$ and the terminal $t^{\mathcal{M}}(g') = t^{\mathcal{N}}(v)$ are determined, the edge g' is uniquely determined because $\mathfrak{L}^{\mathcal{M}}$ is left-resolving, so that the source vertex $s^{\mathcal{M}}(g') \in V^{\mathcal{M}}_{l+1}$ of g' is determined. Since $t^{\mathcal{N}}(w') = s^{\mathcal{M}}(g')$, the edge $w' \in E^{\mathcal{N}}_{l,l+1}$ is uniquely determined. Hence under fixing the vertices $u \in V^{\mathcal{K}}_{l-1}, v \in V^{\mathcal{K}}_{l+1}$, both the edges $g' \in E^{\mathcal{M}}_{l+1,l+2}$ and $w' \in E^{\mathcal{N}}_{l,l+1}$ are uniquely determined by $g \in E^{\mathcal{M}}_{l,l+1}$. Now one has $s^{\mathcal{M}}(g) = s^{\mathcal{N}}(w'), \varphi^{\mathcal{N}}_{l+1}(w') = u$ so that

$$\iota^{\mathcal{M}}_{l-1,l}(s^{\mathcal{N}}(w')) = \iota^{\mathcal{N}}_{l-1,l}(s^{\mathcal{N}}(w')) = s^{\mathcal{N}}(u) \in V^{\mathcal{M}}_{l-1} = V^{\mathcal{N}}_{l-1}.$$

It then follows that $(E^{\mathcal{K}})^{\iota^{\mathcal{K}}}_{l,l+1}(u, v)$ is identified with $\{g \in E^{\mathcal{M}}_{l,l+1} \mid \iota^{\mathcal{M}}_{l-1,l}(s^{\mathcal{M}}(g)) = s^{\mathcal{N}}(u), t^{\mathcal{M}}(g) = s^{\mathcal{N}}(v)\}$, that is $(E^{\mathcal{M}})^{\iota^{\mathcal{M}}}_{l,l+1}(s^{\mathcal{N}}(u), s^{\mathcal{N}}(v))$. By the local property of $\mathfrak{L}^{\mathcal{M}}$, one has a label preserving bijection between

$$(E^{\mathcal{M}})^{\iota^{\mathcal{M}}}_{l,l+1}(s^{\mathcal{N}}(u), s^{\mathcal{N}}(v)) \quad \text{and} \quad (E^{\mathcal{M}})^{l-1,l}_{\iota^{\mathcal{M}}}(s^{\mathcal{N}}(u), s^{\mathcal{N}}(v))$$

that yields a a label preserving bijection between

$$(E^{\mathcal{K}})^{\iota^{\mathcal{K}}}_{l,l+1}(u, v) \quad \text{and} \quad (E^{\mathcal{K}})^{l-1,l}_{\iota^{\mathcal{K}}}(u, v).$$

This means that $\mathfrak{L}^{\mathcal{K}} = (V^{\mathcal{K}}, E^{\mathcal{K}}, \lambda^{\mathcal{K}}, \iota^{\mathcal{K}})$ is a λ-graph system over $\Sigma^{\mathcal{K}}$.

Define a homomorphism $p : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ and a one-shift homomorphism $q : \mathfrak{L}^{\mathcal{K}} \longrightarrow \mathfrak{L}^{\mathcal{M}}$ by setting

$$\begin{aligned} p^E &: (f', f, e, e') \in E^{\mathcal{K}}_{l,l+1} \longrightarrow e \in E^{\mathcal{M}}_{l,l+1}, \\ p^V &: f \in V^{\mathcal{K}}_l = E^{\mathcal{N}}_{l,l+1} \longrightarrow s^{\mathcal{N}}(f) \in V^{\mathcal{N}}_l, \\ p^\Sigma &: (\lambda^{\mathcal{N}}(f'), \lambda^{\mathcal{N}}(f), \lambda^{\mathcal{M}}(e), \lambda^{\mathcal{M}}(e')) \in \Sigma^{\mathcal{K}} \longrightarrow \lambda^{\mathcal{M}}(e) \in \Sigma^{\mathcal{M}} \end{aligned}$$

and

$$\begin{aligned} q^E &: (f', f, e, e') \in E^{\mathcal{K}}_{l,l+1} \longrightarrow e' \in E^{\mathcal{M}}_{l+1,l+2}, \\ q^V &: f' \in V^{\mathcal{K}}_l = E^{\mathcal{N}}_{l,l+1} \longrightarrow t^{\mathcal{N}}(f') \in V^{\mathcal{N}}_{l+1}, \\ q^\Sigma &: (\lambda^{\mathcal{N}}(f'), \lambda^{\mathcal{N}}(f), \lambda^{\mathcal{M}}(e), \lambda^{\mathcal{M}}(e')) \in \Sigma^{\mathcal{K}} \longrightarrow \lambda^{\mathcal{M}}(e') \in \Sigma^{\mathcal{M}} \end{aligned}$$

respectively. Since for $\alpha = (f', f, e, e') \in E^{\mathcal{K}}_{l,l+1}$, one has

$$(s^{\mathcal{K}}(\alpha), t^{\mathcal{K}}(\alpha), p^E(\alpha), q^E(\alpha)) = (f', f, e, e')$$

the square $(s^{\mathcal{K}}(\alpha), t^{\mathcal{K}}(\alpha), p^E(\alpha), q^E(\alpha))$ determines α, and the quadruple

$$(\lambda^{\mathcal{N}}(s^{\mathcal{K}}(\alpha)), \lambda^{\mathcal{N}}(t^{\mathcal{K}}(\alpha)), \lambda^{\mathcal{M}}(p^E(\alpha)), \lambda^{\mathcal{M}}(q^E(\alpha)))$$

determines $\lambda^{\mathcal{K}}(\alpha)$. Hence one has a textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}} = (p, q : \mathfrak{L}^{\mathcal{K}} \to \mathfrak{L}^{\mathcal{M}})$ through the specified equivalences (5.1). □

We call this textile λ-graph system an LR textile λ-graph system, following Nasu's terminology for sofic textile systems ([19]).

Lemma 5.2 (cf. [19; Fact.6.14]). *An LR textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ is nondegenerate.*

Proof. Let $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ be an LR textile λ-graph system defined by (5.1). Keep the notation as in the previous proposition. We will prove that $p_X : X_{\mathfrak{L}^{\mathcal{K}}} \longrightarrow X_{\mathfrak{L}^{\mathcal{M}}}$ is surjective. We set for $l \in \mathbb{Z}_+, n \in \mathbb{N}$

$$\begin{aligned} E^{\mathcal{M}}_{l,l+n} = &\{(e_1, \dots, e_n) \in E^{\mathcal{M}}_{l,l+1} \times E^{\mathcal{M}}_{l+1,l+2} \times \cdots \times E^{\mathcal{M}}_{l+n-1,l+n} \mid \\ &t^{\mathcal{M}}(e_i) = s^{\mathcal{M}}(e_{i+1}), i = 1, 2, \dots, n-1\} \end{aligned}$$

and similarly $E^{\mathcal{K}}_{l,l+n}$. Since $X_{\mathfrak{L}^{\mathcal{K}}}$ is compact, it suffices to show that for $(e_1, \dots, e_n) \in E^{\mathcal{M}}_{l,l+n}$ there exists $(g_1, \dots, g_n) \in E^{\mathcal{K}}_{l,l+n}$ such that $p_X(g_i) = e_i, i = 1, 2, \dots, n$. Take $f_n \in E^{\mathcal{N}}_{l+n,l+n+1}$ such that $t^{\mathcal{M}}(e_n) = s^{\mathcal{N}}(f_n)$. Since $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ is LR, there uniquely exists $f_{n-1} \in E^{\mathcal{N}}_{l+n-1,l+n}$ and $e'_n \in E^{\mathcal{M}}_{l+n,l+n+1}$ such that the quadruple

$(f_{n-1}, f_n, e_n, e'_n)$ denoted by g_n gives rise to an element of $E^{\mathcal{K}}_{l+n-1,l+n}$. One may inductively find $f_k \in E^{\mathcal{N}}_{l+k,l+k+1}$, $e'_k \in E^{\mathcal{M}}_{l+k,l+k+1}$ for $k = 1, 2, \ldots, n$ and $f_0 \in E^{\mathcal{N}}_{l,l+1}$ such that the quadruple $(f_{k-1}, f_k, e_k, e'_k)$ denoted by g_k gives rise to an element of $E^{\mathcal{K}}_{l+k-1,l+k}$ for $k = 1, 2, \ldots, n$. They satisfy $(g_1, \ldots, g_n) \in E^{\mathcal{K}}_{l+1,l+n+1}$ and $p_X(g_i) = e_i, i = 1, 2, \ldots, n$. One also sees that $q_X : X_{\mathfrak{L}^{\mathcal{K}}} \longrightarrow X_{\mathfrak{L}^{\mathcal{M}}_0}$ is surjective in a smilar way. □

PROPOSITION 5.3. *Let $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ be an LR textile λ-graph system. For $k, n \geq 1$, we have*

$$(\check{\mathcal{U}}^{(k,n)}_{\mathcal{T}}, \check{\sigma}^{(k,n)}_{\mathcal{T}}) = (\Lambda_{\mathcal{N}^k\mathcal{M}^n}, \sigma_{\mathcal{N}^k\mathcal{M}^n}), \qquad (\hat{\mathcal{U}}^{(k,n)}_{\mathcal{T}}, \hat{\sigma}^{(k,n)}_{\mathcal{T}}) = (\Lambda_{\mathcal{M}^n\mathcal{N}^k}, \sigma_{\mathcal{M}^n\mathcal{N}^k}).$$

Proof. We will prove that the map $\check{\Theta}^{(k,n)}_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{N}^k\mathcal{M}^n}$ is surjective so that the first equality holds. Take an arbitrary sequence $(a_i)_{i\in\mathbb{Z}} \in \Lambda_{\mathcal{N}^k\mathcal{M}^n}$. Since $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ is LR, there exists a two dimensional configuration $(\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \prod_{(i,j)\in\mathbb{Z}^2} \Sigma^{\mathcal{K}}$ such that by putting

$$\alpha^h_i = (\alpha_{i,j})_{j\in\mathbb{Z}}, \qquad \alpha^v_j = (\alpha_{i,j})_{i\in\mathbb{Z}}$$

α^h_i belongs to $\Lambda_{\mathcal{K}}$ for all $i \in \mathbb{Z}$ and α^v_j belongs to $\Lambda_{\mathcal{K}^*}$ for all $j \in \mathbb{Z}$, that satisfy

$$\xi(\alpha^h_i) = \eta(\alpha^h_{i-1}) \quad \text{for } i \in \mathbb{Z}, \qquad \xi^*(\alpha^v_j) = \eta^*(\alpha^v_{j-1}) \quad \text{for } j \in \mathbb{Z},$$

and

$$a_i = (p^*(\alpha_{ki,ni}), \ldots, p^*(\alpha_{(k+1)i-1,ni}), q(\alpha_{(k+1)i-1,ni}), \ldots, q(\alpha_{(k+1)i-1,(n+1)i-1}))$$

for $i \in \mathbb{Z}$. For $m \in \mathbb{N}$ and $((k+1)m-1, (n+1)m-1) \in \triangle$, we may take an edge

$$(e_{ki,ni}, \ldots, e_{(k+1)i-1,ni}, e_{(k+1)i-1,ni}, \ldots, e_{(k+1)i-1,(n+1)i-1}) \tag{5.2}$$

that belongs to

$$E^{\mathcal{K}}_{ki+ni,ki+ni+1} \times \cdots \times E^{\mathcal{K}}_{(k+1)i+ni-1,(k+1)i+ni} \times E^{\mathcal{K}}_{(k+1)i+ni-1,(k+1)i+ni} \times \cdots \times E^{\mathcal{K}}_{(k+1)i+(n+1)i-2,(k+1)i+(n+1)i-1}, \quad i = 0, 1, \ldots, m$$

such that

$$\begin{aligned}&(\lambda^{\mathcal{K}}(e_{ki,ni}), \ldots, \lambda^{\mathcal{K}}(e_{(k+1)i-1,ni}), \lambda^{\mathcal{K}}(e_{(k+1)i-1,ni}), \ldots, \lambda^{\mathcal{K}}(e_{(k+1)i-1,(n+1)i-1}))\\ =&(\alpha_{ki,ni}, \ldots, \alpha_{(k+1)i-1,ni}, \alpha_{(k+1)i-1,ni}, \ldots, \alpha_{(k+1)i-1,(n+1)i-1}), \quad i = 0, 1, \ldots, m.\end{aligned}$$

Since $\mathfrak{L}^{\mathcal{K}}$ and $\mathfrak{L}^{\mathcal{K}^*}$ are both left-resolving, edges of $\mathfrak{L}^{\mathcal{K}}$ located in $\triangle_{((k+1)m-1,(n+1)m-1)}$ are uniquely determined by the edges (5.2) and the labels $(\alpha_{i,j})_{(i,j)\in\triangle_{((k+1)m-1,(n+1)m-1)}}$. Hence we know that

$$(\alpha_{i,j})_{(i,j)\in\triangle_{((k+1)m-1,(n+1)m-1)}} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}(\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}, \triangle_{((k+1)m-1,(n+1)m-1)}).$$

Since m is arbitrary, we have $(\alpha_{i,j})_{(i,j)\in\triangle} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$. By applying the above discussion to the sequences $(a_{i-K})_{i\in\mathbb{Z}} \in \Lambda_{\mathcal{N}^k\mathcal{M}^n}$ for $K \in \mathbb{Z}$, one has that $(\alpha_{i,j})_{(i,j)\in\triangle} \in \mathcal{U}^{\infty}_{\mathfrak{L}^{\mathcal{K}}}$, and $(\alpha_{i-K,j})_{(i,j)\in\triangle} \in \mathcal{U}^{\infty}_{\mathfrak{L}^{\mathcal{K}}}$ and $(\alpha_{i,j-K})_{(i,j)\in\triangle} \in \mathcal{U}^{\infty}_{\mathfrak{L}^{\mathcal{K}}}$ for $K \in \mathbb{Z}$. Thus we have $(\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \mathcal{U}_{\mathcal{T}}$ and $\check{\Theta}^{(k,n)}_{\mathcal{T}}((\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2}) = (a_i)_{i\in\mathbb{Z}}$. Therefore we conclude that

$$\check{\mathcal{U}}^{(k,n)}_{\mathcal{T}} = \Lambda_{\mathcal{N}^k\mathcal{M}^n} \qquad \text{and} \qquad \check{\sigma}^{(k,n)}_{\mathcal{T}} = \sigma_{\mathcal{N}^k\mathcal{M}^n}.$$

The other equality is similarly proved. □

Proposition 5.4 (cf. [19; Lemma 6.2]). *Let $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ be an LR textile λ-graph system. Then for $k, n \geq 1$ the map $\check{\Theta}^{(k,n)}_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \check{\mathcal{U}}^{(k,n)}_{\mathcal{T}}$ is injective. Hence it gives rise to a topological conjugacy between $(\mathcal{U}_{\mathcal{T}}, \sigma^{(k,n)}_{\mathcal{T}})$ and $(\check{\mathcal{U}}^{(k,n)}_{\mathcal{T}}, \check{\sigma}^{(k,n)}_{\mathcal{T}})$. Similarly we have a topological conjugacy between $(\mathcal{U}_{\mathcal{T}}, \sigma^{(k,n)}_{\mathcal{T}})$ and $(\hat{\mathcal{U}}^{(k,n)}_{\mathcal{T}}, \hat{\sigma}^{(k,n)}_{\mathcal{T}})$.*

Proof. By a similar way to the proof of [19, Lemma 6.2], we can show that for $(a_i)_{i\in\mathbb{Z}} \in \check{\mathcal{U}}^{(k,n)}_{\mathcal{T}}$ there uniquely exists $(\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \mathcal{U}_{\mathcal{T}}$ such that $\check{\Theta}^{(k,n)}_{\mathcal{T}}((\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2}) = (a_i)_{i\in\mathbb{Z}}$. □

We note that if $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ is LR, then $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}{}^*$ is LR. We provide the following lemma.

Lemma 5.5. *A 1-1 LR textile λ-graph system is surjective.*

Proof. Let $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ be a 1-1 LR textile λ-graph system. Since $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}{}^*$ is LR, the both $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ and $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}{}^*$ are nondegenerate. We will prove that the map

$$\Phi_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{K}}$$

is surjective. For $(a_j)_{j\in\mathbb{Z}} \in \Lambda_{\mathcal{K}}$, take $(e_j)_{j\in\mathbb{Z}_+} \in X_{\mathfrak{L}^{\mathcal{K}}} = \{(e_j)_{j\in\mathbb{Z}_+} \mid e_j \in E^{\mathcal{K}}_{j,j+1}, t(e_j) = s(e_{j+1}), j \in \mathbb{Z}_+\}$ such that $a_j = \lambda^{\mathcal{K}}(e_j), j \in \mathbb{Z}_+$. Recall $\triangle = \{(i,j) \in \mathbb{Z}^2 \mid i + j \geq 0\}$. We set

$$\begin{aligned}
\triangle_{r,u} &= \{(i,j) \in \triangle \mid i \leq 0, 0 \leq j\},\\
\triangle_{l,d} &= \{(i,j) \in \triangle \mid 1 \leq i, j \leq -1\},\\
\square_{r,d} &= \{(i,j) \in \triangle \mid 1 \leq i, 0 \leq j\}.
\end{aligned}$$

Now $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ is 1-1 so that there uniquely exists $\alpha_{i,j} \in \Sigma^{\mathcal{K}}$ for $(i,j) \in \mathbb{Z}^2$ such that by putting $\alpha_i = (\alpha_{i,j})_{j\in\mathbb{Z}}$ one has $\alpha_i \in \Lambda_{\mathcal{K}}$, $\alpha_0 = (a_j)_{j\in\mathbb{Z}}$ and $\eta(\alpha_i) = \xi(\alpha_{i+1})$ for $i \in \mathbb{Z}$. Take an arbitrary $(n,k) \in \triangle$. We set

$$\begin{aligned}
\triangle_{r,u}(k) &= \{(i,j) \in \triangle_{r,u} \mid j \leq k\},\\
\triangle_{l,d}(n) &= \{(i,j) \in \triangle_{l,d} \mid i \leq n\},\\
\square_{r,d}(n,k) &= \{(i,j) \in \square_{r,d} \mid i \leq n, j \leq k\}.
\end{aligned}$$

Take $f_k(i) \in E^{\mathcal{N}}_{i+k+1,i+k+2}, i = 1, 2, \ldots, n$ such that there exist $e_{i,j} \in E^{\mathcal{K}}_{i+j,i+j+1}$ for $(i,j) \in \square_{r,d}(n,k) \cup \triangle_{l,d}(n)$ satisfying

$$\begin{aligned}(e_{i,j})_{(i,j)\in\square_{r,d}(n,k)\cup\triangle_{l,d}(n)} &\in X(\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}; \square_{r,d}(n,k) \cup \triangle_{l,d}(n)),\\ p^E(e_{1,j}) &= q^E(e_j) \qquad \text{for } j = 1, 2, \ldots, k,\\ t(e_{i,k}) &= f_k(i) \qquad \text{for } i = 1, 2, \ldots, n,\\ \alpha_{i,j} &= \lambda^{\mathcal{K}}(e_{i,j}) \qquad \text{for } (i,j) \in \square_{r,d}(n,k) \cup \triangle_{l,d}(n)).\end{aligned}$$

As $\mathfrak{L}^{\mathcal{K}}$ is left-resolving, such edges $e_{i,j} \in E^{\mathcal{K}}_{i+j,i+j+1}$ for $(i,j) \in \square_{r,d}(n,k) \cup \triangle_{l,d}(n)$ are unique for $f_k(i) \in E^{\mathcal{N}}_{i+k+1,i+k+2}, i = 1, 2, \ldots, n$. We set $e_{0,j} = e_j$ for $j = 0, 1, \ldots, k$. Since $\mathfrak{L}^{\mathcal{K}^*}$ is left-resolving, the vertices $p^E(e_j), j = 0, 1, \ldots k$ of $\mathfrak{L}^{\mathcal{K}^*}$ and labels $(\alpha_{i,j})_{(i,j)\in\triangle_{r,u}(k)}$ uniquely determine edges $e_{i,j} \in E^{\mathcal{K}}_{i+j,i+j+1}$ for $(i,j) \in \triangle_{r,u}(k)$ such that $t^{\mathcal{K}^*}(e_{1,j}) = s^{\mathcal{K}^*}(e_j), j = 1, \ldots, k$ and $\lambda^{\mathcal{K}}(e_{i,j}) = \alpha_{i,j}$ for $(i,j) \in \triangle_{r,u}(k)$. Hence we have

$$(e_{i,j})_{(i,j)\in\triangle_{r,u}(k)} \in X(\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}; \triangle_{r,u}(k)) \quad \text{and hence} \quad (e_{i,j})_{(i,j)\in\triangle} \in X(\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}; \triangle)$$

so that

$$(\alpha_{i,j})_{(i,j)\in\triangle(n,k)} \in \mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}(\triangle(n,k))$$

By Proposition 4.3, the configuration $(\alpha_{i,j})_{(i,j)\in\triangle}$ belongs to $\mathcal{U}_{\mathfrak{L}^{\mathcal{K}}}$. By applying this argument to the configurations $(a_{i+k,j})_{(i,j)\in\mathbb{Z}^2}$ and $(a_{i,j+k})_{(i,j)\in\mathbb{Z}^2}$ for $k \in \mathbb{Z}$, we know that $(\alpha_{i,j})_{(i,j)\in\triangle}$ belongs to $\mathcal{U}^{\infty}_{\mathfrak{L}^{\mathcal{K}}}$ and to $\mathcal{U}_{\mathcal{T}}$. Since $\Phi_{\mathcal{T}}((\alpha_{i,j})_{(i,j)\in\mathbb{Z}^2}) = (a_j)_{j\in\mathbb{Z}}$, the map $\Phi_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \to \Lambda_{\mathcal{K}}$ is surjective. □

Therefore we obtain

Theorem 5.6. *Let $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{N}}}$ be a 1-1 LR textile λ-graph system defined by a specified equivalence:*

$$\mathcal{M}_{l,l+1}\mathcal{N}_{l+1,l+2} \overset{\kappa}{\simeq} \mathcal{N}_{l,l+1}\mathcal{M}_{l+1,l+2}, \qquad l \in \mathbb{Z}_+.$$

Then the dynamical system $(\Lambda_{\mathcal{M}}, \varphi^k_{\mathcal{T}}\sigma^n_{\mathcal{M}})$, $k \geq 0, n \geq 1$ is topologically conjugate to the subshift $(\Lambda_{\mathcal{N}^k\mathcal{M}^n}, \sigma_{\mathcal{N}^k\mathcal{M}^n})$ presented by the symbolic matrix system $(\mathcal{N}^k\mathcal{M}^n, I^{\mathcal{N}^k\mathcal{M}^n})$, defined by

$$\begin{aligned}(\mathcal{N}^k\mathcal{M}^n)_{l,l+1} &= \mathcal{N}_{l(k+n),l(k+n)+1} \cdots \mathcal{N}_{l(k+n)+n-1,l(k+n)+n} \cdot\\ &\quad \cdot \mathcal{M}_{l(k+n)+n,l(k+n)+n+1} \cdots \mathcal{M}_{(l+1)(k+n)-1,(l+1)(k+n)},\\ I^{\mathcal{N}^k\mathcal{M}^n}{}_{l,l+1} &= I^{\mathcal{N}}_{l(k+n),l(k+n)+1} \cdots I^{\mathcal{N}}_{l(k+n)+n-1,l(k+n)+n} \cdot\\ &\quad \cdot I^{\mathcal{M}}_{l(k+n)+n,l(k+n)+n+1} \cdots I^{\mathcal{M}}_{(l+1)(k+n)-1,(l+1)(k+n)}, \qquad l \in \mathbb{Z}_+.\end{aligned}$$

Proof. Since $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ is nondegenerate, for the case when $k = 0$ the assertion is clear. We may assume that $k \geq 1$. Since $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ is 1-1 and LR, it is surjective by Lemma 5.4 so that $(\Lambda_{\mathcal{M}}, \varphi_{\mathcal{T}}^k \sigma_{\mathcal{M}}^n)$ is conjugate to $(\mathcal{U}_{\mathcal{T}}, \sigma_{\mathcal{T}}^{(k,n)})$ by Proposition 4.8. As $\mathcal{T}_{\mathcal{K}_{\mathcal{N}}^{\mathcal{M}}}$ is LR and $k, n \geq 1$, one has that $(\mathcal{U}_{\mathcal{T}}, \sigma_{\mathcal{T}}^{(k,n)})$ is conjugate to $(\check{\mathcal{U}}_{\mathcal{T}}^{(k,n)}, \check{\sigma}_{\mathcal{T}}^{(k,n)})$ by Proposition 5.4. Hence by Proposition 5.3, we obtain the assertion. □

6. LR textile systems and properly strong shift equivalences

Let $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ be symbolic matrix systems over Σ and Σ' respectively.

Definition ([15]). $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *properly strong shift equivalent in 1-step* if there exist alphabets C, D and specifications $\kappa : \Sigma \to CD$, $\kappa' : \Sigma' \to DC$ and increasing sequences $n(l), n'(l)$ on $l \in \mathbb{Z}_+$ such that for each $l \in \mathbb{Z}_+$, there exist an $n(l) \times n'(l+1)$ matrix $\mathcal{P}_l$ over C, an $n'(l) \times n(l+1)$ matrix $\mathcal{Q}_l$ over D, an $n(l) \times n(l+1)$ matrix X_l over $\{0,1\}$ and an $n'(l) \times n'(l+1)$ matrix Y_l over $\{0,1\}$ satisfying the following equations:

$$\begin{aligned} \mathcal{M}_{l,l+1} &\overset{\kappa}{\simeq} \mathcal{P}_{2l}\mathcal{Q}_{2l+1}, \qquad & \mathcal{M}'_{l,l+1} &\overset{\kappa'}{\simeq} \mathcal{Q}_{2l}\mathcal{P}_{2l+1}, \\ I_{l,l+1} &= X_{2l}X_{2l+1}, \qquad & I'_{l,l+1} &= Y_{2l}Y_{2l+1} \end{aligned}$$

and

$$X_l\mathcal{P}_{l+1} = \mathcal{P}_l Y_{l+1}, \qquad Y_l\mathcal{Q}_{l+1} = \mathcal{Q}_l X_{l+1}.$$

We write this situation as $(\mathcal{P}, \mathcal{Q}, X, Y) : (\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}', I')$.

We in particular consider the case when $(\mathcal{M}', I') = (\mathcal{M}, I)$.

Lemma 6.1. *Suppose that* $(\mathcal{P}, \mathcal{Q}, X, Y) : (\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}, I)$. *Put*

$$\begin{aligned} \mathcal{P}_{l,l+1} &= \mathcal{P}_{2l}Y_{2l+1} (= X_{2l}\mathcal{P}_{2l+1}), \qquad & I^{\mathcal{P}}_{l,l+1} &= I_{l,l+1}, \\ \mathcal{Q}_{l,l+1} &= \mathcal{Q}_{2l}X_{2l+1} (= Y_{2l}\mathcal{Q}_{2l+1}), \qquad & I^{\mathcal{Q}}_{l,l+1} &= I_{l,l+1}, \qquad l \in \mathbb{Z}_+. \end{aligned}$$

Then we have

(i) $(\mathcal{P}, I^{\mathcal{P}}) = (\mathcal{P}_{l,l+1}, I^{\mathcal{P}}_{l,l+1})_{l\in\mathbb{Z}_+}$ *and* $(\mathcal{Q}, I^{\mathcal{Q}}) = (\mathcal{Q}_{l,l+1}, I^{\mathcal{Q}}_{l,l+1})_{l\in\mathbb{Z}_+}$ *are symbolic matrix systems over* C *and* D *respectively.*

(ii) *The pair* $\mathfrak{L}^{\mathcal{M}}$ *and* $\mathfrak{L}^{\mathcal{P}}$, *and the pair* $\mathfrak{L}^{\mathcal{M}}$ *and* $\mathfrak{L}^{\mathcal{Q}}$ *both form squares such that*

$$\mathcal{M}_{l,l+1}\mathcal{P}_{l+1,l+2} \overset{\kappa^{\mathcal{P}}}{\simeq} \mathcal{P}_{l,l+1}\mathcal{M}_{l+1,l+2}, \qquad l \in \mathbb{Z}_+, \tag{6.1}$$

$$\mathcal{M}_{l,l+1}\mathcal{Q}_{l+1,l+2} \overset{\kappa^{\mathcal{Q}}}{\simeq} \mathcal{Q}_{l,l+1}\mathcal{M}_{l+1,l+2}, \qquad l \in \mathbb{Z}_+ \tag{6.2}$$

for some specifications $\kappa^{\mathcal{P}} : \Sigma C \longrightarrow C\Sigma$ *and* $\kappa^{\mathcal{Q}} : \Sigma D \longrightarrow D\Sigma$.

Proof. We will prove the assertions for $(\mathcal{P}, I^{\mathcal{P}})$. The assertions for the other one are symmetric.

(i) The equality $\mathcal{P}_{l,l+1}I^{\mathcal{P}}_{l+1,l+2} = I^{\mathcal{P}}_{l,l+1}\mathcal{P}_{l+1,l+2}$ is easily shown. Hence $(\mathcal{P}, I^{\mathcal{P}})$ $= (\mathcal{P}_{l,l+1}, I^{\mathcal{P}}_{l,l+1})_{l\in\mathbb{Z}_+}$ is a symbolic matrix system over C.

(ii) One has

$$\begin{aligned}\mathcal{M}_{l,l+1}\mathcal{P}_{l+1,l+2} &\overset{\kappa}{\simeq} \mathcal{P}_{2l}\mathcal{Q}_{2l+1}X_{2l+2}\mathcal{P}_{2l+3}\\ &= \mathcal{P}_{2l}Y_{2l+1}\mathcal{Q}_{2l+2}\mathcal{P}_{2l+3} \overset{\kappa'^{-1}}{\simeq} \mathcal{P}_{l,l+1}\mathcal{M}_{l+1,l+2}.\end{aligned}$$

For $\alpha \in \Sigma, c \in C$, by putting $\kappa^{\mathcal{P}}(\alpha c) = c_\alpha \kappa'^{-1}(d_\alpha c)$ where $\kappa(\alpha) = c_\alpha d_\alpha \in CD$, the specification $\kappa^{\mathcal{P}} : \Sigma C \longrightarrow C\Sigma$ yields the desired specified equivalence (6.1). □

By this lemma with Proposition 5.1, the relations (6.1) and (6.2) yield LR textile λ-graph systems $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}$ and $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{Q}}}$ respectively.

Lemma 6.2. *Suppose that* $(\mathcal{P}, \mathcal{Q}, X, Y) : (\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}, I)$. *Keep the notations as in the preceding lemma. The LR textile* λ*-graph systems* $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}$ *and* $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{Q}}}$ *are both 1-1 and hence surjective.*

Proof. We will prove that $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}$ is 1-1. The LR textile system defined by the specified equivalence (6.1) comes from the specified equivalence

$$\mathcal{P}_{2l}\mathcal{Q}_{2l+1} \cdot \mathcal{P}_{2l+2}Y_{2l+3} \overset{\kappa^{\mathcal{P}}}{\simeq} \mathcal{P}_{2l}Y_{2l+1} \cdot \mathcal{Q}_{2l+2}\mathcal{P}_{2l+3}. \tag{6.3}$$

Let $(\beta_l)_{l\in\mathbb{Z}} \in \Lambda_{\mathfrak{L}^\kappa}$ be such that $\xi((\beta_l)_{l\in\mathbb{Z}}) = (\alpha_l)_{l\in\mathbb{Z}}$. We put $\kappa(\alpha_l) = c_l d_l \in CD$ for $l \in \mathbb{Z}$. By (6.3) β_l is uniquely determined by the square:

$$\begin{array}{ccccc} \cdot & \xrightarrow{c_l} & \cdot & \xrightarrow{d_l} & \cdot \\ \downarrow{\scriptstyle c_l} & & & & \downarrow{\scriptstyle c_{l+1}} \\ \cdot & \xrightarrow{d_l} & \cdot & \xrightarrow{c_{l+1}} & \cdot \end{array}$$

Namely, β_l is uniquely determined by the quadruple $(c_l, c_{l+1}, c_l d_l, d_l c_{l+1}) \in \Sigma^{\mathcal{K}}$, that are determined by the sequence $(c_l d_l)_{l\in\mathbb{Z}}$. Hence the code $\xi : \Lambda_{\mathcal{K}} \longrightarrow \Lambda_{\mathcal{M}}$ is one-to-one. We similarly see that $\eta : \Lambda_{\mathcal{K}} \longrightarrow \Lambda_{\mathcal{M}}$ is one-to-one. Hence by Lemma 5.5, $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}$ is surjective. We symmetrically see that $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{Q}}}$ is 1-1 and surjective. □

Following Nasu's notation [18],[19], an automorphism ϕ of a subshift Λ over Σ is called a forward bipartite automorphism if there exist alphabets C, D and

specifications $\kappa : \Sigma \to CD, \kappa' : \Sigma \to DC$ such that ϕ is given by

$$\phi((\alpha_l)_{l\in\mathbb{Z}}) = (\alpha'_l)_{l\in\mathbb{Z}}, \qquad (\alpha_l)_{l\in\mathbb{Z}} \in \Lambda$$

where $\kappa(\alpha_l) = c_l d_l$ for some $c_l \in C$ and $d_l \in D, l \in \mathbb{Z}$ and $\alpha'_l = {\kappa'}^{-1}(d_l c_{l+1}) \in \Sigma$. Hence a properly strong shift equivalence $(\mathcal{P}, \mathcal{Q}, X, Y) : (\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}, I)$ in 1-step yields a forward bipartite automorphism on the subshift presented by $(\mathcal{M}, I)$.

Lemma 6.3. *Let (Λ, σ) be a subshift presented by $(\mathcal{M}, I)$. Let ϕ be a forward bipartite automorphism on (Λ, σ) defined by a properly strong shift equivalence $(\mathcal{P}, \mathcal{Q}, X, Y) : (\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}, I)$ in 1-step. Let $\mathcal{T}^{\mathcal{P}}$ and $\mathcal{T}^{\mathcal{Q}}$ be the LR textile λ-graph systems $\mathcal{T}_{\mathcal{K}_{\mathcal{P}}^{\mathcal{M}}}$ and $\mathcal{T}_{\mathcal{K}_{\mathcal{Q}}^{\mathcal{M}}}$ defined by the relations (6.1) and (6.2) respectively. Then we have*

$$\varphi_{\mathcal{T}^{\mathcal{P}}} = \phi, \qquad \varphi_{\mathcal{T}^{\mathcal{Q}}} = \phi^{-1} \circ \sigma$$

as automorphisms on $\Lambda = \Lambda_{\mathcal{M}}$.

Proof. We will prove that $\phi = \varphi_{\mathcal{T}^{\mathcal{P}}}$. For $(\alpha_l)_{l\in\mathbb{Z}} \in \Lambda$, put $c_l d_l = \kappa(\alpha_l) \in CD, l \in \mathbb{Z}$. By setting $\alpha'_l = {\kappa'}^{-1}(d_l c_{l+1}) \in \Sigma$, one has $\phi((\alpha_l)_{l\in\mathbb{Z}}) = (\alpha'_l)_{l\in\mathbb{Z}}$. Put $\beta_l = (c_l, c_{l+1}, \kappa^{-1}(c_l d_l), {\kappa'}^{-1}(d_l c_{l+1})) \in \Sigma^{\mathcal{K}}, l \in \mathbb{Z}$ so that one has $(\beta_l)_{l\in\mathbb{Z}} \in \Lambda_{\mathcal{K}}$ and

$$\xi((\beta_l)_{l\in\mathbb{Z}}) = (\kappa^{-1}(c_l d_l))_{l\in\mathbb{Z}}, \qquad \eta((\beta_l)_{l\in\mathbb{Z}}) = ({\kappa'}^{-1}(d_l c_{l+1}))_{l\in\mathbb{Z}}.$$

It then follows that

$$\phi((\alpha_l)_{l\in\mathbb{Z}}) = (\alpha'_l)_{l\in\mathbb{Z}} = ({\kappa'}^{-1}(d_l c_{l+1}))_{l\in\mathbb{Z}} = \eta((\beta_l)_{l\in\mathbb{Z}}) = \eta \circ \xi^{-1}((\alpha_l)_{l\in\mathbb{Z}}).$$

The equality $\varphi_{\mathcal{T}^{\mathcal{Q}}} = \phi^{-1} \circ \sigma$ is similarly shown. □

We assume that the previously defined metric is equipped with Λ. Then the homeomorphism σ has 1 as its expansive constant. Therefore we have

Theorem 6.4. *Let (Λ, σ) be a subshift presented by a symbolic matrix system $(\mathcal{M}, I)$. Let ϕ be a forward bipartite automorphism on (Λ, σ) defined by a properly strong shift equivalence $(\mathcal{P}, \mathcal{Q}, X, Y) : (\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}, I)$ in 1-step. If ϕ is expansive with $\frac{1}{k}$ as its expansive constant for some $k \in \mathbb{N}$, the dynamical system (Λ, ϕ) is topologically conjugate to the subshift $(\Lambda_{\mathcal{P}_{\mathcal{T}}^{[2k]}}, \sigma_{\mathcal{P}_{\mathcal{T}}^{[2k]}})$ presented by the symbolic matrix system $(\mathcal{P}_{\mathcal{T}}^{[2k]}, I^{\mathcal{P}_{\mathcal{T}}^{[2k]}})$ where $(\mathcal{P}_{\mathcal{T}}^{[2k]}, I^{\mathcal{P}_{\mathcal{T}}^{[2k]}})$ is the $2k$-higher block of the symbolic matrix system $(\mathcal{P}, I^{\mathcal{P}})$ relative to the LR textile λ-graph system $\mathcal{T}_{\mathcal{K}_{\mathcal{P}}^{\mathcal{M}}}$ defined by the specified equivalence*

$$\mathcal{M}_{l,l+1}\mathcal{P}_{l+1,l+2} \overset{\kappa^{\mathcal{P}}}{\simeq} \mathcal{P}_{l,l+1}\mathcal{M}_{l+1,l+2}, \qquad l \in \mathbb{Z}_+,$$

where

$$\mathcal{P}_{l,l+1} = \mathcal{P}_{2l}Y_{2l+1}(= X_{2l}\mathcal{P}_{2l+1}), \qquad I^{\mathcal{P}}_{l,l+1} = I_{l,l+1}, \qquad l \in \mathbb{Z}_+.$$

Proof. Consider the LR textile λ-graph system $\mathcal{T}^{\mathcal{P}} = \mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}$ defined by (6.1), that is 1-1 and surjective by Lemma 6.2. Lemma 6.3 says that $\varphi_{\mathcal{T}^{\mathcal{P}}} = \phi$ on Λ. By the assumption on ϕ and Theorem 4.10 (i), ${\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}}^{[2k]^*}$ is 1-1. Since $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}$ is LR, ${\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}}^{[2k]}$ and hence ${\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}}^{[2k]^*}$ are both LR and nondegenerate. By Lemma 5.5, ${\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}}^{[2k]^*}$ is surjective. By Theorem 4.10 (ii), the topological dynamical system $(\Lambda, \varphi_{\mathcal{T}^{\mathcal{P}}})$ is realized as the subshift $(\Lambda_{\mathcal{P}^{[2k]}_{\mathcal{T}}}, \sigma_{\mathcal{P}^{[2k]}_{\mathcal{T}}})$. Hence one concludes that the dynamical system (Λ, ϕ) is topologically conjugate to the subshift $(\Lambda_{\mathcal{P}^{[2k]}_{\mathcal{T}}}, \sigma_{\mathcal{P}^{[2k]}_{\mathcal{T}}})$. □

We also have

Theorem 6.5. *Let (Λ, σ) be a subshift presented by a symbolic matrix system $(\mathcal{M}, I)$. Let ϕ be a forward bipartite automorphism on (Λ, σ) defined by a properly strong shift equivalence $(\mathcal{P}, \mathcal{Q}, X, Y) : (\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}, I)$ in 1-step. Then the dynamical system $(\Lambda, \phi^k\sigma^n)$ is topologically conjugate to the subshift $(\Lambda_{\mathcal{P}^k\mathcal{M}^n}, \sigma_{\mathcal{P}^k\mathcal{M}^n})$ presented by the symbolic matrix system $(\mathcal{P}^k\mathcal{M}^n, I^{\mathcal{P}^k\mathcal{M}^n})$ for $k \geq 0, n \geq 1$ defined by*

$$\begin{aligned}(\mathcal{P}^k\mathcal{M}^n)_{l,l+1} =& \mathcal{P}_{l(k+n),l(k+n)+1} \cdots \mathcal{P}_{l(k+n)+n-1,l(k+n)+n} \cdot \\ & \cdot \mathcal{M}_{l(k+n)+n,l(k+n)+n+1} \cdots \mathcal{M}_{(l+1)(k+n)-1,(l+1)(k+n)}, \\ {I^{k+n}}_{l,l+1} =& I_{l(k+n),l(k+n)+1} \cdots I_{(l+1)(k+n)-1,(l+1)(k+n)}, \qquad l \in \mathbb{Z}_+\end{aligned}$$

where $\mathcal{P}_{l,l+1} = \mathcal{P}_{2l}Y_{2l+1}(= X_{2l}\mathcal{P}_{2l+1})$ for $l \in \mathbb{Z}_+$. And also $(\Lambda, {(\sigma\phi^{-1})}^k\sigma^n)$ is topologically conjugate to the subshift $(\Lambda_{\mathcal{Q}^k\mathcal{M}^n}, \sigma_{\mathcal{Q}^k\mathcal{M}^n})$ presented by the similarly defined symbolic matrix system $(\mathcal{Q}^k\mathcal{M}^n, I^{\mathcal{Q}^k\mathcal{M}^n})$ for $k \geq 0, n \geq 1$,

Proof. By Lemma 6.2, the LR textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}$ is 1-1 and surjective. Hence by Theorem 5.6, the dynamical system $(\Lambda, \varphi^k_{\mathcal{T}}\sigma^n)$, $k \geq 0, n \geq 1$ is topologically conjugate to the subshift $(\Lambda_{\mathcal{N}^k\mathcal{M}^n}, \sigma_{\mathcal{N}^k\mathcal{M}^n})$ presented by the symbolic matrix system $(\mathcal{N}^k\mathcal{M}^n, I^{\mathcal{N}^k\mathcal{M}^n})$. Now $\mathcal{T}_{\mathcal{K}^{\mathcal{M}}_{\mathcal{P}}}$ is LR so that it is nondegenerate. By Lemma 6.3 one sees that $\varphi_{\mathcal{T}^{\mathcal{P}}} = \phi$ so that the dynamical system $(\Lambda, \phi^k_{\mathcal{T}}\sigma^n)$ is topologically conjugate to the subshift $(\Lambda_{\mathcal{N}^k\mathcal{M}^n}, \sigma_{\mathcal{N}^k\mathcal{M}^n})$. It is similarly shown that $(\Lambda, {(\sigma\phi^{-1})}^k\sigma^n)$ is topologically conjugate to the subshift $(\Lambda_{\mathcal{Q}^k\mathcal{M}^n}, \sigma_{\mathcal{Q}^k\mathcal{M}^n})$ for $k \geq 0, n \geq 1$. □

7. Subshift-identifications of automorphisms of subshifts

Two symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are said to be *properly strong shift equivalent* if there exists a finite sequence $(\mathcal{M}^{(1)}, I^{(1)}), ldots, (\mathcal{M}^{(N-1)}, I^{(N-1)})$ of symbolic matrix systems such that

$$(\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) \underset{1-pr}{\approx} \cdots \underset{1-pr}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{1-pr}{\approx} (\mathcal{M}', I').$$

In [15], the following theorem has been proved:

Theorem 7.1 ([15]). *If two symbolic matrix systems $(\mathcal{M}, I)$ and $(\mathcal{M}', I')$ are properly strong shift equivalent, then their respect presented subshifts $\Lambda_{\mathcal{M}}$ and $\Lambda_{\mathcal{M}'}$ are topologically conjugate. Furthermore, two subshifts Λ and Λ' are topologically conjugate if and only if their canonical symbolic matrix systems $(\mathcal{M}^{\Lambda}, I^{\Lambda})$ and $(\mathcal{M}^{\Lambda'}, I^{\Lambda'})$ are properly strong shift equivalent.*

In particular, an automorphism of a subshift Λ is given by a properly strong shift equivalence from a symbolic matrix system that presents the subshift to itself. Let $(\mathcal{M}, I)$ be a symbolic matrix system over Σ. Let us consider a properly strong shift equivalence from $(\mathcal{M}, I)$ to itself. Hence we consider symbolic matrix systems $(\mathcal{M}^{(k)}, I^{(k)})$ over $\Sigma^{(k)}, k = 0, 1, \ldots, N$, where $(\mathcal{M}^{(0)}, I^{(0)}) = (\mathcal{M}^{(N)}, I^{(N)}) = (\mathcal{M}, I)$ and $\Sigma^{(0)} = \Sigma^{(N)} = \Sigma$ such that there exist alphabets $C^{(k)}, D^{(k)}$ and specifications $\kappa_0^{(k)} : \Sigma^{(k-1)} \to C^{(k)}D^{(k)}$, $\kappa_1^{(k)} : \Sigma^{(k)} \to D^{(k)}C^{(k)}$ and increasing sequences $n_0^{(k)}(l), n_1^{(k)}(l)$ on $l \in \mathbb{Z}_+$ such that for each $l \in \mathbb{Z}_+$, there exist an $n_0^{(k)}(l) \times n_1^{(k)}(l+1)$ matrix $\mathcal{P}_l^{(k)}$ over $C^{(k)}$, an $n_1^{(k)}(l) \times n_0^{(k)}(l+1)$ matrix $\mathcal{Q}_l^{(k)}$ over $D^{(k)}$, an $n_0^{(k)}(l) \times n_0^{(k)}(l+1)$ matrix $X_l^{(k)}$ over $\{0,1\}$ and an $n_1^{(k)}(l) \times n_1^{(k)}(l+1)$ matrix $Y_l^{(k)}$ over $\{0,1\}$ satisfying the following equations:

$$\begin{cases} \mathcal{M}_{l,l+1}^{(k-1)} \overset{\kappa_0^{(k)}}{\simeq} \mathcal{P}_{2l}^{(k)}\mathcal{Q}_{2l+1}^{(k)}, & \mathcal{M}_{l,l+1}^{(k)} \overset{\kappa_1^{(k)}}{\simeq} \mathcal{Q}_{2l}^{(k)}\mathcal{P}_{2l+1}^{(k)}, \\ I_{l,l+1}^{(k-1)} = X_{2l}^{(k)}X_{2l+1}^{(k)}, & I_{l,l+1}^{(k)} = Y_{2l}^{(k)}Y_{2l+1}^{(k)}, \\ X_l^{(k)}\mathcal{P}_{l+1}^{(k)} = \mathcal{P}_l^{(k)}Y_{l+1}^{(k)}, & Y_l^{(k)}\mathcal{Q}_{l+1}^{(k)} = \mathcal{Q}_l^{(k)}X_{l+1}^{(k)}. \end{cases} \tag{7.1}$$

The equations (7.1) are simply written as

$$(\mathcal{P}^{(k)}, \mathcal{Q}^{(k)}, X^{(k)}, Y^{(k)}) : (\mathcal{M}^{(k-1)}, I^{(k-1)}) \underset{1-pr}{\approx} (\mathcal{M}^{(k)}, I^{(k)}), \qquad k = 1, \ldots, N.$$

Lemma 7.2. *Keep the above notations. Put $m(l) \times m(l+N)$ matrices*

$$\begin{aligned} \mathcal{P}_{l,l+N} &= \mathcal{P}_{2l}^{(1)}Y_{2l+1}^{(1)}\mathcal{P}_{2l+2}^{(2)}Y_{2l+3}^{(2)}\cdots\mathcal{P}_{2l+2N-2}^{(N)}Y_{2l+2N-1}^{(N)}, \\ \mathcal{Q}_{l,l+N} &= \mathcal{Q}_{2l}^{(1)}X_{2l+1}^{(1)}\mathcal{Q}_{2l+2}^{(2)}X_{2l+3}^{(2)}\cdots\mathcal{Q}_{2l+2N-2}^{(N)}X_{2l+2N-1}^{(N)}, \\ I_{l,l+N} &= I_{l,l+1}I_{l+1,l+2}\cdots I_{l+N-1,l+N}, \qquad l \in \mathbb{Z}_+. \end{aligned}$$

(i) *The equalities*

$$\begin{aligned}\mathcal{P}_{l,l+N}I_{l+N,l+N+1} &= I_{l,l+1}\mathcal{P}_{l+1,l+N+1},\\ \mathcal{Q}_{l,l+N}I_{l+N,l+N+1} &= I_{l,l+1}\mathcal{Q}_{l+1,l+N+1}, \qquad l \in \mathbb{Z}\end{aligned}$$

and hence

$$\begin{aligned}\mathcal{P}_{l,l+N}I_{l+N,l+2N} &= I_{l,l+N}\mathcal{P}_{l+N,l+2N},\\ \mathcal{Q}_{l,l+N}I_{l+N,l+2N} &= I_{l,l+N}\mathcal{Q}_{l+N,l+2N}, \qquad l \in \mathbb{Z}\end{aligned}$$

hold.

(ii) *There exist specifications*

$$\begin{aligned}\kappa_{\mathcal{P}} &: \Sigma \cdot C^{(1)}C^{(2)}\cdots C^{(N)} \longrightarrow C^{(1)}C^{(2)}\cdots C^{(N)} \cdot \Sigma,\\ \kappa_{\mathcal{Q}} &: \Sigma \cdot D^{(1)}D^{(2)}\cdots D^{(N)} \longrightarrow D^{(1)}D^{(2)}\cdots D^{(N)} \cdot \Sigma\end{aligned}$$

such that

$$\mathcal{M}_{l,l+1}\mathcal{P}_{l+1,l+N+1} \overset{\kappa_{\mathcal{P}}}{\simeq} \mathcal{P}_{l,l+N}\mathcal{M}_{l+N,l+N+1}, \tag{7.2}$$

$$\mathcal{M}_{l,l+1}\mathcal{Q}_{l+1,l+N+1} \overset{\kappa_{\mathcal{Q}}}{\simeq} \mathcal{Q}_{l,l+N}\mathcal{M}_{l+N,l+N+1}, \qquad l \in \mathbb{Z}_+. \tag{7.3}$$

Proof. (i) We note that the equalities

$$I_{l,l+1} = I^{(0)}_{l,l+1} = X^{(1)}_{2l}X^{(1)}_{2l+1} = I^{(N)}_{l,l+1} = Y^{(N)}_{2l}Y^{(N)}_{2l+1}, \qquad l \in \mathbb{Z}_+$$

hold. It then follows that

$$\begin{aligned}&\mathcal{P}_{l,l+N}I_{l+N,l+N+1}\\ =&\mathcal{P}^{(1)}_{2l}Y^{(1)}_{2l+1}\mathcal{P}^{(2)}_{2l+2}Y^{(2)}_{2l+3}\cdots Y^{(N-1)}_{2l+2N-3}X^{(N)}_{2l+2N-2}\mathcal{P}^{(N)}_{2l+2N-1}Y^{(N)}_{2l+2N}Y^{(N)}_{2l+2N+1}\\ =&\mathcal{P}^{(1)}_{2l}Y^{(1)}_{2l+1}\mathcal{P}^{(2)}_{2l+2}Y^{(2)}_{2l+3}\cdots Y^{(N-1)}_{2l+2N-3}X^{(N)}_{2l+2N-2}X^{(N)}_{2l+2N-1}\mathcal{P}^{(N)}_{2l+2N}Y^{(N)}_{2l+2N+1}\\ =&\mathcal{P}^{(1)}_{2l}Y^{(1)}_{2l+1}\mathcal{P}^{(2)}_{2l+2}Y^{(2)}_{2l+3}\cdots Y^{(N-1)}_{2l+2N-3}\cdot I^{(N-1)}_{l+N-1,l+N}\mathcal{P}^{(N)}_{2l+2N}Y^{(N)}_{2l+2N+1}\end{aligned}$$

and hence inductively

$$\begin{aligned}&\mathcal{P}_{l,l+N}I_{l+N,l+N+1}\\ =&\mathcal{P}^{(1)}_{2l}Y^{(1)}_{2l+1}I^{(1)}_{l+1,l+2}\cdot\mathcal{P}^{(2)}_{2l+4}Y^{(2)}_{2l+5}\cdots\mathcal{P}^{(N-1)}_{2l+2N-2}Y^{(N-1)}_{2l+2N-1}\mathcal{P}^{(N)}_{2l+2N}Y^{(N)}_{2l+2N+1}.\end{aligned}$$

Since

$$\mathcal{P}^{(1)}_{2l}Y^{(1)}_{2l+1}I^{(1)}_{l+1,l+2} = X^{(1)}_{2l}\mathcal{P}^{(1)}_{2l+1}\cdot Y^{(1)}_{2l+2}Y^{(1)}_{2l+3} = I_{l,l+1}\mathcal{P}^{(1)}_{2l+2}Y^{(1)}_{2l+3},$$

one has

$$\mathcal{P}_{l,l+N}I_{l+N,l+N+1} = I_{l,l+1}\mathcal{P}_{l+1,l+N+1}.$$

One then inductively gets the equalities

$$\mathcal{P}_{l,l+N}I_{l+N,l+2N} = I_{l,l+N}\mathcal{P}_{l+N,l+2N}, \qquad l \in \mathbb{Z}_+.$$

The other equalities

$$\mathcal{Q}_{l,l+N}I_{l+N,l+N+1} = I_{l,l+1}\mathcal{Q}_{l+1,l+N+1}, \qquad l \in \mathbb{Z}_+$$

are similarly proved.

(ii) It follows that

$$\begin{aligned}
& \mathcal{M}_{l,l+1}\mathcal{P}_{l+1,l+N+1} \\
= \;& \mathcal{M}^{(0)}_{l,l+1}\mathcal{P}^{(1)}_{2(l+1)}Y^{(1)}_{2(l+1)+1}\mathcal{P}^{(2)}_{2(l+1)+2}Y^{(2)}_{2(l+1)+3}\cdots\mathcal{P}^{(N)}_{2l+2N}Y^{(N)}_{2l+2N+1} \\
\overset{\kappa_0^{(1)}}{\simeq} \;& \mathcal{P}^{(1)}_{2l}\mathcal{Q}^{(1)}_{2l+1}X^{(1)}_{2(l+1)}\mathcal{P}^{(1)}_{2(l+1)+1}X^{(2)}_{2(l+1)+2}\mathcal{P}^{(2)}_{2(l+1)+3}\cdots X^{(N)}_{2l+2N}\mathcal{P}^{(N)}_{2l+2N+1} \\
= \;& \mathcal{P}^{(1)}_{2l}Y^{(1)}_{2l+1}\mathcal{Q}^{(1)}_{2(l+1)}\mathcal{P}^{(1)}_{2(l+1)+1}X^{(2)}_{2(l+1)+2}\mathcal{P}^{(2)}_{2(l+1)+3}\cdots X^{(N)}_{2l+2N}\mathcal{P}^{(N)}_{2l+2N+1} \\
\overset{\kappa_1^{(1)-1}}{\simeq} \;& \mathcal{P}^{(1)}_{2l}Y^{(1)}_{2l+1}\cdot\mathcal{M}^{(1)}_{l+1,l+2}\mathcal{P}^{(2)}_{2(l+1)+2}Y^{(2)}_{2(l+1)+3}\cdots\mathcal{P}^{(N)}_{2l+2N}Y^{(N)}_{2l+2N+1}
\end{aligned}$$

and similarly

$$\begin{aligned}
& \mathcal{M}^{(1)}_{l+1,l+2}\mathcal{P}^{(2)}_{2(l+1)+2}Y^{(2)}_{2(l+1)+3}\mathcal{P}^{(3)}_{2(l+1)+4}Y^{(3)}_{2(l+1)+5}\cdots\mathcal{P}^{(N)}_{2l+2N}Y^{(N)}_{2l+2N+1} \\
\overset{\kappa_1^{(2)-1}\kappa_0^{(2)}}{\simeq} \;& \mathcal{P}^{(2)}_{2l+2}Y^{(2)}_{2l+3}\cdot\mathcal{M}^{(2)}_{l+2,l+3}\mathcal{P}^{(3)}_{2(l+1)+4}Y^{(3)}_{2(l+1)+5}\cdots\mathcal{P}^{(N)}_{2l+2N}Y^{(N)}_{2l+2N+1}.
\end{aligned}$$

Hence we inductively have

$$\begin{aligned}
& \mathcal{M}_{l,l+1}\mathcal{P}_{l+1,l+N+1} \\
\overset{(\kappa_1^{(N)})^{-1}\kappa_0^{(N)}\cdots(\kappa_1^{(1)})^{-1}\kappa_0^{(1)}}{\simeq} \;& \mathcal{P}^{(1)}_{2l}Y^{(1)}_{2l+1}\cdots\mathcal{P}^{(N)}_{2l+2N-2}Y^{(N)}_{2l+2N-1}\mathcal{M}^{(N)}_{l+N,l+N+1} \\
=\;& \mathcal{P}_{l,l+N}\mathcal{M}_{l+N,l+N+1}.
\end{aligned}$$

By putting

$$\kappa_{\mathcal{P}} = (\kappa_1^{(N)})^{-1}\kappa_0^{(N)}\cdots(\kappa_1^{(1)})^{-1}\kappa_0^{(1)} : \Sigma C^{(1)}C^{(2)}\cdots C^{(N)} \longrightarrow C^{(1)}C^{(2)}\cdots C^{(N)}\Sigma$$

one has

$$\mathcal{M}_{l,l+1}\mathcal{P}_{l+1,l+N+1} \overset{\kappa_{\mathcal{P}}}{\simeq} \mathcal{P}_{l,l+N}\mathcal{M}_{l+N,l+N+1}.$$

□

We set,

$$\mathcal{M}^{[1]}_{l,l+1} = \mathcal{M}_{l,l+1}, \qquad I^{[1]}_{l,l+1} = I_{l,l+1}$$

and for $N \geq 2$

$$\mathcal{M}^{[N]}_{l,l+1} = \mathcal{M}_{Nl,Nl+1} I_{Nl+1,Nl+2} \cdots I_{N(l+1)-1,N(l+1)}, \qquad I^{[N]}_{l,l+1} = I_{Nl,Nl+N}.$$

Then the pair $(\mathcal{M}^{[N]}, I^{[N]})$ is a symbolic matrix system over Σ. Since one has

$$\begin{aligned}\mathcal{M}^{[N]}_{l,l+m} &= \mathcal{M}^{[N]}_{l,l+1}\mathcal{M}^{[N]}_{l+1,l+2}\cdots\mathcal{M}^{[N]}_{(l+m)-1,(l+m)} \\ &= \mathcal{M}_{Nl,Nl+m} I_{Nl+m,N(l+m)},\end{aligned}$$

a word $(a_1, \ldots, a_m) \in \Sigma^m$ is admissible for the subshift $\Lambda_{\mathcal{M}^{[N]}}$ presented by $(\mathcal{M}^{[N]}, I^{[N]})$ if and only if it is admissible for the subshift $\Lambda_{\mathcal{M}}$ presented by $(\mathcal{M}, I)$. Hence the subshifts $\Lambda_{\mathcal{M}^{[N]}}$ and $\Lambda_{\mathcal{M}}$ coincide.

Lemma 7.3. *Keep the above notations. Put*

$$\mathcal{P}^{[N]}_{l,l+1} = \mathcal{P}_{Nl,Nl+N}, \qquad \mathcal{Q}^{[N]}_{l,l+1} = \mathcal{Q}_{Nl,Nl+N}, \qquad l \in \mathbb{Z}_+.$$

Then both $(\mathcal{P}^{[N]}, I^{[N]})$ and $(\mathcal{Q}^{[N]}, I^{[N]})$ are symbolic matrix systems such that

(i) *the pair $(\mathcal{M}^{[N]}, I^{[N]})$ and $(\mathcal{P}^{[N]}, I^{[N]})$, and the pair $(\mathcal{M}^{[N]}, I^{[N]})$ and $(\mathcal{Q}^{[N]}, I^{[N]})$ both form squares, and*

(ii) *they satisfy the relations:*

$$\mathcal{M}^{[N]}_{l,l+1}\mathcal{P}^{[N]}_{l+1,l+2} \overset{\kappa_{\mathcal{P}}}{\simeq} \mathcal{P}^{[N]}_{l,l+1}\mathcal{M}^{[N]}_{l+1,l+2}, \tag{7.4}$$

$$\mathcal{M}^{[N]}_{l,l+1}\mathcal{Q}^{[N]}_{l+1,l+2} \overset{\kappa_{\mathcal{Q}}}{\simeq} \mathcal{Q}^{[N]}_{l,l+1}\mathcal{M}^{[N]}_{l+1,l+2}, \qquad l \in \mathbb{Z}_+. \tag{7.5}$$

Hence the pair $(\mathcal{M}^{[N]}, I^{[N]})$ and $(\mathcal{P}^{[N]}, I^{[N]})$, and the pair $(\mathcal{M}^{[N]}, I^{[N]})$ and $(\mathcal{Q}^{[N]}, I^{[N]})$ both give rise to LR textile λ-graph systems.

Proof. The assertion (i) is clear. We will show the assertion (ii). By (7.2), one sees that

$$\mathcal{M}_{Nl,Nl+1}\mathcal{P}_{Nl+1,N(l+1)+1} \overset{\kappa_{\mathcal{P}}}{\simeq} {\mathcal{P}^{[N]}}_{l,l+1}\mathcal{M}_{N(l+1),N(l+1)+1}$$

so that

$$\begin{aligned}&\mathcal{M}_{Nl,Nl+1}\mathcal{P}_{Nl+1,N(l+1)+1} I_{N(l+1)+1,N(l+2)} \\ &\qquad \overset{\kappa_{\mathcal{P}}}{\simeq} {\mathcal{P}^{[N]}}_{l,l+1}\mathcal{M}_{N(l+1),N(l+1)+1} I_{N(l+1)+1,N(l+2)}.\end{aligned}$$

Hence we get

$$\mathcal{M}^{[N]}_{l,l+1}\mathcal{P}^{[N]}_{l+1,l+2} \overset{\kappa_{\mathcal{P}}}{\simeq} \mathcal{P}^{[N]}_{l,l+1}\mathcal{M}^{[N]}_{l+1,l+2}.$$

We similarly have $\mathcal{M}^{[N]}_{l,l+1}\mathcal{Q}^{[N]}_{l+1,l+2} \overset{\kappa_{\mathcal{Q}}}{\simeq} \mathcal{Q}^{[N]}_{l,l+1}\mathcal{M}^{[N]}_{l+1,l+2}$. □

Lemma 7.4. *The LR-textile λ-graph systems $\mathcal{T}^{\mathcal{P}^{[N]}} = \mathcal{T}_{\mathcal{K}^{\mathcal{M}^{[N]}}_{\mathcal{P}^{[N]}}}$ and $\mathcal{T}^{\mathcal{Q}^{[N]}} = \mathcal{T}_{\mathcal{K}^{\mathcal{M}^{[N]}}_{\mathcal{Q}^{[N]}}}$ defined by the relations (7.4) and (7.5) are both 1-1.*

Proof. For $(a_i)_{i\in\mathbb{Z}} \in \Lambda_{\mathcal{M}^{[N]}}$, suppose that a_l appears in a component of $\mathcal{M}^{[N]}_{l,l+1}$. Since $\mathcal{M}^{[N]}_{l,l+1} = I_{Nl,N(l+1)-1}\mathcal{M}^{(0)}_{N(l+1)-1,N(l+1)}$, the symbol a_l appears in a component of $\mathcal{M}^{(0)}_{N(l+1)-1,N(l+1)}$. By the specified equivalence $\mathcal{M}^{(0)}_{N(l+1)-1,N(l+1)} \overset{\kappa^{(1)}_0}{\simeq} \mathcal{P}^{(1)}_{2\{N(l+1)-1\}}\mathcal{Q}^{(1)}_{2\{N(l+1)-1\}+1}$, a symbol $c^{(1)}_{2\{N(l+1)-1\}}d^{(1)}_{2\{N(l+1)-1\}+1} = \kappa^{(1)}_0(a_l) \in CD$ appears in a component of $\mathcal{P}^{(1)}_{2\{N(l+1)-1\}}\mathcal{Q}^{(1)}_{2\{N(l+1)-1\}+1}$. For a_{l+1}, the corresponding symbol $c^{(1)}_{2\{N(l+2)-1\}}$ appears in a component of $\mathcal{P}^{(1)}_{2\{N(l+2)-1\}}$. Since $I^{(0)}_{N(l+1),N(l+2)-1}\mathcal{P}^{(1)}_{2\{N(l+2)-1\}} = \mathcal{P}^{(1)}_{2N(l+1)}X^{(1)}_{2N(l+1)+1}\cdots X^{(1)}_{2\{N(l+2)-2\}+1}$ $X^{(1)}_{2\{N(l+2)-1\}}$ the corresponding symbol to $c^{(1)}_{2\{N(l+2)-1\}}$ appears in a component of $\mathcal{P}^{(1)}_{2N(l+1)}$, that is denoted by $c^{(1)}_{2N(l+1)}$. As one sees that

$$\begin{aligned}
&\mathcal{M}^{[N]}_{l,l+1}\mathcal{P}^{[N]}_{l+1,l+2}\\
=&I_{Nl,N(l+1)-1}\mathcal{M}^{(0)}_{N(l+1)-1,N(l+1)}\mathcal{P}_{N(l+1),N(l+2)}\\
\overset{\kappa^{(1)}_0}{\simeq}& I_{Nl,N(l+1)-1}\mathcal{P}^{(1)}_{2\{N(l+1)-1\}}\mathcal{Q}^{(1)}_{2\{N(l+1)-1\}+1}\mathcal{P}^{(1)}_{2N(l+1)}Y^{(1)}_{2N(l+1)+1}\\
&\cdot\mathcal{P}^{(2)}_{2N(l+1)+2}Y^{(2)}_{2N(l+1)+3}\cdots\mathcal{P}^{(N)}_{2N(l+1)+2N-2}Y^{(N)}_{2N(l+1)+2N-1}
\end{aligned}$$

the symbol $d^{(1)}_{2N(l+1)-1}c^{(1)}_{2N(l+1)}$ appears in a component of $\mathcal{Q}^{(1)}_{2N(l+1)-1}\mathcal{P}^{(1)}_{2N(l+1)}$. As

$$\begin{aligned}
&\mathcal{Q}^{(1)}_{2N(l+1)-1}\mathcal{P}^{(1)}_{2N(l+1)}Y^{(1)}_{2N(l+1)+1}\\
=&Y^{(1)}_{2N(l+1)-1}\mathcal{Q}^{(1)}_{2N(l+1)}\mathcal{P}^{(1)}_{2N(l+1)+1} \overset{{\kappa^{(1)}_1}^{-1}}{\simeq} Y^{(1)}_{2N(l+1)-1}\mathcal{M}^{(1)}_{N(l+1),N(l+1)+1}
\end{aligned}$$

$d^{(1)}_{2N(l+1)-1}c^{(1)}_{2N(l+1)}$ appears in a component of $\mathcal{Q}^{(1)}_{2N(l+1)}\mathcal{P}^{(1)}_{2N(l+1)+1}$, that is written as $d^{(1)}_{2N(l+1)}c^{(1)}_{2N(l+1)+1}$. Hence ${\kappa^{(1)}_1}^{-1}(d^{(1)}_{2N(l+1)}c^{(1)}_{2N(l+1)+1})$ appears in a component of $\mathcal{M}^{(1)}_{N(l+1),N(l+1)+1}$. This procedure shows that for a given $(a_i)_{i\in\mathbb{Z}} \in \Lambda_{\mathcal{M}^{[N]}}$, by starting from a_l in a component of $\mathcal{M}^{(0)}_{N(l+1)-1,N(l+1)}$ a symbol $c^{(1)}_{2N(l+1)}$ in a component of $\mathcal{P}^{(1)}_{2N(l+1)}$ is determined and also ${\kappa^{(1)}_1}^{-1}(d^{(1)}_{2N(l+1)}c^{(1)}_{2N(l+1)+1})$ in a component of $\mathcal{M}^{(1)}_{N(l+1),N(l+1)+1}$ is determined. One may next find a symbol in a component of $\mathcal{P}^{(2)}_{2N(l+1)+2}$ and a symbol in a component of $\mathcal{M}^{(2)}_{N(l+1)+1,N(l+1)+2}$. One inductively finds corresponding symbols in $\mathcal{P}^{(1)}_{2N(l+1)}$,$\mathcal{P}^{(2)}_{2N(l+1)+2}$,$\ldots$,$\mathcal{P}^{(N)}_{2N(l+1)+2N-3}$.

Hence one finds a symbol in $\mathcal{P}^{(N)}_{l+1,l+2}$. That is, a given sequence $(a_i)_{i\in\mathbb{Z}} \in \Lambda_{\mathcal{M}^{[N]}}$ determines a symbol in $\mathcal{P}^{(N)}_{l+1,l+2}, l \in \mathbb{Z}_+$ so that through the relation (7.4) the labeled squares in the LR textile λ-graph system are determined. Hence we conclude that ξ is injective, and similarly see that η is injective. □

As stated in the begining of this section, an automorphism of the subshift Λ presented by $(\mathcal{M}, I)$ is given by a properly strong shift equivalence

$$(\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) \underset{1-pr}{\approx} \cdots \underset{1-pr}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{1-pr}{\approx} (\mathcal{M}, I)$$

in N-step for some N, and conversely a properly strong shift equivalence from $(\mathcal{M}, I)$ to itself gives rise to an automorphism of the subshift. Put for $k = 1, \ldots, N$

$$\Lambda_{C^{(k)}D^{(k)}} = \{(c_i d_i)_{i\in\mathbb{Z}} \mid c_i \in C^{(k)}, d_i \in D^{(k)}, i \in \mathbb{Z}\},$$
$$\Lambda_{D^{(k)}C^{(k)}} = \{(d_i c_i)_{i\in\mathbb{Z}} \mid c_i \in C^{(k)}, d_i \in D^{(k)}, i \in \mathbb{Z}\}.$$

Define $\zeta^{(k)}_+ : \Lambda_{C^{(k)}D^{(k)}} \longrightarrow \Lambda_{D^{(k)}C^{(k)}}$ and $\zeta^{(k)}_- : \Lambda_{C^{(k)}D^{(k)}} \longrightarrow \Lambda_{D^{(k)}C^{(k)}}$ by setting $\zeta^{(k)}_+((c_i d_i)_{i\in\mathbb{Z}}) = (d_i c_{i+1})_{i\in\mathbb{Z}}$ and $\zeta^{(k)}_-((c_i d_i)_{i\in\mathbb{Z}}) = (d_{i-1} c_i)_{i\in\mathbb{Z}}$ respectively. Then $\zeta^{(k)}_+$ is called a forward bipartite conjugacy and $\zeta^{(k)}_-$ is called a backward bipartite conjugacy ([19]). Nasu's result ([18], [19]) says that any automorphism ϕ is factorized as follows:

$$\phi = ((\kappa^{(N)}_1)^{-1} \circ \zeta^{(N)}_\pm \circ \kappa^{(N)}_0) \circ \cdots \circ ((\kappa^{(1)}_1)^{-1} \circ \zeta^{(1)}_\pm \circ \kappa^{(1)}_0)$$

where $\zeta^{(N)}_\pm, \ldots, \zeta^{(1)}_\pm$ are forward or backward bipartite conjugacies. Since properly strong shift equivalence corresponds exactly to bipartite codes of Nasu, the above factorization of ϕ is so called Nasu's $\kappa - \zeta$ factorization ([18], [19]). Following Nasu, an automorphism ϕ is said to be forward if $\zeta^{(N)}_\pm, \ldots, \zeta^{(1)}_\pm$ are all forward bipartite conjugacies $\zeta^{(N)}_+, \ldots, \zeta^{(1)}_+$.

Lemma 7.5. *Let (Λ, σ) be a subshift presented by $(\mathcal{M}, I)$. Let ϕ be a forward automorphism on (Λ, σ) defined by a properly strong shift equivalence*

$$(\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) \underset{1-pr}{\approx} \cdots \underset{1-pr}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{1-pr}{\approx} (\mathcal{M}, I)$$

in N-step. Let $\mathcal{T}^{\mathcal{P}^{[N]}}$ and $\mathcal{T}^{\mathcal{Q}^{[N]}}$ be the LR textile λ-graph systems defined by the relations (7.4) and (7.5) respectively. Then we have

$$\varphi_{\mathcal{T}^{\mathcal{P}^{[N]}}} = \phi, \qquad \varphi_{\mathcal{T}^{\mathcal{Q}^{[N]}}} = \phi^{-1} \circ \sigma$$

as automorphisms on $\Lambda = \Lambda_{\mathcal{M}}$ under the identification $\Lambda_{\mathcal{M}^{[N]}} = \Lambda_{\mathcal{M}}$.

Proof. Keep the notations as in the proof of the previous lemma. For $(a_i)_{i\in\mathbb{Z}} \in \Lambda_{\mathcal{M}^{[N]}}$, by putting

$$c^{(1)}_{2\{N(l+1)-1\}} d^{(1)}_{2\{N(l+1)-1\}+1} = \kappa_0^{(1)}(a_l) \in CD, \qquad l \in \mathbb{Z}_+$$

the symbol $c^{(1)}_{2N(l+1)-2} d^{(1)}_{2N(l+1)-1}$ is written as $d^{(1)}_{2N(l+1)} c^{(1)}_{2N(l+1)+1}$ and the symbol $(\kappa_1^{(1)})^{-1}(d^{(1)}_{2N(l+1)} c^{(1)}_{2N(l+1)+1})$ defines a symbol of a component of $\mathcal{M}^{(1)}_{N(l+1),N(l+1)+1}$. This procedure is nothing but to apply the map $(\kappa_1^{(1)})^{-1} \circ \zeta_+^{(1)} \circ \kappa_0^{(1)}$. We next do this procedure to the symbol $(\kappa_1^{(1)})^{-1}(d^{(1)}_{2N(l+1)} c^{(1)}_{2N(l+1)+1})$ that corresponds to apply the map $(\kappa_1^{(2)})^{-1} \circ \zeta_+^{(2)} \circ \kappa_0^{(2)}$ and get the symbols $((\kappa_1^{(2)})^{-1} \circ \zeta_+^{(2)} \circ \kappa_0^{(2)}) \circ ((\kappa_1^{(1)})^{-1} \circ \zeta_+^{(1)} \circ \kappa_0^{(1)})((a_l))$. We continue this procedures and finally get the element

$$((\kappa_1^{(N)})^{-1} \circ \zeta_+^{(N)} \circ \kappa_0^{(N)}) \cdots ((\kappa_1^{(2)})^{-1} \circ \zeta_+^{(2)} \circ \kappa_0^{(2)}) \circ ((\kappa_1^{(1)})^{-1} \circ \zeta_+^{(1)} \circ \kappa_0^{(1)})((a_l))$$

in $\mathcal{M}^{(N)}_{N(l+1)+N-1,N(l+1)+N}, l \in \mathbb{Z}_+$. The elements lie in the bottoms of the squares arising from the relation (7.4), and hence that are the element $\eta \circ \xi^{-1}((a_l))$. Hence we have

$$\varphi_{\mathcal{T}^{\mathcal{P}^{[N]}}} = ((\kappa_1^{(N)})^{-1} \circ \zeta_+^{(N)} \circ \kappa_0^{(N)}) \cdots ((\kappa_1^{(2)})^{-1} \circ \zeta_+^{(2)} \circ \kappa_0^{(2)}) \circ ((\kappa_1^{(1)})^{-1} \circ \zeta_+^{(1)} \circ \kappa_0^{(1)}).$$

□

We assume that the previously defined metric is equipped with Λ. Then the homeomorphism σ has 1 as its expansive constant. Therefore we have

Theorem 7.6. *Let (Λ, σ) be a subshift presented by a symbolic matrix system $(\mathcal{M}, I)$. Let ϕ be a forward automorphism on (Λ, σ) defined by a properly strong shift equivalence*

$$(\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) \underset{1-pr}{\approx} \cdots \underset{1-pr}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{1-pr}{\approx} (\mathcal{M}, I)$$

in N-step. If ϕ is expansive with $\frac{1}{k}$ as its expansive constant for some $k \in \mathbb{N}$, the dynamical system (Λ, ϕ) is topologically conjugate to the subshift $(\Lambda_{\mathcal{P}^{[N]}{}^{[2k]}_{\mathcal{T}}}, \sigma_{\mathcal{P}^{[N]}{}^{[2k]}_{\mathcal{T}}})$ presented by the $2k$-higher block $(\mathcal{P}^{[N]}{}^{[2k]}_{\mathcal{T}}, I^{\mathcal{P}^{[N]}{}^{[2k]}_{\mathcal{T}}})$ of the symbolic matrix system $(\mathcal{P}^{[N]}, I^{\mathcal{P}^{[N]}})$ relative to the LR textile λ-graph system $\mathcal{T}_{\mathcal{K}^{\mathcal{M}^{[N]}}_{\mathcal{P}^{[N]}}}$ defined by the specification

$$\mathcal{M}^{[N]}_{l,l+1} \mathcal{P}^{[N]}_{l+1,l+2} \overset{\kappa^{\mathcal{P}^{[N]}}}{\simeq} \mathcal{P}^{[N]}_{l,l+1} \mathcal{M}^{[N]}_{l+1,l+2}, \qquad l \in \mathbb{Z}_+,$$

where

$$\mathcal{P}^{[N]}_{l,l+1} = \mathcal{P}^{(1)}_{2Nl}Y^{(1)}_{2Nl+1}\mathcal{P}^{(2)}_{2Nl+2}Y^{(2)}_{2Nl+3}\cdots\mathcal{P}^{(N)}_{2Nl+2N-2}Y^{(N)}_{2Nl+2N-1},$$
$$I^{[N]}_{l,l+1} = I_{Nl,Nl+1}I_{Nl+1,Nl+2}\cdots I_{N(l+1)-1,N(l+1)}, \qquad l \in \mathbb{Z}_+$$

and $\mathcal{P}^{(i)}_{2Nl+2(i-1)}, Y^{(i)}_{2Nl+2i-1}, i = 1, \ldots, N$ *are matrices appearing in the properly strong shift equivalence in (7.1).*

Proof. Since the LR textile λ-graph system $\mathcal{T}^{\mathcal{P}^{[N]}}$ is nondegenerate, 1-1, and surjective, by Lemma 7.5 the assertion is proved in a similar way to the proof of Theorem 6.4. □

For $k \geq 0, n \geq 1$, let $(\mathcal{P}^k\mathcal{M}^n, I^{kN+n})$ be the symbolic matrix system defined by setting

$$\begin{aligned}
&(\mathcal{P}^k\mathcal{M}^n)_{l,l+1}\\
=&\mathcal{P}_{l(kN+n),l(kN+n)+N}\mathcal{P}_{l(kN+n)+N,l(kN+n)+2N}\cdots\mathcal{P}_{l(kN+n)+(k-1)N,l(kN+n)+kN}\cdot\\
&\cdot\mathcal{M}_{l(kN+n)+kN,l(kN+n)+kN+1}\mathcal{M}_{l(kN+n)+kN+1,l(kN+n)+kN+2}\\
&\cdots\mathcal{M}_{(l+1)(kN+n)-1,(l+1)(kN+n)},\ I^{kN+n}_{l,l+1}\\
=&I_{l(kN+n),l(kN+n)+1}I_{l(kN+n)+1,l(kN+n)+2}\cdots I_{(l+1)l(kN+n)-1,(l+1)(kN+n)},\ l\in\mathbb{Z}_+.
\end{aligned}$$

Lemma 7.7. *The subshift* $(\Lambda_{\mathcal{P}^{[N]^k}\mathcal{M}^{[N]^n}}, \sigma_{\mathcal{P}^{[N]^k}\mathcal{M}^{[N]^n}})$ *presented by the symbolic matrix system* $(\mathcal{P}^{[N]^k}\mathcal{M}^{[N]^n}, I^{[N]^{kN+n}})$ *coincides with the subshift* $(\Lambda_{\mathcal{P}^k\mathcal{M}^n}, \sigma_{\mathcal{P}^k\mathcal{M}^n})$ *presented by the symbolic matrix system* $(\mathcal{P}^k\mathcal{M}^n, I^{kN+n})$.

Proof. It is easy to see that the admissible words of the subshift $(\Lambda_{\mathcal{P}^{[N]^k}\mathcal{M}^{[N]^n}}, \sigma_{\mathcal{P}^{[N]^k}\mathcal{M}^{[N]^n}})$ coincides with the admissible words of the subshift $(\Lambda_{\mathcal{P}^k\mathcal{M}^n}, \sigma_{\mathcal{P}^k\mathcal{M}^n})$. □

We reach our main theorem.

Theorem 7.8. *Let* (Λ, σ) *be a subshift presented by a symbolic matrix system* $(\mathcal{M}, I)$. *Let* ϕ *be a forward automorphism on* (Λ, σ) *defined by a properly strong shift equivalence*

$$(\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}^{(1)}, I^{(1)}) \underset{1-pr}{\approx} \cdots \underset{1-pr}{\approx} (\mathcal{M}^{(N-1)}, I^{(N-1)}) \underset{1-pr}{\approx} (\mathcal{M}, I)$$

in N*-step. Then the dynamical system* $(\Lambda, \phi^k\sigma^n)$ *is topologically conjugate to the subshift* $(\Lambda_{\mathcal{P}^k\mathcal{M}^n}, \sigma_{\mathcal{P}^k\mathcal{M}^n})$ *presented by the symbolic matrix system* $(\mathcal{P}^k\mathcal{M}^n, I^{kN+n})$

for $k \geq 0, n \geq 1$, defined by

$$
\begin{aligned}
&(\mathcal{P}^k\mathcal{M}^n)_{l,l+1}\\
=&\mathcal{P}_{l(kN+n),l(kN+n)+N}\mathcal{P}_{l(kN+n)+N,l(kN+n)+2N}\cdots\mathcal{P}_{l(kN+n)+(k-1)N,l(kN+n)+kN}\cdot\\
&\cdot\mathcal{M}_{l(kN+n)+kN,l(kN+n)+kN+1}\mathcal{M}_{l(kN+n)+kN+1,l(kN+n)+kN+2}\\
&\cdots\mathcal{M}_{(l+1)(kN+n)-1,(l+1)(kN+n)},\ I^{kN+n}_{l,l+1}\\
=&I_{l(kN+n),l(kN+n)+1}I_{l(kN+n)+1,l(kN+n)+2}\cdots I_{(l+1)(kN+n)-1,(l+1)(kN+n)},\ l\in\mathbb{Z}_+
\end{aligned}
$$

where $\mathcal{P}_{l,l+1}=\mathcal{P}^{(1)}_{2l}Y^{(1)}_{2l+1}\mathcal{P}^{(2)}_{2l+2}Y^{(2)}_{2l+3}\cdots\mathcal{P}^{(N)}_{2l+2N-2}Y^{(N)}_{2l+2N-1}$ and $\mathcal{P}^{(i)}_{2l+2(i-1)}, Y^{(i)}_{2l+2i-1}$, $i=1,\dots,N$ are matrices appearing in the properly strong shift equivalence in (7.1).

And also $(\Lambda,(\sigma\phi^{-1})^k\sigma^n)$ is topologically conjugate to the subshift $(\Lambda_{\mathcal{Q}^k\mathcal{M}^n}, \sigma_{\mathcal{Q}^k\mathcal{M}^n})$ presented by the similarly defined symbolic matrix system $(\mathcal{Q}^k\mathcal{M}^n, I^{kN+n})$ for $k\geq 0, n\geq 1$.

Proof. By a similar discussion to the proof of Theorem 6.5, the dynamical system $(\Lambda,\phi^k\sigma^n)$ is topologically conjugate to the subshift $(\Lambda_{\mathcal{P}^{[N]^k}\mathcal{M}^{[N]^n}}, \sigma_{\mathcal{P}^{[N]^k}\mathcal{M}^{[N]^n}})$, that is $(\Lambda_{\mathcal{P}^k\mathcal{M}^n}, \sigma_{\mathcal{P}^k\mathcal{M}^n})$ by Lemma 7.7. The assertion for the dynamical system $(\Lambda,(\sigma\phi^{-1})^k\sigma^n)$ is similarly shown. □

8. An application

Let ϕ be an automorphism of a subshift Λ over Σ. We say that ϕ is given by a specification π of a symbolic matrix system $(\mathcal{M}, I)$ if $(\mathcal{M}, I)$ presents the subshift Λ and there exists a specification $\pi:\Sigma\to\Sigma$ such that π gives rise to a specified equivalence

$$\mathcal{M}_{l,l+1}\overset{\pi}{\simeq}\mathcal{M}_{l,l+1}\qquad\text{for } l\in\mathbb{Z}_+,$$

and ϕ is given by the symbolic automorphism of Λ induced by π. The automorphism ϕ is written as ϕ_π. We note that the induced automorphism of the λ-graph system $\mathfrak{L}$ for $(\mathcal{M}, I)$ by the specification π fixes the vertices of $\mathfrak{L}$.

Definition. An automorphism ϕ of a subshift Λ is called a *simple* automorphism if there exist an automorphism ϕ_π of a subshift $\Lambda_{\mathcal{M}}$ that is given by a specification π of a symbolic matrix system $(\mathcal{M}, I)$, and a topological conjugacy $\psi:\Lambda\to\Lambda_{\mathcal{M}}$ such that

$$\phi=\psi^{-1}\circ\phi_\pi\circ\psi.$$

The notion of a simple automorphism of a sofic shift has been introduced by M. Nasu in [19].

As an application of our results, we see the following proposition.

PROPOSITION 8.1. *Let (Λ, σ) be a subshift over Σ. If an automorphism ϕ_π of (Λ, σ) is given by a specification π of a symbolic matrix system $(\mathcal{M}, I)$, the topological dynamical system $(\Lambda, \phi_\pi \circ \sigma^n)$ is topologically conjugate to the n-th power (Λ, σ^n) of (Λ, σ) for $n \in \mathbb{Z}, n \neq 0$.*

Proof. By assumption, the automorphism ϕ_π is given by the one-block code $\phi_\pi((x_i)_{i\in\mathbb{Z}}) = (\pi(x_i))_{i\in\mathbb{Z}}$ for $(x_i)_{i\in\mathbb{Z}} \in \Lambda$. We will realize ϕ_π to be a forward automorphism defined by a properly strong shift equivalence from $(\mathcal{M}, I)$ to itself. Let $\mathbb{I}$ be an arbitrary fixed symbol. Put the alphabets

$$C = \{\mathbb{I}\}, \qquad D = \Sigma.$$

Define the specifications κ from Σ to $C \cdot D$ and κ' from Σ to $D \cdot C$ by setting

$$\kappa(\gamma) = \mathbb{I} \cdot \pi(\gamma), \qquad \kappa'(\gamma) = \gamma \cdot \mathbb{I}, \qquad \gamma \in \Sigma.$$

Suppose that the both matrices $\mathcal{M}_{l,l+1}, I_{l,l+1}$ are $m(l) \times m(l+1)$ matrices. Let $I_l(\mathbb{I})$ and $I_l(1)$ be the $m(l) \times m(l)$ diagonal matrices with diagonal entries $\mathbb{I}$ and 1 respectively. Put $n(2l) = n(2l-1) = m(l)$ for $l \in \mathbb{N}$, and $n'(2l) = n'(2l+1) = m(l)$ for $l \in \mathbb{Z}_+$. Define matrices $\mathcal{P}_l, \mathcal{Q}_l, X_l, Y_l$ for $l \in \mathbb{Z}_+$ by setting

$$\mathcal{P}_{2l} = I_l(\mathbb{I}), \qquad \mathcal{P}_{2l+1} = I_{l+1}(\mathbb{I}), \qquad \mathcal{Q}_{2l} = \mathcal{Q}_{2l+1} = \mathcal{M}_{l,l+1}$$

and

$$X_{2l} = Y_{2l+1} = I_{l,l+1}, \qquad X_{2l+1} = I_{l+1}(1), \qquad Y_{2l} = I_l(1).$$

By noticing that the matrices X_{2l+1}, Y_{2l} are identity matrices, the above matrices give rise to a properly strong shift equivalence

$$(\mathcal{P}, \mathcal{Q}, X, Y) : (\mathcal{M}, I) \underset{1-pr}{\approx} (\mathcal{M}, I)$$

in 1-step from $(\mathcal{M}, I)$ to itself. It is then direct to see that the automorphism ϕ_π is the forward automorphism of the above properly strong shift equivalence. Put

$$\mathcal{P}_{l,l+1} = \mathcal{P}_{2l} Y_{2l+1} = I_l(\mathbb{I}) I_{l,l+1}, \qquad l \in \mathbb{Z}_+.$$

For $n \in \mathbb{Z}$ with $n > 0$, we set

$$\begin{aligned}(\mathcal{P}\mathcal{M}^n)_{l,l+1} =& \mathcal{P}_{(n+1)l,(n+1)l+1} \cdots \mathcal{P}_{(n+1)l+n-1,(n+1)l+n} \\ & \cdot \mathcal{M}_{(n+1)l+n,(n+1)l+n+1} \cdots \mathcal{M}_{(n+1)(l+1)-1,(n+1)(l+1)}, \\ I^{n+1}_{l,l+1} =& I_{(n+1)l,(n+1)l+1} \cdots I_{(n+1)(l+1)-1,(n+1)(l+1)}.\end{aligned}$$

Then by Theorem 6.5, the topological dynamical system $(\Lambda, \phi_\pi \circ \sigma^n)$ is realized as the subshift $(\Lambda_{\mathcal{PM}^n}, \sigma_{\mathcal{PM}^n})$ presented by the symbolic matrix system $(\mathcal{PM}^n, I^{n+1})$. Since the symbolic matrix system $(\mathcal{PM}^n, I^{n+1})$ does not depend on the choice of the specification π on Σ, we have $(\Lambda, \phi_\pi \circ \sigma^n)$ is topologically conjugate to $(\Lambda, \phi_{\mathrm{id}} \circ \sigma^n)$ where ϕ_{id} is the automorphism coming from the identity permutation. Hence $(\Lambda, \phi_\pi \circ \sigma^n)$ is topologically conjugate to the n-th power (Λ, σ^n) of (Λ, σ). For $n \in \mathbb{Z}$ with $n < 0$, the above argument says that the dynamical system $(\Lambda, \phi_{\pi^{-1}} \circ \sigma^{-n})$ is topologically conjugate to (Λ, σ^{-n}). This implies that $(\Lambda, \phi_\pi \circ \sigma^n)$ is topologically conjugate to (Λ, σ^n). □

Thanks to this proposition, one has the following theorem.

Theorem 8.2. *If an automorphism ϕ of a subshift (Λ, σ) is a simple automorphism, the dynamical system $(\Lambda, \phi \circ \sigma^n)$ is topologically conjugate to the n-th power (Λ, σ^n) of the subshift (Λ, σ) for $n \in \mathbb{Z}, n \neq 0$.*

Proof. As ϕ is a simple automorphism of Λ, there exist an automorphism ϕ_π of a subshift $\Lambda_{\mathcal{M}}$ that is given by a specification π of a symbolic matrix system $(\mathcal{M}, I)$, and a topological conjugacy $\psi : \Lambda \to \Lambda_{\mathcal{M}}$ such that

$$\phi = \psi^{-1} \circ \phi_\pi \circ \psi.$$

Hence

$$\psi \circ (\phi \circ \sigma^n) \circ \psi^{-1} = \phi_\pi \circ {\sigma_{\mathcal{M}}}^n.$$

By Proposition 8.1, $(\Lambda, \phi_\pi \circ \sigma^n)$ is topologically conjugate to $(\Lambda_{\mathcal{M}}, {\sigma_{\mathcal{M}}}^n)$. Hence $(\Lambda, \phi \circ \sigma^n)$ is topologically conjugate to (Λ, σ^n). □

We finally give an example. Let us consider the full shift $\Sigma^{\mathbb{Z}}$ with alphabet Σ. It is easy to see that any permutation π on Σ yields a simple automorphism ϕ_π of $\Sigma^{\mathbb{Z}}$. Hence we have

Corollary 8.3. *Let ϕ_π be the automorphism of the full shift $\Sigma^{\mathbb{Z}}$ defined by a permutation π on the symbols Σ. Then the topological dynamical system $(\Sigma^{\mathbb{Z}}, \phi_\pi \circ \sigma)$ is realized as the original full shift $(\Sigma^{\mathbb{Z}}, \sigma)$. That is, $\phi_\pi \circ \sigma$ is topologically conjugate to the original full shift σ.*

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