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PERIODICITY FOR SPACE-INHOMOGENEOUS QUANTUM WALKS ON THE CYCLE

By

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Abstract. In this paper, we consider periodicity for space-inhomogeneous quantum walks on the cycle. For isospectral coin cases, we propose a spectral analysis. Based on the analysis, we extend the result for periodicity for Hadamard walk to some isospectral coin cases. For non-isospectral coin cases, we consider the the system that uses only one general coin at the origin and the identity coin at the other sites. In this case, we show that the periodicity of the general coin at the origin determines the periodicity for the whole system.

1. Introduction

In the last two decades, the theory of quantum walk (QW) has bees extensively studied by many researchers. There exist good reviews for this development, for example, Kempe [5], Kendon [6], Venegas-Andraca [12, 13], Konno [7], Manouchehri and Wang [9], and Portugal [10]. In the present paper, we focus on periodicity of the time evolution operator of two-state discrete-time QWs (DTQWs) on the cycle graph. The periodicity of the Hadamard walk on the cycle graph was determined by Dukes [3] and Konno et al. [8]. Note that the word periodicity is also used in the theory of perfect state transfer [4, 2] but we consider little bit stronger version of periodicity in this paper.

The rest of this paper is organized as follows. In Sect. 2, we give the definitions of our DTQWs and periodicity. Sections 3 and 4 are devoted to spectral analysis of the time evolution operator of our DTQWs. We note that the spectral analysis is viewed as a generalization of that of Segawa [11]. Corollary 4.6 is an extension of the results given by Dukes [3] and Konno et al. [8] for space-inhomogeneous coin cases. In Sect. 5, we deal with periodic arranged coin cases which is motivated by Chou and Ho [1].

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2. Definition of the DTQWs on the cycle graph

In this paper, we consider DTQWs on the cycle graph $C_n = (V_n, E_n)$ with the vertex set $V_n = \{0, 1, \ldots, n-1\}$ and the edge set $E_n = \{(i, i+1) : i \in V_n \pmod n\}$. The Hilbert space of DTQWs is defined by $\mathcal{H}_n = \mathrm{span}\{|i, L\rangle, |i, R\rangle : i \in V_n\}$ with state vectors $|i, J\rangle = |i\rangle \otimes |J\rangle$ $(i \in V_n, J \in \{L, R\})$ given by the tensor product of elements of two orthonormal bases: $\{|i\rangle : i \in V_n\}$ for position of the walker, and $\{|L\rangle = {}^T[1, 0], |R\rangle = {}^T[0, 1]\}$ for the chirality (direction) of the motion of the walker. Here ${}^T\!A$ denotes the transpose of a matrix A.

Now we define two types of time evolution operators $U_n^{MS} = S_n^{MS}\mathcal{C}_n$ and $U_n^{FF} = S_n^{FF}\mathcal{C}_n$ on $\mathcal{H}_n$ with the coin operator $\mathcal{C}_n$, the moving shift operator S_n^{MS} and the flip-flop shift operator S_n^{FF} defined as follows:

$$\mathcal{C}_n = \sum_{i=0}^{n-1} |i\rangle\langle i| \otimes C_i,$$

$$S_n^{MS}|i, J\rangle = \begin{cases} |i+1, R\rangle \pmod n & \text{if } J = R, \\ |i-1, L\rangle \pmod n & \text{if } J = L, \end{cases}$$

$$S_n^{FF}|i, J\rangle = \begin{cases} |i+1, L\rangle \pmod n & \text{if } J = R, \\ |i-1, R\rangle \pmod n & \text{if } J = L, \end{cases}$$

where C_i $(i = 0, \ldots, n-1)$ are 2×2 unitary matrices.

Let $X_t^{(n)} \in V_n$ be the position of our quantum walker driven by the time evolution operator U_n $(= U_n^{MS}$ or $U_n^{FF})$ at time t. The probability that the walker with an initial state (unit vector) $|\psi\rangle \in \mathcal{H}_n$ is found at time t and the position x is defined by

$$\mathbb{P}_{|\psi\rangle}(X_t^{(n)} = x) = \left\| \left(\langle x| \otimes I_2\right) U^t |\psi\rangle \right\|^2.$$

In this paper, we consider periodicity of the DTQWs. In order to define periodicity, we use the following notation:

$$T_n(U) = \inf\left\{t : U^t = I_n \otimes I_2\right\}. \tag{2.1}$$

We will investigate the period $T_n(U_n^{MS})$ and $T_n(U_n^{FF})$. We should remark the following fact:

Remark 2.1. Let $\lambda_1, \ldots, \lambda_{2n}$ be the eigenvalues of the time evolution operator U_n $(= U_n^{MS}$ or $U_n^{FF})$ then $U_n^t = I_n \otimes I_2 \iff \lambda_1^t = \cdots = \lambda_{2n}^t = 1$.

By Remark 2.1, the spectral structure of the time evolution operators are important. Here we show a connection between Spec U_n^{MS} and Spec U_n^{FF}.

Lemma 2.2. *Let* $\sigma_x = |R\rangle\langle L| + |R\rangle\langle L|$ *and* $\mathcal{C}_n\sigma_x = \sum_{i=0}^{n-1} |i\rangle\langle i| \otimes C_i\sigma_x$*. We denote* $U_n^{MS}(\mathcal{C}_n) = S^{MS}\mathcal{C}_n$ *and* $U_n^{FF}(\mathcal{C}_n) = S^{FF}\mathcal{C}_n$*. Then we have* $\mathrm{Spec}\ U_n^{FF}(\mathcal{C}_n) = \mathrm{Spec}\ U_n^{MS}(\mathcal{C}_n\sigma_x)$

Proof of Lemma 2.2.. By the definition, we have $S_n^{FF} = (I_n \otimes \sigma_x)S_n^{MS}$. Then by using $(I_n \otimes \sigma_x)^2 = (I_n \otimes I_2)$, we obtain

$$\begin{aligned} U_n^{FF}(\mathcal{C}_n) &= S^{FF}\mathcal{C}_n = (I_n \otimes \sigma_x)S_n^{MS}\mathcal{C}_n = (I_n \otimes \sigma_x)S_n^{MS}\mathcal{C}_n(I_n \otimes \sigma_x)^2 \\ &= (I_n \otimes \sigma_x)S_n^{MS}\mathcal{C}_n\sigma_x(I_n \otimes \sigma_x) = (I_n \otimes \sigma_x)U_n^{MS}(\mathcal{C}_n\sigma_x)(I_n \otimes \sigma_x). \end{aligned}$$

This completes the proof. □

Lemma 2.2 shows that $T_n(U_n^{MS}) = T_n(U_n^{FF})$ whenever we consider a pair of DTQWs defined by $U_n^{MS}(\mathcal{C}_n\sigma_x)$ and $U_n^{FF}(\mathcal{C}_n)$. Note that the coin operator $\mathcal{C}_n\sigma_x$ is given by exchanging column of all C_i in $\mathcal{C}_n = \sum_{i=0}^{n-1} |i\rangle\langle i| \otimes C_i$.

3. Jacobi matrix

Before we investigate periodicity of quantum walks defined in Sect.2, it is helpful to consider a related Jacobi matrix. Let $\nu_{1,i}, \nu_{2,i}$ and $|w_{1,i}\rangle, |w_{2,i}\rangle$ be the eigenvalues and the corresponding orthonormal eigenvectors of C_i $(i = 0, \ldots, n-1)$. We consider the spectral decomposition of each unitary matrix C_i as follows:

$$\begin{aligned} C_i &= \nu_{1,i}|w_{1,i}\rangle\langle w_{1,i}| + \nu_{2,i}|w_{2,i}\rangle\langle w_{2,i}| \\ &= \nu_{1,i}|w_{1,i}\rangle\langle w_{1,i}| + \nu_{2,i}\left(I_2 - |w_{1,i}\rangle\langle w_{1,i}|\right) \\ &= \left(\nu_{1,i} - \nu_{2,i}\right)|w_{1,i}\rangle\langle w_{1,i}| + \nu_{2,i}I_2, \end{aligned} \tag{3.2}$$

where I_k is the $k \times k$ identity matrix. Here we use the relation $I_2 = |w_{1,i}\rangle\langle w_{1,i}| + |w_{2,i}\rangle\langle w_{2,i}|$ coming from unitarity of C_i. This shows that we can represent C_i without $|w_{2,i}\rangle$.

We define the $n \times n$ Jacobi matrix J_n^{QW} for the DTQW as follows:

$$(J_n^{QW})_{i,j} = \overline{(J_n^{QW})_{j,i}} = \begin{cases} \overline{w_i(R)}w_j(L) & \text{if } j = i+1 \pmod{n}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

where $|w_{1,i}\rangle = {}^T[w_i(L), w_i(R)]$ and $\overline{z}$ means the complex conjugate of $z \in \mathbb{C}$. In this setting, the corresponding Jacobi matrix is the following:

$$J_n^{QW} = \begin{bmatrix} 0 & \overline{w_0(R)}w_1(L) & & & \overline{w_0(L)}w_{n-1}(R) \\ \overline{w_1(L)}w_0(R) & 0 & \ddots & & O \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & \overline{w_{n-2}(R)}w_{n-1}(L) \\ \overline{w_{n-1}(R)}w_0(L) & O & & \overline{w_{n-1}(L)}w_{n-2}(R) & 0 \end{bmatrix}. \tag{3.4}$$

As we will point out at the next line of Eq. (4.7), each eigenvalue of J_n^{QW} becomes inner product of two unit vectors. It means that $\mathrm{Spec}(J_n^{QW}) \subseteq [-1,1]$. By direct calculation, we obtain the following lemma for the characteristic polynomial of the Jacobi matrix J_n^{QW}:

Lemma 3.1. *Let*

$$K_{i,j}^{QW}(\lambda) = \begin{bmatrix} \lambda & -\overline{w_i(R)}w_{i+1}(L) & & & \\ -\overline{w_{i+1}(L)}w_i(R) & \lambda & \ddots & & O \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \lambda & -\overline{w_j(R)}w_{j+1}(L) \\ O & & & -\overline{w_{j+1}(L)}w_j(R) & \lambda \end{bmatrix}.$$

Then

$$\begin{aligned} &\det(\lambda I_n - J_n^{QW}) \\ &\quad = \lambda \det(K_{1,n-2}^{QW}(\lambda)) \\ &\quad - |w_0(R)|^2|w_1(L)|^2 \det(K_{2,n-2}^{QW}(\lambda)) - |w_{n-1}(R)|^2|w_0(L)|^2 \det(K_{1,n-3}^{QW}(\lambda)) \\ &\quad + (-1)^n \cdot 2\Re\left(\prod_{i=0}^{n-1} \overline{w_i(R)}w_i(L)\right), \end{aligned} \tag{3.5}$$

where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$.

In addition, we have

$$\begin{aligned} \det(K_{i,j}^{QW}(\lambda)) &= \lambda \det(K_{i,j-1}^{QW}(\lambda)) \\ &\quad - |w_j(R)|^2|w_{j+1}(L)|^2 \det(K_{i,j-2}^{QW}(\lambda)),\ (j \geq i+1), \\ \det(K_{i,i}^{QW}(\lambda)) &= \lambda^2 - |w_i(R)|^2|w_{i+1}(L)|^2, \end{aligned}$$

with a convention $\det(K^{QW}_{i,i-1}(\lambda)) = \lambda$. This leads to the following lemma:

Lemma 3.2. $\det(K^{QW}_{i,j}(\lambda))$ *is a polynomial with real coefficients. If we define* $p_i = |w_i(R)|^2$ *and* $q_i = |w_i(L)|^2$ *for* $i \in V_n$ *then the coefficients of* $\det(K^{QW}_{i,j}(\lambda))$ *are determined by* $p_i, \ldots, p_j, q_i, \ldots, q_{j+1}$.

4. Isospectral coin cases

Now we give a framework of spectral analysis for DTQWs with flip-flop shift on C_n. In order to do so, we restrict the coin operator as follows:

Assumption 4.1. We assume all the local coins are isospectral. Thus we use

$$\mathcal{C}_n = \sum_{i=0}^{n-1} |i\rangle\langle i| \otimes \{(\nu_1 - \nu_2)|w_i\rangle\langle w_i| + \nu_2 I_2\}, \tag{4.6}$$

as the coin operator.

Let λ_m $(m = 0, \ldots n-1)$ be the eigenvalues and $|v_m\rangle$ $(m = 0, \ldots n-1)$ be the corresponding (orthonormal) eigenvectors of J^{QW}_n. For each λ_m and $|v_m\rangle$, we define two vectors

$$\mathbf{a}_m = \sum_{i=0}^{n-1} v_m(i)|i\rangle \otimes |w_i\rangle,$$
$$\mathbf{b}_m = S^{FF}_n \mathbf{a}_m,$$

where $|v_m\rangle = {}^T[v_m(0) \ldots v_m(n-1)]$. By using $(S^{FF}_n)^2 = I_n \otimes I_2$, it is easy to see that $\mathcal{C}_n \mathbf{a}_m = \nu_1 \mathbf{a}_m$ and then $U^{FF}_n \mathbf{a}_m = \nu_1 \mathbf{b}_m$. Also we have $\mathcal{C}_n \mathbf{b}_m = (\nu_1 - \nu_2)\lambda_m \mathbf{a}_m + \nu_2 \mathbf{b}_m$ and $U^{FF}_n \mathbf{b}_m = \nu_2 \mathbf{a}_m + (\nu_1 - \nu_2)\lambda_m \mathbf{b}_m$. So we have the following relationship:

$$U^{FF}_n \begin{bmatrix} \mathbf{a}_m \\ \mathbf{b}_m \end{bmatrix} = \begin{bmatrix} 0 & \nu_1 \\ \nu_2 & (\nu_1 - \nu_2)\lambda_m \end{bmatrix} \begin{bmatrix} \mathbf{a}_m \\ \mathbf{b}_m \end{bmatrix}. \tag{4.7}$$

We also obtain $|\mathbf{a}_m| = |\mathbf{b}_m| = 1$ and the inner product $(\mathbf{a}_m, \mathbf{b}_m) = \lambda_m$. This shows that if $\lambda_m = \pm 1$ then $\mathbf{b}_m = \pm \mathbf{a}_m$. Therefore if $\lambda_m = \pm 1$ then $U^{FF}_n \mathbf{a}_m = \pm \nu_1 \mathbf{a}_m$.

For cases with $\lambda_m \neq \pm 1$, we see from Eq. (4.7) that the operator U^{FF}_n is a linear operator acting on the linear space Span $(\mathbf{a}_m, \mathbf{b}_m)$. In order to obtain the eigenvalues and eigenvectors, we take a vector $\alpha \mathbf{a}_m + \beta \mathbf{b}_m \in$ Span $(\mathbf{a}_m, \mathbf{b}_m)$. The eigen equation for U_n^{FF} is given by $U_n^{FF}(\alpha\mathbf{a}_m+\beta\mathbf{b}_m)=\mu(\alpha\mathbf{a}_m+\beta\mathbf{b}_m)$. From Eq. (4.7), this is equivalent to

$$\begin{bmatrix}0 & \nu_2\\ \nu_1 & (\nu_1-\nu_2)\lambda_m\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\mu\begin{bmatrix}\alpha\\ \beta\end{bmatrix}.$$

Therefore we can obtain two eigenvalues $\mu_{\pm m}$ of U_n^{FF} which are related to the eigenvalue λ_m of J_n^{QW} as solutions of the following quadratic equation:

$$\mu^2-(\nu_1-\nu_2)\lambda_m\mu-\nu_1\nu_2=0.$$

Also we have the corresponding eigenvectors $\nu_2\mathbf{a}_m+\mu_{\pm m}\mathbf{b}_m$ by setting $\alpha=\nu_2,\beta=\mu_{\pm m}$. As a consequence, we obtain the following lemma:

Lemma 4.2. *Let λ_m $(m=0,\dots,n-1)$ be the eigenvalues of J_n^{QW}, then the corresponding eigenvalues $\mu_{\pm m}$ and the eigenvectors $\mathbf{u}_{\pm m}$ of U_n^{FF} are the following:*

1. *If $\lambda_m=\pm1$ then $\mu_m=\pm\nu_1$ and $\mathbf{u}_m=\mathbf{a}_m$.*
2. *If $\lambda_m\neq\pm1$ then $\mu_{\pm m}$ are the solutions of the following quadratic equation:*

$$\mu^2-(\nu_1-\nu_2)\lambda_m\mu-\nu_1\nu_2=0,$$

and $\mathbf{u}_{\pm m}=\nu_2\mathbf{a}_m+\mu_{\pm m}\mathbf{b}_m$.

Remark 4.3. The quadratic equation in Lemma 4.2 is rearranged to

$$\left\{i\overline{\nu_1}^{1/2}\overline{\nu_2}^{1/2}\mu\right\}^2+2\Im(\nu_1^{1/2}\overline{\nu_2}^{1/2})\lambda_m\left\{i\overline{\nu_1}^{1/2}\overline{\nu_2}^{1/2}\mu\right\}+1=0.$$

Thus we have

$$\begin{aligned} i\overline{\nu_1}^{1/2}\overline{\nu_2}^{1/2}\mu_{\pm m} &= -\Im(\nu_1^{1/2}\overline{\nu_2}^{1/2})\lambda_m\pm i\sqrt{1-\left(\Im(\nu_1^{1/2}\overline{\nu_2}^{1/2})\lambda_m\right)^2}\\ \mu_{\pm m} &= \left(-\nu_1\nu_2\right)^{1/2}e^{\pm i\theta_m}, \end{aligned}$$

where $\cos\theta_m=-\Im(\nu_1^{1/2}\overline{\nu_2}^{1/2})\lambda_m$. Therefore if we put $\nu_j=e^{i\phi_j}$ then the eigenvalues $\mu_{\pm m}$ are given by the following procedure:

1. Rescale the eigenvalue λ_m of J_n^{QW} as $-\Im(\nu_1^{1/2}\overline{\nu_2}^{1/2})\lambda_m=-\sin[(\phi_1-\phi_2)/2]\times\lambda_m$.
2. Map the rescaled eigenvalue upward and downward to the unit circle on the complex plane.
3. Take $[(\phi_1+\phi_2-\pi)/2]$-rotation of the mapped eigenvalues.

For usual Szegedy walk cases, i.e., $\nu_1=1,\nu_2=-1$ case, we have $\phi_1=0,\phi_2=\pi$. Thus we can omit 1 and 3 of the procedure because $-\sin[(\phi_1-\phi_2)/2]=1,[(\phi_1+\phi_2-\pi)/2]=0$.

Remark 4.4. According to Lemma 4.2, if all n numbers of eigenvalues of J_n^{QW} are not equal to ± 1 then we obtain all $2n$ numbers of eigenvalues of U_n^{FF}. But if there exist s numbers of the $\lambda_m = \pm 1$ eigenvalues of J_n^{QW} then we can only obtain $2n - s$ numbers of eigenvalues of U_n^{FF}.

In this case, for every $\lambda_m = \pm 1$, we construct the following two vectors:

$$\tilde{\mathbf{a}}_m = \sum_{i=0}^{n-1} v_m(i)|i\rangle \otimes |w_{2,i}\rangle,$$
$$\tilde{\mathbf{b}}_m = S_n^{FF}\tilde{\mathbf{a}}_m,$$

where $|w_{2,i}\rangle$ is the eigenvector corresponding to the eigenvalue ν_2 of C_i in Eq. (3.2). By the definition, we have $|\tilde{\mathbf{a}}_m| = |\tilde{\mathbf{b}}_m| = 1$. Also we obtain the inner product $(\tilde{\mathbf{a}}_m, \mathbf{a}_m) = 0$ from orthogonality and $(\tilde{\mathbf{b}}_m, \mathbf{a}_m) = (\tilde{\mathbf{b}}_m, \pm\mathbf{b}_m) = \pm(S_n^{FF}\tilde{\mathbf{a}}_m, S_n^{FF}\mathbf{a}_m) = 0$ from $\lambda_m = \pm 1$ and $(S_n^{FF})^2 = I_n \otimes I_2$. Since $\mathbf{a}_m$ belongs to the eigensystem of ν_1 of $\mathcal{C}_n$, this shows that both $\tilde{\mathbf{a}}_m$ and $\tilde{\mathbf{b}}_m$ belong to the eigensystem of ν_2 of $\mathcal{C}_n$. This implies that

$$U_n^{FF}\begin{bmatrix}\tilde{\mathbf{a}}_m\\ \tilde{\mathbf{b}}_m\end{bmatrix} = \begin{bmatrix}0 & \nu_2\\ \nu_2 & 0\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{a}}_m\\ \tilde{\mathbf{b}}_m\end{bmatrix}.$$

Therefore $U_n^{FF}(\tilde{\mathbf{a}}_m \pm \tilde{\mathbf{b}}_m) = \pm\nu_2(\tilde{\mathbf{a}}_m \pm \tilde{\mathbf{b}}_m)$. These are the candidates of eigenvalues and eigenvectors.

On the other hand, the two sets $\mathcal{H}_n^{(\pm)} = \mathrm{span}\{|i+1, L\rangle \pm |i, R\rangle : i \in V_n (\mathrm{mod}\ n)\}$ are subspaces of whole Hilbert space $\mathcal{H}_n$ with $\dim\mathcal{H}_n^{(\pm)} = n$, i.e., $\mathcal{H}_n = \mathcal{H}_n^{(+)} \oplus \mathcal{H}_n^{(-)}$. Note that $\mathbf{a}_m \pm \mathbf{b}_m \in \mathcal{H}_n^{(\pm)}$. If $\lambda_m = \pm 1$ then $\mathbf{a}_m = \pm\mathbf{b}_m$. This implies that if $\lambda_m = \pm 1$ then the dimension of $\mathcal{H}_n^{(\mp)} \cap \mathsf{Span}\ (\mathbf{a}_m, \mathbf{b}_m)$ decreases by 1. Therefore if $\lambda_m = 1$ then we can only choose $U_n^{FF}(\tilde{\mathbf{a}}_m - \tilde{\mathbf{b}}_m) = \nu_2(\tilde{\mathbf{a}}_m - \tilde{\mathbf{b}}_m)$. In the same way, if $\lambda_m = -1$ then we can only choose $U_n^{FF}(\tilde{\mathbf{a}}_m + \tilde{\mathbf{b}}_m) = -\nu_2(\tilde{\mathbf{a}}_m + \tilde{\mathbf{b}}_m)$. Using these procedure, we have remaining s numbers of eigenvalues and eigenvectors.

As a consequence of Lemmas 3.1, 3.2, 4.2 and Remark 4.4, we have the following result:

Theorem 4.5. *Under the Assumption 4.1, let* $w_j(R) = \sqrt{p_j}e^{i\theta_R(j)}$ *and* $w_j(L) = \sqrt{q_j}e^{i\theta_L(j)}$ *for* $j \in V_n$ *where* $p_j = |w_j(R)|^2$ *and* $q_j = |w_i(L)|^2$. *Also let* $\tilde{U}_n^{FF}$ *which is defined by the coin operator* $\tilde{\mathcal{C}}_n$ *with* $\tilde{w}_j(R) = \sqrt{p_j}e^{i(\theta_R(j)+\tilde{\theta}_R(j))}$ *and* $\tilde{w}_j(L) = \sqrt{q_j}e^{i(\theta_L(j)+\tilde{\theta}_L(j))}$ *for* $j \in V_n$. *If* $\sum_{j=1}^{n-1}(\tilde{\theta}_L(j) - \tilde{\theta}_R(j)) = 2\pi k$ $(k \in \mathbb{Z})$ *then* $T_n(U_n^{FF}) = T_n(\tilde{U}_n^{FF})$.

Proof of Theorem 4.5.. From Eq. (3.5), if $\sum_{j=1}^{n-1}(\tilde{\theta}_L(j) - \tilde{\theta}_R(j)) = 2\pi k$ $(k \in \mathbb{Z})$ then $\Re\left(\prod_{j=0}^{n-1}\overline{w_j(R)}w_j(L)\right) = \Re\left(\prod_{j=0}^{n-1}\overline{\tilde{w}_j(R)}\tilde{w}_j(L)\right)$. Then from Lemmas 3.1, 3.2, 4.2 and Remark 4.4, we have Spec U_n^{FF} = Spec $\tilde{U}_n^{FF}$. Therefore we have $T_n(U_n^{FF}) = T_n(\tilde{U}_n^{FF})$. □

Theorem 4.5 provides a classification of our DTQW from the point of the periodicity. Indeed, $T_n(U_n^{FF})$ depends only on the sequence $\{p_j\}_{0\le j\le n-1}$ and a value $\sum_{j=1}^{n-1}(\tilde{\theta}_L(j) - \tilde{\theta}_R(j))$. Therefore we can identify DTQWs having the same set of these values. The next corollary provides "Hadamard class" of periodicity.

Corollary 4.6. *Let* $\mathcal{C}'_n = \sum_{j=0}^{n-1} |j\rangle\langle j| \otimes C'_j$ *with* $C'_j = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\tilde{\theta}(j)} & 1 \\ 1 & -e^{-i\tilde{\theta}(j)} \end{bmatrix}$. *If* $\sum_{j=0}^{n-1}\tilde{\theta}(j) = 2\pi k$ $(k \in \mathbb{Z})$ *then*

$$T_n(U_n^{MS}(\mathcal{C}'_n)) = \begin{cases} 2, & (n=2) \\ 8, & (n=4) \\ 24 & (n=8) \\ \infty & (n \neq 2,4,8). \end{cases}$$

Proof of Corollary 4.6.. Let $\mathcal{C}_n = \sum_{j=0}^{n-1} |j\rangle\langle j| \otimes H$ with $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, i.e., the Hadamard walk case. The periodicity for this case is as follows [3, 8]:

$$T_n(U_n^{MS}(\mathcal{C}_n)) = \begin{cases} 2, & (n=2) \\ 8, & (n=4) \\ 24 & (n=8) \\ \infty & (n \neq 2,4,8). \end{cases}$$

From Lemma 2.2, we have $T_n(U_n^{MS}(\mathcal{C}_n)) = T_n(U_n^{FF}(\mathcal{C}_n\sigma_x))$. So we consider $H\sigma_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ case. By direct calculation, we obtain

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} = \frac{1+i}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix} = e^{i\pi/4}\begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix},$$
$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix} = \frac{1-i}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix} = e^{-i\pi/4}\begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}.$$

Therefore the spectral decomposition of the coin operator $H\sigma_1$ is

$$H\sigma_1 = (e^{i\pi/4} - e^{-i\pi/4})\begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} + e^{-i\pi/4} I_2.$$

We consider the coin operator $\tilde{\mathcal{C}}_n = \sum_{j=0}^{n-1} |j\rangle\langle j| \otimes \widetilde{(H\sigma_1)}_j$ with

$$\begin{aligned}
\widetilde{(H\sigma_1)}_j &= (e^{i\pi/4} - e^{-i\pi/4}) \begin{bmatrix} 1e^{i\tilde{\theta}_L(j)}/\sqrt{2} \\ ie^{i\tilde{\theta}_R(j)}/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1e^{-i\tilde{\theta}_L(j)}/\sqrt{2} & -ie^{-i\tilde{\theta}_R(j)}/\sqrt{2} \end{bmatrix} + e^{-i\pi/4} I_2 \\
&= \frac{i}{\sqrt{2}} \begin{bmatrix} 1 & -ie^{i(\tilde{\theta}_L(j)-\tilde{\theta}_R(j))} \\ ie^{-i(\tilde{\theta}_L(j)-\tilde{\theta}_R(j))} & 1 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1-i & 0 \\ 0 & 1-i \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i(\tilde{\theta}_L(j)-\tilde{\theta}_R(j))} \\ -e^{-i(\tilde{\theta}_L(j)-\tilde{\theta}_R(j))} & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{i\tilde{\theta}(j)} \\ -e^{-i\tilde{\theta}(j)} & 1 \end{bmatrix}.
\end{aligned}$$

Using Theorem 4.5, we have $T_n(U_n^{FF}(\tilde{\mathcal{C}}_n)) = T_n(U_n^{FF}(\mathcal{C}_n))$ if $\sum_{j=0}^{n-1} \tilde{\theta}(j) = 2\pi k$ ($k \in \mathbb{Z}$). Noting that $\mathcal{C}_n' = \tilde{\mathcal{C}}_n\sigma_1$, we obtain the desired result by Lemma 2.2. □

Remark 4.7. By the same arguments of the proof of Corollary 4.6, we obtain the following result:

Let $\mathcal{C}_n = \sum_{j=0}^{n-1} |j\rangle\langle j| \otimes C$ with $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\tilde{\mathcal{C}}_n = \sum_{j=0}^{n-1} |j\rangle\langle j| \otimes \tilde{C}_j$ with $\tilde{C}_j = \begin{bmatrix} ae^{i\tilde{\theta}(j)} & b \\ c & de^{-i\tilde{\theta}(j)} \end{bmatrix}$. If $\sum_{j=0}^{n-1} \tilde{\theta}(j) = 2\pi k$ ($k \in \mathbb{Z}$) then $T_n(U_n^{MS}(\mathcal{C}_n)) = T_n(U_n^{MS}(\tilde{\mathcal{C}}_n))$.

5. Non-isospectral coin cases

In this section, we consider several types of DTQWs with non-isospectral coin and the moving shift. In order to define periodic coin operator, we introduce a notation $[C : l, \tilde{C} : m]$ which denotes

$$C_i = \begin{cases} C & \text{if } 0 \le i \le l-1 \ (\mod (l+m)), \\ \tilde{C} & \text{if } l \le i \le l+m-1 \ (\mod (l+m)), \end{cases}$$

in the coin operator $\mathcal{C}_n = \sum_{i=0}^{n-1} |i\rangle\langle i| \otimes C_i$. In this section, we consider $[C : 1, I_2 : m]$ model with $n = 0 \ (\mod m+1)$ for 2×2 unitary matrix C.

At the beginning, we consider $m = n-1$ cases. In this cases, the coin operator is defined by $\mathcal{C}_n = |0\rangle\langle 0| \otimes C + \sum_{i=1}^{n-1} |i\rangle\langle i| \otimes I_2$ then the time evolution operator

$U_n^{MS} = S_n^{MS}\mathcal{C}_n$ is given by

$$\begin{aligned} U_n^{MS} &= |0\rangle\langle 1| \otimes |L\rangle\langle L|I_2 + |0\rangle\langle n-1| \otimes |R\rangle\langle R|I_2 \\ &\quad + |1\rangle\langle 2| \otimes |L\rangle\langle L|I_2 + |1\rangle\langle 0| \otimes |R\rangle\langle R|C \\ &\quad + \sum_{i=2}^{n-2}(|i\rangle\langle i+1| \otimes |L\rangle\langle L|I_2 + |i\rangle\langle i-1| \otimes |R\rangle\langle R|I_2) \\ &\quad + |n-1\rangle\langle 0| \otimes |L\rangle\langle L|C + |n-1\rangle\langle n-2| \otimes |R\rangle\langle R|I_2 \\ &= |0\rangle\langle 1| \otimes |L\rangle\langle L| + |0\rangle\langle n-1| \otimes |R\rangle\langle R| \\ &\quad + |1\rangle\langle 2| \otimes |L\rangle\langle L| + |1\rangle\langle 0| \otimes |R\rangle\langle R|C \\ &\quad + \sum_{i=2}^{n-2}(|i\rangle\langle i+1| \otimes |L\rangle\langle L| + |i\rangle\langle i-1| \otimes |R\rangle\langle R|) \\ &\quad + |n-1\rangle\langle 0| \otimes |L\rangle\langle L|C + |n-1\rangle\langle n-2| \otimes |R\rangle\langle R|. \end{aligned}$$

Thus we have

$$\begin{aligned} (\langle 0| \otimes I_2)\left(U_n^{MS}\right)^{kn}(|0\rangle \otimes I_2) &= |L\rangle\langle L|C^k + |R\rangle\langle R|C^k = (|L\rangle\langle L| + |R\rangle\langle R|)C^k \\ &= I_2C^k = C^k, \end{aligned}$$

for $k = 1, 2, \ldots$. Also for $i \neq 0$, we obtain

$$\begin{aligned} (\langle i| \otimes I_2)\left(U_n^{MS}\right)^{kn}(|i\rangle \otimes I_2) &= (|L\rangle\langle L|)^{n-i-1}(|L\rangle\langle L|C)\,C^{k-1}(|L\rangle\langle L|)^i \\ &\quad + (|R\rangle\langle R|)^{i-1}(|R\rangle\langle R|C)\,C^{k-1}(|R\rangle\langle R|)^{n-i} \\ &= |L\rangle\langle L|C^k|L\rangle\langle L| + |R\rangle\langle R|C^k|R\rangle\langle R|, \end{aligned}$$

for $k = 1, 2, \ldots$. Using this observation, we can reach the following result:

Theorem 5.1. *For the $[C : 1, I_2 : n-1]$ model for 2×2 unitary matrix C, let λ_1, λ_2 be the pair of eigenvalues of C. If $\lambda_1 = \exp[2\pi i(L_1/N_1)]$ and $\lambda_2 = \exp[2\pi i(L_2/M_2)]$ where L_1/N_1 and L_2/N_2 are reduced rational numbers, we take $M = l.c.m.(M_1, M_2)$ then $T_n(U_n^{FF}) = Mn$, where $l.c.m.(a, b)$ denotes the least common multiple of two integers a and b.*

From the above discussion, we can see the vertex which has the coin I_2 just through the coin state. Therefore we have the following result for general $[C : 1, I_2 : m]$ model with $n = 0 \pmod{m+1}$ for 2×2 unitary matrix C:

Corollary 5.2. *For the $[C : 1, I_2 : m]$ model with $n = 0 \pmod{m+1}$ for 2×2 unitary matrix C, let $T_{n/(m+1)}^{FF}$ be the period of the time evolution operator with coin operator $\mathcal{C}_{n/(m+1)} = \sum_{i=0}^{n/(m+1)-1} |i\rangle\langle i| \otimes C$ and flip-flop shift operator $S_{n/(m+1)}^{FF}$. Then we have $T_n(U_n^{FF}) = (m+1)T_{n/(m+1)}^{FF}$.*

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