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DISTINGUISHING COLORINGS OF TRUNCATED 3-REGULAR POLYHEDRA

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Abstract. A *distinguishing k-coloring* of a graph G is defined as a proper coloring with k colors such that no automorphism of G other than the identity map preserves the colors. We call a 3-connected planar graph embedded on the sphere a *polyhedron* and its *truncation* is obtained from it by replacing a small local part around each vertex with a cycle. We shall show that any truncated 3-regular polyhedron has a distinguishing 3-coloring.

Introduction

Let G be a graph without loops and multiple edges. A function $c : V(G) \to \{1, 2, \ldots, k\}$ is called a *coloring* or a *k-coloring* if $c(u) \neq c(v)$ for any two adjacent vertices u and v. Furthermore, if there is no automorphism, other than the identity map, which preserves the colors, then c is said to be *distinguishing*. A graph G is said to be *distinguishing k-colorable* if it has a distinguishing k-coloring. The *distinguishing chromatic number* of G is defined as the minimum number k such that G is distinguishing k-colorable and is denoted by $\chi_D(G)$ while $\chi(G)$ denotes the usual chromatic number of G. The notion of distinguishing chromatic numbers has been introduced in [2].

Consider a graph G embedded on a closed surface F^2. Such a graph with its specified embedding is called a *map* on F^2 with an underlying graph G and is denoted by $M(G)$ to distinguish it from the abstract graph G itself. Then a subgroup in the automorphism group $\mathrm{Aut}(G)$ of G acts on the surface F^2 so that each member carries faces to faces. The maximal one among such subgroups is called the *automorphsim group* of $M(G)$ and is denoted by $\mathrm{Aut}(M(G))$. We often call each member of $\mathrm{Aut}(M(G))$ a *map automorphism* of $M(G)$ or of G.

We can define the *distinguishing chromatic number* $\chi_D(M(G))$ of a map $M(G)$ by replacing automorphisms of G with map automorphisms of $M(G)$ in the definition of $\chi_D(G)$. Then it is clear that $\chi(G) \leq \chi_D(M(G)) \leq \chi_D(G)$. In

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particular, we can conclude that $\chi_D(M(G)) = \chi_D(G)$ for any 3-connected planar graph G embedded on the sphere since such a graph can be uniquely and faithfully embedded on the sphere, as is pointed out in [7].

By recent works on this topic [3, 8, 9, 10, 11, 15, 16], we have already known that the distinguishing chromatic number of a map $M(G)$ is very close to the chromatic number of its underlying graph G in most of cases. In particular, the first author has proved the following theorem:

Theorem 1. (Negami [9]) *Every 3-regular map on a closed surface has a distinguishing 4-coloring with color 4 used for at most one vertex unless it is one of the followings:*

- *The 3-cube Q_3 on the sphere*
- *$K_{3,3}$ on the torus with three hexagonal faces*
- *$K_{4,4} - 4K_2$ on the torus with four hexagonal faces*

This leads us to a conjecuture that 3-regular maps have a distinguishing 3-coloring with a few exceptions. As a partial answer to this conjecture, we shall show a special class of 3-regular maps on the sphere which are distinguishing 3-colorable, as follows.

Let G be a 3-connected planar graph embedded on the sphere, which is often called a *polyhedron*. Replace a small local part around each vertex with a cycle; this corresponding to cutting off each vertex from a polyhedron by a plane. The resulting graph is called the *truncation* of G or "the *truncated G*". Every truncated polyhedron is 3-regular. In particular, a truncated 3-regular polyhedron consists of triangles and edges joining them. All faces except such triangles are bounded by cycles of even length. We shall prove the following theorem:

Theorem 2. *Every truncated 3-regular polyhedron is distinguishing 3-colorable.*

We shall show a method to generate all truncated 3-regular polyhedra in Section 1 and discuss a special type of 3-colorings, called "good 3-colorings", which will be distinguishing, in Section 2. Combining these, we shall give an inductive proof for Theorem 2 and discuss its generalization in Section 3.

1. Generating truncated 3-regular polyhedra

Let G be a 3-regular graph and take two distinct edges e_1 and e_2, possibly having a common end. Subdivide them by their middle points u_1 and u_2, and add a new edge u_1u_2 to obtain another 3-regular graph G'. This deformation of

G into G' is called a *bridging*. Negami [6] discussed generation of 3-connected graphs using "bridgings" in more general situation to establish what is called "a splitter theorem". The following lemma can be obtained easily from his general result as its corollary:

Lemma 3. (Negami [6]) *Let G be a 3-connected 3-regular graph. If G contains a subdivision of another 3-connected 3-regular graph H, then G can be obtained from H by a sequence of bridgings.*

It is easy to see that any 3-connected 3-regular graph G contains a subdivision of K_4 and hence it can be obtained from K_4 by a sequence of bridgings. Suppose that G is embedded on the sphere as a polyhedron. Consider the inverse process of bridgings to construct G from K_4, erasing the added edges in order. Finally, we get K_4 embedded on the sphere as the tetrahedron. This observation implies the following lemma:

Lemma 4. *Every 3-regular polyhedron can be obtained from the tetrahedron K_4 by a sequence of bridgings across faces.* ■

Now let G be a truncated 3-regular polyhedron and choose two distinct edges e_1 and e_2 which do not belong to any triangle and which meet a common face A. It should be noticed that there is no other common face A' which e_1 and e_2 meet; otherwise, G would not be 3-connected. Make small triangles at the middle points of e_1 and e_2, and join these two triangles with a new edge across A, as shown in Figure 1. We call this deformation a *truncated bridging* or a *bridging* for a truncated 3-regular polyhedron.

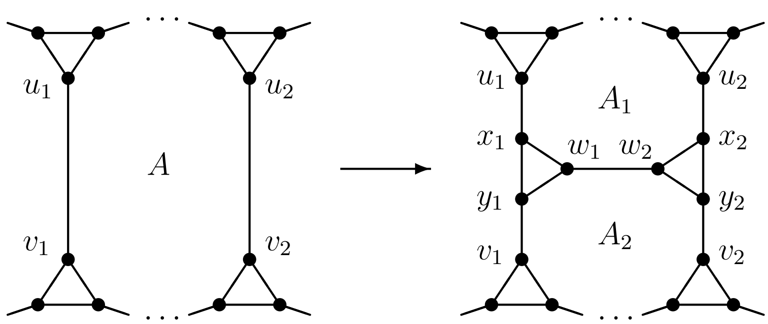

Figure 1 A truncated bridging

Lemma 4 for 3-regular polyhedra can be immediately translated into one for truncated 3-regular polyhedra as follows:

Lemma 5. *Every truncated 3-regular polyhedron can be obtained from the trun-*

cated tetrahedron K_4 by a sequence of truncated bridgings. ■

2. Good 3-colorings

Let G be a truncated 3-regular polyhedron. By Brooks' Theorem, G has a 3-coloring $c : V(G) \to \{1, 2, 3\}$ with colors 1, 2 and 3. We call c a *good* 3*-coloring* if the following conditions hold:

(i) There is only one non-triangular face of G such that color 3 appears along its boundary cycle exactly once.

(ii) Any other non-triangular face has two or more vertices with color 3 along its boundary cycle.

Call a face of the first type a *distinguished face* here. For example, the truncated tetrahedron has a good 3-coloring as shown in Figure 2. The hexagonal face lying at the bottom is a distinguished face.

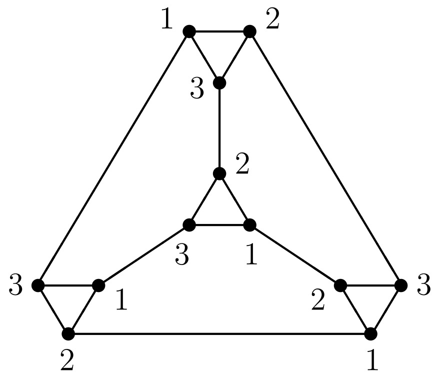

Figure 2 A good coloring of the truncated tetrahedron

Lemma 6. *Any good* 3*-coloring is distinguishing.*

Proof. Let $c : V(G) \to \{1, 2, 3\}$ be a good 3-coloring of a truncated 3-regular polyhedron G and suppose that c is not distinguishing. Then there is an automorphism $\sigma : G \to G$ which preserves the colors of vertices given by c. Since there is only one distinguished face A of G, σ sends the boundary cycle of A to itself, fixing the unique vertex v with color 3. Thus, σ must swap two edges incident to v on the boundary cycle. However, it cannot do since one of them is contained in a triangle but the other is not, a contradiction. Therefore, c is distinguishing. ■

It is important to discuss the configuration of vertices with color 3 to discuss good 3-colorings. The following lemma is so useful to do it since we do not need to care about the vertices with colors 1 and 2.

Lemma 7. *Let G be a truncated 3-regular polyhedron. If there is an independent set S in $V(G)$ which contains one vertex of each triangle in G, then there is a 3-coloring $c : V(G) \to \{1,2,3\}$ with $c(x) = 3$ for all $x \in S$ and $c(x) \in \{1,2\}$ for the others.*

Proof. It is clear that $G - S$ consists of only vertices of degree 1 or 2 and that any path or cycle in $G - S$ passes through edges lying and not lying on triangles alternately. This implies that $G - S$ is bipartite and hence $G - S$ has a 2-coloring with colors 1 and 2. Assigning color 3 to each vertex in S, we obtain a 3-coloring as we want. ■

The following lemma forms an essential part of our proof for Theorem 2:

Lemma 8. *If a truncated 3-regular polyhedron G has a good 3-coloring, then so does any truncated 3-regular polyhedron obtained from G by a truncated bridging.*

Proof. Let G be a truncated 3-regular polyhedron and suppose that G has a good coloring $c : V(G) \to \{1,2,3\}$. Put S be the set of vertices of G with color 3, which is independent. Let A be a face of G where an extra edge w_1w_2 is added by a truncated bridging joining two edges u_1v_1 and u_2v_2 on its boundary cycle C. That is, two other vertices x_i and y_i are added along each edge u_iv_i and $w_ix_iy_i$ forms a triangle for $i = 1, 2$. Let G' be the resulting polyhedron by this truncated bridging.

We may assume that u_1, x_1, y_1, v_1, v_2, y_2, x_2 and u_2 lie along C in this order, as shown in Figure 1. Then C decomposes into a path P_1 joining u_1 and u_2, a path P_2 joining v_1 and v_2, and two edges u_1v_1 and u_2v_2. The face A is divided into two faces A_1 and A_2 by bridging, where P_i runs along the boundary cycle of A_i for $i = 1, 2$. Let B_i be another face of G sharing the edge u_iv_i with A for $i = 1, 2$. Since G is a 3-connected graph embedded on the sphere, these two faces B_1 and B_2 are distinct.

Case 1: *The face A is distinguished.* Then the boundary cycle of A contains only one vertex in S and hence we may assume that P_1 contains the unique vertex in S while P_2 contains no vertex in S. Add y_1 and y_2 to S and construct a 3-coloring $c' : V(G') \to \{1,2,3\}$ with the new S by Lemma 7. Then A_1 is a distinguished face for c' and A_2 is not since $\{y_1, y_2\} \subset S$. The other faces are not still distinguished. Therefore, c' is a good 3-coloring.

Case 2: *The face A is not distinguished.* Then its boundary cycle C contains at least two vertices with color 3. Without loss of generality, we may assume that $|P_1 \cap S| \geq |P_2 \cap S|$. Under this assumption, P_2 may contain no vertex in S. Since two faces B_1 and B_2 are different, one of them is not distinguished, say B_2.

Add w_1 to S. If $c(v_2) = 3$, that is, if v_2 belongs to S, then u_2 does not and hence we can add x_2 to S, keeping S independent. Otherwise, we can add y_2 to S, instead of x_2. Construct a 3-coloring $c' : V(G') \to \{1, 2, 3\}$ with the new S by Lemma 7.

Since S contains $\{w_1, v_2\}$ or $\{w_1, y_2\}$, the boundary cycle of A_2 contains at least two vertices in S in either case. Since A is not distinguished, $|(P_1 \cup P_2) \cap S| \geq 2$ and we have $|P_1 \cap S| \geq |P_2 \cap S|$ under our assumption. This implies that $|P_1 \cap S| \geq 1$ and the boundary cycle of A_1 contains at least two vertices in the new S, including w_1. The number of vertices in S along the boundary cycle of B_2 increased by one since one of x_2 and y_2 was added to S. The number of such vertices for no other faces changed. Thus, the unique distinguished face is still distinguished and the others are not distinguished and have at least two vertices with color 3 along its boundary cycle. Therefore, c' is a good coloring.

In either case, we have constructed a good 3-coloring for G'. ■

3. Proof and generalization

Combining lemmas in the previous sections, we can prove our main theorem easily:

Proof of Theorem 2. Le G be any truncated 3-regular polyhedron. By Lemma 5, G can be obtained from the truncated tetrahedron by a sequence of truncated bridgings. Since the truncated tetrahedron has a good 3-coloring, we can show inductively that each of truncated 3-reguler polyhedra appearing in the sequence has a good 3-coloring by Lemma 8. Therefore, G has a good 3-coloring, which is distinguishing by Lemma 6, and hence it is distinguishing 3-colorable. ■

At the end, we shall note some arguments to generalize Theorem 2 for maps on other surfaces. In general, a map $M(G)$ on a closed surface is said to be *polyhedral* if it satisfies the following three conditions:

(i) Its underlying graph G is 3-connected.
(ii) Each of its faces is a 2-cell bounded by a cycle.
(iii) Two distinct faces share at most one vertex or one edge.

Clearly, any polyhedron, that is, a map of any 3-connected planar graph on the sphere is polyhedral. It is easy to see that if a map $M(G)$ on a closed surface is polyhedral, then so is its truncation.

In fact, the same arguments as in the proofs of Lemmas 6, 7 and 8 hold for 3-regular polyhedral maps $M(G)$ on a given closed surface. Thus, it would be concluded that a truncated 3-regular polyhedral map on a closed surface F^2 is distinguishing 3-colorable if we could carry out the following two:

- Determine the set of those 3-regular polyhedral maps on F^2 from which all others are obtained by bridgings.
- Show that the truncation of each member in the set has a good 3-coloring,

Call each 3-regular polyhedral map in the set *minimal*. For example, the only tetrahedron is a minimum 3-regular polyhedral map on the sphere and its truncation has a good 3-coloring, as shown in Figure 2.

Take the *dual* of a 3-regular polyhedral map on a closed surface F^2 to connect our arguments with a well developed theory on "irreducible triangulations" on closed surfaces. A *triangulation* on F^2 is a simple graph embedded on F^2 so that each face is bounded by a cycle of length 3 and that any two face meet along at most one edge. It is clearly that the dual of a 3-regular map is a graph embedded on the same surface with only triangular faces, but it might not be simple. The polyhedrality of a 3-regular map guarantees the simpleness of its dual to be a triangulation.

Shrink an edge uv in a triangulation on a closed surface to be a point $u = v$ and eliminate two pairs of multiple edges to erase the resulting digonal regions. This deformation is called a *contraction* of uv. We allow ourselves to carry out a contraction of an edge and call it a *contractible edge* only when the resulting graph is a triangulation on the same surface. A triangulation on a closed surface F^2 is said to be *irreducible* if it contains no contractible edge. There have been already classified the irreducible triangulations on the sphere [12], the projective plane [1], the torus [4] and the Klein bottle [5, 14]; they are 1, 2, 21 and 29 in number.

It is easy to see that the inverse process of a bridging in a 3-regular polyhedral map $M(G)$ on F^2 corresponds to the contraction of an edge in its dual, which crosses the edge in $M(G)$ added by the bridging and that the dual of a minimal 3-regular polyhedral map on F^2 is nothing but one of irreducible triangulations on F^2. This responds to the first item in the above. On the other hand, there has never been any theoretical method to answer to the second yet, but we have already done it for the projective plane, the torus and the Klein bottle in [13], finding good 3-colorings concretely.

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