Yokohama Mathematical
Journal Vol. 63, 2017

QUADRANGULATIONS ON THE PROJECTIVE PLANE WITH A $K_{3,4}$-MINOR

By

Naoki Matsumoto, Atsuhiro Nakamoto and Shin-ichi Yonekura

(Received September 29, 2017; Revised December 6, 2017)

Abstract. A *quadrangulation* G on a closed surface $\mathbb{F}$ is a map of a simple graph on F with each face quadrilateral. In this paper, as an application for the result of Matsumoto et al. [10], we give a forbidden structure of a bipartite quadrangulation on the projective plane to have a $K_{3,4}$-minor.

1. Introduction

Let G be a graph and let e be an edge of G. An *edge deletion* of e is to remove e from G, and an *edge contraction* of e is to remove e from G and identify the two endpoints of e. If these operations destroy the simpleness of graphs, then we do not apply them. A *deletion of an isolate vertex* is to remove an isolate vertex. A graph H is a *minor* of G or G has an H*-minor*, if H is obtained from G by edge contractions, edge deletions and deletions of isolate vertices. The three operations are called *minor operations*. If we deal with only connected graphs, we never need to use the third operation under careful use of the first two. Therefore, we only use the first two as minor operations. A graph G is H*-minor-free* if G does not have H as a minor. A vertex of degree k is a k*-vertex* and a cycle of length k is a k*-cycle*.

A *surface* $\mathbb{F}$ is a compact 2-dimensional manifold without boundary, and a *map* G on $\mathbb{F}$ is a fixed embedding of a simple graph on $\mathbb{F}$. For a 2-cell region R in G with boundary walk W, *inner* vertices and edges are ones in R but not contained in W. (An *inner* vertex or edge of a plane graph is similarly defined as one not contained in the boundary walk of the infinite face.) A *triangulation* (resp., *quadrangulation*) on $\mathbb{F}$ is a map of a simple graph on $\mathbb{F}$ such that each face is bounded by a 3-cycle (resp., 4-cycle). A simple closed curve ℓ on a non-spherical surface is *essential* if ℓ does not bound a 2-cell on the surface. The *representativity* of a map G on $\mathbb{F}$ is defined as the minimum number of intersecting points of G and a closed curve ℓ, where ℓ ranges over all essential

2010 Mathematics Subject Classification: 05C10
Key words and phrases: Quadrangulation, Planar graph, Minor, Projective plane

simple closed curves on $\mathbb{F}$. A map is *k-representative* if it has representativity at least k.

We introduce two local operations for quadrangulations. Suppose that a quadrangulation G has a 2-vertex v with neighbors a_1 and a_2. A *2-vertex deletion* of v is to delete v from G, as shown in the left of Figure 1. Suppose that a quadrangulation G has a 3-vertex x with neighbors a_1, a_2, a_3. Let $a_1b_1a_2b_2a_3b_3$ be the boundary walk of the region consisting of all faces incident to x. A *hexagonal contraction* of x at a_2, a_3 is to delete x, identify a_2 and a_3, and replace a pair of multiple edges a_2b_2, a_3b_2 with a single edge, as shown in the right of Figure 1. We do not apply these reductions if the resulting graph is not simple. It is easy to see that each of the two operations is obtained by a sequence of edge contractions and edge deletions. For these operations, the inverse operations are a *2-vertex addition* and a *hexagonal splitting*, respectively.

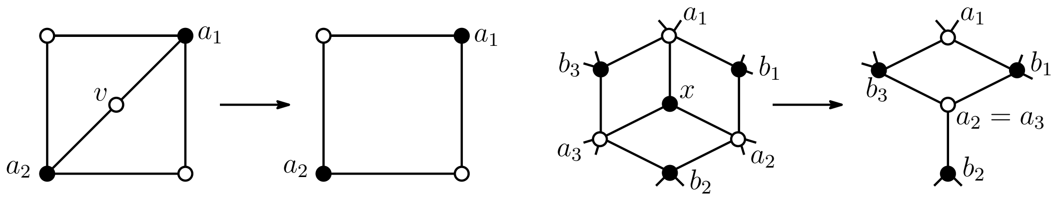


Figure 1 The 2-vertex deletion and the hexagonal contraction

Bau, Matsumoto, Nakamoto and Zheng proved the following result for quadrangulations on the sphere by those operations.

THEOREM 1. (Bau et al. [3]) *Every quadrangulation of order at least 5 on the sphere can be reduced to a 4-cycle C_4 by a sequence of 2-vertex deletions and hexagonal contractions.*

Matsumoto, Nakamoto and Yonekura obtained an analogy for quadrangulations on the projective plane by the same operations, where $K_{4,4}^{--}$ is a quadrangulation on the projective plane whose graph is $K_{4,4}$ with two independent edges deleted, as follows.

THEOREM 2. (Matsumoto et al. [10]) *Every bipartite quadrangulation on the projective plane can be reduced to either $K_{3,4}$ or $K_{4,4}^{--}$ by a sequence of 2-vertex deletions and hexagonal contractions, through bipartite ones. Every non-bipartite quadrangulation can be reduced to K_4 by those operations, through non-bipartite ones. (Figure 2 shows two bipartite quadrangulation isomorphic to $K_{3,4}$ and $K_{4,4}^{--}$, respectively.)*

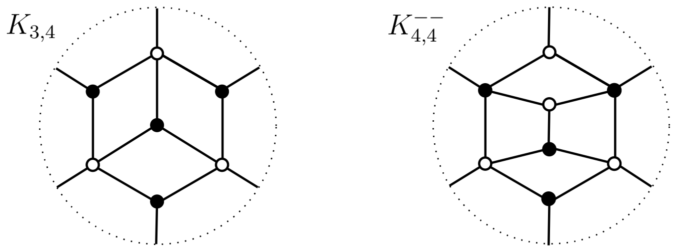


Figure 2 The graphs $K_{3,4}$ and $K_{4,4}^{--}$

We have already known several theorems generating quadrangulations [2, 4, 15]. However, Theorem 2 has given a new direction for a minor relation of quadrangulations. Since each of the 2-vertex deletion and the hexagonal contraction is obtained by a combination of minor operations, Theorem 2 asserts that every bipartite quadrangulation on the projective plane contains $K_{3,4}$ or $K_{4,4}^{--}$ as a minor, and that every non-bipartite quadrangulation on the projective plane contains K_4 as a minor. (For K_4- and K_5-minors in quadrangulations on surfaces, see a related result [5].)

In this paper, we consider an application of Theorem 2 to give a forbidden structure of a bipartite quadrangulation on the projective plane to have a $K_{3,4}$-minor.

Let G be a graph and let H be a subgraph of G. An *H-bridge* in G is a subgraph of G which is either an edge not in H but with both ends in H, or a connected component of $G - V(H)$ together with all edges which have one end in this component and the other end in H. Let B be an H-bridge. The elements of $V(B) \cap V(H)$ are called its *attachments*. (The above two definitions can be found in [11, Page 7].)

Definition 3. (Q-structure) *A quadrangulation G on the projective plane has a Q-*structure *when G has a subgraph Q (shown in Figure 3) with faces $R_1, \ldots, R_5$ such that: R_1 and R_2 are hexagonal regions of G, R_4 and R_5 are faces of G (as well as Q) and every Q-bridge in R_3 has at most two attachments.*

Remark. Note that if G is 3-connected, then R_3 has no Q-bridge, and that $K_{4,4}^{--}$ has a Q-structure which is obtained from the map shown in Figure 3 by adding two edges uv and $u'v'$. For examples, the quadrangulation in the left of Figure 4 has a Q-structure depicted by bold edges, but the right of Figure 4 does not since it has Q as a subgraph but the Q-bridge in R_3 has three attachments a, c and d.

The following is our main result in this paper.

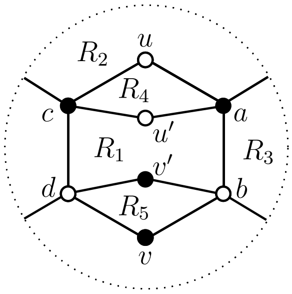
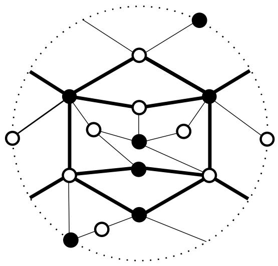


Figure 3 The Q-structure



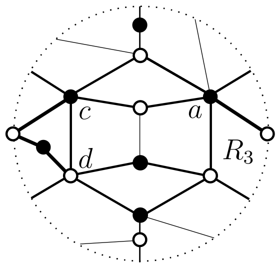


Figure 4 Examples of bipartite quadrangulations with (or without) a Q-structure

Theorem 4. *Let G be a bipartite quadrangulation on the projective plane. If G does not have a Q-structure, then G has a $K_{3,4}$-minor.*

In Section 2, we introduce some lemmas, and then we prove Theorem 4 by those lemmas. In Section 3, we give some remarks for Theorem 4.

2. Proof of the theorem

We first prepare two lemmas to prove Theorem 4.

Lemma 5. *Let Q be a simple plane quadrangulation with outer cycle $C = v_0v_1v_2v_3$. If every inner vertex of Q has at most two disjoint paths to C, then* $\deg(v_i) = \deg(v_{i+2}) = 2$ *for some i.*

Proof. Observe that if $\deg(v_i) = \deg(v_{i+1}) = 2$ for some i, then Q has no inner vertex, and hence $\deg(v_i) = 2$ for $i = 0, 1, 2, 3$. Thus, if the lemma does not hold, then we may suppose that v_0, v_1 and v_2 have degree at least 3.

Let $v_0, x_1, \dots, x_k, v_2$ be the neighbors of v_1 appearing in the rotation of v_1 in this order, where $k \geq 1$ and we note that each x_i is an inner vertex since Q is bipartite. Since $\deg(v_0) \geq 3$, we can find a quadrilateral face $v_0v_1x_1y_0$ for some inner vertex y_0. (Note that if y_0 is contained in C, then $y_0 = v_3$, since Q is bipartite. In this case, $\deg(v_0) = 2$, a contradiction.) Similarly, since $\deg(v_2) \geq 3$, we can find a quadrilateral face $v_1v_2y_kx_k$ for some inner vertex y_k. Moreover, if $k \geq 2$, take $k-1$ faces $x_iv_1x_{i+1}y_i$ for some y_i, for $i = 1, \ldots, k-1$. Note that $y_0, \ldots, y_n$ are not necessarily distinct, and that we might have $y_i = v_3$ for some $i \in \{1, \ldots, k-1\}$.

Suppose that $y_p = v_3$ for some $p \in \{1, \ldots, k-1\}$, where we take p as small as possible. Then x_p has three neighbors y_{p-1}, v_1, v_3. By the definition of y_p, all vertices in the walk $y_0x_1 \cdots x_{p-1}y_{p-1}x_p$ are inner vertices, and hence we can find a path P from y_{p-1} to v_0 without intersecting v_1 and v_3. Hence x_p has three internally disjoint paths from x_p to v_1, v_3 and v_0 along the path P (i.e., the three paths share x_p only). Suppose that all y_i are inner vertices. If y_0 is distinct from all other y_i for $i = 1, \ldots, k$, then by taking a path P from y_1 to v_2 in the walk $y_1x_2 \cdots x_ky_kv_2$, we can take three internally disjoint paths from x_1 to v_0, v_1 and v_2 along the path P. If y_0 coincides with other y_i's, then we let $y_p = y_0$ with p the largest. Let P' be a path from the vertex $y_0 = y_p$ to v_2 in the walk $y_px_{p+1}y_{p+1} \cdots y_kv_2$, which intersects none of v_0, v_1, v_3. Then the vertex $y_0 = y_p$ has three internally disjoint paths from y_0 to v_0, v_1 and v_2 along the path P'.

Consequently, we can find three internally disjoint paths from some inner vertex, a contradiction. ■

Lemma 6. *Let Q be a bipartite plane graph with outer 6-cycle $C = v_0v_1v_2v_3v_4v_5$ such that every inner face is quadrilateral. Suppose that Q has neither edge v_0v_3 nor v_2v_5. Then Q has an internal (v_1, v_4)-path, i.e., a path P joining v_1 and v_4 such that $V(P) \cap V(C) = \{v_1, v_4\}$, unless Q has an inner face incident to both v_0 and v_2, or an inner face incident to both v_3 and v_5.*

Proof. Adding two new vertices x to v_0, v_1, v_2, and y to v_3, v_4, v_5, we get a new plane graph Q'. If we can find a path P' in Q' joining x and y but intersecting none of v_0, v_2, v_3 and v_5, then we will have a required path from v_1 to v_4. By using Menger's theorem, we find three internally disjoint paths from x to y. Since Q' is a plane graph with $\deg(x) = \deg(y) = 3$, one of the three paths is a required path. If Q' has no such three paths, then Q' has a separator separating x and y. Since Q' is a plane graph, we can find a face f of Q which contains (i) v_0, v_2, (ii) v_3, v_5, (iii) v_0, v_3 or (iv) v_2, v_5. Since Q is bipartite and each inner face is quadrilateral, we have an edge v_0v_3 in (iii), and an edge v_2v_4 in (iv). However, by the assumption, these cases do not happen. In the cases (i) and (ii), we have the conclusion. ■

Now we shall prove Theorem 4.

Proof of Theorem 4. Let G be a bipartite quadrangulation on the projective plane with no Q-structure. For contradiction, we suppose that G has no $K_{3,4}$-minor. By Theorem 2, G can be reduced to $K_{4,4}^{--}$ by 2-vertex deletions and hexagonal contractions. That is, there exists a sequence of bipartite quadrangulations $G_0, G_1, \dots, G_k$ with $G = G_k$ such that

(i) $G_0 \cong K_{4,4}^{--}$, and
(ii) G_{i+1} is obtained from G_i by either a single 2-vertex addition or a single hexagonal splitting, for each $i \in \{0, 1, \cdots, k-1\}$.

By Remark of Definition 3, $K_{4,4}^{--}$ has a Q-structure, and hence, there exists an index k such that G_k contains a Q-structure but G_{k+1} does not. If we can prove that G_{k+1} has a $K_{3,4}$-minor, then so does G. Therefore, we let $G = G_{k+1}$ and $G' = G_k$, and we shall prove that G has a $K_{3,4}$-minor. By the assumption, we note that G is obtained from G' by either a single 2-vertex addition or a hexagonal splitting, and that G' contains the Q-structure. We suppose that vertices in G' are labeled as in Figure 3, where for $i = 1, 2, 3, 4, 5$, R_i denotes the 2-cell region in G' with the Q-structure, or the plane subgraph of G contained in the interior and the boundary cycle of R_i. Moreover, we may be assumed that there is only one face corresponding to R_4 and R_5.

Case I. G is obtained from G' by a 2-vertex addition.

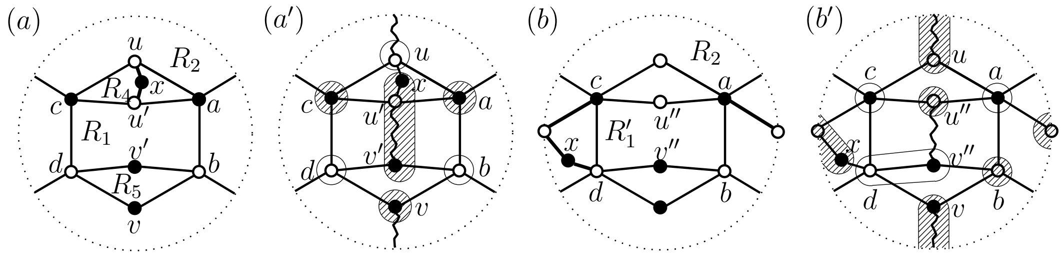


Figure 5 Case I

Let x be the added 2-vertex. We first consider the case when we put a 2-vertex in one of the two faces R_4 and R_5, say R_4, as in Figure 5 (a). In this case, the region R_1 has no face incident to both a and c. For otherwise, i.e., if R_1 has such a face f, then regarding f as R_4, we see that G still has a Q-structure, a contradiction. Therefore, by Lemma 6, R_1 (resp., R_2) has an internal (u', v')-path (resp., (u, v)-path). Thus, regarding four shaded parts and three circled parts showing Figure 5 (a') as seven vertices of $K_{3,4}$, we can find a $K_{3,4}$-minor in G.

We next consider the case when G has a Q-bridge B with two attachments

a and c and x is adjacent to a vertex of B and d as in Figure 5 (b). Observe that R_1 has a hexagonal region R_1' bounded by $abv''dcu''$ which has neither a face incident to both a and c nor a face incident to b and d, even if R_1 has such faces. Similarly, we may suppose that R_2 also has no such face. Hence, by Lemma 6, since R_1' (resp., R_2) has an internal (u'', v'')-path (resp., (u, v)-path), we can find a $K_{3,4}$-minor in G, as in Figure 5 (b').

Case II. G is obtained from G' by a hexagonal splitting.

Let $\mathcal{A}$ be the submap of G' consisting of a quadrilateral face $f = xa_2a_1a_3$ and an edge xy, shown in the left of Figure 6 which is applied a hexagonal splitting to get G.

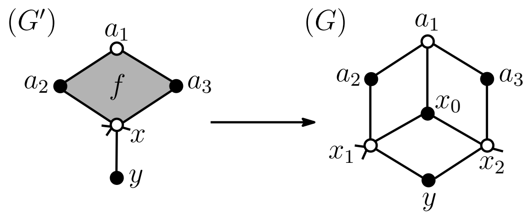


Figure 6 The submap $\mathcal{A}$ and a hexagonal splitting

Observe that if $\mathcal{A}$ satisfies one of the following conditions (see Figure 7):

1. $V(\mathcal{A}) \subseteq V(R_i)$ and $E(\mathcal{A}) \subseteq E(R_i)$ for some $i \in \{1, 2, 3\}$, or
2. f and R_4 (or R_5) coincide and xy is on the boundary of R_i for some $i \in \{1, 2, 3\}$,

then the resulting map by the hexagonal splitting of $\mathcal{A}$ still contains a Q-structure.

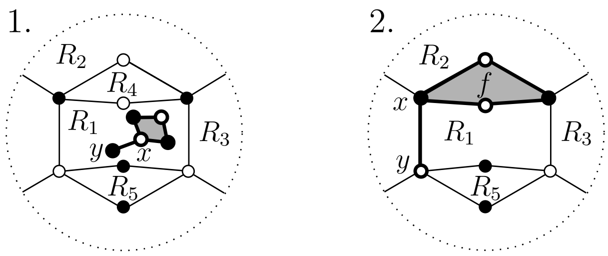


Figure 7 Examples for two conditions

Hence we suppose that $\mathcal{A}$ satisfies none of the above conditions, and so by symmetry, it suffices to consider the following three subcases.

Subcase 1. f and R_4 coincide and xy is an inner edge of R_i for some $i \in \{1, 2, 3\}$.

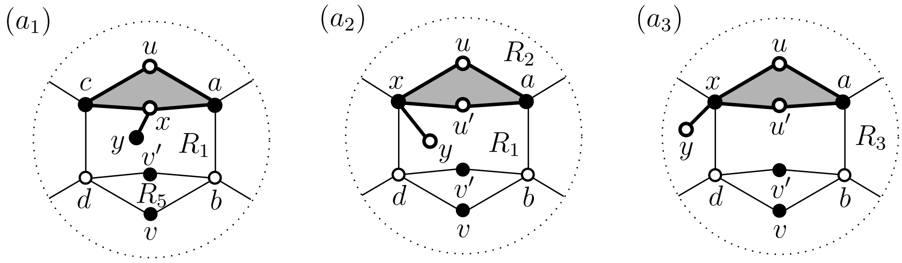


Figure 8 Subcase 1

We have three cases (see Figure 8): Now the edge xy is by symmetry contained in either R_1 as shown in (a_1) and (a_2) or alternatively R_3 as shown in (a_3). However, we consider only the cases (a_1) and (a_3), since the hexagonal splitting in the case (a_2) does not break the Q-structure.

Focusing on the case (a_1), we consider the resulting graph G by applying a hexagonal splitting to the $\mathcal{A}$ in G'. Note that $V(G) = (V(G') \setminus \{x\}) \cup \{x_0, x_1, x_2\}$, where x_i's are vertices corresponding to them shown in Figure 6. In this case, x_0 is adjacent to x_1, x_2 and the common neighbor of a and c on the boundary of f in G' (as well as G) which is not deleted by the above hexagonal splitting. (Note that x is excluded.) Moreover, $x_1c, x_2a \in E(G)$ and by Lemma 6, G has an internal path in R_1 joining x_1 (or x_2) and the common neighbor of b and d shared by boundaries of R_1 and R_5. Therefore, by symmetry, we can find a $K_{3,4}$-minor as shown in Figure 11 (A). Similarly, for the case (a_3), we can find a $K_{3,4}$-minor by Lemma 6, as shown in Figure 11 (D).

Subcase 2. f is contained in R_1 and $xy \notin E(R_1)$.

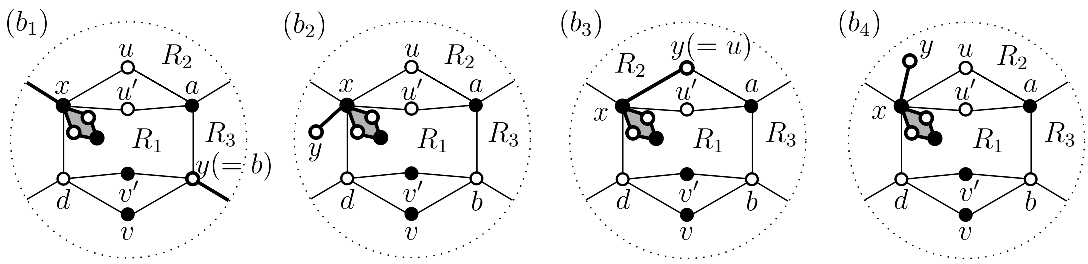


Figure 9 Subcase 2

By the assumption, we have $x \in \{a, b, c, d\}$, say $x = c$. Then we consider the following four cases (see Figure 9): (b_1) $xy = cb$, (b_2) xy is an inner edge of R_3,

(b_3) $xy = cu$ and (b_4) xy is an inner edge of R_2. Observe that R_1 and R_2 admits an internal (u', v')-path and (u, v)-path, respectively. In the case (b_2), we can find an internal (x, a)-path in R_3, where $c = x$. Therefore, for the case (b_2), G has a $K_{3,4}$-minor as shown in Figure 11 (B), and in Figure 11 (C) for other cases. (Note that the hexagonal splitting does not break the (u', v')-path as shown in Figure 11.)

Subcase 3. f is contained in R_3 and $xy \notin E(R_3)$.

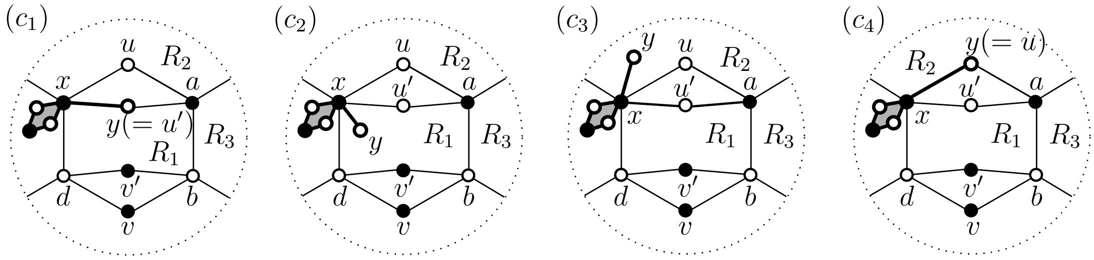


Figure 10 Subcase 3

Similarly to Subcase 2, we have $x = c$. Hence we have four cases (see Figure 10): (c_1) $xy = cu'$, (c_2) xy is an inner edge of R_1, (c_3) xy is an inner edge of R_2 and (c_4) $xy = cu$. Observe that (c_2) and (c_3) are equivalent by symmetry. Similarly to the above subcases, using internal paths, we can find a $K_{3,4}$-minor as shown in Figure 11 (B) (resp., (E)) for the cases (c_1), (c_2) and (c_3) (resp., (c_4)).

Therefore, we can find a $K_{3,4}$-minor in G in all possible cases, and hence we are done. ■

3. Some Remarks

For triangulations on surfaces, we can find several results on a characterization of ones containing a complete graph as a minor. Such results might affirmatively support the well-known Hadwiger's Conjecture, which states that every graph with no K_k-minor is $(k-1)$-colorable, and has been solved for $k \leq 6$, but is still open for $k \geq 7$. It is easy to see that every triangulation on any surface has a K_4-minor, and that every triangulation on any non-spherical surface has a K_5-minor, where we note that every graph on the sphere cannot have a K_5-minor, by the Kuratowski-Wagner's result [18]. Using the complete list of "minimal triangulations" on surfaces with respect to the minor relations, called *irreducible*

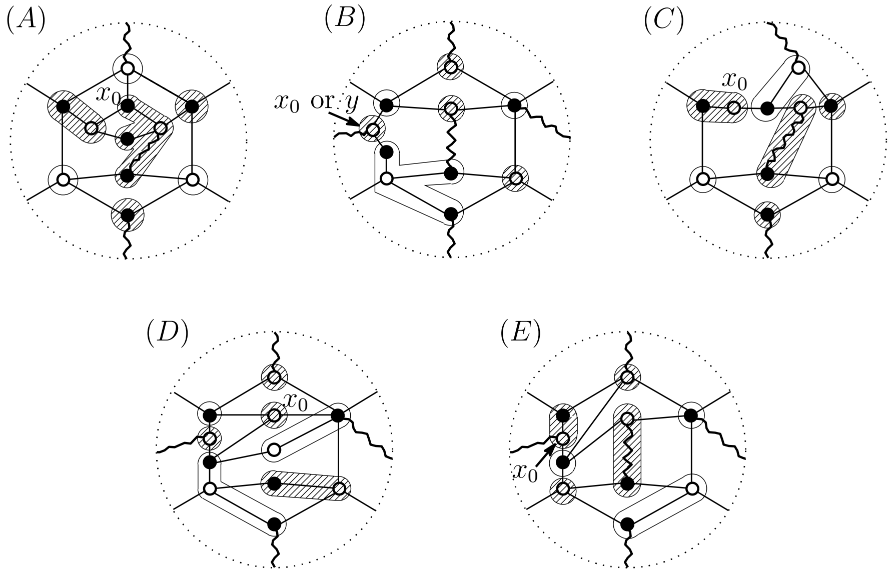


Figure 11 Five cases with a $K_{3,4}$-minor

triangulations [1, 7, 8, 16, 17], Mukae et al. characterized triangulations with no K_6-minor [6, 12, 13, 14].

In this paper, we have given a sufficient condition for a bipartite quadrangulation on the projective plane $\mathbb{P}$ to have a $K_{3,4}$-minor by a single forbidden structure called the Q-structure, using the fact that every bipartite quadrangulation on $\mathbb{P}$ contains $K_{3,4}$ or $K_{4,4}^{--}$ as a minor [10]. So we wonder if avoiding the Q-structure can also be a necessary condition for a bipartite quadrangulation on $\mathbb{P}$ to have a $K_{3,4}$-minor.

For this problem, Maharry and Slilaty [9] gave a constructive characterization of projective planar maps with no $K_{3,4}$-minor, using the notion of a "patch graph". A *patch* is a plane graph with quadrilateral outer cycle. A map G on $\mathbb{P}$ is a *patch graph* if it is obtained by the following procedures:

- Step 1: Start with the *initial map* (a) shown in Figure 12, where the quadrilateral region P_0 is a patch.
- Step 2: Apply *patch operations*, each of which replaces a single patch with a configuration H or Y containing two or one patch(es) ((b), (c) in Figure 12, where the regions P_1, P_2 and P_3 are patches). Repeat this operation several times.
- Step 3: Finally, replace each of the remaining patches with one of the two configurations X and I ((d), (e) in Figure 12), called the *terminal patch*.

We note that a patch graph is always a triangulation on $\mathbb{P}$, and *almost* 4-*connected* (i.e., it is 3-connected and every 3-cut separates the graph into one with an isolated vertex). Actually, Maharry and Slilaty proved that an almost 4-connected map $G \neq K_6$ on $\mathbb{P}$ has no $K_{3,4}$-minor if and only if G is a subgraph of a patch graph.

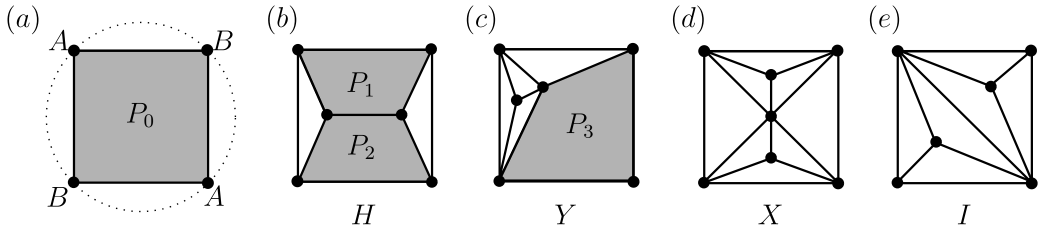


Figure 12 The initial patch graph (a) and four patch operations H, Y, X and I

Let T be a patch graph. Assign black or white to each vertex of T so that no triangular face has three vertices of the same color in its three corners, and that the subgraph of T induced by the black-white edges is simple. We see that T has such a vertex-coloring only if the first patch operation applied to the initial patch for constructing T is H. Deleting the edges of T joining the same color, we obtain a spanning bipartite quadrangulation G of T. Then we can see that G has a Q-structure since the first patch operation is H. Hence by Maharry-Slilaty's theorem, if G does not have a Q-structure, then G has a $K_{3,4}$-minor. This is our main claim in this paper, but on the other hand, we would like to know whether the converse is true or not. However, we have a counterexample G shown in the left of Figure 13, which has a Q-structure but also has a $K_{3,4}$-minor, as in the right.

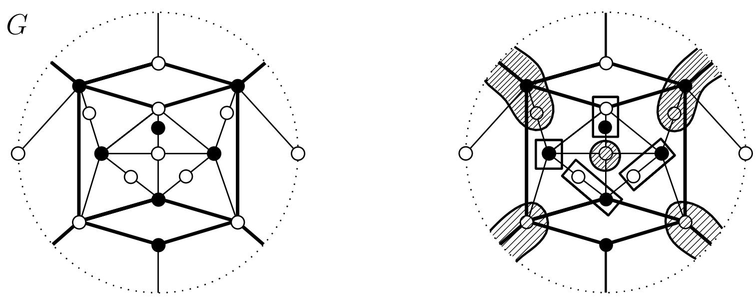


Figure 13 Bipartite quadrangulation G on the projective plane having a $K_{3,4}$-minor and a Q-structure

Let T be the patch graph which is obtained from the initial map by the single patch operation H. Then T has exactly two patches, say P_1 and P_2. From T, we can get infinitely many patch graphs $T_1, T_2, \ldots$ by applying patch operations to P_1 and P_2. Unless both P_1 and P_2 are replaced with the terminal patches, each T_i

contains a spanning bipartite quadrangulation, say G_i, and by Maharry-Slilaty's theorem, all G_i's have no $K_{3,4}$-minor. However, the bipartite quadrangulation G in Figure 13 is constructed as a spanning subgraph of a triangulation obtained from T by replacing P_1 and P_2 with some dense graphs, and hence the failure of the patch operations in T causes the existence of a $K_{3,4}$-minor in G.

Finally, we would like to know whether we can obtain a necessary and sufficient condition for a bipartite quadrangulation on $\mathbb{P}$ to have a $K_{3,4}$-minor, by analyzing the structure more carefully than what we did in this paper. Even if it can be, then the argument will be so long and complicated, and moreover, it might be nothing but tracing Maharry-Slilaty's proof through bipartite quadrangulations.

References

[1] D. W. Barnette, Generating the triangulations of the projective plane, *J. Combin. Theory Ser. B*, **33** (1982), 222–230.

[2] V. Batagelj, An inductive definition of the class of 3-connected quadrangulations of the plane, *Discrete Math.*, **78** (1989), 45–53.

[3] S. Bau, N. Matsumoto, A. Nakamoto, and L. Zheng, Minor relations for quadrangulations on the sphere, *Graphs Combin.*, **31** (2015), 2029–2036.

[4] G. Brinkmann, S. Greenberg, C. Greenhill, B. D. McKay, R. Thomas, and P. Wollan, Generation of simple quadrangulations of the sphere, *Discrete Math.*, **305** (2005), 33–54.

[5] G. Fijavž, and A. Nakamoto, Odd minors in even embeddings on surfaces, *Discrete Math.*, **339** (2016), 165–178.

[6] K. Kawarabayashi, R. Mukae, and A. Nakamoto, K_6-minors in triangulations on the Klein bottle, *SIAM J. Discrete. Math.*, **23** (2008), 96–108.

[7] S. A. Lawrencenko, Irreducible triangulations of the torus, *J. Sov. Math.*, **51** (1990), 2537–2543, translation from *Ukrain. Geom. Sb.* **30** (1987), 52–62.

[8] S. Lawrencenko, and S. Negami, Irreducible triangulations of the Klein bottle, *J. Combin. Theory Ser. B*, **70** (1997), 265–291.

[9] J. Maharry, and D. Slilaty, Projective-planar graphs with no $K_{3,4}$-minor, *J. Graph Theory*, **70** (2012), 121–134.

[10] N. Matsumoto, A. Nakamoto, and S. Yonekura, Minor relation for quadrangulations on the projective plane, *Discrete Appl. Math.*, **209** (2016), 296–302.

[11] B. Mohar, and C. Thomassen, *Graphs on surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, 2001.

[12] R. Mukae, and A. Nakamoto, K_6-minors in triangulations and complete quadrangulations, *J. Graph Theory*, **60** (2009), 302–312.

[13] R. Mukae, A. Nakamoto, Y. Oda, and Y. Suzuki, K_6-minors in triangulations on the nonorientable surface of genus 3, *Graphs Combin.*, **26** (2010), 559–570.

[14] R. Mukae, A. Nakamoto, Y. Oda, and Y. Suzuki, K_6-minors in triangulations on the nonorientable surface of genus 4, preprint.

[15] A. Nakamoto, Generating quadrangulations of surfaces with minimum degree at least 3, *J. Graph Theory*, **30** (1999), 223–234.

[16] A. Nakamoto, Y. Oda, and K. Ota, K_6-minors in triangulations on the double torus, *Congr. Numer.*, **188** (2007), 150–160.

[17] E. Steinitz, and H. Rademacher, Vorlesungenüber die Theorie der Polyeder, Springer, Berlin, 1934.

[18] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.*, **114** (1937), 570–590.

Naoki Matsumoto
Faculty of Science and Technology,
Department of Computer and Information Science,
Seikei University,
3-3-1 Kichijoji-Kitamachi, Musashino-Shi,
Tokyo 180-8633, Japan
E-mail: naoki-matsumoto@st.seikei.ac.jp,

Atsuhiro Nakamoto
Faculty of Environment of Information Science,
Yokohama National University
79-2 Tokiwadai, Hodogaya-Ku,
Yokohama 240-8501, Japan
E-mail: nakamoto@ynu.ac.jp,

Shin-ichi Yonekura
Graduate School of Environment of Information Science,
Yokohama National University
79-2 Tokiwadai, Hodogaya-Ku,
Yokohama 240-8501, Japan
E-mail: yonekura-shinichi-by@ynu.jp