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TRUNCATED 3-REGULAR CONNECTED GRAPHS ARE DISTINGUISHING 3-COLORABLE

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Abstract. A graph is said to be *distinguishing k-colorable* if it has a proper k-coloring such that no automorphism other than the identity map preserves the colors. A *truncated* 3*-regular graph* is one obtained from a 3-regular graph by replacing a small part around each vertex with a triangle. We shall show that any truncated 3-regular connected graph is distinguishing 3-colorable.

Introduction

A simple graph G is said to be *distiguishing k-colorable* if there is a k-coloring $c : V(G) \to \{1, \dots, k\}$ such that no automorphism of G other than the identity map preserves the colors given by c. Such a coloring is called a *distinguishing k-coloring* of G. The *distinguishing chromatic number* of G is defined as the minimum number k such that G is distinguishing k-colorable and is denoted by $\chi_D(G)$. These notions for abstract graphs have been introduced in [1].

In general, the gap between the chromatic number and the distinguishing chromatic number of a graph can be arbitrarily large; $\chi(K_{n,n}) = 2 < \chi_D(K_{n,n}) = n+n$ for example. However, if we embed a graph G on a closed surface and restrict its automorphisms to its map automorphisms, which exhibit the symmetry over the surface, then the distinguishing chromatic number of a map defined similarly is so close to its chromatic number, as is shown in [2, 3, 4, 5, 6, 7, 8, 9, 10]. This suggests that 3-regular maps on closed surfaces might be distinguishing 3-colorable with few exceptions; there actually exist some exceptions and the distinguishing chromatic number of the cube is equal to 4 for example.

As one of evidences supporting it, Negami [4] has already shown that 3-regular maps on closed surfaces have distinguishing 4-coloring with color 4 used only for one vertex, with three exceptions. Recently, Negami and Sugihara [5] have proved that any truncated 3-regular polyhedron is distinguishing 3-colorable, developing a method to discuss it in more general situation from a point of view of topological

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graph theory. In this paper, we shall prove their result in a combinatorial way and generalize it to a theorem for "truncated 3-regular graphs" defined below.

Let $\bar{G}$ be a 3-regular graph. *Truncation* of a vertex v in $\bar{G}$ is to replace a small part around v with a triangle xyz; remove v as a point and cut off a sufficiently short segment on each edge incident to v. The endpoints of such three segments will be x, y and z, which form a triangle with three short edges xy, yz and zx added. A *truncated* 3*-regular graph* G is one obtained from a 3-regular graph $\bar{G}$ by truncating each of all vertices in $\bar{G}$. This is actually 3-regular, but the reason why we call it a truncated "3-regular graph" is that its underlying graph $\bar{G}$ is a "3-regular graph". Note that if we contract each of the triangles in G to a point, then we will recover $\bar{G}$ itself.

Theorem 1. *Any truncated* 3*-regular connected graph has a distinguishing* 3*-coloring.*

This is our main theorem in this paper, which states a fact on the distinguishing chromatic number of an abstract graph. Our arguments below do not need any assumption on topological properties, embeddings or the polyhedrality of maps, so well as in Negami and Sugihara's result [5].

1. Modifying colorings

Let G be a truncated 3-regular connected graph, that is, one obtained from another 3-regular connected graph $\bar{G}$ by truncating each vertex v; denote the triangle corresponding to v by Δ_v. Since G is 3-regular and is not isomorphic to K_4, it has a 3-coloring $c : V(G) \to \{1, 2, 3\}$ by Brooks Theorem, which is well known and can be found in a standard texbook of graph theory.

We identify the color set $\{1, 2, 3\}$ with $\mathbb{Z}_3$ to use the addition of integers modulo 3. Then we can give a direction from x to y for each edge xy of G so that $c(y) \equiv c(x) + 1 \pmod 3$. Since three vertices forming $\Delta_v = xyz$ have three different colors, one orientation along the cycle Δ_v is induced by directions of edges xy, yz and zx. There are other three edges incident to x, y and z. If one of them has the direction from Δ_v to another vertex not on Δ_v, we assign "+" to it; otherwise, assign "−". Counting the number of "+" and "−" around Δ_v, we call Δ_v *Type* $(+ + -)$ for exampe. There are four types, namely $(+ + +)$, $(+ + -)$, $(+ - -)$ and $(- - -)$.

Here, we shall consider three modifications of the 3-coloring $c : V(G) \to \mathbb{Z}_3$ of G. We may assume that $c(y) \equiv c(x) + 1$ and $c(z) \equiv c(x) - 1 \pmod 3$. Let x', y' and z' be the neighbors of x, y and z, respecively, not lying on the triangle $\Delta_v = xyz$. We can define another 3-coloring $c' : V(G) \to \mathbb{Z}_3$ by $c'(s) \equiv -c(s)$

(mod 3) and call it the *reverse* of c. The directions of all edges induced by c' are opposite to those by c.

Suppose that Δ_v is of Type $(+ + +)$. Then x', y' and z' have three different colors since $c(x') \equiv c(x)+1$, $c(y') \equiv c(y)+1$ and $c(z') \equiv c(z)+1 \pmod 3$. Change the colors of only x, y and z in c by $c_-(x) \equiv c(x) - 1$, $c_-(y) \equiv c(y) - 1$ and $c_-(z) \equiv c(z) - 1 \pmod 3$ in order to define another 3-coloring $c_- : V(G) \to \mathbb{Z}_3$. Since $c_-(x') = c(x') \equiv (c_-(x) + 1) + 1 \equiv c_-(x) - 1 \pmod 3$ and so one, the type of Δ_v changes from $(+ + +)$ to $(- - -)$. We call this modification the *rotation* along Δ_v. Similarly, if Δ_v of Type $(- - -)$, then we can define the rotation along Δ_v to get another 3-coloring $c_+ : V(G) \to \mathbb{Z}_3$ with $c_+(s) \equiv c(s) + 1 \pmod 3$ $(s \in \{x, y, z\})$, which makes Δ_v be of Type $(+ + +)$.

We shall modify the 3-coloring of G, step by step, to adapt it for use in the later proof of Theorem 1.

Lemma 2. *Any truncated 3-regular connected graph has a 3-coloring such that there is at least one triangle of Type $(+ + +)$.*

Proof. Let G be a truncated 3-regular connected graph and choose a triangle $\Delta_v = xyz$ which corresponding to a vertex v in the underlying 3-regular graph $\bar{G}$. Let G_v be the graph obtained from G by shrinking Δ_v to a vertex v. Since G_v is 3-regular and is not isomorphic to K_4, G_v has a 3-coloring $c_v : V(G_v) \to \mathbb{Z}_3$. We shall construct a 3-coloring $c : V(G) \to \mathbb{Z}_3$ of G, modifying c_v as follows.

Let x', y' and z' be the neighbors of x, y and z not lying in Δ_v, respectively, which are the three neighbors of v in G_v. We may assume that $c_v(v) = 0$. First suppose that all of x', y' and z' have the same color, say 1, in the coloring c_v. Consider the subgraph of G_v induced by the vertices colored by 1 and 2 and let $H_{1,2}$ be its component containing z', called the $(1, 2)$-*Kempe chain* from z'.

Since each triangle in G_v consists of three vertices with three colors 0, 1 and 2, $H_{1,2}$ forms a path which goes along edges lying and not lying on triangles alternately. Each of edges in $H_{1,2}$ lying on triangles is colored by 1 and 2 in this order while each of those not on triangles are colored by 2 and 1. This implies that $H_{1,2}$ contains neither x' nor y' and we can exchange colors 1 and 2 along the path $H_{1,2}$ to obtain a new 3-coloring of G_v. Thus, we may assume that $c_v(x') = c_v(y') = 1$ and $c_v(z') = 2$ after modifying colors.

Now consider the $(1, 0)$-Kempe chain $H_{1,0}$ from x' in $G_v - v$, but not in G_v. As in the previous, $H_{1,0}$ forms a path and contians neither y' nor z'. Thus, we can exchange colors 1 and 0 along $H_{1,0}$ to obtain a 3-coloring $c : V(G_v - v) \to \mathbb{Z}_3$ with $c(x') = 0$, $c(y') = 1$ and $c(z) = 2$. This 3-coloring c of $G_v - v = G - \Delta_v$ extends to a 3-coloring of G so that $c(x) \equiv c(x') - 1$, $c(y) \equiv c(y') - 1$ and $c(z) \equiv c(z') - 1 \pmod 3$. Therefore, Δ_v is of Type $(+ + +)$ in the 3-coloring $c : V(G) \to \mathbb{Z}_3$

finally obtained. ■

Lemma 3. *Any truncated 3-regular connected graph has a 3-coloring such that there is exactly one triangle of Type* $(+++)$.

Proof. Let $c : V(G) \to \mathbb{Z}_3$ be a 3-coloring of G which has a triangle Δ_{v_0} of Type $(+++)$, which exists by Lemma 2. Put $V_0 = \{v_0\}$ and let V_i $(i \geq 1)$ be the subset of $V(\bar{G})$ which consists of the vertices w such that the distance between v_0 and w in $\bar{G}$ is equal to i. We say that the direction from V_{i-1} to V_i is "downward".

Let n_i $(i \geq 1)$ denote the number of edges uv in $\bar{G}$ such that $u \in V_{i-1}$, $v \in V_i$ and that Δ_v is joined to Δ_u by a directed edge in G, which induces the "upward" direction from v to u. Consider the *complexity* $n(c) = (n_1, n_2, \ldots)$ for a 3-coloring $c : V(G) \to \mathbb{Z}_3$ with the order "$\leq$" defined as:

$$(n_1, n_2, \ldots) \leq (n'_1, n'_2, \ldots) \iff n_1 = n'_1, \ \ldots, \ n_{i-1} = n'_{i-1} \\ \text{and} \quad n_i \leq n'_i \text{ for some } i \geq 1.$$

We may assume that $n(c)$ is the minimum under this order taken over all 3-colorings of G with Δ_{v_0} being of Type $(+++)$.

Suppose that there is another triangle Δ_v of Type $(+++)$ with $v \in V_i$ for some $i \geq 1$. Then there is a vertex $u \in V_{i-1}$ such that Δ_v is joined to Δ_u by an upward edge. Two or three candidates for u may exist and Δ_v must be joined to all of their triangles by upward edges. Since Δ_v is of Type $(+++)$, we can apply the rotation along Δ_v so that such upward edges turn into downward edges. This would decrease the value of n_i by at least 1 although n_{i+1} might be bigger. That is, $n(c)$ would be smaller for the modified c, which is contrary to our assumption of the minimality of $n(c)$. Therefore, there is no triangle of Type $(+++)$ other than Δ_{v_0} for this c. ■

2. Proof of Theorem

We have already prepare what we need to prove our main theorem. It suffices to discuss the 3-coloring guaranteed by Lemma 3 with automorphisms, as follows:

Proof of Theorem 1. Let G be a truncated 3-regular connected graph. By Lemma 3, G has a 3-coloring $c : V(G) \to \mathbb{Z}_3$ such that there is exactly one triangle Δ_{v_0} of Type $(+++)$. It is easy to see that any automorphism $\tau \in \mathrm{Aut}(G)$ carries each of triangles created by truncation to one of them and that it preserves their types if it preserves the colors given by c. Since Δ_{v_0} is a unique triangle of Type $(+++)$, any color-preserving automorphism τ must fix Δ_{v_0}.

Let $v_0v_1\cdots v_k$ be any path in the underlying 3-regular graph $\bar{G}$ and assume that τ fixes all vertices lying on $\Delta_{v_0}\cup\cdots\cup\Delta_{v_{k-1}}$, where $\Delta_{v_i} = x_iy_iz_i$ is the triangle corresponding to v_i and is assumed to be joined to $\Delta_{v_{i-1}}$ by an edge x_iz_{i-1}. We have already known that $\tau(x_{k-1}) = x_{k-1}$, $\tau(y_{k-1}) = y_{k-1}$ and $\tau(z_{k-1}) = z_{k-1}$ and x_k is a unique neighbor of z_{k-1} whose image by τ has been never decided yet. Thus, we must have $\tau(x_k) = x_k$. Furthermore, the colors given by c force each of y_k and z_k to be carried to itself by τ since $\{c(x_k), c(y_k), c(z_k)\} = \{0, 1, 2\}$.

Since $\bar{G}$ is connected, any vertex v can be joined to v_0 by a path and we conclude inductively that $\tau(x) = x$, $\tau(y) = y$ and $\tau(z) = z$ for any triangle $\Delta_v = xyz$, using the path as above. Since the triangles Δ_v $(v \in V(\bar{G}))$ cover $V(G)$, τ must be the identity map over G and hence c is a distinguishing 3-coloring of G. ■

Note that we assume the simpleness of a truncated 3-regular graph implicitly, but its underlying 3-regular graph may have multiple edges without self-loops. For example. the triangular prism $K_3\times K_2$ can be regarded as a truncation of a 3-regular graph with two vertices joined by three multiple edges. This has a unique 3-coloring, up to exchanging colors, and such a 3-coloring is a distinguishing 3-coloring of $K_3 \times K_2$.

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