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ITERATIVE APPROXIMATION WITH ERRORS OF ZERO POINTS OF MONOTONE OPERATORS IN A BANACH SPACE

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Abstract. In this paper, we study an iterative scheme for a generalized resolvent of a monotone operator defined on a Banach space. We obtain an iterative approximation of a zero point of a monotone operator generated by the shrinking projection method with errors in a Banach space. Using our result, we discuss some applications.

1. Introduction

Let H be a real Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Then, the zero point problem is to find $u \in H$ such that

$$0 \in Au. \tag{1.1}$$

Such a $u \in H$ is called a zero point (or a zero) of A. The set of zero points of A is denoted by $A^{-1}0$. This problem is connected with many problems in Nonlinear Analysis and Optimization, for example, convex minimization problems, variational inequality problems, equilibrium problems and so on. A well-known method for solving (1.1) is the proximal point algorithm: $x_1 \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, \dots, \tag{1.2}$$

where $\{r_n\} \subset]0, \infty[$ and $J_{r_n} = (I + r_n A)^{-1}$. This algorithm was first introduced by Martinet [20]. In 1976, Rockafellar [26] proved that if $\inf_n r_n > 0$ and $A^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ defined by (1.2) converges weakly to a solution of the zero point problem. Later, many researchers have studied this problem; see [6, 7, 10, 14, 18, 19] and others.

On the other hand, Kimura [15] introduced the following iterative scheme for finding a fixed point of nonexpansive mappings by the shrinking projection method with errors in a Hilbert space:

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Theorem 1.1. ([15]) *Let C be a bounded closed convex subset of a Hilbert space H and let $T : C \to H$ be a nonexpansive mapping having a fixed point. Let $\{\epsilon_n\}$ be a nonnegative real sequence such that $\epsilon_0 = \limsup_n \epsilon_n < \infty$. For a given point $u \in H$, generate an iterative sequence $\{x_n\}$ as follows: $x_1 \in C$ such that $\|x_1 - u\| < \epsilon_1$, $C_1 = C$,*

$$C_{n+1} = \{z \in C : \|z - Tx_n\| \leq \|z - x_n\|\} \cap C_n,$$
$$x_{n+1} \in C_{n+1} \text{ such that } \|u - x_{n+1}\|^2 \leq d(u, C_{n+1})^2 + \epsilon_{n+1}^2$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq 2\epsilon_0.$$

Further, if $\epsilon_0 = 0$, then $\{x_n\}$ converges strongly to $P_{F(T)}u \in F(T)$.

We remark that the original result of the theorem above deals with a family of nonexpansive mappings, and the shrinking projection method was first introduced by Takahashi, Takeuchi and Kubota [28]. This result was extended to more general Banach spaces by Kimura [16] (see also Ibaraki and Kimura [9]). Recently, Ibaraki [7] study an iterative scheme for two different types of a resolvent of a monotone operator in a Banach space by the shrinking projection method with errors.

In this paper, we study an iterative scheme for a generalized resolvent of a monotone operator which is different type of two resolvents in [7] by the shrinking projection method with errors. We first obtain an iterative approximation of a zero point of a monotone operator generated by the shrinking projection method with errors in a Banach space. Using our result, we discuss some applications.

2. Preliminaries

Let E be a real Banach space with its dual E^*. We denote strong convergence and weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We also denote the weak* convergence of a sequence $\{x_n^*\}$ to x^* in E^* by $x_n^* \overset{*}{\rightharpoonup} x^*$. The closed ball is defined by $B_r := \{x \in E : \|x\| \leq r\}$, where r is a positive real number. A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ whenever $x, y \in E$ satisfies $\|x\| = \|y\| = 1$ and $x \neq y$. E is said to be uniformly convex if for each $\epsilon \in]0, 2]$, there exists $\delta > 0$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$ implies $\|x + y\|/2 \leq 1 - \delta$.

A Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S(E) = \{z \in E : \|z\| = 1\}$. In this case, the norm of E is said to be Gâteaux differentiable. The norm of E is said to be Fréchet differentiable if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$. The norm of E is said to be uniformly Fréchet differentiable (or E is said to be uniformly smooth) if the limit (2.1) is attained uniformly for $x, y \in S(E)$; see [27] for more details. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E converges strongly to x_0 whenever it satisfies $x_n \rightharpoonup x_0$ and $\|x_n\| \to \|x_0\|$.

The normalized duality mapping J from E into E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for each $x \in E$. An operator $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x-y, x^*-y^* \rangle \geq 0$ for all $(x, x^*), (y, y^*) \in A$. An operator A is said to be strictly monotone if $\langle x - y, x^* - y^* \rangle > 0$ for all $(x, x^*), (y, y^*) \in A$ $(x \neq y)$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is a maximal monotone operator, then the zero point set $A^{-1}0$ is closed and convex. If E is smooth, reflexive, and strictly convex, then a monotone operator A is maximal if and only if $R(J + rA) = E^*$ for each $r > 0$; see [4, 27] for more details.

We also know the following properties; see [4, 27] for more details.

(1) $Jx \neq \emptyset$ for each $x \in E$;
(2) if E is reflexive, then J is surjective;
(3) If E is smooth, then the duality mapping J is single valued;
(4) if E is strictly convex, then J is one-to-one and strictly monotone;
(5) if E is reflexive, smooth and strictly convex, then the duality mapping J_* on E^* is the inverse of J, that is, $J_* = J^{-1}$;
(6) E is reflexive, strictly convex, and has the Kadec-Klee property if and only if E^* has a Fréchet differentiable norm;
(7) if E uniformly smooth, then the duality mapping J is uniformly norm to norm continuous on each bounded set of E.

Let E be a smooth Banach space and consider the following function $V : E \times E \to \mathbb{R}$ defined by

$$V(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for each $x, y \in E$. We know the following properties (see [1, 11, 14] for more details):

(1) $(\|x\|-\|y\|)^2 \le V(x,y) \le (\|x\|+\|y\|)^2$ for each $x, y \in E$;

(2) $V(x,y)+V(y,x) = 2\langle x-y, Jx-Jy\rangle$ for each $x, y \in E$; (2.2)

(3) if E is additionally assumed to be strictly convex, then$V(x,y)=0$ if and only if $x=y$.

Let D be a nonempty subset of a smooth Banach space E and let T be a mapping from D into E. We denote by $F(T)$ the set of fixed points of T. A mapping $T : D \to E$ is said to be generalized nonexpansive [11] if $F(T) \neq \emptyset$ and

$$V(Tx,p) \le V(x,p)$$

for each $x \in D$ and $p \in F(T)$. A mapping $T : D \to E$ is said to be of firmly generalized nonexpansive type [12, 3] if

$$\langle (x-Tx)-(y-Ty), JTx-JTy\rangle \ge 0$$

for each $x, y \in D$. We know that T is a generalized nonexpansive if T is a firmly generalized nonexpansive type with $F(T) \neq \emptyset$ (see [12] for more details). Let E be a smooth Banach space. Then, a point p in C is said to be a generalized asymptotic fixed point [13] of T if C contains a sequence $\{x_n\}$ such that $Jx_n \overset{*}{\rightharpoonup} Jp$ and $Jx_n - JTx_n \to 0$. The set of generalized asymptotic fixed points of T is denoted by $\check{F}(T)$. We know the following results.

Lemma 2.1. ([13]) *Let E be a reflexive, smooth and strictly convex Banach space, let C be a nonempty subset of E such that JC is closed and convex and let $T : C \to C$ be a generalized nonexpansive mapping. Then $JF(T)$ is closed and convex*

In [13], Lemma 2.1 was proved in case of $C = E$. However, we know that Lemma 2.1 holds under the above condition by s similar proof of [13].

Theorem 2.2. ([3, 12]) *Let E be a reflexive and smooth Banach space whose dual space has a uniformly Gâteaux differentiable norm, let D be a nonempty subset of E and let $T : D \to E$ be a firmly generalized nonexpansive type mapping. If $F(T)$ is nonempty, then $\check{F}(T) = F(T)$.*

A mapping $R : E \to D$ is said to be sunny if

$$R(Rx + t(x-Rx)) = Rx$$

for all $x \in E$ and $t \ge 0$. A mapping $R : E \to D$ is said to be a retraction if $R^2 = R$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto D is uniquely determined if it exists; see [11]. Then, such a sunny generalized nonexpansive retraction of E onto D is denoted by R_D.

A nonempty subset D of E is called a sunny generalized nonexpansive retract of E if there exists a sunny generalized nonexpansive retraction of E onto D. Obviously, the set of fixed points of a sunny generalized nonexpansive retraction of E onto D is D; see [11] for more details. We recall the following results for sunny generalized nonexpansive retractions and sunny generalized nonexpansive retracts.

Lemma 2.3. ([11]) *Let D be a nonempty subset of a smooth and strictly convex Banach space E. Let R_D be a retraction of E onto D. Then R_D is sunny and generalized nonexpansive if and only if*

$$\langle x - R_D x, JR_D x - Jy\rangle \geq 0.$$

for each $x \in E$ and $y \in D$.

Theorem 2.4. ([18]) *Let D be a nonempty subset of a smooth, reflexive, and strictly convex Banach space E. Then, the following conditions are equivalent:*

(1) *D is a sunny generalized nonexpansive retract of E;*
(2) *JD is closed and convex.*

In this case, D is closed.

Theorem 2.5. ([18]) *Let D be a nonempty subset of a smooth, reflexive, and strictly convex Banach space E with JD is closed and convex and let $(x, z) \in E \times D$. Then, the following are equivalent:*

(1) *$z = R_D x$;*
(2) *$V(x, z) = \min\{V(x, y) : y \in D\}$.*

We also know the following result. For the exact definition of Mosco limit $\text{M-lim}_n C_n$; see [21].

Theorem 2.6. ([8]) *Let E be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $\{C_n\}$ be a sequence of nonempty sunny generalized nonexpansive retracts of E. Let $u \in E$ and let $\{u_n\}$ be a sequence of E converging strongly to u. If $C_0^* = \text{M-lim}_n JC_n$ exists and is nonempty, then $\{R_{C_n} u_n\}$ converges strongly to $R_{C_0} u$, where $C_0 = J^{-1}C_0^*$.*

One of the simplest example of the sequence $\{C_n\} \subset E$ satisfying the condition in this theorem above is a decreasing sequence with respect to inclusion;

$C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$. In this case, $\text{M-lim}\, JC_n = \cap_{n=1}^{\infty} JC_n$ (see [16, 17, 21] for more details).

The following results show that the existence of mappings $\underline{g}_r$, $\overline{g}_r$, $\underline{g}_r^*$ and $\overline{g}_r^*$, which are related to the convex and smooth structures of a Banach space E.

Theorem 2.7. ([30]) *Let E be a Banach space and $r \in\,]0, \infty[$. Then,*

(i) *if E is uniformly convex, then there exists a continuous, strictly increasing and convex function $\underline{g}_r : [0, 2r] \to [0, \infty[$ with $\underline{g}_r(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\underline{g}_r(\|x-y\|)$$

for all $x, y \in B_r$ and $\alpha \in [0, 1]$;

(ii) *if E is uniformly smooth, then there exists a continuous, strictly increasing and convex function $\overline{g}_r : [0, 2r] \to [0, \infty[$ with $\overline{g}_r(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \ge \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\overline{g}_r(\|x-y\|)$$

for all $x, y \in B_r$ and $\alpha \in [0, 1]$.

From this theorem, we can show the following result; for the proof see [16]

Theorem 2.8. ([16]) *Let E be a uniformly smooth and uniformly convex Banach space and let $r > 0$. Then, the function $\underline{g}_r$ and $\overline{g}_r$ in Theorem 2.7 satisfies*

$$\underline{g}_r(\|x-y\|) \le V(x, y) \le \overline{g}_r(\|x-y\|)$$

for all $x, y \in B_r$.

As a direct consequence of Theorem 2.7, we obtain the following result.

Theorem 2.9. *Let E be a reflexive, smooth and strictly convex Banach space, $r \in\,]0, \infty[$. Then,*

(i) *if E is uniformly smooth, then there exists a continuous, strictly increasing and convex function $\underline{g}_r^* : [0, 2r] \to [0, \infty[$ with $\underline{g}_r^*(0) = 0$ such that*

$$\|\alpha Jx + (1-\alpha)Jy\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\underline{g}_r^*(\|Jx-Jy\|)$$

for all $x, y \in B_r$ and $\alpha \in [0, 1]$;

(ii) *if E is uniformly convex, then there exists a continuous, strictly increasing and convex function $\overline{g}_r^* : [0, 2r] \to [0, \infty[$ with $\overline{g}_r^*(0) = 0$ such that*

$$\|\alpha Jx + (1-\alpha)Jy\|^2 \ge \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\overline{g}_r^*(\|Jx-Jy\|)$$

for all $x, y \in B_r$ and $\alpha \in [0, 1]$.

From this theorem, we can show the following result; for the proof see [9]

Theorem 2.10. ([9]) *Let E be a uniformly smooth and uniformly convex Banach space and let $r > 0$. Then, the function $\underline{g}_r^*$ and $\overline{g}_r^*$ in Theorem 2.9 satisfies*

$$\underline{g}_r^*(\|Jx - Jy\|) \leq V(x, y) \leq \overline{g}_r^*(\|Jx - Jy\|)$$

for all $x, y \in B_r$.

3. Approximation theorem for the generalized resolvents

We consider an iterative scheme of resolvents of a monotone operator in a Banach space. Let C be a nonempty subset of a reflexive, smooth and strictly convex Banach space E such that JC is closed and convex, let $\lambda > 0$ and let $B \subset E^* \times E$ be a monotone operator satisfying

$$D(BJ) \subset C \subset R(I + \lambda BJ). \tag{3.1}$$

It is known that if B is a maximal monotone operator, then $R(I + \lambda BJ) = E$ (see [11, Proposition 4.1]). Hence, if B is maximal monotone, then (3.1) holds for $C = J^{-1}\overline{D(B)}$, where $\overline{K}$ is the closure of K. We also know that $\overline{D(B)}$ is convex; see [25]. If B satisfies (3.1) for $\lambda > 0$, then we can define the generalized resolvent $J_\lambda : C \to D(BJ)$ of B by

$$J_\lambda x = \{z \in E : x \in z + \lambda BJz\} \tag{3.2}$$

for all $x \in C$. In other words, $J_\lambda x = (I + \lambda BJ)^{-1}x$ for all $x \in C$. We know the following; see, for instance, [3, 11, 12]:

(1) J_λ is of firmly generalized nonexpansive type from C into $D(BJ)$;
(2) $(x - J_\lambda x)/\lambda \in BJJ_\lambda x$ for all $x \in C$;
(3) $F(J_\lambda) = (BJ)^{-1}0$.

Before showing our result, we need the following lemmas.

Lemma 3.1. *Let E be a reflexive, smooth and strictly convex Banach space and let $B \subset E^* \times E$ be an operator. Then the following holds.*

(1) *$D(BJ) = J^{-1}D(B)$;*
(2) *if the norms of E and E^* are Fréchet differentiable, then $\overline{D(BJ)} = J^{-1}\overline{D(B)}$.*

Proof. (1) It is easy to see that $D(BJ) = J^{-1}D(B)$. In fact,

$$x \in D(BJ) \Leftrightarrow BJx \neq \emptyset \Leftrightarrow Jx \in D(B) \Leftrightarrow x \in J^{-1}D(B).$$

(2) We first show that $\overline{D(BJ)} \subset J^{-1}\overline{D(B)}$. For each $z \in \overline{D(BJ)}$, there exists a sequence $\{z_n\} \subset D(BJ)$ such that $z_n \to z$. By (1), we obtain $\{Jz_n\} \subset D(B)$. Since E has a Fréchet differential norm, the duality mapping J on E is norm to norm continuous and hence we obtain $Jz_n \to Jz$. So, we have $Jz \in \overline{D(B)}$ and hence we get $z \in J^{-1}\overline{D(B)}$. This implies that $\overline{D(BJ)} \subset J^{-1}\overline{D(B)}$.

Next we show that $\overline{D(BJ)} \supset J^{-1}\overline{D(B)}$. Put $z \in J^{-1}\overline{D(B)}$, then $Jz \in \overline{D(B)}$. There exists a sequence $\{z_n^*\} \subset D(B)$ such that $z_n^* \to Jz$. Since E^* has a Fréchet differential norm, the duality mapping J^{-1} on E^* is norm to norm continuous and hence we obtain $J^{-1}z_n^* \to J^{-1}Jz = z$. By (1), we obtain $\{J^{-1}z_n^*\} \subset D(BJ)$ and hence we get $z \in \overline{D(BJ)}$. This implies that $J^{-1}\overline{D(B)} \subset \overline{D(BJ)}$, which completes the proof. □

Lemma 3.2. *Let E be a reflexive, smooth and strictly convex Banach space, and let $B \subset E^* \times E$ be a monotone operator. Let $\lambda > 0$ and let C be a nonempty subset of E satisfying JC is closed and convex, and* (3.1) *for λ. Then the following holds.*

$$V(x, J_\lambda x) + V(J_\lambda x, x) \leq 2\lambda\langle y, Jx - JJ_\lambda x\rangle$$

for all $x \in D(BJ)$ and $y \in BJx$.

Proof. Let $x \in D(BJ)$ and $y \in BJx$. Since $(x - J_\lambda x)/\lambda \in BJJ_\lambda x$, we have

$$\begin{aligned} 0 &\leq \left\langle y - \frac{x - J_\lambda x}{\lambda}, Jx - JJ_\lambda x \right\rangle \\ \Leftrightarrow \quad \left\langle \frac{x - J_\lambda x}{\lambda}, Jx - JJ_\lambda x \right\rangle &\leq \langle y, Jx - JJ_\lambda x\rangle \\ \Leftrightarrow \quad \langle x - J_\lambda x, Jx - JJ_\lambda x\rangle &\leq \lambda\langle y, Jx - JJ_\lambda x\rangle. \end{aligned}$$

From (2.2), we have

$$V(x, J_\lambda x) + V(J_\lambda x, x) = 2\langle x - J_\lambda x, Jx - JJ_\lambda x\rangle \leq 2\lambda\langle y, Jx - JJ_\lambda x\rangle$$

for all $x \in D(BJ)$ and $y \in BJx$. □

We obtain an approximation theorem for a zero point of a monotone operator in a uniformly smooth and uniformly convex Banach space by using the generalized resolvent.

Theorem 3.3. *Let E be a uniformly smooth and uniformly convex Banach space, let $B \subset E^* \times E$ be a monotone operator with $B^{-1}0 \neq \emptyset$ and let $\{\lambda_n\}$ be a positive real numbers such that $\inf_n \lambda_n > 0$. Let C be a nonempty bounded subset of E satisfying JC is closed and convex, and*

$$D(BJ) \subset C \subset R(I + \lambda_n BJ)$$

for all $n \in \mathbb{N}$ and let $r \in]0, \infty[$ such that $C \subset B_r$. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n < \infty$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and

$$\begin{aligned}
&y_n = J_{\lambda_n} x_n,\\
&C_{n+1} = \{z \in C : \langle x_n - y_n, Jy_n - Jz\rangle \geq 0\} \cap C_n,\\
&x_{n+1} \in \{z \in C : V(u, z) \leq \inf\{V(u, v) : v \in C_{n+1}\} + \delta_{n+1}\} \cap C_{n+1}
\end{aligned}$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - y_n\| \leq \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0))).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $R_{(BJ)^{-1}0}u$.

Proof. It follows from Lemma 2.1 and Theorem 2.4 that $F(J_\lambda)$ is a sunny generalized nonexpansive retract of E for each $\lambda > 0$. Since $F(J_\lambda) = (BJ)^{-1}0$ for each $\lambda > 0$, $(BJ)^{-1}0$ is a sunny generalized nonexpansive retract of E.

We first show that JC_n is a closed convex subset of E and $(BJ)^{-1}0 \subset C_n$ for all $n \in \mathbb{N}$ by induction. From the assumption for C, it is obvious that JC_1 is a closed convex subset of E and $(BJ)^{-1}0 \subset C_1$. From the surjectivity of J, we have

$$\begin{aligned}
JC_{n+1} &= J\big(\{z \in C : \langle x_n - y_n, Jy_n - Jz\rangle \geq 0\} \cap C_n\big),\\
&= J\{z \in C : \langle x_n - y_n, Jy_n - Jz\rangle \geq 0\} \cap JC_n,\\
&= \{z^* \in JC : \langle x_n - y_n, Jy_n - z^*\rangle \geq 0\} \cap JC_n
\end{aligned}$$

for all $n \in \mathbb{N}$. Suppose that JC_k is a closed convex subset of E and $(BJ)^{-1}0 \subset C_k$ for some $k \in \mathbb{N}$. Let

$$D_k^* = \{z^* \in JC : \langle x_k - y_k, Jy_k - z^*\rangle \geq 0\}.$$

It is obvious that D_k^* is closed and convex. Therefore $JC_{k+1} = D_k^* \cap JC_k$ is also closed and convex. Let $p \in (BJ)^{-1}0$. Since J_{λ_k} is a firmly generalized nonexpansive type mapping and $F(J_{\lambda_k}) = (BJ)^{-1}0$, we have

$$0 \leq \langle (x_k - y_k) - (p - J_{\lambda_k}p), Jy_k - JJ_{\lambda_k}p\rangle = \langle x_k - y_k, Jy_k - Jp\rangle$$

and thus $p \in C_{k+1}$. Hence JC_n is a closed convex subset of E and $(BJ)^{-1}0 \subset C_n$ for all $n \in \mathbb{N}$. Since $(BJ)^{-1}0$ is nonempty, C_n is also nonempty for all $n \in \mathbb{N}$. Let $p_n = R_{C_n}u$ for all $n \in \mathbb{N}$. Then, since $\{C_n\}$ is decreasing with respect to inclusion, by Theorem 2.6, $\{p_n\}$ converges strongly to $p_0 = R_{C_0}u$, where $C_0 = J^{-1}\left(\bigcap_{n=1}^\infty JC_n\right)$. Since $x_n \in C_n$, it follows from Theorem 2.5 that

$$V(u, x_n) \leq V(u, p_n) + \delta_n$$

for every $n \in \mathbb{N} \setminus \{1\}$. Since JC_n is closed and convex, then for each $\alpha \in]0,1[$, we have

$$\alpha Jp_n + (1-\alpha)Jx_n \in JC_n$$

and hence

$$J^{-1}(\alpha Jp_n + (1-\alpha)Jx_n) \in C_n$$

for every $n \in \mathbb{N} \setminus \{1\}$. From Theorem 2.9 (i), we have

$$\begin{aligned} V(u,p_n) &\leq V\left(u, J^{-1}(\alpha Jp_n + (1-\alpha)Jx_n)\right) \\ &= \|u\|^2 - 2\langle u, \alpha Jp_n + (1-\alpha)Jx_n\rangle + \|\alpha Jp_n + (1-\alpha)Jx_n\|^2 \\ &\leq \|u\|^2 - 2\alpha\langle u, Jp_n\rangle - 2(1-\alpha)\langle u, Jx_n\rangle \\ &\quad + \alpha\|p_n\|^2 + (1-\alpha)\|x_n\|^2 - \alpha(1-\alpha)\underline{g}_r^*(\|Jx_n - Jp_n\|) \\ &= \alpha V(u,p_n) + (1-\alpha)V(u,x_n) - \alpha(1-\alpha)\underline{g}_r^*(\|Jx_n - Jp_n\|) \end{aligned}$$

and thus

$$\alpha \underline{g}_r^*(\|Jx_n - Jp_n\|) \leq V(u,x_n) - V(u,p_n) \leq \delta_n$$

for every $n \in \mathbb{N} \setminus \{1\}$. Tending $\alpha \to 1$, we obtain $\underline{g}_r^*(\|Jx_n - Jp_n\|) \leq \delta_n$ and thus $\|Jx_n - Jp_n\| \leq \underline{g}_r^{*-1}(\delta_n)$. Using the definition of p_n, we have that $p_{n+1} \in C_{n+1}$ and thus

$$\langle x_n - y_n, Jy_n - Jp_{n+1}\rangle \geq 0$$

for every $n \in \mathbb{N} \setminus \{1\}$. From the property of the function V, we have

$$\begin{aligned} 0 &\leq 2\langle x_n - y_n, Jy_n - Jp_{n+1}\rangle \\ &= V(x_n, p_{n+1}) - V(x_n, y_n) - V(y_n, p_{n+1}) \\ &\leq V(x_n, p_{n+1}) - V(x_n, y_n) \end{aligned}$$

for every $n \in \mathbb{N} \setminus \{1\}$. By Theorem 2.10, we obtain

$$\begin{aligned} V(x_n,y_n) &\leq V(x_n, p_{n+1}) \\ &= V(x_n,p_n) + V(p_n,p_{n+1}) + 2\langle x_n - p_n, Jp_n - Jp_{n+1}\rangle \\ &\leq \overline{g}_r^*(\|Jx_n - Jp_n\|) + V(p_n,p_{n+1}) + 2\langle x_n - p_n, Jp_n - Jp_{n+1}\rangle \\ &\leq \overline{g}_r^*(\underline{g}_r^{*-1}(\delta_n)) + V(p_n,p_{n+1}) + 2\langle x_n - p_n, Jp_n - Jp_{n+1}\rangle. \end{aligned}$$

for every $n \in \mathbb{N}\setminus\{1\}$. Since E is uniformly convex, J is norm to norm continuous. Therefore, from $\limsup_n \delta_n = \delta_0$ and $p_n \to p_0$, we have

$$\limsup_{n\to\infty} V(x_n,y_n) \leq \overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0)).$$

Therefore, by Theorem 2.8, we have

$$\limsup_{n\to\infty} \|x_n - y_n\| \le \limsup_{n\to\infty} \underline{g}_r^{-1}(V(x_n, y_n)) \le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0))).$$

For the latter part of the theorem, suppose that $\delta_0 = 0$. Then we have

$$\limsup_{n\to\infty} \|x_n - y_n\| \le \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(0))) = 0$$

and

$$\limsup_{n\to\infty} \underline{g}_r(\|Jx_n - Jp_n\|) \le \limsup_{n\to\infty} \delta_n = 0.$$

Therefore, we obtain

$$\lim_{n\to\infty} \|x_n - y_n\| = 0 \text{ and } \lim_{n\to\infty} \|Jx_n - Jp_n\| = 0. \tag{3.3}$$

Hence, we also obtain

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = p_0 \quad \text{and} \quad \lim_{n\to\infty} Jx_n = \lim_{n\to\infty} Jy_n = Jp_0. \tag{3.4}$$

From Lemma 3.2 and we have

$$V(y_n, J_{\lambda_1} y_n) \le V(y_n, J_{\lambda_1} y_n) + V(J_{\lambda_1} y_n, y_n) \le 2\lambda_1 \langle z, Jy_n - JJ_{\lambda_1} y_n \rangle$$

for all $z \in BJy_n$. From y_n, $J_{\lambda_1} y_n \in D(BJ) \subset C \subset B_r$ and $(x_n - y_n)/\lambda_n \in BJy_n$, we have

$$\begin{aligned} V(y_n, J_{\lambda_1} y_n) &\le 2\lambda_1 \left\langle \frac{x_n - y_n}{\lambda_n}, Jy_n - JJ_{\lambda_1} y_n \right\rangle \\ &\le 2\lambda_1 \left\| \frac{x_n - y_n}{\lambda_n} \right\| \|Jy_n - JJ_{\lambda_1} y_n\| \\ &\le 2\lambda_1 \left\| \frac{x_n - y_n}{\lambda_n} \right\| \big(\|y_n\| + \|J_{\lambda_1} y_n\|\big) \le 4\lambda_1 r \left\| \frac{x_n - y_n}{\lambda_n} \right\|. \end{aligned}$$

Since $\inf_n \lambda_n > 0$ and (3.3), we obtain

$$\limsup_{n\to\infty} V(y_n, J_{\lambda_1} y_n) \le 0.$$

This implies that $\lim_n V(y_n, J_{\lambda_1} y_n) = 0$. From Theorem 2.10, we have

$$\limsup_{n\to\infty} \|Jy_n - JJ_{\lambda_1} y_n\| \le \limsup_{n\to\infty} \underline{g}_r^{*-1}\left(V(y_n, J_{\lambda_1} y_n)\right) = \underline{g}_r^{*-1}(0) = 0.$$

This implies that $\lim_n \|Jy_n - JJ_{\lambda_1} y_n\| = 0$. Then, by Theorem 2.2 and (3.4) we have $p_0 \in \check{F}(J_{\lambda_1}) = F(J_{\lambda_1}) = (BJ)^{-1}0$. Since $(BJ)^{-1}0 \subset C_0$, we have $p_0 = R_{C_0} u = R_{(BJ)^{-1}0} u$, which completes the proof. □

4. Applications

In this section, we give some applications of Theorem 3.3. We first study the approximation of fixed points for mappings of firmly nonexpansive type.

4.1 Fixed Point Problem

Let C be a nonempty subset of a Banach space E and let T be a mapping from E to C. Then the fixed point problem is to find $x_0 \in C$ such that

$$x_0 = Tx_0.$$

Before we show our applications, we need the following results.

Proposition 4.1. ([3]) *Let E be a reflexive, smooth and strictly convex Banach space, let C be a nonempty subset of E, let $T : C \to E$ be a mapping, and let $B_T \subset E^* \times E$ be an operator defined by $B_T = (T^{-1} - I)J^{-1}$. Then, T is of firmly nonexpansive type if and only if B_T is monotone. In this case $T = (I + B_T J)^{-1}$.*

As a direct consequence of Theorem 3.3 and Proposition 4.1, we obtain the following result.

Corollary 4.2. *Let E be a uniformly smooth and uniformly convex Banach space, let C be a nonempty bounded closed subset of E satisfying JC is closed and convex and let $r \in \,]0, \infty[$ such that $C \subset B_r$. Let $T : C \to C$ be a firmly generalized nonexpansive type with $F(T) \neq \emptyset$, let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n < \infty$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$\begin{aligned} C_{n+1} &= \{z \in C : \langle x_n - Tx_n, JTx_n - Jz\rangle \geq 0\} \cap C_n, \\ x_{n+1} &\in \{z \in C : V(u, z) \leq \inf\{V(u, v) : v \in C_{n+1})\} + \delta_{n+1}\} \cap C_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0))).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $R_{F(T)}u$.

Proof. Put $B_T = (T^{-1} - I)J^{-1}$ and $\lambda_n = 1$ for all $n \in \mathbb{N}$. It follows from Proposition 4.1 that T is the generalized resolvent of B_T for 1 and

$$D(B_T J) = R(T) \subset C = D(T) = R(I + B_T J).$$

Therefore, we obtain the desired result by Theorem 3.3. □

4.2 Convex Minimization Problem

Let E be a reflexive, smooth and strictly convex Banach space with its dual E^* and let $f : E^* \to]-\infty, \infty]$ be a proper lower semicontinuous convex function.

Then, the convex minimization problem is to find

$$x_0^* \in E^* \text{ such that } f^*(x_0^*) = \min_{z^* \in E^*} f^*(z^*).$$

The subdifferential ∂f^* of f^* is defined as follows:

$$\partial f^*(x^*) = \{x \in E : f^*(x^*) + \langle x, y^* - x^* \rangle \leq f^*(y^*),\ \forall y^* \in E^*\}$$

for all $x^* \in E^*$. By Rockafellar's theorem [22, 23], the subdifferential $\partial f^* \subset E^* \times E$ is maximal monotone. It is easy to see that $(\partial f^*)^{-1}0 = \operatorname{argmin}\{f^*(z^*) : z^* \in E^*\}$. It is also known that

$$D(\partial f^*) \subset D(f^*) \subset \overline{D(\partial f^*)}; \tag{4.1}$$

see, for instance, [4, 25]. Before showing our application, we need the following lemma.

Lemma 4.3. *Let E be a reflexive, smooth and strictly convex Banach space and let $f : E^* \to]-\infty, \infty]$ be a proper function. Then the following holds.*

(1) *$D(f^*J) = J^{-1}D(f^*)$;*
(2) *if E and E^* has a Fréchet differential norm, then $\overline{D(f^*J)} = J^{-1}\overline{D(f*)}$.*

Proof. In the same way as Lemma 3.1, we have the desired result. □

As a direct consequence of Theorems 3.3, we can show the following corollary.

Corollary 4.4. *Let E be a uniformly smooth and uniformly convex Banach space and let $f : E^* \to]-\infty, \infty]$ be a proper lower semicontinuous convex function with $D(f^*)$ being bounded, and let $r \in]0, \infty[$ such that $D(f^*) \subset B_r^* := \{z^* \in E^* : \|z^*\| \leq r\}$. Let $\{\lambda_n\}$ be a positive real numbers such that $\inf_n \lambda_n > 0$, let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n < \infty$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in \overline{D(f^*J)}$, $C_1 = \overline{D(f^*J)}$, and*

$$\begin{aligned}
&y_n = J^{-1} \operatorname*{argmin}_{y^* \in E^*} \left\{ f^*(y^*) + \frac{1}{2\lambda_n}\|y^*\|^2 - \frac{1}{\lambda_n}\langle x_n, y^* \rangle \right\}, \\
&C_{n+1} = \{z \in \overline{D(f^*J)} : \langle x_n - y_n, Jy_n - Jz \rangle \geq 0\} \cap C_n, \\
&x_{n+1} \in \{z \in \overline{D(f^*J)} : V(u, z) \leq \inf\{V(u, v) : v \in C_{n+1})\} + \delta_{n+1}\} \cap C_{n+1}
\end{aligned}$$

for all $n \in \mathbb{N}$. If $(\partial f^*)^{-1}0 \neq \emptyset$, then,

$$\limsup_{n\to\infty} \|x_n - y_n\| \leq \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0))).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $R_{(\partial f^* J)^{-1}0}u$.

Proof. Put $C = \overline{D(f^*J)}$. Since the subdifferential $\partial f^* \subset E^* \times E$ is maximal monotone, then $E = R(I + \lambda\partial f^* J)$ for all $\lambda > 0$. By (4.1) and Lemmas 3.1 and 4.3, we have

$$D(\partial f^* J) = J^{-1}D(\partial f^*) \subset J^{-1}\overline{D(\partial f^*)} = J^{-1}\overline{D(f^*)} = C \subset E = R(I + \lambda\partial f^* J)$$

for all $\lambda > 0$. Fix $\lambda > 0$ and $z \in C$. Let J_λ be the generalized resolvent of ∂f^*, then we also know that; see [10, 13]

$$JJ_\lambda z = \operatorname*{argmin}_{y^* \in E^*} \left\{ f^*(y^*) + \frac{1}{2\lambda}\|y^*\|^2 - \frac{1}{\lambda}\langle z, y^* \rangle \right\}.$$

Therefore, we obtain the desired result by Theorem 3.3. □

4.3 Variational Inequality Problem

Let E be a Banach space with its dual E^*, let C be a nonempty closed subset of E such that JC is closed and convex. Let T^* be a single valued operator of JC to E. Then the variational inequality problem is to find

$$x \in C \text{ such that } \langle T^* Jx, Jy - Jx \rangle \geq 0 \tag{4.2}$$

for each $y \in JC$. The set of solutions is denoted by $VI(C, T^*)$. A single-valued operator T^* is said to be hemicontinuous if T^* is continuous from each line segment of JC into E with weak topology.

As a direct consequence of Theorems 3.3, we can show the following corollary.

Corollary 4.5. *Let E be a uniformly smooth and uniformly convex Banach space, let C be a nonempty bounded closed subset of E satisfying JC is closed and convex and let $r \in \,]0, \infty[$ such that $C \subset B_r$. Let $T^* : JC \to E$ be a single valued, monotone and hemicontinuous operator and let $\{\lambda_n\}$ be a positive real numbers such that $\inf_n \lambda_n > 0$. Let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n < \infty$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$\begin{aligned}
&y_n = VI\left(C, T^* + \frac{1}{\lambda_n}(J^{-1} - x_n)\right) \\
&C_{n+1} = \{z \in C : \langle x_n - y_n, Jy_n - Jz \rangle \geq 0\} \cap C_n, \\
&x_{n+1} \in \{z \in C : V(u, z) \leq \inf\{V(u, v) : v \in C_{n+1})\} + \delta_{n+1}\} \cap C_{n+1}
\end{aligned}$$

for all $n \in \mathbb{N}$. If $VI(C, T^*) \neq \emptyset$, then,

$$\limsup_{n \to \infty} \|x_n - y_n\| \leq \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0))).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $R_{V(C,T^*)}u$.

Proof. Let $N_{JC}(z^*)$ be the normal cone to JC at $z^* \in JC$, i.e.

$$N_{JC}(z^*) := \{z \in E : \langle z, z^* - y^* \rangle \geq 0, \ \forall y^* \in JC\}$$

and let

$$B_{T^*}z^* := \begin{cases} T^*z^* + N_{JC}(z^*), & z^* \in JC, \\ \emptyset, & z^* \notin JC. \end{cases}$$

Then, from Rockafellar [24], B is a maximal monotone operator and $B_{T^*}^{-1}0 = JVI(C, T^*)$ and hence we have

$$D(B_{T^*}J) = J^{-1}D(B_{T^*}) = J^{-1}D(T^*) = C \subset E = R(I + \lambda B_{T^*}J)$$

for all $\lambda > 0$. Fix $\lambda > 0$ and $z \in C$ and let J_λ be the generalized resolvent of B_{T^*}. Then we have

$$z \in J_\lambda z + \lambda B_{T^*}JJ_\lambda z$$

and hence,

$$-T^*JJ_\lambda z + \frac{1}{\lambda}(z - J_\lambda z) \in N_{JC}(JJ_\lambda z).$$

Thus, we have

$$\left\langle T^*JJ_\lambda z + \frac{1}{\lambda}(J_\lambda z - z), y^* - JJ_\lambda z \right\rangle \geq 0$$

for each $y^* \in JC$ and hence we obtain

$$\left\langle \left(T^* + \frac{1}{\lambda}\left(J^{-1} - z\right)\right)(JJ_\lambda z), Jy - JJ_\lambda z \right\rangle \geq 0$$

for each $y \in C$, that is,

$$J_\lambda z = VI\left(C, T^* + \frac{1}{\lambda}(J^{-1} - z)\right).$$

Therefore, we obtain the desired result by Theorem 3.3. □

4.4 Equilibrium Problem

Let E be a smooth Banach space with its dual E^* and let C be a nonempty closed subset of E such that JC is closed and convex. Let f^* be a bifunction from $JC \times JC$ to $\mathbb{R}$. Then, the equilibrium problem for $f^* : JC \times JC \to \mathbb{R}$ is to find

$$x_0 \in C \text{ such that } f^*(Jx_0, Jy) \geq 0, \ \forall y \in C.$$

The set of such solutions is denoted by $EP(f^*)$; see [29] for more details.

Let C be a nonempty closed subset of a reflexive, smooth and strictly convex Banach space E such that JC is closed and convex. For solving the equilibrium problem, let us assume that a bifunction $f^* : JC \times JC \to \mathbb{R}$ satisfies the following conditions:

(A1) $f^*(x^*, x^*) = 0$ for all $x^* \in JC$;
(A2) f^* is monotone, i.e., $f^*(x^*, y^*) + f(y^*, x^*) \leq 0$ for all $x^*, y^* \in JC$;
(A3) for all $x^*, y^*, z^* \in JC$,

$$\limsup_{t \downarrow 0} f^*(tz^* + (1-t)x^*, y^*) \leq f^*(x^*, y^*);$$

(A4) for all $x^* \in JC$, $f(x^*, \cdot)$ is convex and lower semicontinuous.

We know the following results for such a bifunction f^*.

Lemma 4.6. ([5]) *Let C be a nonempty closed subset of a reflexive, smooth and strictly convex Banach space E such that JC is closed and convex, let f^* be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, let $\lambda > 0$ and let $x \in E$. Then, there exists $z \in C$ such that*

$$f^*(Jz, Jy) + \frac{1}{\lambda}\langle z - x, Jy - Jz\rangle \geq 0$$

for all $y \in C$.

Lemma 4.7. ([29]) *Let C be a nonempty closed subset of a uniformly smooth and strictly convex Banach space E such that JC is closed and convex and let f^* be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. For $\lambda > 0$ and $x \in E$, define a mapping $F_\lambda : E \to C$ as follows:*

$$F_\lambda(x) = \left\{ z \in C : f^*(Jz, Jy) + \frac{1}{\lambda}\langle z - x, Jy - Jz\rangle \geq 0 \text{ for all } y \in C \right\} \quad (4.3)$$

for all $x \in E$. Then, the following hold:

(1) *F_λ is single-valued;*
(2) *$F(F_\lambda) = EP(f^*)$;*
(3) *$JEP(f^*)$ is closed and convex.*

The following theorem is essentially due to Aoyama, Kimura and Takahashi [2, Theorem 3.5].

Theorem 4.8. *Let C be a nonempty closed subset of a reflexive, smooth and strictly convex Banach space E such that JC is closed and convex, let f^* be a* *bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. Let B_{f^*} be a multi-valued mapping of E^* into E defined by*

$$B_{f^*}x^* = \begin{cases} \{x \in E : f^*(x^*, y^*) \geq \langle x, y^* - x^* \rangle, \ \forall y^* \in JC\} & x^* \in JC, \\ \emptyset & x^* \notin JC, \end{cases}$$

Then, the following hold:

(1) $EP(f^*) = (B_{f^*}J)^{-1}0$;
(2) $B_{f^*} \subset E^* \times E$ *is maximal monotone;*
(3) $F_r = (I + \lambda B_{f^*}J)^{-1}$ *for each* $\lambda > 0$.

Theorem 4.9. *Let C be a nonempty bounded closed subset of a uniformly smooth and uniformly convex Banach space E such that JC is closed and convex and let $r \in]0, \infty[$ such that $C \subset B_r$. Let f^* be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, let $\{\lambda_n\}$ be a positive real numbers such that $\inf_n \lambda_n > 0$ and let $\{\delta_n\}$ be a nonnegative real sequence and let $\delta_0 = \limsup_n \delta_n < \infty$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by $x_1 = x \in C$, $C_1 = C$, and*

$$\begin{aligned} &y_n = F_{\lambda_n} x_n, \\ &C_{n+1} = \{z \in C : \langle x_n - y_n, Jy_n - Jz \rangle \geq 0\} \cap C_n, \\ &x_{n+1} \in \{z \in C : V(u, z) \leq \inf\{V(u, v) : v \in C_{n+1})\} + \delta_{n+1}\} \cap C_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. If $EP(f^*) \neq \emptyset$, then

$$\limsup_{n \to \infty} \|x_n - y_n\| \leq \underline{g}_r^{-1}(\overline{g}_r^*(\underline{g}_r^{*-1}(\delta_0))).$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $R_{EP(f^*)}u$.

Proof. Let B_{f^*} be defined as in Theorem 4.8. From Theorem 4.8, B_{f^*} is maximal monotone, F_{λ_n} is the generalized resolvent of B_{f^*} for λ_n and $EP(f^*) = (B_{f^*}J)^{-1}0$. We also have

$$D(B_{f^*}J) = J^{-1}D(B_{f^*}) = C \subset E = R(I + B_{f^*}J).$$

Therefore, we obtain the desired result by Theorem 3.3. □

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