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N-FLIPS IN EVEN TRIANGULATIONS ON THE PROJECTIVE PLANE PRESERVING CHROMATIC NUMBER

By

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Abstract. An *even triangulation* on a surface F^2 is a triangular embedding of a loopless graph on F^2, possibly with multiple edges, such that each vertex has even degree. In this paper, we prove that any two even triangulations on the projective plane with the same number of vertices and the same chromatic number can be transformed into each other by two specified local transformations, an N-flip and a D_2-flip, preserving the chromatic number.

1. Introduction

A *surface* F^2 is a connected compact 2-dimensional manifold without boundaries. A *map* G on F^2 is a fixed embedding of a loopless graph on F^2. A *face* of G is a component of $F^2 - G$. For a vertex v of G, the *link* of v is the boundary walk of the region consisting of all faces of G incident to v. A closed curve γ on F^2 is *contractible* (resp., *essential*) if γ does (resp., does not) bound a 2-cell on F^2. This definition is also applied for a closed walk or cycle of a map on F^2. A *triangulation* on F^2 is a map on F^2 such that each face is triangular. A triangulation is *even* if each vertex has even degree. If a graph has neither multiple edges nor loops, then it is *simple*. A *k-vertex* means a vertex of degree k, and a *k-cycle* or a *k-path* is one with length k. When we use subscripts for symbols, we take them by suitable modulus.

A *k-coloring* of a graph G is a color-assignment $c : V(G) \to \{1, ..., k\}$ such that for any $xy \in E(G)$, $c(x) \neq c(y)$, and G is *k-colorable* if G admits a k-coloring. The *chromatic number* of G, denoted $\chi(G)$, is the minimum integer $k \geq 1$ such that G is k-colorable. In particular, if $\chi(G) = k$, then G is *k-chromatic*.

Let G be a triangulation on a surface F^2, and let $e \in E(G)$. A *diagonal flip* of e is to replace e with another diagonal in the quadrilateral region formed by two faces of G sharing e. Wagner proved that any two simple triangulations on the

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sphere with the same number of vertices can be transformed by diagonal flips, preserving the simpleness of graphs [16]. This theorem was extended to simple triangulations on the torus [2], the projective plane and the Klein bottle [13]. After those results, Negami proved that any two simple triangulations on any surface with the same and sufficiently large number of vertices can be transformed by diagonal flips, preserving the simpleness of graphs [14]. These results with some others for diagonal flips in triangulations have been summarized in a survey [15].

An *even embedding* on a surface F^2 is a map on F^2 with each face bounded by a closed walk of even length, and a *quadrangulation* on F^2 is an even embedding with each face quadrilateral. It is known that every even embedding on the sphere is bipartite (i.e., 2-colorable), but every non-spherical surface admits non-bipartite ones. For quadrangulations, a *diagonal slide* is to slide an edge as shown in the left of Figure 1, and a *diagonal rotation* is to rotate a 2-path containing a 2-vertex in a middle, as shown in the right of Figure 1. Note that both operations preserve the bipartiteness of quadrangulations. The second author pointed out that any two homotopic closed walks in a quadrangulation have the same parity of length, and that the diagonal slide and rotation preserve, not only the bipartiteness of graphs, but also the parity of length of closed walks with any fixed homotopy type in the quadrangulation. Extending this argument, he proved that any two simple quadrangulations G and G' on any surface F^2 with the same and sufficiently large number of vertices can be transformed by diagonal slides and rotations, preserving the simpleness of graphs, if and only if for each homotopy type l on F^2, a closed walk of G and that of G' homotopic to l have the same parity of length [7, 8]. Such a property can be described by a homomorphism from the fundamental group of F^2 to $\mathbb{Z}_2$, and this notion is called the *cycle parity*. See [9] for the detail.

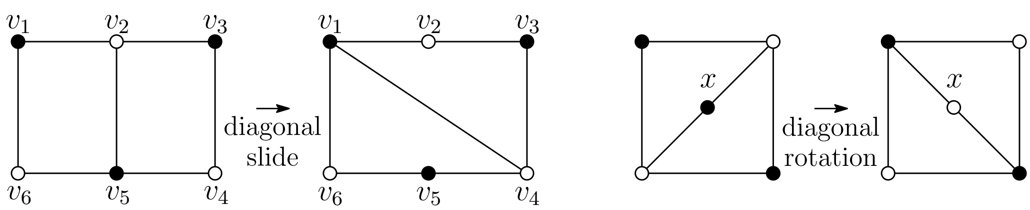

Figure 1 A diagonal slide and a diagonal rotation

Now we consider even triangulations on surfaces. It is well known that every even triangulation on the sphere is 3-chromatic, but any other surface admits non-3-colorable ones.

For an even triangulation G, an *N-flip* in G is to flip three consecutive edges

v_1v_3, v_3v_6 and v_6v_4 forming "N", as shown in the left of Figure 2, where the edge v_3v_6 is the *middle edge* of the 3-path. Suppose that G has adjacent 4-vertices x, y. A *P_2-flip* of $\{x, y\}$ in G is one shown in the right of Figure 2. It is easy to see that both of the operations preserve the 3-chromaticity of even triangulations.

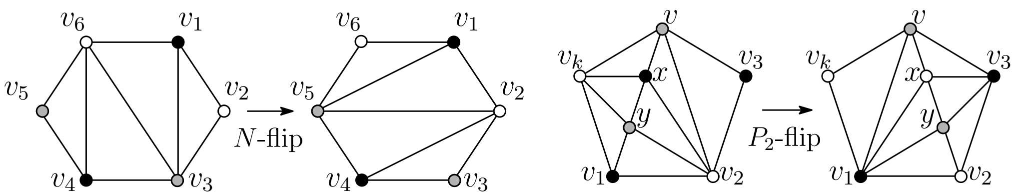

Figure 2 An N-flip and a P_2-flip

Using these two operations, Nakamoto et al. [11] proved the following.

Theorem 1. (Nakamoto et al. [11]) *Any two simple even triangulations on the sphere with the same number of vertices can be transformed into each other by N- and P_2-flips, preserving the simpleness of graphs.*

It was also proved in [3] that two even triangulations on any surface with the same and sufficiently large number of vertices can be transformed by these two operations, preserving the simpleness of graphs, if and only if their *monodromies* coincide, where the monodromy is a homomorphism from the fundamental group of the surface to the symmetric group of degree 3, and its definition should be referred to [3]. Roughly speaking, the monodromy classifies even triangulations into several distinct classes, depending on their homological structures, in which the 3-chromatic ones have the trivial monodromy. (Its definition is similar to the cycle parity for quadrangulations on surfaces.)

In this paper, we deal with even triangulations on the projective plane $\mathbb{P}$, where we let $\mathbb{P}$ denote the projective plane throughout the paper. The projective plane is known to admit two distinct types of monodromies, which are of 3-chromatic ones and of non-3-chromatic ones, respectively. For even triangulations on $\mathbb{P}$, Nakamoto and Suzuki [12] proved the following.

Theorem 2. (Nakamoto and Suzuki [12]) *Any two simple even triangulations G and G' on the projective plane with $|V(G)| = |V(G')| \geq 14$ can be transformed into each other by N- and P_2-flips, preserving the simpleness of graphs, if and only if G and G' are simultaneously 3-chromatic or not.*

In this paper, we would like to consider the following problem:

Can any two even triangulations G and G' on the projective plane $\mathbb{P}$

with $|V(G)| = |V(G')| \geq 14$ and $\chi(G) = \chi(G')$ be transformed by N- and P_2-flips, preserving the chromatic number?

The motivation comes from the corresponding result for quadrangulations. Nakamoto and Negami [10] pointed out that any two simple quadrangulations G and G' on the projective plane $\mathbb{P}$ with the same number of vertices can be transformed by diagonal slides and rotations if and only if both or neither are bipartite. The key is the fact that any quadrangulation on $\mathbb{P}$ is known to be either 2- or 4-chromatic [17]. Moreover, using the similar fact for chromatic numbers of quadrangulations on other surfaces [1], they proved in [10] that if two quadrangulations G and G' on a surface with high *representativity* can be transformed by diagonal slides and rotations, then $\chi(G) = \chi(G')$ (where the representativity of G on a surface F^2 is the minimum number of intersecting points of G and ℓ, where ℓ ranges over all essential closed curves on F^2).

However, for the above-mentioned problem, we encounter a counterexample for *simple* even triangulations, as in the following proposition (which will be proved in Section 2):

PROPOSITION 3. *There exists a pair of simple even triangulations G and G' on the projective plane $\mathbb{P}$ with $|V(G)| = |V(G')|$ and $\chi(G) = \chi(G')$ which cannot be transformed by N- and P_2-flips, preserving the simpleness and the chromatic number of graphs.*

By Proposition 3, we focus on even triangulations on $\mathbb{P}$ allowed to have multiple edges. For those even triangulations, we introduce the following transformation instead of a P_2-flip. Let x be a 2-vertex with neighbors v and v_1. The *D_2-flip* of x is to remove x and a single edge between v and v_1, and add a new 2-vertex and a single edge between v and v_2, where vv_2 is an edge contained in a facial triangle vv_1v_2 with $v_2 \neq x$ (see Figure 3).

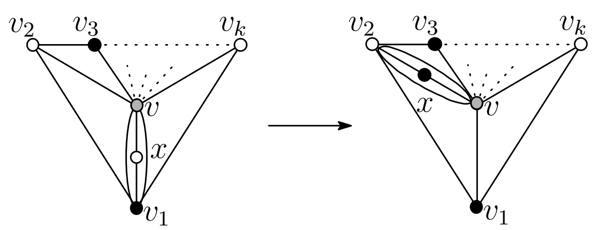

Figure 3 A *D_2-flip*

Our main theorem is as follows:

Theorem 4. *Let G and G' be two even triangulations on the projective plane $\mathbb{P}$ with the same number of vertices. If G and G' have the same chromatic number, then G and G' can be transformed into each other by N- and D_2-flips, preserving the chromatic number.*

It has been known that Theorem 4 does not hold for even triangulations on other surfaces. For example, the first author has proved that there exists a pair of 6-chromatic even triangulations on the torus which cannot be transformed into each other by the two transformations, preserving the 6-chromaticity [4].

In Section 2, we prove Proposition 3. In Section 3, we give some preliminaries for proving Theorem 4, and in Section 4, we finally prove Theorem 4.

2. Proof of Proposition 3

Let G be a map on a surface F^2. The *face subdivision* of G, denoted $\mathrm{FS}(G)$, is the triangulation on F^2 obtained from G by adding a new vertex into each face of G and joining it to all vertices on the corresponding boundary walk. It is easy to see that $\mathrm{FS}(G)$ is an even triangulation on F^2 if G is an even embedding. The set U of the vertices of $\mathrm{FS}(G)$ added to G is the *color factor* of $\mathrm{FS}(G)$. Since U is an independent set, we have $\chi(G) \leq \chi(\mathrm{FS}(G)) \leq \chi(G)+1$.

For even triangulations on $\mathbb{P}$, the following theorem is known.

Theorem 5. (Mohar [6]) *Let G be an even triangulation on the projective plane. Then G is the face subdivision of some even embedding K, and $\chi(G) \in \{3,4,5\}$. In particular,*

(i) *if $\chi(G) \geq 4$, then K is non-bipartite, and the color factor of G can be uniquely taken, and*

(ii) *$\chi(G)=5$ if and only if K includes a non-bipartite quadrangulation Q as a subgraph.*

By Theorem 5, if G is an even triangulation on $\mathbb{P}$ with $\chi(G) \geq 4$, then we have a unique expression $G = \mathrm{FS}(K)$ for some *non-bipartite* even embedding K on $\mathbb{P}$. (If K is bipartite with partite sets X and Y, then $(V(G)-V(K), X, Y)$ is a tripartition of $V(G)$, and hence G is 3-colorable, contrary to $\chi(G) \geq 4$.)

Let us consider what happens in K when an N-flip is applied in G to transform it into another even triangulation G'. Suppose that three consecutive edges e_1, e_2, e_3 of G in this order are moved by an N-flip. Then the middle edge e_2 of the path $e_1 \cup e_2 \cup e_3$ is either contained in K, or joining a vertex of K and one in the color factor. The former and the latter cases are shown in the left and right of Figure 4, respectively. Then, by the N-flip, the even embedding K is deformed into another even embedding, say K', by the two operations, a diagonal slide or an *edge wipe*. In this case, we see that $G' = \mathrm{FS}(K')$. The edge wipe of an edge e in K is to move $e = v_1v_{2k}$ to $e' = v_1v_{2k+2}$, where e is shared by two faces f and f' of K bounded by closed walks $v_1v_2\cdots v_{2k}v_1$ and $v_1v_{2k}v_{2k+1}\cdots v_{2m}v_1$ with $1 \leq k \leq m-1$, respectively.

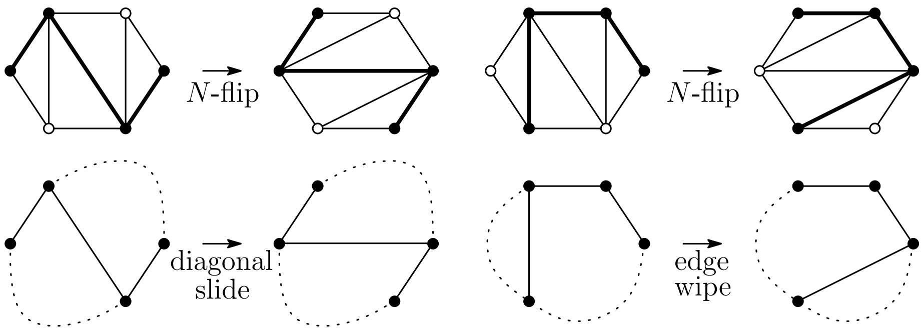

Figure 4 A diagonal slide and an edge wipe in K corresponding to an N-flip in G

Before proving Proposition 3, we prepare the following lemma.

Lemma 6. *Let G be an even triangulation on the projective plane, and let G' be an even triangulation obtained from G by a single N-flip. Then we have:*

(i) *G is 3-chromatic if and only if so is G'.*
(ii) *Suppose that G (resp., G') is the face subdivision of a non-bipartite even embedding K (resp., K'). Let e be an edge of G which is not moved by the N-flip. Then $e \in E(K)$ if and only if $e \in E(K')$.*

Proof. Figure 2 shows that (i) holds. The statement (ii) follows from the explanation before the lemma, since the expression $G' = \mathrm{FS}(K')$ is also unique. ■

Let G be a triangulation and let f be a face of G bounded by a 3-cycle $x_0x_1x_2$. The *octahedron addition* to f is to put a 3-cycle $a_0a_1a_2$ in the interior of f, add edges a_ix_i, a_ix_{i+1} for $i = 0, 1, 2$ where $x_3 = x_0$, as shown in Figure 5. In particular, the subgraph of the resulting graph induced by those six vertices is the *octahedron*. The inverse operation is the *octahedron removal*.

Now we prove Proposition 3.

Proof of Proposition 3. Let T be the even triangulation on $\mathbb{P}$ shown in the center of Figure 6, where we label the vertices of T as in the figure. (The hexagon

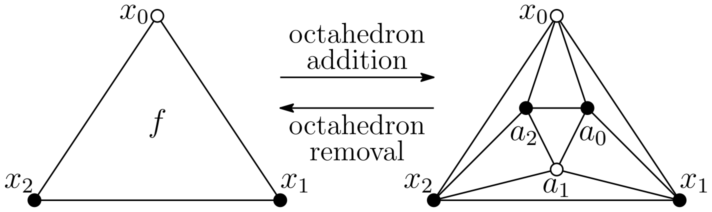

Figure 5 An octahedron addition and an octahedron removal

with each pair of antipodal points identified stands for $\mathbb{P}$.) Then T is the face subdivision of a non-bipartite quadrangulation on $\mathbb{P}$, say Q, isomorphic to K_4, and then $\chi(T) = 5$, by Theorem 5(ii). Let T_1 be the even triangulation on $\mathbb{P}$ obtained from T by applying a octahedron addition to the face abf and subdividing the edge be to put six new vertices so that all of them have degree 4 by adding edges. Then T_1 is 5-chromatic by Theorem 5(ii). Let $\mathcal{T}$ be the set of all even triangulations on $\mathbb{P}$ obtained from T by a single octahedron addition to each of f_1, f_2 and f_3, where f_1, f_2 and f_3 are triangular faces of T chosen from the quadrilateral regions $abcd$, $abdc$ and $acbd$ of Q, respectively. Let $T_2 \in \mathcal{T}$ be shown in Figure 6. Then T_2 is also a 5-chromatic even triangulation on $\mathbb{P}$ with $|V(T_1)| = |V(T_2)|$, since T_2 includes T as a subgraph.

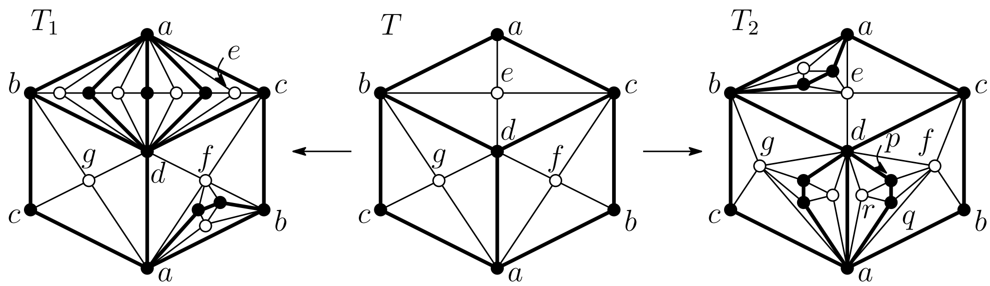

Figure 6 Two simple even triangulations T_1 and T_2 on $\mathbb{P}$ obtained from T

We shall prove that T_1 and T_2 cannot be transformed into each other by N- and P_2-flips, preserving the 5-chromaticity and the simpleness of graphs. So, for contradictions, we suppose that T_2 admits a sequence of N- and P_2-flips to transform it into T_1, preserving the 5-chromaticity and the simpleness of graphs.

We first observe that any pair of adjacent 4-vertices in T_2 is contained in one of the added octahedra, and hence every P_2-flip in T_2 yields multiple edges. Therefore, a P_2-flip cannot be applied in T_2, and so, in the sequence, we must apply an N-flip in T_2 at first. Let G' be the resulting simple even triangulation on $\mathbb{P}$ by the N-flip.

By Theorem 5 (i), T_2 has a unique even embedding, say K, such that $\mathrm{FS}(K) = T_2$. Then the N-flip in T_2 corresponds to either a diagonal slide or an edge wipe in K. Observe that those transformations in K must be forbidden if the resulting map K' is not simple or has a vertex of degree less than 2. Let ε be the edge in K moved by either the diagonal slide or the edge wipe, and let ε' be the edge added to K' instead of ε by the transformation.

Suppose that ε is an edge of $Q = K_4$ in K. We can check that if we move each of the six edges of Q in K by a diagonal slide or an edge wipe, then the resulting even embedding K' no longer contains any quadrangulation as a subgraph, or has multiple edges. (For example, if we apply a diagonal slide of the edge ad of Q in K, then the resulting map K' has no quadrilateral region containing the face bounded by $abcdpq$. Moreover, if we apply the edge wipe of the edge cd to put cq, then the face of K' bounded by $abdpqc$ has no quadrilateral region containing it.) Consequently, since K' contains no quadrangulation as a subgraph, G' is 4-colorable, by Theorem 5 (ii) and Lemma 6 (ii), contrary to the assumption. Therefore we have $\varepsilon \notin E(Q)$.

So we have $\varepsilon \in E(K) - E(Q)$. We consider only the case when ε is either pq or dp, since other cases are similar. If we apply an diagonal slide or an edge wipe to $\varepsilon = pq$ in K, then the resulting graph K' must have a 1-vertex, and hence the even triangulation G' have multiple edges, contrary to the simpleness of graphs. So we may suppose that $\varepsilon = dp$, and we apply the edge wide of dp so that the new edge is bp. However, the face subdivision G' of the resulting map K' is also an even triangulation on $\mathbb{P}$ obtained from T by adding a single octahedron to the triangular faces abe, adg and abf, and hence we have $G' \in \mathcal{T}$.

So T_2 can be transformed only into one in $\mathcal{T}$ by an N-flip, if we preserve the simpleness and 5-chromaticity of the graphs. Moreover, the above argument also follows for every even triangulation in $\mathcal{T}$. Therefore, since $T_1 \notin \mathcal{T}$, there exists no sequence of N- and P_2-flips to transform T_1 into T_2 preserving the conditions. ∎

3. Reductions in even triangulations

We first note that even triangulations in this and the following sections might have multiple edges but no loops, though we dealt with simple even triangulations in the proof of Proposition 3.

Let G be an even triangulation on a surface F^2, and let v be a 4-vertex of G with link $v_1v_2v_3v_4$. The 4-*contraction* of v at $\{v_2, v_4\}$ is to remove v, identify v_2 and v_4, and replace two pairs of multiple edges by two single edges respectively, as shown in the left of Figure 7. If the resulting graph has a loop or a pinched point of the surface, then we do not apply it. Let x be a 2-vertex of G with neighbors a, b. Then x is contained in the digonal region bounded by multiple edges ab. A 2-*contraction* of x is to remove x and replace the two edges ab by a single edge, as shown in the right of Figure 7. We call the inverse operation of a 2-contraction a 2-*addition*.

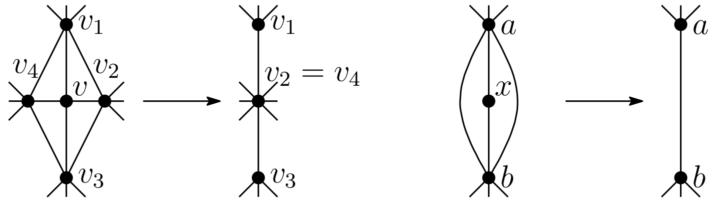

Figure 7 A 4-contraction at $\{v_2, v_4\}$ and a 2-contraction of x

Three graphs I_3, I_4, I_5 in Figure 8 are even triangulations on $\mathbb{P}$, and for $k = 3, 4, 5$, I_k is k-chromatic. The following is the main result in this section, which has an independent interest.

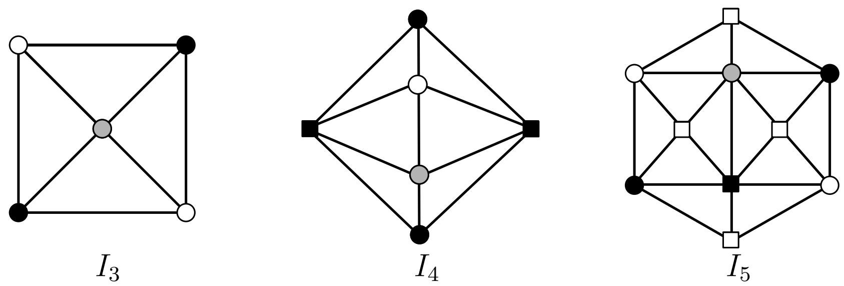

Figure 8 Even triangulations I_3, I_4, I_5 on $\mathbb{P}$

THEOREM 7. *Let G be an even triangulation on the projective plane with $\chi(G) = k$, where $k \in \{3, 4, 5\}$. Then G can be reduced into I_k by* 4- *and* 2-*contractions preserving the k-chromaticity, where I_k is shown in Figure* 8.

We use the following facts to prove Theorem 7.

LEMMA 8. *Let G be an even triangulation on a surface and let H be a graph obtained from G by a single* 4-*contraction. Then $\chi(G) \leq \chi(H)$. In particular, if $\chi(G) = 3$, then $\chi(H) = 3$.*

Proof. Let v be a 4-vertex of G with link $v_1v_2v_3v_4$. Suppose that H is obtained from G by a 4-contraction of v at $\{v_1, v_3\}$, where we let $v' \in V(H)$ be the image of the identification $v_1 = v_3$ by the 4-contraction of v. Suppose $\chi(H) = k$, and let $c' : V(H) \to \{1, \ldots, k\}$ be a k-coloring of H, where $k \in \{3, 4, 5\}$ by Theorem 5.

If $4 \leq \chi(H) \leq 5$, then we let $c(x) = c'(x)$ for any $x \in V(G) - \{v, v_1, v_3\}$, and let $c(v_1) = c(v_3) = c'(v')$. Finally we let $c(v)$ be a color distinct from the three colors $c(v_1) = c(v_3)$, $c(v_2)$ and $c(v_4)$, which is possible since $k \geq 4$. If $\chi(H) = 3$, then $c'(v_2) = c'(v_4)$, since the vertices of the link of v' in H are colored by two colors alternately, and since the distance between v_2 and v_4 on the link of v' in H is even. (For otherwise, v_1 and v_3 would have odd degree in G.) Hence, similarly to the previous case, v can be colored by the third color distinct from $c(v_1) = c(v_3)$ and $c(v_2) = c(v_4)$. Hence $\chi(G) = 3 = \chi(H)$.

Consequently, G is k-colorable and hence $\chi(G) \leq k = \chi(H)$. ■

Lemma 9. *Let Q be a non-bipartite quadrangulation on the projective plane. If Q has no quadrangulation as a proper subgraph, then Q is simple and has no vertex of degree less than* 3.

Proof. If Q has a vertex v of degree 1 or 2, then we get a smaller quadrangulation Q' by removing v (and replacing multiple edges by a single edge suitably). This contradicts the assumption.

So we suppose that Q is not simple. Since Q is loopless by the definition, Q has multiple edges xy to form a 2-cycle C. If C is contractible, then the digonal region bounded by C can be replaced with a single edge to get a smaller quadrangulation, contrary to the assumption. So suppose that C is essential. Then cut open Q on $\mathbb{P}$ along C to get a plane quadrangulation Q' with outer quadrilateral $xyx'y'$, where $x = x'$ and $y = y'$ in Q. Since every plane quadrangulation is known to be bipartite, Q' has a unique bipartition where $\{x, x'\}$ and $\{y, y'\}$ belong to distinct partite sets. Here the bipartition of $V(Q')$ must give a bipartition of $V(Q)$, since $x = x'$ and $y = y'$ in Q. This contradicts that Q is non-bipartite.

Therefore Q is simple and has no vertex of degree less than 3. ■

Proof of Theorem 7. Figure 9 shows three even triangulations T_1, T_2^k and T_3, where T_2^k consists of exactly k configurations of the same structure. It was proved in [5] that every even triangulation on $\mathbb{P}$ with no 2-vertex can be reduced to one of T_1, T_2^k, T_3 by a sequence of 4-contractions and octahedron removals without introducing any 2-vertex, any pinched point and any loop. It is easy to see that T_2^k with $k \geq 2$ is reduced to T_2^{k-1} by a 4-contraction followed by a 2-contraction, and that the octahedron removal is also obtained by a 4-contraction followed by a 2-contraction. Hence every even triangulation on $\mathbb{P}$ can be reduced to one of I_3, I_4 and I_5 by 4-contractions and 2-contractions without making loops and pinched points, where $T_1 = I_3$, $T_2^1 = I_4$ and $T_3 = I_5$.

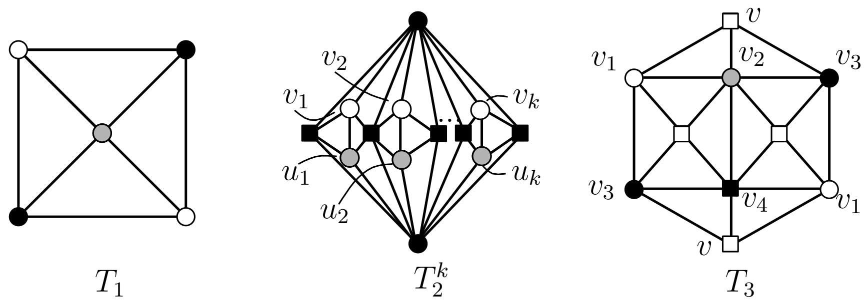

Figure 9 Even triangulations T_1, T_2^k, T_3 on $\mathbb{P}$

We prove that for $k = 3, 4, 5$, any k-chromatic even triangulation can be transformed into I_k preserving the k-chromaticity. Clearly, every 2-contraction preserves the chromatic number, and so we consider only a 4-contraction. By Lemma 8, no 4-contraction decreases the chromatic number, and hence every 5-chromatic even triangulation on $\mathbb{P}$ can be reduced to I_5 by 4- and 2-contractions, preserving the 5-chromaticity. Moreover, since a 4-contraction always preserves the 3-chromaticity by Lemma 8, we have only to prove that every 4-chromatic even triangulation G on $\mathbb{P}$ with no 2-vertex admits a 4-contraction preserving the 4-colorability.

Let c be a 4-coloring of G, and let v be a 4-vertex of G with link $v_1v_2v_3v_4$. Since G is 4-colored, we have $c(v_1) = c(v_3)$ or $c(v_2) = c(v_4)$, say the former. Apply the 4-contraction of v at $\{v_1, v_3\}$ in G, and let H be the resulting graph. If it is applicable, then H has a natural 4-coloring in which the vertex $v_1 = v_3$ in H has the color $c(v_1) = c(v_3)$, and we are done. So suppose that the 4-contraction of v at $\{v_1, v_3\}$ is not applicable in G. Since $c(v_1) = c(v_3)$, H has no loop incident to the vertex $v_1 = v_3$, and hence in G, we must have $v_1 = v_3$.

We first suppose that the 2-cycle $C = v_1vv_3$ of G with $v_1 = v_3$ is essential. We apply a 4-contraction of v at $\{v_2, v_4\}$ in G, and let H' be the resulting graph. (If we also have $v_2 = v_4$ in G, then G contains the map I_3 in Figure 8, and hence G is 3-colorable, contrary to the assumption $\chi(G) = 4$. Hence H' actually exists.) If H' is 5-chromatic, H' is the face subdivision of an even embedding K including a non-bipartite quadrangulation Q, by Theorem 5 (ii). By Lemma 9, we may suppose that Q is a simple graph 2-cell embedded in $\mathbb{P}$ so that each face is bounded by a cycle. So H' can have no essential 2-cycle, since G is the face subdivision of K, contrary to that C is essential. Therefore, H' is 4-chromatic.

Secondly we suppose that C is contractible, and that v_2 is contained in the interior of the digonal region R bounded by C. If $c(v_2) = c(v_4)$, then we can identify v_2 and v_4 by a 4-contraction of v, since $v_2 \neq v_4$ by the obstruction of the contractible 2-cycle v_1vv_3. So we may suppose that $c(v_2) \neq c(v_4)$. In this case, we recolor the vertices of R so that v_2 and v_4 have the same color, exchanging the two colors except $c(v)$ and $c(v_1) = c(v_3)$ in the interior of R. In the new 4-coloring $\tilde{c}$ of G, we have $\tilde{c}(v_2) = \tilde{c}(v_4)$, and we can apply a 4-contraction of v at $\{v_2, v_4\}$ preserving the 4-colorability. ■

4. Proof of Theorem 4

Let H be an even triangulation and let $H + D_2(m)$ denote an even triangulation obtained from H by a repeated application of 2-additions m times. It is easy to see that all even triangulations expressed as $H + D_2(m)$ have the same chromatic number, and that they can be transformed into each other by D_2-flips, preserving the chromatic number.

Lemma 10. *Let $k \in \{3, 4, 5\}$ and let G be a k-chromatic even triangulation on $\mathbb{P}$ but $G \neq I_k$. Then there exists a k-chromatic even triangulation H on $\mathbb{P}$ such that*

(i) *H is obtained from G by a single 2- or 4-contraction, and*
(ii) *G can be transformed into $H + D_2(m)$ by N- and D_2-flips, preserving the k-chromaticity, where $m = |V(G)| - |V(H)| \in \{1, 2\}$.*

Proof. By Theorem 7, G admits a 2-contraction or a 4-contraction preserving the k-chromaticity. In the former case, if we let H be the resulting graph by the 2-contraction, then we immediately have $G = H + D_2(1)$ and $\chi(G) = \chi(H) = k$, and so we are done. So consider the latter case, in which we suppose that G admits a 4-contraction at $\{v_1, v_3\}$ preserving the k-chromaticity, where $v_1v_2v_3v_4$ is the link of v, and we let $vv_4p_1q_1p_2q_2 \cdots q_{m-1}p_mv_2$ be the link of v_1. In the following, we suppose that H is the k-chromatic even triangulation on $\mathbb{P}$ obtained from G by the 4-contraction.

If $\deg_G(v_1) = 2$, then we can apply the 2-contraction of v_1. So we let $H = G - v_1$ and apply the same argument as above. Hence we may suppose that $\deg_G(v_1) \geq 4$, and apply induction on $\deg_G(v_1)$. Let G' be the even triangulation obtained from G by an N-flip of the path $vv_4v_1p_1$. Since the 4-contraction of v at $\{v_1, v_3\}$ is applicable, p_1 and q_1 do not coincide with v_3 in G, and hence G' has no loops and no pinched vertices. (See Figure 10. In the figure, G'' is obtained from G' by an N-flip which is similar to the N-flip between G and G'. By their several N-flips, finally, we can obtain $H + D_2(2)$ from G.) If $\chi(G') = k$, then we are done, by induction hypothesis, since G' is a k-chromatic even triangulation with $\deg_{G'}(v_1) < \deg_G(v_1)$, which admits a 4-contraction of v at $\{v_1, v_3\}$. (Note that the map obtained from G' by the 4-contraction at $\{v_1, v_3\}$ is nothing but H.)

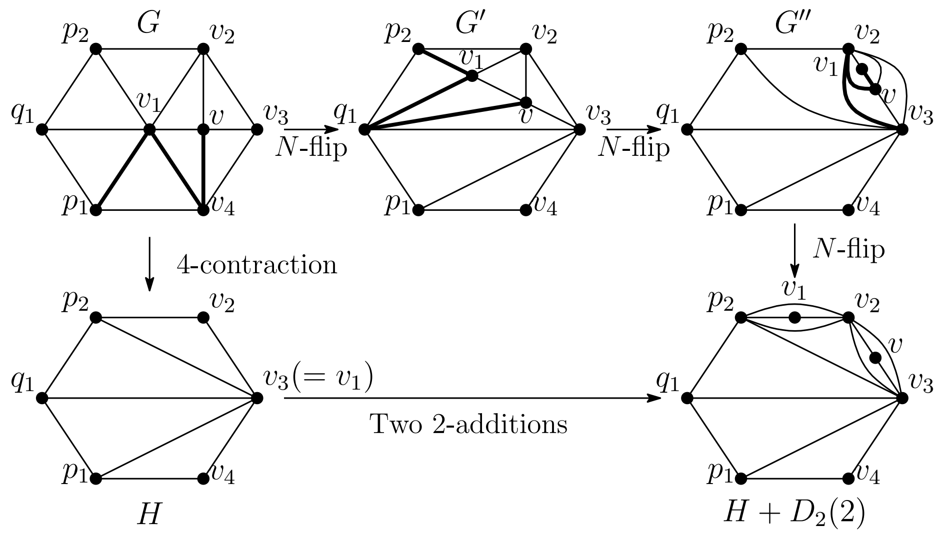

Figure 10 Transformations from G to $H + D_2(2)$

So, for each $k \in \{3, 4, 5\}$, we shall prove that $\chi(G') = k$ when $\deg_G(v_1) \geq 4$, using the assumption that $\chi(H) = k$.

<u>**Case 1** : $k \leq 4$.</u> Since an N-flip always preserves 3-chromaticity and the non-3-colorability, we show that G' is 4-colorable. Observe that $\deg_G(v_1) \geq 4$ by the assumption. Since H has a 4-coloring c_H, G has a 4-coloring c such that

(i) $c(x) = c_H(x)$ for any $x \in V(G) - \{v, v_1, v_3\}$,

(ii) $c(v_1) = c(v_3) = c_H([v_1v_3])$, where $[v_1v_3] \in V(H)$ is the image of the identification $v_1 = v_3$,

(iii) $c(v)$ is a color distinct from the three or two colors $c(v_1) = c(v_3)$, $c(v_2)$ and $c(v_4)$.

Observe that $c(v_3) \neq c(p_1)$ and $c(v_3) \neq c(q_1)$, since c_H is a proper 4-coloring of H. If $c(v) \neq c(q_1)$ in G, then G' has a 4-coloring. Otherwise, i.e., if $c(v) = c(q_1)$ in G, then we can recolor v in G' to construct a 4-coloring of G' so that $c(v)$ has a color distinct from the three colors $c(v_1) = c(v_3)$, $c(v_2)$ and $c(q_1)$, since $c(v_1) = c(v_3)$ by the assumption.

<u>**Case 2** : $k = 5$.</u> In this case, since every even triangulation on $\mathbb{P}$ is 5-colorable (Theorem 5), we prove that the chromatic number does not decrease by the N-flip in G to obtain G'. In order to do so, we carefully choose the 4-vertex v, as follows:

By Theorem 5, G is the face subdivision of an even embedding K, and K includes a non-bipartite quadrangulation Q as a subgraph. We suppose that Q is *minimal*, i.e., Q has no quadrangulation as a proper subgraph. Then, for any 4-vertex $u \in V(G)$, we have $u \notin V(Q)$. For, if $u \in V(Q)$, then $\deg_Q(u) \leq \frac{\deg_G(u)}{2} = 2$, and this contradicts Lemma 9.

We first consider the case when G has a 4-vertex, say v, such that at least one of the four neighbors of v, say v_1, is not contained in Q, where we let $v_1v_2v_3v_4$ be the link of v. Let G' be the even triangulation as before, and let K' be the even embedding on $\mathbb{P}$ such that $G' = \mathrm{FS}(K')$. Since $v, v_1 \notin V(Q)$, $G - \{v, v_1\}$ also contains Q as a subgraph, since the N-flip in G moves only edges incident to v or v_1 to obtain G_1. By Lemma 6, K' has Q as a subgraph, and hence we have $\chi(G') = 5$. Similarly to the previous case, we can apply the induction hypothesis to G'.

Secondly we suppose that every 4-vertex of G has four neighbors in Q. In this case, we see that all the four neighbors are distinct in Q. Hence G is the face subdivision of the quadrangulation $K = Q$. Then G' is the face subdivision of a non-bipartite quadrangulation Q' on $\mathbb{P}$ which is obtained from Q by a single diagonal slide. See Figure 11, where the thick segments stand for edges of Q and Q'. Hence G' is also 5-chromatic, by Theorem 5, and we are done. ■

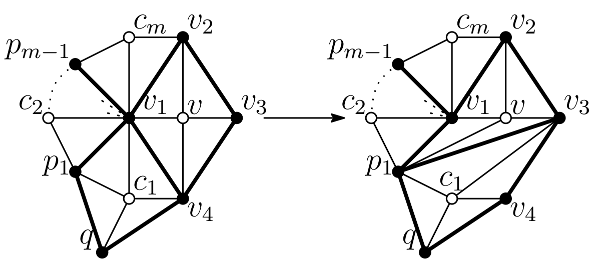

Figure 11 An N-flip preserving 5-chromaticity

Now we shall prove Theorem 4.

Proof of Theorem 4. Let G be an even triangulation on $\mathbb{P}$ with $\chi(G) = k \in \{3, 4, 5\}$. By Theorem 7, G can be transformed into I_k by 4- and 2-contractions preserving the k-chromaticity. Since 2-vertices can be moved to any place preserving the chromatic number, we can apply Lemma 10 repeatedly. (Note that 2-vertices do not disturb N-flips in Lemma 10.) Hence, G can be transformed into $I_k + D_2(m)$ by N- and D_2-flips preserving the chromatic number, where $m = |V(G)| - |V(I_k)|$. Thus, G' can similarly be transformed into $I_k + D_2(m)$, since $|V(G)| = |V(G')|$ and $\chi(G) = \chi(G')$. Therefore, G and G' can be transformed into each other, via the standard form $I_k + D_2(m)$, preserving the k-chromaticity. ■

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