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STATIONARY MEASURES OF THREE-STATE QUANTUM WALKS ON THE ONE-DIMENSIONAL LATTICE

By

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Abstract. In this paper, we consider stationary measures of discrete-time three-state quantum walks including the Fourier and Grover walks in the one-dimensional lattice. We give non-uniform stationary measures by solving the corresponding eigenvalue problem. Our new method is based on a reduced matrix, which is different from the generating function approach in our previous work. As a corollary, the Fourier walk on the cycle has a stationary measure with a periodicity.

1. Introduction

The notion of discrete-time quantum walks was introduced by Aharonov et al. [1] as a quantum analog of the classical one-dimensional random walks. It is known that the long-time asymptotic behavior of the transition probability for quantum walks on the one-dimensional lattice is quite different from that of classical random walks [11]. Recently, the quantum walk is intensively studied in quantum physics and quantum computing [15], [16].

One of the basic interests for quantum walks is to determine stationary measures of quantum walks. The stationary measures of Markov chains have been intensively investigated, however, the corresponding study of quantum walks has not been done enough. In 2013, Konno et al. [13] treated two-state quantum walks with one defect at the origin and showed that a stationary measure with exponential decay with respect to the location for the quantum walk starting from infinite sites is identical to a time-averaged limit measure for the same quantum walk starting from just the origin. Endo et al. [8] got a stationary measure of the quantum walk with one defect whose coin matrices are defined by the Hadamard matrix. Endo and Konno [6] calculated a stationary measure of quantum walk with one defect which was introduced by Wojcik et al. [21]. The

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stationary measure of the two-phase quantum walk with one defect and without defect was obtained by Endo et al. [3] and Endo et al. [7], respectively.

Konno and Takei [14] showed that the set of uniform measures is contained in the set of stationary measures and gave non-uniform stationary measures. Konno [12] obtained stationary measures of the three-state Grover walk. Wang et al. [19] investigated stationary measures of the three-state Grover walk with one defect at the origin. Furthermore, Endo et al. [5] clarified a relation between stationary and limit measures of the three-state Grover walk. Endo et al. [4] obtained stationary measures for the diagonal quantum walks with one defect or without defect including the three-state model. These previous works are studied by using the generating function method. On the other hand, we analyze the three-state quantum walk via the reduced matrix.

This paper is organized as follows. Section 2 is devoted to the definition of three-state discrete-time quantum walks on the one-dimensional integer lattice. In Section 3, we introduce a new method based on a reduced matrix, which is different from the generating function approach in our previous work. In Sections 4 and 5, we obtain stationary measures of Types 1 and 2 by solving the eigenvalue problem, respectively. Moreover, we give their typical examples. Conclusions are given in Section 6.

2. Three-state discrete-time quantum walks

In this section, we give the definition of three-state quantum walk on $\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. It is well known that the long-time asymptotic behavior of the transition probability for the three-state Grover walk on $\mathbb{Z}$ shows *localization* [10]. This phenomenon is one of typical properties for discrete-time quantum walks [9], [10], [18], [20] which is not seen for usual classical random walks.

The discrete-time quantum walk on $\mathbb{Z}$ defined by a unitary matrix;

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

We call this unitary matrix the coin matrix. To consider the time evolution, decompose the matrix A as

$$A = P + R + Q$$

with

$$P = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

A particle in the classical random walk moves at each step either one unit to the right with probability p or one unit to the left with probability q, where $p+q=1$, p, $q>0$. On the other hand, the discrete-time quantum walk describes not only the motion of a particle but also the change of the states of a particle. Let $\mathbb{C}$ be the set of complex numbers. The state at time n and location x can be expressed by a three-dimensional vector:

$$\Psi_n(x) = \begin{bmatrix} \Psi_n^L(x) \\ \Psi_n^O(x) \\ \Psi_n^R(x) \end{bmatrix} \in \mathbb{C}^3 \quad (x \in \mathbb{Z},\ n \in \mathbb{Z}_{\geq}),$$

where $\mathbb{Z}_{\geq} = \{0, 1, 2, \ldots\}$. The time evolution of a quantum walk with a coin matrix A is defined by the unitary operator U_A in the following way:

$$\Psi_{n+1}(x) \equiv (U_A \Psi_n)(x) = P\Psi_n(x+1) + R\Psi_n(x) + Q\Psi_n(x-1).$$

This equation means that the particle moves at each step one unit to the right with matrix P or one unit to the left with matrix Q. The particle stays at the current location with matrix R. For time $n \in \mathbb{Z}_{\geq}$ and location $x \in \mathbb{Z}$, we define the measure $\mu_n(x)$ by

$$\mu_n(x) = \|\Psi_n(x)\|^2_{\mathbb{C}^3},$$

where $\|\cdot\|_{\mathbb{C}^3}$ denotes the standard norm on $\mathbb{C}^3$. Our model here is considered on the set of all the $\mathbb{C}^3$-valued functions on $\mathbb{Z}$ whose inner product is $\langle \Psi, \Phi \rangle = \sum_{x\in\mathbb{Z}} \langle \Psi(x), \Phi(x) \rangle_{\mathbb{C}^3}$, where $\langle \cdot, \cdot \rangle_{\mathbb{C}^3}$ denotes the standard inner product on $\mathbb{C}^3$. We do not have any restrictions about the norm. Remark that this is isomorphic to $(\mathbb{C}^3)^{\mathbb{Z}}$. We are interested in the measure ϕ induced by $\Psi \in (\mathbb{C}^3)^{\mathbb{Z}}$ such that $\phi : (\mathbb{C}^3)^{\mathbb{Z}} \longrightarrow (\mathbb{R}_{\geq})^{\mathbb{Z}}$ with $(\phi(\Psi))(x) = \|\Psi(x)\|^2_{\mathbb{C}^3}$. Let $\mathbb{R}_{\geq} = [0, \infty)$. Here we introduce a map $\phi : (\mathbb{C}^3)^{\mathbb{Z}} \longrightarrow (\mathbb{R}_{\geq})^{\mathbb{Z}}$ such that if

$$\Psi_n = {}^T\left[\cdots, \begin{bmatrix} \Psi_n^L(-1) \\ \Psi_n^O(-1) \\ \Psi_n^R(-1) \end{bmatrix}, \begin{bmatrix} \Psi_n^L(0) \\ \Psi_n^O(0) \\ \Psi_n^R(0) \end{bmatrix}, \begin{bmatrix} \Psi_n^L(1) \\ \Psi_n^O(1) \\ \Psi_n^R(1) \end{bmatrix}, \cdots\right] \in (\mathbb{C}^3)^{\mathbb{Z}},$$

then

$$\phi(\Psi_n) = {}^T\left[\cdots, \sum_{j=L}^R |\Psi_n^j(-1)|^2, \sum_{j=L}^R |\Psi_n^j(0)|^2, \sum_{j=L}^R |\Psi_n^j(1)|^2, \cdots\right] \in (\mathbb{R}_{\geq})^{\mathbb{Z}}.$$

Thus for any $x \in \mathbb{Z}$, we get

$$\phi(\Psi_n)(x) = \phi(\Psi_n(x)) = \sum_{j=L}^{R} |\Psi_n^j(x)|^2 = \mu_n(x).$$

3. Stationary measure and our method

3.1 Definition of stationary measure for quantum walk

In this subsection, we give the definition of stationary measure for the quantum walk. We define a set of measures, $\mathcal{M}_s(U_A)$, by

$$\mathcal{M}_s(U_A) = \Big\{\mu \in (\mathbb{R}_{\geq})^{\mathbb{Z}} \setminus \{\mathbf{0}\};\ \textit{there exists}\ \Psi_0 \in (\mathbb{C}^3)^{\mathbb{Z}}\ \textit{such that}\ \phi\big(U_A^n \Psi_0\big) = \mu\ \big(n = 0, 1, 2, \dots\big)\Big\}.$$

where $\mathbf{0}$ is the zero vector. Here U_A is time evolution operator of quantum walk associated with a unitary matrix A. We call this measure $\mu \in \mathcal{M}_s(U_A)$ the stationary measure for the quantum walk defined by the unitary operator U_A. If $\mu \in \mathcal{M}_s(U_A)$, then $\mu_n = \mu$ for $n \in \mathbb{Z}_{\geq}$, where μ_n is the measure of quantum walk given by U_A at time n.

Next we consider the following eigenvalue problem of the quantum walk determined by U_A:

$$U_A \Psi = \lambda \Psi \quad (\lambda \in \mathbb{C},\ |\lambda| = 1). \tag{3.1}$$

Then we see that $\phi(\Psi) \in \mathcal{M}_s(U_A)$. Our purpose of this paper is to find stationary measures for our three-state quantum walks by using Eq. (3.1).

3.2 Our method

In this subsection, we introduce our new method to obtain the stationary measures for the three-state quantum walks. First, we see that $U_A\Psi = \lambda\Psi$ is equivalent to the following relations:

$$\begin{cases} \lambda\Psi^L(x) = a_{11}\Psi^L(x+1) + a_{12}\Psi^O(x+1) + a_{13}\Psi^R(x+1), \\ \lambda\Psi^O(x) = a_{21}\Psi^L(x) + a_{22}\Psi^O(x) + a_{23}\Psi^R(x), \\ \lambda\Psi^R(x) = a_{31}\Psi^L(x-1) + a_{32}\Psi^O(x-1) + a_{33}\Psi^R(x-1). \end{cases} \tag{3.2}$$

Then we can rewrite Eq. (3.2) as

$$\begin{cases} \Psi^O(x) = \frac{1}{\lambda - a_{22}}\Big\{a_{21}\Psi^L(x) + a_{23}\Psi^R(x)\Big\}, \\ \lambda\Psi^L(x) = \Big(a_{11} + \frac{a_{12}a_{21}}{\lambda - a_{22}}\Big)\Psi^L(x+1) + \Big(a_{13} + \frac{a_{12}a_{23}}{\lambda - a_{22}}\Big)\Psi^R(x+1), \\ \lambda\Psi^R(x) = \Big(a_{31} + \frac{a_{21}a_{32}}{\lambda - a_{22}}\Big)\Psi^L(x-1) + \Big(a_{33} + \frac{a_{23}a_{32}}{\lambda - a_{22}}\Big)\Psi^R(x-1). \end{cases} \tag{3.3}$$

Since $\Psi^O(x)$ is expressed by $\Psi^L(x)$ and $\Psi^R(x)$ from Eq. (3.3), we consider only $\Psi^L(x)$ and $\Psi^R(x)$. Thus we get

$$\begin{aligned} \lambda \begin{bmatrix} \Psi^L(x) \\ \Psi^R(x) \end{bmatrix} &= \begin{bmatrix} a_{11} + \frac{a_{12}a_{21}}{\lambda - a_{22}} & a_{13} + \frac{a_{12}a_{23}}{\lambda - a_{22}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Psi^L(x+1) \\ \Psi^R(x+1) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 \\ a_{31} + \frac{a_{21}a_{32}}{\lambda - a_{22}} & a_{33} + \frac{a_{23}a_{32}}{\lambda - a_{22}} \end{bmatrix} \begin{bmatrix} \Psi^L(x-1) \\ \Psi^R(x-1) \end{bmatrix}. \end{aligned} \tag{3.4}$$

Here we introduce a 2×2 reduced matrix $A^{(Re)}$ derived from our 3×3 coin matrix A as follows:

$$A^{(Re)} = \frac{1}{\lambda - a_{22}} \begin{bmatrix} \lambda a_{11} - B & \lambda a_{13} + C \\ \lambda a_{31} + D & \lambda a_{33} - E \end{bmatrix}, \tag{3.5}$$

where we put

$$B = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right), \quad C = \det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}\right),$$
$$D = \det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right), \quad E = \det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right).$$

We should remark that Eq. (3.4) can be expressed as follows by using $A^{(Re)}$:

$$\lambda \begin{bmatrix} \Psi^L(x) \\ \Psi^R(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} A^{(Re)} \begin{bmatrix} \Psi^L(x+1) \\ \Psi^R(x+1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} A^{(Re)} \begin{bmatrix} \Psi^L(x-1) \\ \Psi^R(x-1) \end{bmatrix}.$$

To obtain stationary measures of the quantum walk given by the coin matrix A, we focus on the reduced matrix $A^{(Re)}$ given by Eq. (3.5).

From now on, we treat the two classes of three-state quantum walks, i.e., Type 1 and Type 2. We suppose that

$$a_{ij} \neq 0 \quad (1 \le i, j \le 3), \qquad |a_{22}| \neq 1.$$

- Type 1 : We assume that $\lambda = -\frac{C}{a_{13}} = -\frac{D}{a_{31}}$ with $|\lambda| = 1$. Then the reduced matrix $A^{(Re)}$ is

$$A^{(Re)} = \begin{bmatrix} \tilde{a}_1 & 0 \\ 0 & \tilde{a}_2 \end{bmatrix}, \tag{3.6}$$

where

$$\tilde{a}_1 = a_{11} - \frac{a_{13}a_{21}}{a_{23}}, \quad \tilde{a}_2 = a_{33} - \frac{a_{23}a_{31}}{a_{21}}.$$

- Type 2 : We assume that $\lambda = \frac{B}{a_{11}} = \frac{E}{a_{33}}$ with $|\lambda| = 1$. Then the reduced matrix $A^{(Re)}$ is

$$A^{(Re)} = \begin{bmatrix} 0 & \tilde{a}_1 \\ \tilde{a}_2 & 0 \end{bmatrix},$$

where

$$\tilde{a}_1 = a_{13} - \frac{a_{11}a_{23}}{a_{21}}, \quad \tilde{a}_2 = a_{31} - \frac{a_{21}a_{33}}{a_{23}}.$$

4. Result of Type 1

In the previous section, we introduced a reduced matrix $A^{(Re)}$ to obtain stationary measures of quantum walks for Type 1 and Type 2. In this section, by using $A^{(Re)}$, we present a solution of eigenvalue problem, $U_A\Psi = \lambda\Psi$, for the three-state quantum walk of Type 1. Then, we have the following result which gives an explicit form of the solution Ψ of the eigenvalue problem.

Theorem 4.1. *We assume that $\lambda = -\frac{C}{a_{13}} = -\frac{D}{a_{31}}$ with $|\lambda| = 1$. Then we get*

$$\Psi(x) = \begin{bmatrix} \left(\tilde{a}_1^{-1}\lambda\right)^x \varphi_1 \\ -\frac{a_{13}}{a_{12}a_{23}}\left\{a_{21}\left(\tilde{a}_1^{-1}\lambda\right)^x \varphi_1 + a_{23}\left(\tilde{a}_2\lambda^{-1}\right)^x \varphi_3\right\} \\ \left(\tilde{a}_2\lambda^{-1}\right)^x \varphi_3 \end{bmatrix}, \tag{4.7}$$

for arbitrary φ_1, $\varphi_3 \in \mathbb{C}$, where

$$\tilde{a}_1 = a_{11} - \frac{a_{13}a_{21}}{a_{23}}, \qquad \tilde{a}_2 = a_{33} - \frac{a_{23}a_{31}}{a_{21}}.$$

Proof. Suppose that $a_{11} \neq 0$, $a_{33} \neq 0$ and $|\lambda| = 1$. From Eq. (3.4) and assumptions, $\lambda = -\frac{C}{a_{13}} = -\frac{D}{a_{31}}$ with $|\lambda| = 1$, we get

$$\lambda \begin{bmatrix} \Psi^L(x) \\ \Psi^R(x) \end{bmatrix} = \begin{bmatrix} \tilde{a}_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Psi^L(x+1) \\ \Psi^R(x+1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{a}_2 \end{bmatrix} \begin{bmatrix} \Psi^L(x-1) \\ \Psi^R(x-1) \end{bmatrix} = \begin{bmatrix} \tilde{a}_1\Psi^L(x+1) \\ \tilde{a}_2\Psi^R(x-1) \end{bmatrix}.$$

Then we have

$$\Psi^L(x+1) = \left(\tilde{a}_1^{-1}\lambda\right)\Psi^L(x), \quad \Psi^R(x) = \left(\tilde{a}_2\lambda^{-1}\right)\Psi^R(x-1),$$

$$\Psi^O(x) = \frac{1}{\lambda - a_{22}}\left\{a_{21}\Psi^L(x) + a_{23}\Psi^R(x)\right\}.$$

Therefore a solution Ψ of $U_A\Psi = \lambda\Psi$ is given by

$$\Psi(x) = \begin{bmatrix} \left(\tilde{a}_1^{-1}\lambda\right)^x \varphi_1 \\ -\frac{a_{13}}{a_{12}a_{23}}\left\{a_{21}\left(\tilde{a}_1^{-1}\lambda\right)^x \varphi_1 + a_{23}\left(\tilde{a}_2\lambda^{-1}\right)^x \varphi_3\right\} \\ \left(\tilde{a}_2\lambda^{-1}\right)^x \varphi_3 \end{bmatrix} \quad (x \in \mathbb{Z}),$$

where $\Psi^L(0) \equiv \varphi_1$, $\Psi^R(0) \equiv \varphi_3$ and $\varphi_1,\ \varphi_3 \in \mathbb{C}^2$ with $|\varphi_1| + |\varphi_3| > 0$. This completes the proof of Theorem 4.1.

We should remark that Theorem 4.1 implies

$$U_A\Psi = \lambda\Psi.$$

Here Ψ can be rewritten as

$$\Psi(x) = \varphi_1\alpha_1(x) + \varphi_3\alpha_3(x),$$

where

$$\alpha_1(x) = \begin{bmatrix} \left(\tilde{a}_1^{-1}\lambda\right)^x \\ -\frac{a_{13}a_{21}}{a_{12}a_{23}}\left(\tilde{a}_1^{-1}\lambda\right)^x \\ 0 \end{bmatrix}, \qquad \alpha_3(x) = \begin{bmatrix} 0 \\ -\frac{a_{13}}{a_{12}}\left(\tilde{a}_2\lambda^{-1}\right)^x \\ \left(\tilde{a}_2\lambda^{-1}\right)^x \end{bmatrix}. \tag{4.8}$$

Furthermore, we see that

$$U_A\alpha_j = \lambda\alpha_j \qquad (j = 1, 3).$$

In general, we can notice that α_j is a bounded function, however, not necessarily square summable function on $\mathbb{Z}$, that is, $\alpha_j \in \ell^\infty(\mathbb{Z}, \mathbb{C}^3)$ but $\alpha_j \notin \ell^2(\mathbb{Z}, \mathbb{C}^3)$.

Next we present some examples by Theorem 4.1.

Example 4.1. We consider the three-state Grover walk given by the 3×3 Grover matrix as

$$A_G = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}. \tag{4.9}$$

When we choose the value of $\lambda = -1$, then $\tilde{a}_1 = \tilde{a}_2 = -1$ and the reduced matrix becomes

$$A_G^{(Re)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.10}$$

From Theorem 4.1 and Eq. (4.10), we get

$$\Psi(x) = \begin{bmatrix} \varphi_1 \\ -(\varphi_1 + \varphi_3) \\ \varphi_3 \end{bmatrix} \quad (x \in \mathbb{Z}).$$

Let $\mathfrak{Re}(z)$ be the real part of complex number z. So we have a stationary measure of the Grover walk as

$$\mu(x) = 2\{|\varphi_1|^2 + |\varphi_3|^2 + \mathfrak{Re}(\varphi_1 \overline{\varphi_3})\}.$$

In this example, the stationary measure is a uniform measure.

Example 4.2. We consider the three-state Fourier walk defined by the 3×3 Fourier matrix as

$$A_F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \tag{4.11}$$

where $\omega = e^{\frac{2\pi i}{3}}$. When we choose the value of $\lambda = i$, then $\tilde{a}_1 = e^{-\frac{\pi i}{6}}$, $\tilde{a}_2 = -i$ and the reduced matrix becomes

$$A_F^{(Re)} = \begin{bmatrix} e^{-\frac{\pi i}{6}} & 0 \\ 0 & i \end{bmatrix}. \tag{4.12}$$

It follows from Theorem 4.1 and Eq. (4.12) that

$$\Psi(x) = \begin{bmatrix} \omega^x \varphi_1 \\ -(\omega^{x+1} \varphi_1 + \varphi_3) \\ \varphi_3 \end{bmatrix} \quad (x \in \mathbb{Z}). \tag{4.13}$$

Thus a stationary measure of the Fourier walk is given by

$$\mu(x) = 2\{|\varphi_1|^2 + |\varphi_3|^2 + \mathfrak{Re}(\omega^{x+1} \varphi_1 \overline{\varphi_3})\}.$$

If we take $\varphi_1 = \omega$, $\varphi_3 = \omega^2$, then

$$\mu(x) = 2\{2 + \mathfrak{Re}(\omega^x)\} = \begin{cases} 6, & x = 3m \quad (m \in \mathbb{Z}_{\geq}), \\ 3, & x = 3m+1,\ 3m+2 \quad (m \in \mathbb{Z}_{\geq}). \end{cases}$$

Therefore the stationary measure of the Fourier walk is not the uniform measure. Furthermore, the measure has period 3. This is the first result on the Fourier walk that the stationary measure has a periodicity. Moreover, we can apply this example to the three-state Fourier walk on the cycle with $3m$ nodes ($m \in \mathbb{Z}_>$) and get the stationary measure with period 3. More precisely, it is shown that the solution of the eigenvalue problem $U_A\Psi = \lambda\Psi$ for the three-state Fourier walk on $\mathbb{Z}$ given by Eq. (4.13) implies a solution of the eigenvalue problem on C_{3m} ($m \in \mathbb{Z}_>$) with the following boundary conditions:

$$\begin{cases} \sqrt{3}i\Psi^R(0) = \Psi^L(3m-1) + \omega^2\Psi^O(3m-1) + \omega\Psi^R(3m-1), \\ \sqrt{3}i\Psi^L(3m-1) = \Psi^L(0) + \omega^2\Psi^O(0) + \omega\Psi^R(0). \end{cases}$$

Here C_N is the cycle where the number of the vertices is N. Therefore we have

Corollary 4.2. *We consider the three-state Fourier walk defined by Eq.*(4.11) *on a cycle with* $3m$ ($m \in \mathbb{Z}_\geq$) *nodes. Then the stationary measure of this quantum walk has the stationary measure with period* 3.

Note that as in the case of $\mathbb{Z}$, we see

$$U_A\Psi = i\Psi,$$

where Ψ is a linear combination of two eigenfunctions $\alpha_1(x)$ and $\alpha_3(x)$ given by Eq. (4.8) whose eigenvalues are i.

Example 4.3. We consider a class of quantum walks determined by the 3×3 unitary matrices $A_1(\eta)$ ($\eta \in [0, 2\pi)$) introduced by Stefanak et al. [17] as

$$A_1(\eta) = \frac{1}{6}\begin{bmatrix} -1-e^{2i\eta} & 2(1+e^{2i\eta}) & 5-e^{2i\eta} \\ 2(1+e^{2i\eta}) & 2(1-2e^{2i\eta}) & 2(1+e^{2i\eta}) \\ 5-e^{2i\eta} & 2(1+e^{2i\eta}) & -1-e^{2i\eta} \end{bmatrix}, \tag{4.14}$$

where $\eta \in [0, 2\pi)$. Note that the quantum walk determined by $A_1(0)$ becomes the Grover walk. When we choose the value of

$$\lambda = \frac{10 - 26\cos(2\eta) - 24i\sin(2\eta)}{26 - 10\cos(2\eta)} \equiv e^{i\xi}, \qquad \cos\xi = \frac{10 - 26\cos(2\eta)}{26 - 10\cos(2\eta)},$$

then $\tilde{a}_1 = \tilde{a}_2 = -1$ and the reduced matrix becomes

$$A_1(\eta)^{(Re)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

From Theorem 4.1, we obtain

$$\Psi(x) = \begin{bmatrix} (-\lambda)^x\varphi_1 \\ -\left(1 - \frac{3}{2}\tan\eta\cdot i\right)\left\{(-\lambda)^x\varphi_1 + \left(-\overline{\lambda}\right)^x\varphi_3\right\} \\ (-\overline{\lambda})^x\varphi_3 \end{bmatrix} \quad (x \in \mathbb{Z}).$$

Especially, we set $\varphi_1 = \varphi_3$ and $T_x(\cos\xi) = \cos(x\xi)$, where $T_x(u)$ is the Chebyshev polynomial of the first kind. Then we have

$$\begin{aligned}|\Psi^O(x)|^2 &= \left|-\left(1-\frac{3}{2}\tan\eta\cdot i\right)(-1)^x\left(\lambda^x+\overline{\lambda}^x\right)\varphi_1\right|^2 \\ &= \left|-\left(2-3\tan\eta\cdot i\right)(-1)^x T_x\left(\cos\xi\right)\varphi_1\right|^2 \\ &= \left(4+9\tan^2\eta\right)T_x^2\left(\cos\xi\right)\left|\varphi_1\right|^2,\end{aligned}$$

Therefore we get

$$\mu(x) = \Big\{2 + (4+9\tan^2\eta)T_x^2(\cos\xi)\Big\}|\varphi_1|^2.$$

Example 4.4. We consider a class of quantum walks determined by the 3×3 unitary matrices $A_2(\rho)$ ($\rho \in (0,1)$) introduced by Stefanak et al. [17] as

$$A_2(\rho) = \begin{bmatrix} -\rho^2 & \rho\sqrt{2(1-\rho^2)} & 1-\rho^2 \\ \rho\sqrt{2(1-\rho^2)} & 2\rho^2-1 & \rho\sqrt{2(1-\rho^2)} \\ 1-\rho^2 & \rho\sqrt{2(1-\rho^2)} & -\rho^2 \end{bmatrix}, \qquad (4.15)$$

where $\rho \in (0,1)$. Remark that the quantum walk determined by the matrix $A_2(1/\sqrt{3})$ becomes the Grover walk.

When we choose the value of $\lambda = -1$, then $\tilde{a}_1 = \tilde{a}_2 = -1$ and the reduced matrix becomes

$$A_2(\rho)^{(Re)} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

So we have

$$\Psi(x) = \begin{bmatrix} \varphi_1 \\ -\frac{\sqrt{1-\rho^2}}{\sqrt{2}\rho}(\varphi_1+\varphi_3) \\ \varphi_3 \end{bmatrix} \quad (x \in \mathbb{Z}).$$

Therefore we obtain

$$\mu(x) = \frac{1+\rho^2}{2\rho^2}\Big(|\varphi_1|^2 + |\varphi_3|^2\Big) + \frac{1-\rho^2}{\rho^2}\mathfrak{Re}(\varphi_1\overline{\varphi_3}).$$

We see that this stationary measure is a uniform measure.

5. Result of Type 2

In the previous section, we obtained the stationary measure for the three-state quantum walk (Type 1) determined by the reduced matrix $A^{(Re)}$ whose off-diagonal component is zero (i.e., diagonal matrix). This section deals with the stationary measure for the three-state quantum walk (Type 2) given by the reduced matrix $A^{(Re)}$ whose diagonal component is zero. By using $A^{(Re)}$, we present a solution of $U_A\Psi = \lambda\Psi$, for the quantum walk of Type 2 for $\Psi^L(x) = \varphi_x \in \mathbb{C}$ $(x \in \mathbb{Z})$ with $\varphi \not\equiv \mathbf{0}$. Here $\varphi \equiv \mathbf{0}$ means that $\varphi_x = 0$ $(x \in \mathbb{Z})$. The following result for Type 2 is a counterpart of Theorem 4.1 for Type 1.

Theorem 5.1. *We assume that* $\lambda = \frac{B}{a_{11}} = \frac{E}{a_{33}}$ *with* $|\lambda| = 1$ *and* $\lambda^2 = \tilde{a}_1\tilde{a}_2$. *Let* $\{\varphi_x\}_{x\in\mathbb{Z}}$ *be arbitrary sequence of complex numbers except for* $\varphi \equiv \mathbf{0}$. *Then we get*

$$\Psi(x) = \begin{bmatrix} \varphi_x \\ -\frac{a_{11}}{a_{12}a_{21}}\left\{a_{21}\varphi_x + a_{23}\left(\tilde{a}_1^{-1}\lambda\right)\varphi_{x-1}\right\} \\ \left(\tilde{a}_1^{-1}\lambda\right)\varphi_{x-1} \end{bmatrix} \quad (x \in \mathbb{Z}),$$

where

$$\tilde{a}_1 = a_{13} - \frac{a_{11}a_{23}}{a_{21}}, \qquad \tilde{a}_2 = a_{31} - \frac{a_{21}a_{33}}{a_{23}}.$$

Proof. Suppose that $\lambda = \frac{B}{a_{11}} = \frac{E}{a_{33}}$ with $|\lambda| = 1$. From Eq. (3.4) and assumptions $\lambda = \frac{B}{a_{11}} = \frac{E}{a_{33}}$ with $|\lambda| = 1$, we get

$$\lambda \begin{bmatrix} \Psi^L(x) \\ \Psi^R(x) \end{bmatrix} = \begin{bmatrix} 0 & \tilde{a}_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Psi^L(x+1) \\ \Psi^R(x+1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \tilde{a}_2 & 0 \end{bmatrix} \begin{bmatrix} \Psi^L(x-1) \\ \Psi^R(x-1) \end{bmatrix} = \begin{bmatrix} \tilde{a}_1\Psi^R(x+1) \\ \tilde{a}_2\Psi^L(x-1) \end{bmatrix}.$$

Then we have

$$\Psi^L(x) = \lambda^{-1}\tilde{a}_1\Psi^R(x+1), \quad \Psi^R(x) = \lambda^{-1}\tilde{a}_2\Psi^L(x-1). \tag{5.16}$$

From Eq. (5.16), we see that this quantum walk (Type 2) must satisfy the condition $\lambda^2 = \tilde{a}_1\tilde{a}_2$. Let $\{\varphi_x\}_{x\in\mathbb{Z}}$ be arbitrary sequence of complex numbers except for $\varphi \equiv \mathbf{0}$. Put $\Psi^L(x) = \varphi_x$. Therefore we obtain

$$\Psi(x) = \begin{bmatrix} \varphi_x \\ -\frac{a_{11}}{a_{12}a_{21}}\left\{a_{21}\varphi_x + a_{23}\left(\tilde{a}_1^{-1}\lambda\right)\varphi_{x-1}\right\} \\ \left(\tilde{a}_1^{-1}\lambda\right)\varphi_{x-1} \end{bmatrix} \quad (x \in \mathbb{Z}),$$

where

$$\tilde{a}_1 = a_{13} - \frac{a_{11}a_{23}}{a_{21}}, \quad \tilde{a}_2 = a_{31} - \frac{a_{21}a_{33}}{a_{23}}.$$

The proof of Theorem 5.1 is complete.

We should remark that Ψ in Theorem 5.1 can be rewritten as

$$\Psi(x) = \varphi_x \gamma_1 + \varphi_{x-1}\gamma_3,$$

where

$$\gamma_1 = \begin{bmatrix} 1 \\ -\frac{a_{11}}{a_{12}} \\ 0 \end{bmatrix}, \qquad \gamma_3 = \begin{bmatrix} 0 \\ -\frac{a_{11}a_{23}}{a_{12}a_{21}}\left(\tilde{a}_1^{-1}\lambda\right) \\ \left(\tilde{a}_1^{-1}\lambda\right) \end{bmatrix}.$$

For fixed $x_* \in \mathbb{Z}$, we let

$$\beta_{x_*}(x) = \delta_{x_*}(x)\gamma_1 + \delta_{x_*+1}(x)\gamma_3,$$

where $\delta_{x_*}$is the Dirac delta function. Thus we have

$$\beta_{x_*} \in \ell^2(\mathbb{Z}, \mathbb{C}^3),$$

since

$$\text{supp }\ (\beta_{x_*}) = \{x_*, x_* + 1\}.$$

Then the ℓ^2-element of the eigenspace $\langle \beta_{x_*}; x_* \in \mathbb{Z}\rangle$ spanned by $\{\beta_{x_*}; x_* \in \mathbb{Z}\}$ causes localization for the three-state quantum walks given by Type 2.

Example 5.1. We consider the Grover walk whose coin matrix is determined by the 3×3 unitary matrix A_G given by the matrix (4.9). When we choose the value of $\lambda = 1$, then $\tilde{a}_1 = \tilde{a}_2 = 1$ and the reduced matrix becomes

$$A_G^{(Re)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let $\{\varphi_x\}_{x\in\mathbb{Z}}$ be arbitrary sequence of complex numbers except for $\varphi \equiv \mathbf{0}$. Then we get

$$\Psi(x) = \begin{bmatrix} \varphi_x \\ \frac{1}{2}\Big(\varphi_x + \varphi_{x-1}\Big) \\ \varphi_{x-1} \end{bmatrix} \quad (x \in \mathbb{Z}). \tag{5.17}$$

This form was obtained by Cantero et al. [2] via a different approach; i.e., the CGMV method. From Eq. (5.17), we have

$$\mu(x) = \frac{5}{4}\Big(|\varphi_x|^2 + |\varphi_{x-1}|^2\Big) + \frac{1}{2}\mathfrak{Re}(\varphi_x\overline{\varphi_{x-1}}).$$

EXAMPLE 5.2. We consider a class of quantum walks determined by the 3×3 unitary matrix $A_1(\eta)$ given by the matrix (4.14). When we choose the value of $\lambda=1$, then $\tilde{a}_1=\tilde{a}_2=1$ and the reduced matrix becomes

$$A_1(\eta)^{(Re)}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

Let $\{\varphi_x\}_{x\in\mathbb{Z}}$ be arbitrary sequence of complex numbers except for $\varphi\equiv\mathbf{0}$. Then we obtain

$$\Psi(x)=\begin{bmatrix}\varphi_x\\ \frac{1}{2}\Big(\varphi_x+\varphi_{x-1}\Big)\\ \varphi_{x-1}\end{bmatrix}\quad(x\in\mathbb{Z}).$$

Therefore we get

$$\mu(x)=\frac{5}{4}\Big(|\varphi_x|^2+|\varphi_{x-1}|^2\Big)+\frac{1}{2}\mathfrak{Re}(\varphi_x\overline{\varphi_{x-1}}).$$

Interestingly, the stationary measure is independent of the parameter $\eta\in[0,2\pi)$.

EXAMPLE 5.3. We consider a class of quantum walks determined by the 3×3 unitary matrix $A_2(\rho)$ given by the matrix (4.15). When we choose the value of $\lambda=1$, then $\tilde{a}_1=\tilde{a}_2=1$ and the reduced matrix becomes

$$A_2(\rho)^{(Re)}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

Let $\{\varphi_x\}_{x\in\mathbb{Z}}$ be arbitrary sequence of complex numbers except for $\varphi\equiv\mathbf{0}$. Then we get

$$\Psi(x)=\begin{bmatrix}\varphi_x\\ \frac{\rho}{\sqrt{2(1-\rho^2)}}\Big(\varphi_x+\varphi_{x-1}\Big)\\ \varphi_{x-1}\end{bmatrix}\quad(x\in\mathbb{Z}).$$

Therefore we obtain

$$\mu(x)=\frac{2-\rho^2}{2(1-\rho^2)}\Big(|\varphi_x|^2+|\varphi_{x-1}|^2\Big)+\frac{\rho^2}{1-\rho^2}\mathfrak{Re}(\varphi_x\overline{\varphi_{x-1}}).$$

In the previous example (Example 5.2), the stationary measure does not depend on the parameter $\eta\in[0,2\pi)$, but the stationary measure in this example depends on the parameter $\rho\in(0,1)$.

In the rest of this section, we consider the 3×3 Fourier matrix defined by Eq. (4.11). Then we get

$$\lambda = -\omega^2 i, \quad \tilde{a}_1 = \tilde{a}_2 = -e^{\frac{\pi i}{6}}.$$

Thus we have $\lambda^2 \neq \tilde{a}_1\tilde{a}_2$ which does not satisfy the assumption $\lambda^2 = \tilde{a}_1\tilde{a}_2$ in Theorem 5.1. So we do not apply Theorem 5.1 to the Fourier walk. In fact, we get

$$\Psi^L(x) = \omega^2\Psi^R(x+1), \quad \Psi^L(x) = \omega\Psi^R(x+1).$$

Therefore the first equation is contradictory to the second one.

6. Conclusion

In this paper, we obtained the stationary measures for the three-state quantum walks including the Fourier and Grover walks on $\mathbb{Z}$ by using the corresponding reduced matrix $A^{(Re)}$. As a special case, we found a stationary measure with periodicity. Moreover, this periodic stationary measure is also stationary measure on cycles. Our results are summarized in the following two tables.

Table 1 Results of Type 1

Type 1	*Grover matrix*	*Fourier matrix*	$A_1(\eta)$	$A_2(\rho)$
reduced matrix	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} e^{-\frac{\pi i}{6}} & 0 \\ 0 & i \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
eigenvalue of U_A	-1	i	$e^{i\xi}$	-1
stationary measure	*uniform*	*non-uniform* (*with periodicity*)	*non-uniform*	*uniform*

Table 2 Results of Type 2

Type 2	*Grover matrix*	*Fourier matrix*	$A_1(\eta)$	$A_2(\rho)$
reduced matrix	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & e^{i\frac{\pi}{6}} \\ e^{i\frac{\pi}{6}} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
eigenvalue of U_A	1	*not applicable*	1	1
stationary measure	*non-uniform*	*not applicable*	*non-uniform*	*non-uniform*

It means that "stationary measure" in the tables is a measure given by the eigenfunction with eigenvalue of the unitary operator U_A. We note that the Fourier walk of Type 1 in Table 1 has a non-trivial stationary measure with periodicity, while the Fourier walk of Type 2 in Table 2 does not have the stationary measure given by solving the corresponding eigenvalue problem. Therefore, the meaning of "*not applicable*" in Table 2 is that we are not able to apply our new method based on a reduced matrix to the Fourier walk.

We should remark that localization of the three-state quantum walk with Type 1 does not occur, but localization with Type 2 occurs. Because amplitude Ψ in Theorem 4.1 for the three-state quantum walk with Type 1 is not a square summable function but a bounded function on $\mathbb{Z}$ in general. On the other hand, Ψ in Theorem 5.1 for the three-state quantum walk with Type 2 is a square summable function.

One of the future interesting problems would be to investigate the stationary measure for the general N-state quantum walk by using our new method introduced here.

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