Yokohama Mathematical
Journal Vol. 63, 2017

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A CLASS OF A SINGULAR FRACTIONAL INITIAL VALUE PROBLEM

By

Toshiharu Kawasaki and Masashi Toyoda

(Received March 6, 2017)

Abstract. In this paper we show the existence and uniqueness of solutions of the Cauchy problem

$$\begin{cases} D^{\nu}_{0+}u(t) = f(t, u(t)), \\ \lim\limits_{t\to 0+} u(t) = 0, \ \lim\limits_{t\to 0+} u'(t)t^{2-\nu} = (\nu - 1)\lambda, \end{cases}$$

where f is a mapping of $[0, 1] \times (0, \infty)$ into $\mathbb{R}$, $\lambda \in \mathbb{R}$ with $\lambda > 0$, $\nu \in \mathbb{R}$ with $1 < \nu \leq 2$ and D^{ν}_{0+} is the ν-th Riemann-Liouville fractional derivative.

1. Introduction and Preliminaries

In [8], Knežević-Miljanović considered the Cauchy problem for singular differential equations

$$\begin{cases} u''(t) = p(t)t^a u(t)^{\sigma}, \\ \lim\limits_{t\to 0+} u(t) = 0, \ \lim\limits_{t\to 0+} u'(t) = \lambda, \end{cases} \tag{1.1}$$

where p is continuous, $a, \sigma, \lambda \in \mathbb{R}$ with $\sigma < 0$ and $\lambda > 0$. She proved that if p satisfies $\int_0^1 |p(t)|t^{a+\sigma}dt < \infty$, then the problem has a unique solution. For related results of the Cauchy problem for singular differential equations (1.1), see [3, 4] and the references therein.

On the other hand, fractional differential equations have been studied. For example, in [1] and [10], the authors considered the fractional differential equation

$$D^{\nu}_{0+}u(t) + f(t, u(t)) = 0$$

where $1 < \nu \leq 2$ and D^{ν}_{0+} is the ν-th Riemann-Liouville fractional derivative. The ν-th Riemann-Liouville fractional derivative of a function u is given by

$$D^{\nu}_{0+}u(t) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\nu-1} u(s)ds$$

2010 Mathematics Subject Classification: 34A08, 34A12
Key words and phrases: Cauchy problem, fixed point theorem, fractional differential equation

where n is an integer with $n-1 \le \nu < n$ and $\Gamma(\cdot)$ is the gamma function. For singular fractional differential equations, see [9, 11, 12]. However in the obtained results, the Cauchy problem (1.1) cannot be treated. Therefore, in [5], the authors showed the existence and uniqueness of solutions of the Cauchy problem

$$(1.2) \qquad \begin{cases} D^{\nu}_{0+}u(t) = p(t)t^a u(t)^{\sigma}, \\ \lim\limits_{t\to 0+} u(t) = 0, \ \lim\limits_{t\to 0+} u'(t)t^{2-\nu} = (\nu-1)\lambda, \end{cases}$$

where p is continuous, $a, \sigma, \lambda \in \mathbb{R}$ with $\sigma < 0$ and $\lambda > 0$. If $\nu = 2$, then the Cauchy problem (1.2) is the problem (1.1).

In this paper we consider the Cauchy problem

$$(1.3) \qquad \begin{cases} D^{\nu}_{0+}u(t) = f(t, u(t)), \\ \lim\limits_{t\to 0+} u(t) = 0, \ \lim\limits_{t\to 0+} u'(t)t^{2-\nu} = (\nu-1)\lambda, \end{cases}$$

where f is a mapping of $[0,1] \times (0,\infty)$ into $\mathbb{R}$, $\lambda \in \mathbb{R}$ with $\lambda > 0$ and $\nu \in \mathbb{R}$ with $1 < \nu \le 2$. If $f(t,u) = p(t)t^a u^{\sigma}$, the Cauchy problem (1.3) is the problem (1.2). We show the existence and uniqueness of solutions of the problem (1.3).

2. Decreasing cases

In this section, we consider the case that the mapping $f(t,u)$ in the Cauchy problem (1.3) is decreasing for u. First we derive the integral equation which is equivalent to the problem (1.3).

We use the following; see, for example, [1, 7]. See [6] also.

Lemma 2.1. *Let $\nu > 0$. Let u be a Lebesgue integrable function of $[0,1]$ into $\mathbb{R}$ such that $D^{\nu}_{0+}u$ is also Lebesgue integrable. Then*

$$I^{\nu}_{0+}D^{\nu}_{0+}u(t) = u(t) + C_1 t^{\nu-1} + C_2 t^{\nu-2} + \cdots + C_n t^{\nu-n}$$

for some $C_1, C_2, \ldots, C_n \in \mathbb{R}$ and an integer n with $n-1 \le \nu < n$, where the ν-th Riemann-Liouville fractional integral $I^{\nu}_{0+}u$ of a function u is defined by

$$I^{\nu}_{0+}u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s)ds.$$

Let f be a mapping of $[0,1] \times (0,\infty)$ into $\mathbb{R}$. A mapping f is said to satisfy the Carathéodory conditions if for each $u \in (0,\infty)$, $t \longmapsto f(t,u)$ is measurable and for almost every $t \in [0,1]$, $u \longmapsto f(t,u)$ is continuous. A function u is said to be a solution of the Cauchy problem (1.3) if there exists $h > 0$ such that

$u \in C[0,h]$ and u satisfies the equation $D^{\nu}_{0+}u(t) = f(t,u(t))$ for almost all t in $[0,h]$ and the conditions $\lim_{t\to 0+} u(t) = 0$ and $\lim_{t\to 0+} u'(t)t^{2-\nu} = (\nu-1)\lambda$, where $C[0,h]$ is the set of all continuous functions of $[0,h]$ to $\mathbb{R}$. It is noted that $C[0,h]$ is a Banach space by the maximum norm

$$\|u\| = \max\{|u(t)| \mid t \in [0,h]\}.$$

Lemma 2.2. *Let $1 < \nu \le 2$ and $\lambda > 0$. Let f be a mapping of $[0,1]\times(0,\infty)$ into $\mathbb{R}$ satisfying the Carathéodory conditions. Suppose that the mapping f satisfies the following:*

(a1) *There exists $h \in \mathbb{R}$ with $0 < h \le 1$ such that $|f(t,u_1)| \ge |f(t,u_2)|$ for almost every $t \in [0,h]$ and for any $u_1, u_2 \in (0,\infty)$ with $u_1 \le u_2$.*

(b1) *There exists $\alpha \in \mathbb{R}$ with $0 < \alpha < \lambda$ such that*

$$\lim_{t\to 0+} t\int_0^1 (1-s)^{\nu-2}|f(st,\alpha(st)^{\nu-1})|ds = 0.$$

Then u is a solution of the Cauchy problem (1.3) if and only if u is a solution of the equation

$$u(t) = \lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} f(s,u(s))ds \tag{2.1}$$

under the assumption $\alpha t^{\nu-1} \le u(t)$ for any $t \in (0,h]$.

Proof. Let u be a solution of the Cauchy problem (1.3) under the assumption $\alpha t^{\nu-1} \le u(t)$ for any $t \in (0,h]$. We will show that u is a solution of the equation (2.1). Since u is a solution of (1.3), $u \in C[0,h]$ and u satisfies the equation $D^{\nu}_{0+}u(t) = f(t,u(t))$ for almost every t in $[0,h]$ and the conditions $\lim_{t\to 0+} u(t) = 0$ and $\lim_{t\to 0+} u'(t)t^{2-\nu} = (\nu-1)\lambda$. Since $u \in C[0,h]$, u is Lebesgue integrable. Moreover, by the condition (b1), there exists $0 < h_0 \le h$ such that

$$h_0\int_0^1 (1-s)^{\nu-2}|f(sh_0,\alpha(sh_0)^{\nu-1})|ds < \infty.$$

Since

$$\begin{aligned}\int_0^{h_0} |f(s,u(s))|ds &\le \int_0^{h_0}\left(1-\frac{s}{h_0}\right)^{\nu-2}|f(s,u(s))|ds\\ &\le \int_0^{h_0}\left(1-\frac{s}{h_0}\right)^{\nu-2}|f(s,\alpha s^{\nu-1})|ds\\ &= h_0\int_0^1 (1-s)^{\nu-2}|f(sh_0,\alpha(sh_0)^{\nu-1})|ds < \infty\end{aligned}$$

and $D^{\nu}_{0+}u(t) = f(t, u(t))$, $D^{\nu}_{0+}u$ is Lebesgue integrable. By Lemma 2.1, we have the equation

$$u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s, u(s))ds + C_1 t^{\nu-1} + C_2 t^{\nu-2}$$

for some C_1 and C_2. The condition $\lim_{t\to 0+} u(t) = 0$ implies $C_2 = 0$. In fact, by the conditions (a1) and (b1), we have

$$\begin{aligned}
\left| \int_0^t (t-s)^{\nu-1} f(s, u(s))ds \right| &\leq \int_0^t (t-s)^{\nu-1} |f(s, u(s))| ds \\
&\leq \int_0^t (t-s)^{\nu-1} |f(s, \alpha s^{\nu-1})| ds \\
&= t^{\nu} \int_0^1 (1-s)^{\nu-1} |f(st, \alpha (st)^{\nu-1})| ds \\
&\leq t \int_0^1 (1-s)^{\nu-2} |f(st, \alpha (st)^{\nu-1})| ds \to 0
\end{aligned}$$

as $t \to 0+$. Thus we have

$$u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s, u(s))ds + C_1 t^{\nu-1}.$$

Then we have

$$u'(t) = \frac{1}{\Gamma(\nu-1)} \int_0^t (t-s)^{\nu-2} f(s, u(s))ds + (\nu-1) C_1 t^{\nu-2}.$$

By the condition (a1), we have

$$\begin{aligned}
\left| u'(t) t^{2-\nu} - (\nu-1) C_1 \right| &\leq \frac{1}{\Gamma(\nu-1)} \int_0^t \left(1 - \frac{s}{t}\right)^{\nu-2} |f(s, u(s))| ds \\
&\leq \frac{1}{\Gamma(\nu-1)} \int_0^t \left(1 - \frac{s}{t}\right)^{\nu-2} |f(s, \alpha s^{\nu-1})| ds \\
&= \frac{t}{\Gamma(\nu-1)} \int_0^1 (1-s)^{\nu-2} |f(st, \alpha (st)^{\nu-1})| ds.
\end{aligned}$$

By the condition (b1), we have

$$\lim_{t\to 0+} u'(t) t^{2-\nu} = (\nu-1) C_1.$$

Thus $C_1 = \lambda$. Therefore u is a solution on $[0, h_0]$ of the equation (2.1).

Let u be a solution of the equation (2.1) under the assumption $\alpha t^{\nu-1} \leq u(t)$ for any $t \in (0, h]$. We will show that u is a solution of the Cauchy problem

(1.3). Since u satisfies the equation (2.1), we have $D^{\nu}_{0+}u(t) = f(t, u(t))$. By the condition (a1), we have

$$\begin{aligned}|u(t)| &\le \lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|f(s,u(s))|ds\\ &\le \lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|f(s,\alpha s^{\nu-1})|ds\\ &= \lambda t^{\nu-1} + \frac{t^{\nu}}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-1}|f(st,\alpha (st)^{\nu-1})|ds\\ &\le \lambda t^{\nu-1} + \frac{t}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-2}|f(st,\alpha (st)^{\nu-1})|ds.\end{aligned}$$

By the condition (b1), $\lim_{t\to 0+} u(t) = 0$. Since

$$u'(t) = \frac{1}{\Gamma(\nu-1)}\int_0^t (t-s)^{\nu-2} f(s,u(s))ds + (\nu-1)\lambda t^{\nu-2},$$

we have

$$\begin{aligned}\left|u'(t)t^{2-\nu} - (\nu-1)\lambda\right| &\le \frac{1}{\Gamma(\nu-1)}\int_0^t \left(1-\frac{s}{t}\right)^{\nu-2}|f(s,u(s))|ds\\ &\le \frac{1}{\Gamma(\nu-1)}\int_0^t \left(1-\frac{s}{t}\right)^{\nu-2}|f(s,\alpha s^{\nu-1})|ds\\ &= \frac{t}{\Gamma(\nu-1)}\int_0^1 (1-s)^{\nu-2}|f(st,\alpha (st)^{\nu-1})|ds\end{aligned}$$

by the condition (a1). By the condition (b1), we have $\lim_{t\to 0+} u'(t)t^{2-\nu} = (\nu - 1)\lambda$. Therefore u is a solution on $[0, h]$ of the Cauchy problem (1.3). □

Theorem 2.1. *Let $1 < \nu \le 2$ and $\lambda > 0$. Let f be a mapping of $[0, 1] \times (0,\infty)$ into $\mathbb{R}$ satisfying the Carathéodory conditions. Suppose that the mapping f satisfies* (a1) *and* (b1). *Moreover suppose that the mapping f satisfies the following:*

(c) *There exists $\beta \in \mathbb{R}$ with $\beta > 0$ such that*

$$\left|\frac{\partial f}{\partial u}(t,u)\right| \le \frac{\beta|f(t,u)|}{u}$$

for almost every $t \in [0, h]$, where h is in (a1), *and for any $u \in (0,\infty)$.*

Then there exist $h_0 \in \mathbb{R}$ with $0 < h_0 \le h$ and a unique solution $u : (0, h_0] \to \mathbb{R}$ of the Cauchy problem (1.3) satisfying $\alpha t^{\nu-1} \le u(t)$ for any $t \in (0, h_0]$.

Proof. By Lemma 2.2, we consider the integral equation (2.1). By the condition (b1), there exists $h_0 \in \mathbb{R}$ with $0 < h_0 \le h$ such that

$$\sup_{t\in(0,h_0]} t \int_0^1 (1-s)^{\nu-2}|f(st, \alpha(st)^{\nu-1})|ds < \min\left\{\lambda-\alpha, \frac{\alpha}{\beta}\right\}\Gamma(\nu).$$

Let X be a subset of $C[0,h_0]$ defined by

$$X = \left\{ u \;\middle|\; \begin{array}{l} u(0)=0,\ \alpha t^{\nu-1} \le u(t) \text{ for any } t\in[0,h_0], \\ u \text{ is differentiable on a right-hand neighboorhood of } 0, \\ \lim_{t\to 0+} u'(t)t^{2-\nu} = (\nu-1)\lambda \end{array} \right\}.$$

Since a mapping $t \longmapsto \lambda t^{\nu-1}$ belongs to X, $X \neq \emptyset$. Let A be an operator of X defined by

$$Au(t) = \lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} f(s,u(s))ds.$$

Then $A(X) \subset X$. Indeed since f satisfies the Carathéodory conditions, $Au \in C[0,h_0]$. By the condition (a1), we have

$$\begin{aligned} |Au(t)| &= \left|\lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} f(s,u(s))ds\right| \\ &\le \lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|f(s,\alpha s^{\nu-1})|ds \\ &= \lambda t^{\nu-1} + \frac{t^{\nu}}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-1}|f(st,\alpha(st)^{\nu-1})|ds \\ &\le \lambda t^{\nu-1} + \frac{t}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-2}|f(st,\alpha(st)^{\nu-1})|ds \end{aligned}$$

and hence by the condition (b1), $Au(0) = 0$. By the condition (a1), we have

$$\begin{aligned} \left|(Au)'(t)t^{2-\nu} - (\nu-1)\lambda\right| &\le \frac{1}{\Gamma(\nu-1)}\int_0^t \left(1-\frac{s}{t}\right)^{\nu-2}|f(s,u(s))|ds \\ &\le \frac{1}{\Gamma(\nu-1)}\int_0^t \left(1-\frac{s}{t}\right)^{\nu-2}|f(s,\alpha s^{\nu-1})|ds \\ &= \frac{t}{\Gamma(\nu-1)}\int_0^1 (1-s)^{\nu-2}|f(st,\alpha(st)^{\nu-1})|ds \end{aligned}$$

and hence by the condition (b1), $\lim_{t\to 0+}(Au)'(t)t^{2-\nu} = (\nu-1)\lambda$. By the condition (a1), we have

$$\begin{aligned}Au(t) &= \lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s, u(s)) ds \\ &\geq \lambda t^{\nu-1} - \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |f(s, \alpha s^{\nu-1})| ds \\ &= \lambda t^{\nu-1} - \frac{t^\nu}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-1} |f(st, \alpha (st)^{\nu-1})| ds \\ &\geq \lambda t^{\nu-1} - (\lambda - \alpha) t^{\nu-1} \\ &= \alpha t^{\nu-1}\end{aligned}$$

for any $t \in [0, h_0]$. We will find a fixed point of A. Let φ be an operator of X into $C[0, h_0]$ defined by

$$\varphi[u](t) = \begin{cases} \dfrac{u(t)}{t^{\nu-1}}, & \text{if } t \in (0, h_0], \\ \lambda, & \text{if } t = 0 \end{cases}$$

and

$$\begin{aligned}\varphi[X] &= \{\varphi[u] \mid u \in X\} \\ &= \{v \mid v \in C[0, h_0], v(0) = \lambda \text{ and } \alpha \leq v(t) \text{ for any } t \in [0, h_0]\}.\end{aligned}$$

Then $\varphi[X]$ is a closed subset of $C[0, h_0]$ and hence it is a complete metric space. Let Φ_A be an operator of $\varphi[X]$ into $\varphi[X]$ defined by

$$\Phi_A \varphi[u] = \varphi[Au].$$

By the mean value theorem, for any $u_1, u_2 \in X$ there exists a mapping ξ such that

$$f(t, u_1(t)) - f(t, u_2(t)) = \frac{\partial f}{\partial u}(t, \xi(t))(u_1(t) - u_2(t))$$

and

$$\alpha t^{\nu-1} \leq \min\{u_1(t), u_2(t)\} \leq \xi(t) \leq \max\{u_1(t), u_2(t)\}$$

for any $t \in [0, h_0]$. By the conditions (a1) and (c), we have

$$\begin{aligned}|f(t, u_1(t)) - f(t, u_2(t))| &= \left| \frac{\partial f}{\partial u}(t, \xi(t))(u_1(t) - u_2(t)) \right| \\ &\leq \left| \frac{\beta f(t, \xi(t))}{\xi(t)} \right| |u_1(t) - u_2(t)| \\ &\leq \left| \frac{\beta f(t, \alpha t^{\nu-1})}{\alpha t^{\nu-1}} \right| |u_1(t) - u_2(t)|\end{aligned}$$

for almost every $t \in [0, h_0]$. In the last inequality, it is noted that $\alpha t^{\nu-1} \le \xi(t)$ for any $t \in [0, h_0]$. Therefore we have

$$\begin{aligned}
&|\Phi_A\varphi[u_1](t) - \Phi_A\varphi[u_2](t)| \\
&= \left| \frac{1}{\Gamma(\nu)t^{\nu-1}} \int_0^t (t-s)^{\nu-1}(f(s,u_1(s)) - f(s,u_2(s)))ds \right| \\
&\le \frac{1}{\Gamma(\nu)t^{\nu-1}} \int_0^t (t-s)^{\nu-1} \left| \frac{\beta f(s,\alpha s^{\nu-1})}{\alpha s^{\nu-1}} \right| |u_1(s) - u_2(s)| ds \\
&\le \frac{\beta}{\alpha\Gamma(\nu)} \left(\int_0^t (t-s)^{\nu-1} |f(s,\alpha s^{\nu-1})| ds \right) \|\varphi[u_1] - \varphi[u_2]\| \\
&= \frac{\beta t}{\alpha\Gamma(\nu)} \left(\int_0^1 (1-s)^{\nu-1} |f(st,\alpha(st)^{\nu-1})| ds \right) \|\varphi[u_1] - \varphi[u_2]\| \\
&\le \frac{\beta}{\alpha\Gamma(\nu)} \left(\sup_{t\in(0,h]} t \int_0^1 (1-s)^{\nu-2} |f(st,\alpha(st)^{\nu-1})| ds \right) \|\varphi[u_1] - \varphi[u_2]\|
\end{aligned}$$

for any $t \in (0, h_0]$. Then we have

$$\begin{aligned}
&\|\Phi_A\varphi[u_1] - \Phi_A\varphi[u_2]\| \\
&\le \frac{\beta}{\alpha\Gamma(\nu)} \left(\sup_{t\in(0,h_0]} t \int_0^1 (1-s)^{\nu-2} |f(st,\alpha(st)^{\nu-1})| ds \right) \|\varphi[u_1] - \varphi[u_2]\|.
\end{aligned}$$

Since $\frac{\beta}{\alpha\Gamma(\nu)} \left(\sup_{t\in(0,h_0]} t \int_0^1 (1-s)^{\nu-2} |f(st,\alpha(st)^{\nu-1})| ds \right) < 1$, by the Banach fixed point theorem, there exists a unique mapping $\varphi[u] \in \varphi[X]$ such that $\Phi_A\varphi[u] = \varphi[u]$. Then $Au = u$. The mapping u is a unique solution of the Cauchy problem (1.3). □

Example 2.1. Consider the Cauchy problem

$$\left\{ \begin{array}{l} u''(t) = a(t)u(t)^\sigma, \\ \lim_{t\to 0+} u(t) = 0, \; \lim_{t\to 0+} u'(t) = \lambda, \end{array} \right. \tag{2.2}$$

where a is a mapping of $[0, 1]$ which is continuous on $(0, 1]$, $\int_0^1 |a(t)|t^\sigma dt < \infty$ and $\sigma < 0$, $\lambda > 0$. In this case, $f(t, u) = a(t)u^\sigma$ for $(t, u) \in [0, 1] \times (0, \infty)$. Then the mapping f satisfies the condition (a1). In fact, let $t \in [0, 1]$ and $u_1, u_2 > 0$ with $u_2 \le u_1$. Since $u_1^\sigma \ge u_2^\sigma$, we have $|a(t)u_1^\sigma| \ge |a(t)u_2^\sigma|$. Moreover the mapping f satisfies the condition (b1). In fact, since $\int_0^1 |a(t)|t^\sigma dt < \infty$, we have $\lim_{t\to 0+} \int_0^t |a(s)|s^\sigma ds = 0$. Then we have

$$\begin{aligned}
\int_0^t |a(s)|s^\sigma ds &= \int_0^1 |a(ts)|(ts)^\sigma t ds \\
&= t^{\sigma+1} \int_0^1 |a(ts)|s^\sigma ds \to 0
\end{aligned} \tag{2.3}$$

as $t \to 0+$. Then we have

$$t \int_0^1 |a(st)\,(\alpha st)^\sigma|\,ds = \alpha^\sigma t^{\sigma+1} \int_0^1 |a(st)| s^\sigma ds \to 0$$

as $t \to 0+$ by (2.3). Hence the mapping f satisfies the condition (b1). Since

$$\left|\frac{\partial f}{\partial u}(t,u)\right| = |a(t)\sigma u^{\sigma-1}| = \frac{|\sigma||f(t,u)|}{u},$$

the mapping f satisfies the condition (c). By Theorem 2.1, the Cauchy problem (2.2) has a unique solution.

Example 2.2. In [8], Knežević-Miljanović considered the Cauchy problem (1.1), where p is a continuous function satisfying $\int_0^1 |p(t)|t^{a+\sigma}dt < \infty$, $a, \sigma, \lambda \in \mathbb{R}$ with $\sigma < 0$ and $\lambda > 0$. In the case that $a(t) = p(t)t^a$ in Example 2.1, the problem (1.1) has a unique solution.

3. Increasing cases

In this section, we consider the case that $f(t,u)$ in the Cauchy problem (1.3) is increasing for u. First we show the following lemma. The proof is similar to that of Lemma 2.2. But, for the sake of completeness, we show the proof.

Lemma 3.1. *Let $1 < \nu \le 2$ and $\lambda > 0$. Let f be a mapping of $[0,1]\times(0,\infty)$ into $\mathbb{R}$ satisfying the Carathéodory conditions. Suppose that the mapping f satisfies the following:*

(a2) *There exists $h \in \mathbb{R}$ with $0 < h \le 1$ such that $|f(t,u_1)| \le |f(t,u_2)|$ for almost every $t \in [0,h]$ and for any $u_1, u_2 \in (0,\infty)$ with $u_1 \le u_2$.*

(b2) *There exists $\alpha \in \mathbb{R}$ with $0 < \alpha < \lambda$ such that*

$$\lim_{t\to 0+} t \int_0^1 (1-s)^{\nu-2} |f(st, (2\lambda-\alpha)(st)^{\nu-1})| ds = 0.$$

Then u is a solution of the Cauchy problem (1.3) if and only if u is a solution of the equation (2.1) under the assumption $\alpha t^{\nu-1} \le u(t) \le (2\lambda-\alpha)t^{\nu-1}$ for any $t \in (0,h]$.

Proof. Let u be a solution of the Cauchy problem (1.3) under the assumption $\alpha t^{\nu-1} \le u(t) \le (2\lambda-\alpha)t^{\nu-1}$ for any $t \in (0,h]$. We will show that u is a solution of the equation (2.1). Then $u \in C[0,h]$ and u satisfies the equation $D^\nu_{0+}u(t) = f(t,u(t))$ for almost every t in $[0,h]$ and the conditions $\lim_{t\to 0+} u(t) =$

0 and $\lim_{t\to 0+} u'(t)t^{2-\nu} = (\nu-1)\lambda$. Since $u \in C[0,h]$, u is Lebesgue integrable. Moreover, by the condition (b2), there exists $0 < h_0 \le h$ such that

$$h_0 \int_0^1 (1-s)^{\nu-2} |f(sh_0, (2\lambda-\alpha)(sh_0)^{\nu-1})| ds < \infty.$$

Since

$$\begin{aligned}\int_0^{h_0} |f(s,u(s))| ds &\le \int_0^{h_0} \left(1-\frac{s}{h_0}\right)^{\nu-2} |f(s,u(s))| ds \\ &\le \int_0^{h_0} \left(1-\frac{s}{h_0}\right)^{\nu-2} |f(s,(2\lambda-\alpha)s^{\nu-1})| ds \\ &= h_0 \int_0^1 (1-s)^{\nu-2} |f(sh_0, (2\lambda-\alpha)(sh_0)^{\nu-1})| ds < \infty\end{aligned}$$

and $D^\nu_{0+}u(t) = f(t,u(t))$, $D^\nu_{0+}u$ is Lebesgue integrable. By Lemma 2.1, we have the integral equation

$$u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s,u(s)) ds + C_1 t^{\nu-1} + C_2 t^{\nu-2}$$

for some C_1 and C_2. The condition $\lim_{t\to 0+} u(t) = 0$ implies $C_2 = 0$. In fact, by the conditions (a2) and (b2), we have

$$\begin{aligned}\left|\int_0^t (t-s)^{\nu-1} f(s,u(s)) ds\right| &\le \int_0^t (t-s)^{\nu-1} |f(s,u(s))| ds \\ &\le \int_0^t (t-s)^{\nu-1} |f(s,(2\lambda-\alpha)s^{\nu-1})| ds \\ &= t^\nu \int_0^1 (1-s)^{\nu-1} |f(st,(2\lambda-\alpha)(st)^{\nu-1})| ds \\ &\le t \int_0^1 (1-s)^{\nu-2} |f(st,(2\lambda-\alpha)(st)^{\nu-1})| ds \to 0\end{aligned}$$

as $t \to 0$. Thus we have

$$u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s,u(s)) ds + C_1 t^{\nu-1}.$$

Then we have

$$u'(t) = \frac{1}{\Gamma(\nu-1)} \int_0^t (t-s)^{\nu-2} f(s,u(s)) ds + (\nu-1)C_1.$$

By the condition (a2), we have

$$\begin{aligned}\left|u'(t)t^{2-\nu}-(\nu-1)C_1\right| &\le \frac{1}{\Gamma(\nu-1)}\int_0^t\left(1-\frac{s}{t}\right)^{\nu-2}|f(s,u(s))|ds\\ &\le \frac{1}{\Gamma(\nu-1)}\int_0^t\left(1-\frac{s}{t}\right)^{\nu-2}|f(s,(2\lambda-\alpha)s^{\nu-1})|ds\\ &= \frac{t}{\Gamma(\nu-1)}\int_0^1(1-s)^{\nu-2}|f(st,(2\lambda-\alpha)(st)^{\nu-1})|ds.\end{aligned}$$

By the condition (b2), we have

$$\lim_{t\to 0+}u'(t)t^{2-\nu}=(\nu-1)C_1.$$

Thus $C_1=\lambda$. Therefore u is a solution on $[0,h_0]$ of the equation (2.1).

Let u be a solution of the equation (2.1) under the assumption $\alpha t^{\nu-1}\le u(t)\le (2\lambda-\alpha)t^{\nu-1}$ for any $t\in(0,h]$. We will show that u is a solution of the Cauchy problem (1.3). Since u satisfies the equation (2.1), we have $D_{0+}^{\nu}u(t)=f(t,u(t))$. By (a2), we have

$$\begin{aligned}|u(t)| &\le \lambda t^{\nu-1}+\frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\alpha-1}|f(s,u(s))|ds\\ &\le \lambda t^{\nu-1}+\frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\alpha-1}|f(s,(2\lambda-\alpha)s^{\nu-1})|ds\\ &= \lambda t^{\nu-1}+\frac{t^{\nu}}{\Gamma(\nu)}\int_0^1(1-s)^{\nu-1}|f(st,(2\lambda-\alpha)(st)^{\nu-1})|ds\\ &\le \lambda t^{\nu-1}+\frac{t}{\Gamma(\nu)}\int_0^1(1-s)^{\nu-2}|f(st,(2\lambda-\alpha)(st)^{\nu-1})|ds.\end{aligned}$$

By the condition (b2), $\lim_{t\to 0+}u(t)=0$. Since

$$u'(t)=\frac{1}{\Gamma(\nu-1)}\int_0^t(t-s)^{\nu-2}f(s,u(s))ds+(\nu-1)\lambda t^{\nu-2},$$

we have

$$\begin{aligned}\left|u'(t)t^{2-\nu}-(\nu-1)\lambda\right| &\le \frac{1}{\Gamma(\nu-1)}\int_0^t\left(1-\frac{s}{t}\right)^{\nu-2}|f(s,u(s))|ds\\ &\le \frac{1}{\Gamma(\nu-1)}\int_0^t\left(1-\frac{s}{t}\right)^{\nu-2}|f(s,(2\lambda-\alpha)s^{\nu-1})|ds\\ &= \frac{t}{\Gamma(\nu-1)}\int_0^1(1-s)^{\nu-2}|f(st,(2\lambda-\alpha)(st)^{\nu-1})|ds.\end{aligned}$$

By the condition (b2), we have $\lim_{t\to 0+}u'(t)t^{2-\nu}=(\alpha-1)\lambda$. Therefore u is a solution on $[0,h]$ of the Cauchy problem (1.3). □

Theorem 3.1. *Let* $1 < \nu \leq 2$ *and* $\lambda > 0$. *Let* f *be a mapping of* $[0,1] \times (0,\infty)$ *into* $\mathbb{R}$ *satisfying the Carathéodory conditions. Suppose that the mapping* f *satisfies the conditions* (a2), (b2) *and* (c). *Then there exist* $h_0 \in \mathbb{R}$ *with* $0 < h_0 \leq h$ *and a unique solution* $u : (0, h_0] \to \mathbb{R}$ *of the Cauchy problem (1.3) satisfying* $\alpha t^{\nu-1} \leq u(t) \leq (2\lambda - \alpha)t^{\nu-1}$ *for any* $t \in (0, h_0]$.

Proof. By the condition (b2), there exists $h_0 \in \mathbb{R}$ with $0 < h_0 \leq h$ such that

$$\sup_{t\in(0,h_0]} t\int_0^1 (1-s)^{\nu-2}|f(st,(2\lambda-\alpha)(st)^{\nu-1})|ds < \min\left\{\lambda-\alpha, \frac{\alpha}{\beta}\right\}\Gamma(\nu).$$

Let X be a subset of $C[0, h_0]$ defined by

$$X = \left\{ u \;\middle|\; \begin{array}{l} u(0) = 0,\ \alpha t^{\nu-1} \leq u(t) \leq (2\lambda-\alpha)t^{\nu-1} \text{ for any } t \in [0,h_0] \\ u \text{ is differentiable on a right-hand neighboorhood of } 0, \\ \lim_{t\to 0+} u'(t)t^{2-\nu} = (\nu-1)\lambda \end{array} \right\}.$$

Since a mapping $t \longmapsto \lambda t^{\nu-1}$ belongs to X, $X \neq \emptyset$. Let A be an operator of X defined by

$$Au(t) = \lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} f(s,u(s))ds.$$

Then $A(X) \subset X$. Indeed since f satisfies the Carathéodory conditions, $Au \in C[0, h_0]$. By the condition (a2), we have

$$\begin{aligned} |Au(t)| &= \left|\lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} f(s,u(s))ds\right| \\ &\leq \lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|f(s,(2\lambda-\alpha)s^{\nu-1})|ds \\ &= \lambda t^{\nu-1} + \frac{t^\nu}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-1}|f(st,(2\lambda-\alpha)(st)^{\nu-1})|ds \\ &\leq \lambda t^{\nu-1} + \frac{t}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-2}|f(st,(2\lambda-\alpha)(st)^{\nu-1})|ds \end{aligned}$$

and hence by the condition (b2), $Au(0) = 0$. By the condition (a2), we have

$$\begin{aligned} \left|(Au)'(t)t^{2-\nu} - \lambda(\nu-1)\right| &\leq \frac{1}{\Gamma(\nu-1)}\int_0^t \left(1-\frac{s}{t}\right)^{\nu-2}|f(s,u(s))|ds \\ &\leq \frac{1}{\Gamma(\nu-1)}\int_0^t \left(1-\frac{s}{t}\right)^{\nu-2}|f(s,(2\lambda-\alpha)s^{\nu-1})|ds \\ &= \frac{t}{\Gamma(\nu-1)}\int_0^1 (1-s)^{\nu-2}|f(st,(2\lambda-\alpha)(st)^{\nu-1})|ds \end{aligned}$$

and hence by the condition (b2), $\lim_{t\to 0+}(Au)'(t)t^{2-\nu} = (\nu-1)\lambda$. By the condition (a2), we have

$$\begin{aligned} Au(t) &= \lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} f(s,u(s))ds \\ &\geq \lambda t^{\nu-1} - \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|f(s,(2\lambda-\alpha)s^{\nu-1})|ds \\ &= \lambda t^{\nu-1} - \frac{t^\nu}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-1}|f(st,(2\lambda-\alpha)(st)^{\nu-1})|ds \\ &\geq \lambda t^{\nu-1} - (\lambda-\alpha)t^{\nu-1} \\ &= \alpha t^{\nu-1} \end{aligned}$$

and

$$\begin{aligned} Au(t) &= \lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1} f(s,u(s))ds \\ &\leq \lambda t^{\nu-1} + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}|f(s,(2\lambda-\alpha)s^{\nu-1})|ds \\ &= \lambda t^{\nu-1} + \frac{t^\nu}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-1}|f(st,(2\lambda-\alpha)(st)^{\nu-1})|ds \\ &\leq \lambda t^{\nu-1} + (\lambda-\alpha)t^{\nu-1} \\ &= (2\lambda-\alpha)t^{\nu-1} \end{aligned}$$

for any $t \in [0, h_0]$. We will find a fixed point of A. Let φ be an operator of X into $C[0, h_0]$ defined by

$$\varphi[u](t) = \begin{cases} \dfrac{u(t)}{t^{\nu-1}}, & \text{if } t \in (0, h_0], \\ \lambda, & \text{if } t = 0 \end{cases}$$

and

$$\begin{aligned} \varphi[X] &= \{\varphi[u] \mid u \in X\} \\ &= \{v \mid v \in C[0,h_0], v(0) = \lambda \text{ and } \alpha \leq v(t) \leq 2\lambda - \alpha \text{ for any } t \in [0,h_0]\}. \end{aligned}$$

Then $\varphi[X]$ is a closed subset of $C[0, h_0]$ and hence it is a complete metric space. Let Φ_A be an operator of $\varphi[X]$ into $\varphi[X]$ defined by

$$\Phi_A \varphi[u] = \varphi[Au].$$

By the mean value theorem, for any $u_1, u_2 \in X$ there exists a mapping ξ such that

$$f(t,u_1(t)) - f(t,u_2(t)) = \frac{\partial f}{\partial u}(t,\xi(t))(u_1(t) - u_2(t))$$

and

$$\alpha t^{\nu-1} \le \min\{u_1(t), u_2(t)\} \le \xi(t) \le \max\{u_1(t), u_2(t)\} \le (2\lambda - \alpha)t^{\nu-1}$$

for any $t \in [0, h_0]$. By the conditions (a2) and (c)

$$\begin{aligned} |f(t, u_1(t)) - f(t, u_2(t))| &= \left| \frac{\partial f}{\partial u}(t, \xi(t))(u_1(t) - u_2(t)) \right| \\ &\le \left| \frac{\beta f(t, \xi(t))}{\xi(t)} \right| |u_1(t) - u_2(t)| \\ &\le \left| \frac{\beta f(t, (2\lambda - \alpha)t^{\nu-1})}{\alpha t^{\nu-1}} \right| |u_1(t) - u_2(t)| \end{aligned}$$

for almost every $t \in [0, h_0]$. In the last inequality, it is noted that $\alpha t^{\nu-1} \le \xi(t) \le (2\lambda - \alpha)t^{\nu-1}$ for any $t \in [0, h_0]$. Therefore

$$\begin{aligned} &|\Phi_A\varphi[u_1](t) - \Phi_A\varphi[u_2](t)| \\ &= \left| \frac{1}{\Gamma(\nu)t^{\nu-1}} \int_0^t (t-s)^{\nu-1}(f(s, u_1(s)) - f(s, u_2(s)))ds \right| \\ &\le \frac{1}{\Gamma(\nu)t^{\nu-1}} \int_0^t (t-s)^{\nu-1} \left| \frac{\beta f(s, (2\lambda - \alpha)s^{\nu-1})}{\alpha s^{\nu-1}} \right| |u_1(s) - u_2(s)| ds \\ &\le \frac{\beta}{\alpha\Gamma(\nu)} \left(\int_0^t (t-s)^{\nu-1} |f(s, (2\lambda - \alpha)s^{\nu-1})| ds \right) \|\varphi[u_1] - \varphi[u_2]\| \\ &= \frac{\beta t}{\alpha\Gamma(\nu)} \left(\int_0^1 (1-s)^{\nu-1} |f(st, (2\lambda - \alpha)(st)^{\nu-1})| ds \right) \|\varphi[u_1] - \varphi[u_2]\| \\ &\le \frac{\beta}{\alpha\Gamma(\nu)} \left(\sup_{t\in(0,h]} t \int_0^1 (1-s)^{\nu-2} |f(st, (2\lambda - \alpha)(st)^{\nu-1})| ds \right) \|\varphi[u_1] - \varphi[u_2]\| \end{aligned}$$

for any $t \in (0, h_0]$. Therefore we have

$$\begin{aligned} &\|\Phi_A\varphi[u_1] - \Phi_A\varphi[u_2]\| \\ &\le \frac{\beta}{\alpha\Gamma(\nu)} \left(\sup_{t\in(0,h_0]} t \int_0^1 (1-s)^{\nu-2} |f(st, (2\lambda - \alpha)(st)^{\nu-1})| ds \right) \|\varphi[u_1] - \varphi[u_2]\|. \end{aligned}$$

Since $\frac{\beta}{\alpha\Gamma(\nu)} \left(\sup_{t\in(0,h_0]} t \int_0^1 (1-s)^{\nu-2} |f(st, (2\lambda - \alpha)(st)^{\nu-1})| ds \right) < 1$, by the Banach fixed point theorem, there exists a unique mapping $\varphi[u] \in \varphi[X]$ such that $\Phi_A\varphi[u] = \varphi[u]$. Then $Au = u$. The mapping u is a unique solution of the Cauchy problem (1.3). □

EXAMPLE 3.1. Consider the Cauchy problem (2.2) in the case that $\sigma \ge 0$. In this case, $f(t, u) = a(t)u^\sigma$ for $(t, u) \in [0, 1] \times \mathbb{R}$. Then the mapping f satisfies (a2). In fact, let $t \in [0,1]$ and $u_1, u_2 > 0$ with $u_2 \le u_1$. Since $u_1^\sigma \le u_2^\sigma$, we have $|a(t)u_1^\sigma| \le |a(t)u_2^\sigma|$. Moreover the mapping f satisfies (b2). In fact, since $\int_0^1 |a(t)|t^\sigma dt < \infty$, we have $\lim_{t\to 0+} \int_0^t |a(s)|s^\sigma ds = 0$. Then we have (2.3). Since

$$t\int_0^1 |a(st)\left((2\lambda-\alpha)st\right)^\sigma|\, ds = (2\lambda-\alpha)^\sigma t^{\sigma+1}\int_0^1 |a(st)|s^\sigma ds \to 0$$

as $t \to 0+$, the mapping f satisfies the condition (b2). Since

$$\begin{aligned}\left|\frac{\partial f}{\partial u}(t,u)\right| &= \begin{cases} |a(t)\sigma u^{\sigma-1}| & (u>0)\\ 0 & (u=0)\end{cases}\\ &= \frac{|\sigma||f(t,u)|}{u},\end{aligned}$$

the mapping f satisfies the condition (c). By Theorem 3.1, the Cauchy problem (2.2) has a unique solution.

Example 3.2. Consider the Cauchy problem

$$\begin{cases} u''(t) = \dfrac{l(t^2+1)\tan^{-1}(u(t))}{\sqrt{t}}, \\ \lim\limits_{t\to 0+} u(t) = 0, \ \lim\limits_{t\to 0+} u'(t) = \lambda, \end{cases} \tag{3.1}$$

where $l > 0$ and $\lambda > 0$. In this case, $f(t,u) = \frac{l(t^2+1)\tan^{-1} u}{\sqrt{t}}$ for $(t,u) \in (0,1] \times [0,\infty)$. The equation of (3.1) is considered in [2]. Then the mapping f satisfies the condition (a2). Moreover the mapping f satisfies the condition (b2). In fact, let $\alpha \in (0,\lambda)$. Then we have

$$\begin{aligned}\int_0^1 \frac{((st)^2+1)\tan^{-1}((2\lambda-\alpha)(st))}{\sqrt{s}} ds &\le \frac{\pi}{2}\int_0^1 \frac{(st)^2+1}{\sqrt{s}} ds\\ &= \frac{\pi}{2}\int_0^1 \left(t^2 s^{\frac{3}{2}} + \frac{1}{\sqrt{s}}\right) ds\\ &= \frac{\pi}{5}(t^2+1) \le \frac{2\pi}{5}.\end{aligned}$$

Therefore we have

$$t\int_0^1 |f(st,(2\lambda-\alpha)(st))| ds = t\int_0^1 \frac{l((st)^2+1)\tan^{-1}((2\lambda-\alpha)(st))}{\sqrt{st}} ds \to 0$$

as $t \to 0+$. Hence the mapping f satisfies the condition (b2). Let $\beta > 0$. We have

$$\begin{aligned}\frac{\beta|f(t,u)|}{u} - \left|\frac{\partial f}{\partial u}(t,u)\right| &= \frac{l(t^2+1)}{\sqrt{t}}\left(\beta\frac{\tan^{-1} u}{u} - \frac{1}{1+u^2}\right)\\ &= \frac{l(t^2+1)}{\sqrt{t}} \cdot \frac{\beta(1+u^2)\tan^{-1} u - u}{u(1+u^2)}.\end{aligned}$$

Let $g(u) = \beta(1+u^2)\tan^{-1} u - u$. Then $g(0) = 0$. Moreover we have $g'(u) = \beta + 2\beta u \tan^{-1} u - 1$. So if $\beta \geq 1$, then $g'(u) > 0$ for $u > 0$. Thus for $\beta \geq 1$, we have

$$\frac{\beta |f(t,u)|}{u} - \left| \frac{\partial f}{\partial u}(t,u) \right| \geq 0.$$

Therefore the mapping f satisfies the condition (c). By Theorem 3.1, the Cauchy problem (3.1) has a unique solution.

Acknowledgement. The authors would like to thank the referees for valuable suggestions and comments. We couldn't correct mistakes without their helps.

References

[1] Z. Bai, and H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *Journal of Mathematical Analysis and Applications*, **311** (2005), 495–505.

[2] J. Caballero, J. Harjani, and K. Sadarangani, Positive solutions for a class of singular fractional boundary value problems, *Computers and Mathematics with Applications*, **62** (2011), 1325–1332.

[3] T. Kawasaki, and M. Toyoda, Existence of positive solution for the Cauchy problem for an ordinary differential equation, *Nonlinear Mathematics for Uncertainly and its Applications, Advances in Intelligent and Soft Computing* **100**, Springer-Verlag, Berlin and New York, 2011, 435–441.

[4] T. Kawasaki, and M. Toyoda, Existence of positive solutions of the Cauchy problem for a second-order differential equation, *Journal of Inequalities and Applications*, 2013, 2013:465.

[5] T. Kawasaki, and M. Toyoda, Note on Knežević-Miljanović's theorem in a class of fractional differential equations, *Journal of Nonlinear and Convex Analysis*, **16** (2015), 2235–2241.

[6] T. Kawasaki, and M. Toyoda, Existence of solutions of initial value problems for singular fractional differential equations, to appear in *Sūrikaisekikenkyūsho Kōkyūroku.* (in Japanese)

[7] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, *In North-Holland Mathematics Studies* **204**, Elsevier, Amsterdam, 2006.

[8] J. Knežević-Miljanović, On the Cauchy problem for an Emden-Fowler equation, *Differential Equations*, **45** (2009), 267–270.

[9] C. Kou, H. Zhou, and Y. Yan, Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis, *Nonlinear Analysis*, **74** (2011), 5975–5986.

[10] C. F. Li, X. N. Luo, and Y. Zhou, Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations, *Computers and Mathematics with Applications*, **59** (2010), 1363–1375.

[11] X. Yang, and Y. Liu, Picard iterative processes for initial value problems of singular fractional differential equations, *Advances in Difference Equations* 2014, 2014:102.

[12] S. Zhang, Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives, *Nonlinear Analysis*, **71** (2009), 2087–2093.

Toshiharu Kawasaki
College of Engineering,
Tamagawa University,
6-1-1 Tamagawa-gakuen, Machida-Shi,
Tokyo, 194-8610, Japan
E-mail: toshiharu.kawasaki@nifty.ne.jp

Masashi Toyoda
College of Engineering,
Tamagawa University,
6-1-1 Tamagawa-gakuen, Machida-Shi,
Tokyo, 194-8610, Japan
E-mail: mss-toyoda@eng.tamagawa.ac.jp