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A PROPERTY FOR THE FORMULA OF THE SECTIONAL CLASSES OF CLASSICAL SCROLLS

By

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Abstract. In this note, we investigate the formula of the sectional class of classical scrolls and we give an answer of a conjecture proposed in a previous paper.

1. Introduction

Let (X, L) be a polarized manifold of dimension n. Assume that L is very ample and let $\varphi : X \hookrightarrow \mathbb{P}^N$ be the morphism defined by $|L|$. Then φ is an embedding. In this situation, its dual variety $X^\vee \to (\mathbb{P}^N)^\vee$ is a hypersurface of N-dimensional projective space except some special types. Then the *class* $\mathrm{cl}(X, L)$ of (X, L) is defined by the following.

$$\mathrm{cl}(X, L) = \begin{cases} \deg(X^\vee), & \text{if } X^\vee \text{ is a hypersurface in } (\mathbb{P}^N)^\vee \\ 0, & \text{otherwise.} \end{cases}$$

As a generalization of this notion, in [3], we defined the *ith sectional class* $\mathrm{cl}_i(X, L)$ for any ample line bundle L and every integer i with $0 \le i \le n$ (see Definition 2.2).

Here we note the following fact: Assume that L is very ample. Then there exists a sequence of smooth subvarieties $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that $X_j \in |L_{j-1}|$ and $\dim X_j = n-j$ for every integer j with $1 \le j \le n-i$, where $L_j = L|_{X_j}$ and $L_0 := L$. In particular, X_{n-i} is a smooth projective variety of dimension i and L_{n-i} is a very ample line bundle on X_{n-i}. Then $\mathrm{cl}_i(X, L)$ is equal to the class of (X_{n-i}, L_{n-i}). This is the reason why we call this invariant the ith sectional class.

In [4], we calculated the sectional class of special polarized manifolds. For example, we consider the case where (X, L) is a classical scroll over a smooth

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projective variety Y of dimension m such that $n := \dim X \geq 2m$. Namely, there exists an ample vector bundle $\mathcal{E}$ on Y of rank $r \geq m+1$ such that $(X, L) \cong (\mathbb{P}_Y(\mathcal{E}), H(\mathcal{E}))$, where $H(\mathcal{E})$ is the tautological line bundle. Here we note that we need the assumption $n \geq 2m$ in order to define and compare $\mathrm{cl}_i(X, L)$ and $\mathrm{cl}_{2m-i}(X, L)$ for every integer i with $0 \leq i \leq m$. Then we get the following:
(i) If $m = 1$, then by [4, Example 2.1 (ix)] we have

$$\mathrm{cl}_i(X,L) = \begin{cases} s_1(\mathcal{E}), & \text{if } i = 0, \\ 2g(C) - 2 + 2c_1(\mathcal{E}), & \text{if } i = 1, \\ c_1(\mathcal{E}), & \text{if } i = 2, \\ 0, & \text{if } i \geq 3 \text{ and } n \geq 3. \end{cases} \tag{1}$$

(ii) If $m = 2$, then by [4, Example 2.1 (x)] we have

$$\mathrm{cl}_i(X,L) = \begin{cases} s_2(\mathcal{E}), & \text{if } i = 0, \\ (s_1(\mathcal{E}) + K_S)s_1(\mathcal{E}) + 2s_2(\mathcal{E}), & \text{if } i = 1, \\ c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}), & \text{if } i = 2, \\ (c_1(\mathcal{E}) + K_S)c_1(\mathcal{E}) + 2c_2(\mathcal{E}), & \text{if } i = 3, \\ c_2(\mathcal{E}), & \text{if } i = 4, \\ 0, & \text{if } i \geq 5 \text{ and } n \geq 5. \end{cases} \tag{2}$$

(iii) If $m = 3$, then by [4, Example 2.1] we have

$$\mathrm{cl}_i(X,L) = \begin{cases} s_3(\mathcal{E}), & \text{if } i = 0, \\[2ex] 3s_3(\mathcal{E}) + (s_1(\mathcal{E}) + K_Y)s_2(\mathcal{E}), & \text{if } i = 1, \\[2ex] 3s_3(\mathcal{E}) + 12(s_1(\mathcal{E}) + K_Y)s_2(\mathcal{E}) & \\ +(s_1(\mathcal{E}) + K_Y)s_1(\mathcal{E})^2 + c_2(Y)s_1(\mathcal{E}), & \text{if } i = 2, \\[2ex] -c_3(Y) + 2c_3(\mathcal{E}) - 2c_1(\mathcal{E})c_2(\mathcal{E}) & \\ +4c_1(\mathcal{E})^3 + 3K_Y c_1(\mathcal{E})^2 + 2c_2(Y)c_1(\mathcal{E}), & \text{if } i = 3, \\[2ex] 3c_3(\mathcal{E}) + 12(c_1(\mathcal{E}) + K_Y)c_2(\mathcal{E}) & \\ +(c_1(\mathcal{E}) + K_Y)c_1(\mathcal{E})^2 + c_2(Y)c_1(\mathcal{E}), & \text{if } i = 4, \\[2ex] 3c_3(\mathcal{E}) + (c_1(\mathcal{E}) + K_Y)c_2(\mathcal{E}), & \text{if } i = 5, \\[2ex] c_3(\mathcal{E}), & \text{if } i = 6, \\[2ex] 0, & \begin{array}{l}\text{if } i \geq 7 \\ \text{and } n \geq 7.\end{array} \end{cases} \tag{3}$$

The above equations show that there exists a relation between $\mathrm{cl}_i(X,L)$ and $\mathrm{cl}_{2m-i}(X,L)$. Here we note that for every integer i with $0 \leq i \leq m$, $\mathrm{cl}_i(X,L)$ can be written by the Segre classes $s_1(\mathcal{E}),\dots,s_m(\mathcal{E})$.

DEFINITION 1.1. For every integer i with $0 \leq i \leq m$, we define the polynomial $F_i(t_1,\dots,t_m) \in \mathbb{Z}[t_1,\dots,t_m]$ such that the following equality holds.

$$F_i(s_1(\mathcal{E}),\dots,s_m(\mathcal{E})) = \mathrm{cl}_i(X,L).$$

Then we see from the above that if $m = 1$, 2 and 3, then

$$\mathrm{cl}_j(X,L) = F_{2m-j}(c_1(\mathcal{E}),\dots,c_m(\mathcal{E}))$$

for $m \leq j \leq 2m$. In general, we can prove the following theorem, which was proposed in [4] and is the main result of this paper.

THEOREM 1.1. *Let a polarized manifold (X,L) be a classical scroll over a smooth projective variety Y with $\dim X = n$ and $\dim Y = m$. Let $\mathcal{E}$ be an ample vector bundle on Y such that $X \cong \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$. Let $F_i(t_1,\dots,t_m)$ be the polynomial defined in Definition 1.1 for every integer i with $0 \leq i \leq m$. Assume that $n \geq 2m$. Then for any integer j with $m \leq j \leq 2m$ we have*

$$\mathrm{cl}_j(X,L) = F_{2m-j}(c_1(\mathcal{E}),\dots,c_m(\mathcal{E})).$$

In particular

$$F_m(s_1(\mathcal{E}),\dots,s_m(\mathcal{E})) = F_m(c_1(\mathcal{E}),\dots,c_m(\mathcal{E})).$$

By Theorem 1.1 we can easily calculate $\mathrm{cl}_{2m-i}(X,L)$ (resp. $\mathrm{cl}_i(X,L)$) if we are able to calculate $\mathrm{cl}_i(X,L)$ (resp. $\mathrm{cl}_{2m-i}(X,L)$). By this relation we expect that we can get some useful information about $\mathrm{cl}_{2m-i}(X,L)$ (resp. $\mathrm{cl}_i(X,L)$) from several properties of $\mathrm{cl}_i(X,L)$ (resp. $\mathrm{cl}_{2m-i}(X,L)$). Moreover if $i = m$, then we have $\mathrm{cl}_m(X,L) = F_m(c_1(\mathcal{E}),\dots,c_m(\mathcal{E})) = F_m(s_1(\mathcal{E}),\dots,s_m(\mathcal{E}))$ by Theorem 1.1. So $\mathrm{cl}_m(X,L)$ may have special and interesting properties. We will study these on another occasion.

2. Preliminaries

NOTATION 2.1. For a real number m and a non-negative integer n, let

$$[m]_n := \begin{cases} m(m-1)\cdots(m-n+1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

For any non-negative integer n,

$$n! := \begin{cases} [n]_n & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Assume that m and n are integers. Then we put

$$\binom{m}{n} := \begin{cases} \frac{[m]_n}{n!} & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

We note that $\binom{m}{n} = 0$ if $0 \leq m < n$ or $n < 0$, and $\binom{m}{0} = 1$.

Definition 2.1. (See [1, Definition 3.1].) Let (X, L) be a polarized manifold of dimension n, and i an integer with $0 \leq i \leq n$. Then the *ith sectional Euler number* $e_i(X, L)$ *of* (X, L) is defined by the following:

$$e_i(X, L) := \sum_{l=0}^{i} (-1)^l \binom{n-i+l-1}{l} c_{i-l}(X) L^{n-i+l}.$$

Definition 2.2. (See [3, Definitions 2.8 and 2.9]. See also [3, Remark 2.6].) Let (X, L) be a polarized manifold of dimension n and i an integer with $0 \leq i \leq n$. Then the *ith sectional class* of (X, L) is defined by the following.

$$\mathrm{cl}_i(X, L) = \begin{cases} e_0(X, L), & \text{if } i = 0, \\ (-1)\{e_1(X, L) - 2e_0(X, L)\}, & \text{if } i = 1, \\ (-1)^i\{e_i(X, L) - 2e_{i-1}(X, L) + e_{i-2}(X, L)\}, & \text{if } 2 \leq i \leq n. \end{cases}$$

Definition 2.3. Let Y be a smooth projective variety of dimension m and $\mathcal{E}$ a vector bundle of rank r on Y.

(i) The *Chern polynomial* $c_t(\mathcal{E})$ is defined by $c_t(\mathcal{E}) = \sum_{i \geq 0} c_i(\mathcal{E}) t^i$.

(ii) For every integer j with $j \geq 0$, the *jth Segre class* $s_j(\mathcal{F})$ *of* $\mathcal{F}$ is defined by the following equation: $c_t(\mathcal{F}^\vee) s_t(\mathcal{F}) = 1$, where $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_t(\mathcal{F}) = \sum_{j \geq 0} s_j(\mathcal{F}) t^j$.

Remark 2.1.

(i) Let Y be a smooth projective variety and $\mathcal{F}$ a vector bundle on X. Let $\tilde{s}_j(\mathcal{F})$ be the Segre class which is defined in [5, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.

(ii) For every integer i with $1 \leq i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

Notation 2.2. Let (X, L) be an n-dimensional classical scroll over a smooth projective variety Y of dimension m. Let $\mathcal{E}$ be an ample vector bundle of rank r on Y such that $X = \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$. Let $p : X \to Y$ be the projection. Then $n = m + r - 1$. In this paper we assume that $r \geq m + 1$, that is, $n \geq 2m$.

Proposition 2.1. *Let (X, L) be a classical scroll over a smooth projective variety Y of dimension m. We use notations in Notation 2.2. Then for every integer i with $0 \leq i \leq n$ the following holds.*

$$e_i(X, L) = \sum_{t=0}^{i} \sum_{k=0}^{i-t} (-1)^{i-t} \binom{m-t-2}{i-t-k} c_k(\mathcal{E})\, c_t(Y)\, s_{m-k-t}(\mathcal{E}).$$

Proof. See the first part of the proof in [2, Theorem 3.1]. □

3. Main result

Definition 3.1. Let Y be a smooth projective variety of dimension m and $\mathcal{E}$ a vector bundle on Y. Then for every integer i with $0 \leq i \leq m$ we define the polynomial $t_i(x_0, \ldots, x_i) \in \mathbb{Z}[x_0, \ldots, x_i]$ which satisfies the following.

$$c_i(\mathcal{E}) = t_i(s_0(\mathcal{E}), \ldots, s_i(\mathcal{E})). \tag{4}$$

For example, we see that $t_0(x_0) = 1$, $t_1(x_0, x_1) = x_1$, $t_2(x_0, x_1, x_2) = x_1^2 - x_2$ and so on.

Proposition 3.1. *Let Y be a smooth projective variety of dimension m and $\mathcal{E}$ a vector bundle over Y. For every integer i with $0 \leq i \leq m$, we have $s_i(\mathcal{E}) = t_i(c_0(\mathcal{E}), \ldots, c_i(\mathcal{E}))$.*

Proof. We prove this by induction.
(I) If $i = 0$, then this is true because $c_0(\mathcal{E}) = s_0(\mathcal{E}) = 1$.
(II) Assume that the assertion holds for every i with $i \leq k - 1$. So we consider the case $i = k$. Then by Definition 2.3 (ii)

$$\sum_{\substack{i+j=k \\ i\geq 0, j\geq 0}} (-1)^i c_i(\mathcal{E}) s_j(\mathcal{E}) = 0. \tag{5}$$

Hence by (5) we have

$$\begin{aligned} t_k(s_0(\mathcal{E}), \ldots, s_k(\mathcal{E})) &= c_k(\mathcal{E}) \\ &= (-1)^{k+1} \sum_{\substack{i+j=k \\ j\geq 1}} (-1)^i c_i(\mathcal{E}) s_j(\mathcal{E}) \end{aligned}$$

$$= (-1)^{k+1} \sum_{\substack{i+j=k \\ j\geq 1}} (-1)^i t_i(s_0(\mathcal{E}), \dots, s_i(\mathcal{E}))s_j(\mathcal{E}).$$

In particular, we have

$$t_k(x_0, \dots, x_k) = (-1)^{k+1} \sum_{\substack{i+j=k \\ j\geq 1}} (-1)^i t_i(x_0, \dots, x_i)x_j. \tag{6}$$

On the other hand, we see from the induction hypothesis and (6) that

$$\begin{aligned} s_k(\mathcal{E}) &= -\sum_{\substack{i+j=k \\ i\geq 1}} (-1)^i c_i(\mathcal{E})s_j(\mathcal{E}) \\ &= (-1)^{k+1} \sum_{\substack{i+j=k \\ i\geq 1}} (-1)^j c_i(\mathcal{E})t_j(c_0(\mathcal{E}), \dots, c_j(\mathcal{E})) \\ &= t_k(c_0(\mathcal{E}), \dots, c_k(\mathcal{E})). \end{aligned}$$

So we get the assertion. □

The following theorem which is Theorem 1.1 in Introduction is the main result of this note.

Theorem 3.1. *Let (X, L) be an n-dimensional classical scroll over a smooth projective variety Y of dimension m such that $n \geq 2m$. Let $F_i(t_1, \dots, t_m)$ be the polynomial defined in Definition 1.1 for every integer i with $0 \leq i \leq m$. We use notations in Notation 2.2. Then for any integer j with $m \leq j \leq 2m$ we have*

$$\mathrm{cl}_j(X, L) = F_{2m-j}(c_1(\mathcal{E}), \dots, c_m(\mathcal{E})).$$

In particular

$$F_m(s_1(\mathcal{E}), \dots, s_m(\mathcal{E})) = F_m(c_1(\mathcal{E}), \dots, c_m(\mathcal{E})).$$

Proof. First we prove the following.

Claim 3.1 *For any integer i with $0 \leq i \leq m$, we have*

$$\begin{aligned} & e_{2m-2-i}(X, L) \\ &= \sum_{t=0}^{i} \sum_{l=0}^{i-t} (-1)^{i-t} \binom{m-t-2}{i-t-l} c_{m-t-l}(\mathcal{E}^\vee)c_t(Y)s_l(\mathcal{E}) + (m-i-1)c_m(Y), \\ & e_{2m-1-i}(X, L) \\ &= \sum_{t=0}^{i-1} \sum_{l=0}^{i-1-t} (-1)^{i-1-t} \binom{m-t-2}{i-1-t-l} c_{m-t-l}(\mathcal{E}^\vee)c_t(Y)s_l(\mathcal{E}) + (m-i)c_m(Y), \end{aligned}$$

$$\begin{aligned}&e_{2m-i}(X,L)\\&=\sum_{t=0}^{i-2}\sum_{l=0}^{i-2-t}(-1)^{i-2-t}\binom{m-t-2}{i-2-t-l}c_{m-t-l}(\mathcal{E}^{\vee})c_t(Y)s_l(\mathcal{E})+(m-i+1)c_m(Y).\end{aligned}$$

(*We note that if* $i=0$ (*resp.* $i=0,1$), *then* $\sum_{t=0}^{i-1}\sum_{l=0}^{i-1-t}(-1)^{i-1-t}\binom{m-t-2}{i-1-t-l}$ $c_{m-t-l}(\mathcal{E}^{\vee})c_t(Y)s_l(\mathcal{E})=0$ (*resp.* $\sum_{t=0}^{i-2}\sum_{l=0}^{i-2-t}(-1)^{i-2-t}\binom{m-t-2}{i-2-t-l}$ $c_{m-t-l}(\mathcal{E}^{\vee})c_t(Y)s_l(\mathcal{E})=0$).)

Proof. (A) First we treat $e_{2m-2-i}(X,L)$. Then by Proposition 2.1

$$\begin{aligned}&e_{2m-2-i}(X,L)\\&=\sum_{t=0}^{2m-2-i}\left(\sum_{k=0}^{2m-2-i-t}(-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^{\vee})c_t(Y)s_{m-k-t}(\mathcal{E})\right).\end{aligned}$$

Here we note that

$$2m-2-i-t\geq k. \tag{7}$$

We set

$$E(i,k,t)=(-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^{\vee})c_t(Y)s_{m-k-t}(\mathcal{E}).$$

If $E(i,k,t)\neq 0$, then the following two conditions hold by noting (7).

$$0\leq k\leq m. \tag{8}$$

$$0\leq t\leq m. \tag{9}$$

$$k+t\leq \min\{m,2m-2-i\}. \tag{10}$$

If $m-t-2>0$ and $m-t-2<2m-2-i-t-k$, then $\binom{m-t-2}{2m-2-i-t-k}=0$. Hence if $E(i,k,t)\neq 0$, then $m-t-2\leq 0$ or $m-t-2\geq 2m-2-i-t-k$, that is,

$$t\geq m-2\quad\text{or}\quad k\geq m-i. \tag{11}$$

(A.1) The case where $0\leq i\leq m-3$.
We see from (8), (9), (10) and (11) that $e_{2m-2-i}(X,L)$ is the sum of $E(i,k,t)$ in the range of the following (k,t).

$$\text{(A.1.1)}\begin{cases}k=0,1,2,\quad t=m-2,\\ k=0,1,\quad t=m-1,\\ k=0,\quad t=m.\end{cases}\qquad \text{(A.1.2)}\begin{cases}k=m-i,\quad t=i,i-1,\dots,1,0\\ k=m-i+1,\quad t=i-1,\dots,1,0\\ \qquad\vdots\\ k=m,\quad t=0.\end{cases}$$

The sum of $E(i,k,t)$ in the range of the case (A.1.1) is the following.

$$
\begin{aligned}
(12)\quad &\sum_{t=m-2}^{m}\sum_{k=0}^{m-t}E(i,k,t)\\
&=(-1)^{2m-2-i-m+2-0}\binom{m-(m-2)-2}{2m-2-i-(m-2)-0}c_{m-2}(Y)s_2(\mathcal{E})\\
&\quad+(-1)^{2m-2-i-m+2-1}\binom{m-(m-2)-2}{2m-2-i-(m-2)-1}c_1(\mathcal{E}^\vee)c_{m-2}(Y)s_1(\mathcal{E})\\
&\quad+(-1)^{2m-2-i-m+2-2}\binom{m-(m-2)-2}{2m-2-i-(m-2)-2}c_2(\mathcal{E}^\vee)c_{m-2}(Y)\\
&\quad+(-1)^{2m-2-i-m+1-0}\binom{m-(m-1)-2}{2m-2-i-(m-1)-0}c_{m-1}(Y)s_1(\mathcal{E})\\
&\quad+(-1)^{2m-2-i-m+1-1}\binom{m-(m-1)-2}{2m-2-i-(m-1)-1}c_1(\mathcal{E}^\vee)c_{m-1}(Y)\\
&\quad+(-1)^{2m-2-i-m}\binom{m-m-2}{2m-2-i-m-0}c_m(Y)\\
&=(-1)^{m-i}\binom{0}{m-i}c_{m-2}(Y)s_2(\mathcal{E})\\
&\quad+(-1)^{m-i-1}\binom{0}{m-i-1}c_1(\mathcal{E}^\vee)c_{m-2}(Y)s_1(\mathcal{E})\\
&\quad+(-1)^{m-i-2}\binom{0}{m-i-2}c_2(\mathcal{E}^\vee)c_{m-2}(Y)\\
&\quad+(-1)^{m-i-1}\binom{-1}{m-1-i}c_{m-1}(Y)s_1(\mathcal{E})\\
&\quad+(-1)^{m-i-2}\binom{-1}{m-2-i}c_1(\mathcal{E}^\vee)c_{m-1}(Y)\\
&\quad+(-1)^{m-i-2}\binom{-2}{m-i-2}c_m(Y)\\
&=c_{m-1}(Y)s_1(\mathcal{E})+c_1(\mathcal{E}^\vee)c_{m-1}(Y)+(m-i-1)c_m(Y)\\
&=(m-1-i)c_m(Y).
\end{aligned}
$$

On the other hand, the sum of $E(i,k,t)$ in the range of the case (A.1.2) is the following.

$$
\begin{aligned}
&\sum_{k=m-i}^{m}\sum_{t=0}^{m-k}(-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^\vee)c_t(Y)s_{m-k-t}(\mathcal{E})\\
&=\sum_{t=0}^{i}\sum_{k=m-i}^{m-t}(-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^\vee)c_t(Y)s_{m-k-t}(\mathcal{E}).
\end{aligned}
$$

Here we put $j := k - (m-i)$. Then by $t \leq i \leq m-3$ we have

$$\sum_{t=0}^{i}\sum_{k=m-i}^{m-t}(-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^\vee)c_t(Y)s_{m-k-t}(\mathcal{E})$$
$$=\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E})$$
$$=\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E}).$$

Hence we have

$$\begin{aligned}(13)\qquad & e_{2m-2-i}(X,L)\\ &=\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E})\\ &\quad+(m-1-i)c_m(Y).\end{aligned}$$

(A.2) The case where $i = m-2$.
We see from (8), (9), (10) and (11) that $e_{2m-2-i}(X,L)$ is the sum of $E(m-2,k,t)$ in the range of the following (k,t).

$$\text{(A.2.1)}\begin{cases} k=0,1, & t=m-2,\\ k=0,1, & t=m-1,\\ k=0, & t=m.\end{cases}\quad \text{(A.2.2)}\begin{cases} k=2, & t=m-2,m-3,\dots,1,0\\ k=3, & t=m-3,\dots,1,0\\ & \vdots\\ k=m, & t=0.\end{cases}$$

First we calculate the sum of $E(m-2,k,t)$ in the range of the case (A.2.1).

(14)
$$\sum_{k=0}^{1}E(m-2,k,m-2)+\sum_{t=m-1}^{m}\sum_{k=0}^{m-t}E(m-2,k,t)$$
$$=(-1)^{2m-2-(m-2)-m+2-0}\binom{m-(m-2)-2}{2m-2-(m-2)-(m-2)-0}c_{m-2}(Y)s_2(\mathcal{E})$$
$$+(-1)^{2m-2-(m-2)-m+2-1}\binom{m-(m-2)-2}{2m-2-(m-2)-(m-2)-1}c_1(\mathcal{E}^\vee)c_{m-2}(Y)s_1(\mathcal{E})$$
$$+(-1)^{2m-2-(m-2)-m+1-0}\binom{m-(m-1)-2}{2m-2-(m-2)-(m-1)-0}c_{m-1}(Y)s_1(\mathcal{E})$$

$$+(-1)^{2m-2-(m-2)-m+1-1}\binom{m-(m-1)-2}{2m-2-(m-2)-(m-1)-1}c_1(\mathcal{E}^\vee)c_{m-1}(Y)$$
$$+(-1)^{2m-2-(m-2)-m}\binom{m-m-2}{2m-2-(m-2)-m-0}c_m(Y)$$
$$=(-1)^2\binom{0}{2}c_{m-2}(Y)s_2(\mathcal{E})+(-1)^1\binom{0}{1}c_1(\mathcal{E}^\vee)c_{m-2}(Y)s_1(\mathcal{E})$$
$$+(-1)^1\binom{-1}{1}c_{m-1}(Y)s_1(\mathcal{E})+(-1)^0\binom{-1}{0}c_1(\mathcal{E}^\vee)c_{m-1}(Y)+(-1)^0\binom{-2}{0}c_m(Y)$$
$$=c_{m-1}(Y)s_1(\mathcal{E})+c_1(\mathcal{E}^\vee)c_{m-1}(Y)+c_m(Y)$$
$$=c_m(Y).$$

Next we calculate the sum of $E(m-2,k,t)$ in the range of the case (A.2.2).

$$\sum_{k=2}^{m}\sum_{t=0}^{m-k}E(m-2,k,t) \tag{15}$$
$$=\sum_{t=0}^{m-2}\sum_{k=2}^{m-t}(-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^\vee)c_t(Y)s_{m-k-t}(\mathcal{E})$$
$$=\sum_{t=0}^{m-2}\sum_{j=0}^{m-2-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+2}(\mathcal{E}^\vee)c_t(Y)s_{(m-2)-j-t}(\mathcal{E})$$
$$=\sum_{t=0}^{m-2}\sum_{j=0}^{m-2-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+2}(\mathcal{E}^\vee)c_t(Y)s_{(m-2)-j-t}(\mathcal{E}).$$

(Here we set $j:=k-2$ in the above equations.)

Since $i=m-2$, we see from (14) and (15) that

$$\begin{aligned} e_{2m-2-i}(X,L) &= \sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E}) \\ &\quad +(m-1-i)c_m(Y). \end{aligned} \tag{16}$$

(A.3) The case where $i=m-1$.
We see from (8), (9), (10) and (11) that $e_{2m-2-i}(X,L)$ is the sum of $E(m-1,k,t)$ in the range of the following (k,t).

$$\text{(A.3.1)}\ \begin{cases} k=0, & t=m-2, \\ k=0, & t=m-1. \end{cases} \qquad \text{(A.3.2)}\ \begin{cases} k=1, & t=m-2, m-3, \dots, 1, 0 \\ k=2, & t=m-3, \dots, 1, 0 \\ & \vdots \\ k=m-1, & t=0. \end{cases}$$

The sum of $E(m-1,k,t)$ in the range of the case (A.3.1) is obtained as follows.

(17)

$$\begin{aligned}
&\sum_{t=m-2}^{m-1} E(m-1,0,t)\\
&= (-1)^{2m-2-(m-1)-m+2-0}\binom{m-(m-2)-2}{2m-2-(m-1)-(m-2)-2} c_2(\mathcal{E}^\vee)c_{m-2}(Y)\\
&\quad +(-1)^{2m-2-(m-1)-m+1-0}\binom{m-(m-1)-2}{2m-2-(m-1)-(m-1)-0} c_{m-1}(Y)s_1(\mathcal{E})\\
&= (-1)^1\binom{0}{-1}c_1(\mathcal{E}^\vee)c_{m-2}(Y)s_1(\mathcal{E})\\
&\quad +(-1)^0\binom{-1}{0}c_{m-1}(Y)s_1(\mathcal{E})\\
&= c_{m-1}(Y)s_1(\mathcal{E}).
\end{aligned}$$

On the other hand, we get the sum of $E(m-1,k,t)$ in the range of the case (A.3.2) as follows.

(18)

$$\begin{aligned}
&\sum_{k=1}^{m-1}\sum_{t=0}^{m-1-k} E(m-1,k,t)\\
&= \sum_{t=0}^{m-2}\sum_{k=1}^{m-1-t}(-1)^{2m-2-(m-1)-t-k}\binom{m-t-2}{2m-2-(m-1)-t-k}c_k(\mathcal{E}^\vee)c_t(Y)s_{m-k-t}(\mathcal{E})\\
&= \sum_{t=0}^{m-2}\sum_{j=0}^{m-2-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+1}(\mathcal{E}^\vee)c_t(Y)s_{(m-1)-j-t}(\mathcal{E})\\
&= \sum_{t=0}^{m-1}\sum_{j=0}^{m-1-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+1}(\mathcal{E}^\vee)c_t(Y)s_{(m-1)-j-t}(\mathcal{E}).
\end{aligned}$$

We note that in the final step of the above equalities we use $\binom{m-t-2}{m-2-t-j} = 0$ for $(t,j) = (0,m-1),(1,m-2),\ldots,(m-1,0)$. Moreover

(19)

$$\begin{aligned}
&\sum_{t=0}^{m-1}\sum_{j=0}^{m-1-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+1}(\mathcal{E}^\vee)c_t(Y)s_{(m-1)-j-t}(\mathcal{E})\\
&= \sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E}) - c_1(\mathcal{E})c_{m-1}(Y).
\end{aligned}$$

We also note that in the final step of the above equalities

$$\binom{m-t-2}{m-2-t-j} = \begin{cases} \binom{m-t-2}{j} & \text{if } t \leq m-2, \\ \binom{m-t-2}{j} - 1 & \text{if } t = m-1. \end{cases}$$

(Here we note that if $t = m-1$, then $j = 0$ in this case.)

Hence we see from (17), (18) and (19) that for $i = m-1$

$$\begin{aligned} (20) \qquad & e_{2m-2-i}(X,L) \\ &= \sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j} c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E}) \\ &\quad + (m-1-i)c_m(Y). \end{aligned}$$

(A.4) The case where $i = m$.
We see from (8), (9), (10) and (11) that $e_{2m-2-i}(X,L)$ is the sum of $E(m,k,t)$ in the range of the following (k,t).

$$\begin{cases} k = 0, \quad t = m-2, m-3, \ldots, 1, 0, \\ k = 1, \quad t = m-3, \ldots, 1, 0, \\ \qquad \vdots \\ k = m-2, \quad t = 0. \end{cases}$$

By the same argument as above we get

$$\begin{aligned} (21) \quad & e_{2m-2-i}(X,L) \\ &= \sum_{k=0}^{m-2}\sum_{t=0}^{m-2-k}(-1)^{2m-2-m-t-k}\binom{m-t-2}{2m-2-m-t-k} c_k(\mathcal{E}^\vee)c_t(Y)s_{m-k-t}(\mathcal{E}) \\ &= \sum_{t=0}^{m-2}\sum_{j=0}^{m-2-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j} c_j(\mathcal{E}^\vee)c_t(Y)s_{m-j-t}(\mathcal{E}) \\ &= \sum_{t=0}^{m}\sum_{j=0}^{m-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j} c_j(\mathcal{E}^\vee)c_t(Y)s_{m-j-t}(\mathcal{E}). \end{aligned}$$

We note that in the final step of the above equalities we use $\binom{m-t-2}{m-2-t-j} = 0$ for $(t,j) = (0,m), (1,m-1), \ldots, (m,0), (0,m-1), (1,m-2), \ldots, (m-1,0)$.

On the other hand

$$(22) \qquad \binom{m-t-2}{m-2-t-j} = \begin{cases} \binom{m-t-2}{j} & \text{if } t \leq m-2, \\ \binom{m-t-2}{j} - (-1)^j & \text{if } t = m-1, m. \end{cases}$$

(Here we note that if $t = m-1$ (resp. $t = m$), then $j = 0, 1$ (resp. $j = 0$) in this case.)

Hence we see from (21) and (22) that for $i = m$

$$
\begin{aligned}
(23)\quad & e_{2m-2-i}(X, L) \\
&= \sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+m-i}(\mathcal{E}^{\vee})c_t(Y)s_{i-j-t}(\mathcal{E}) \\
&= \sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^{\vee})c_t(Y)s_{i-j-t}(\mathcal{E}) \\
&\quad + c_{m-1}(Y)s_1(\mathcal{E}) + c_1(\mathcal{E}^{\vee})c_{m-1}(Y) - c_m(Y) \\
&= \sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^{\vee})c_t(Y)s_{i-j-t}(\mathcal{E}) \\
&\quad + (m-i-1)c_m(Y).
\end{aligned}
$$

By (13), (16), (20) and (23) for any i with $0 \leq i \leq m$ we have

$$
\begin{aligned}
e_{2m-2-i}(X, L) &= \sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^{\vee})c_t(Y)s_{i-j-t}(\mathcal{E}) \\
&\quad + (m-i-1)c_m(Y).
\end{aligned}
$$

Furthermore we set $l := i-t-j$. Then

$$
\begin{aligned}
&\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^{\vee})c_t(Y)s_{i-j-t}(\mathcal{E}) \\
&= \sum_{t=0}^{i}\sum_{l=0}^{i-t}(-1)^{m-2-i+l}\binom{m-t-2}{i-t-l}c_{m-t-l}(\mathcal{E}^{\vee})c_t(Y)s_l(\mathcal{E}) \\
&= \sum_{t=0}^{i}\sum_{l=0}^{i-t}(-1)^{i-t}\binom{m-t-2}{i-t-l}c_{m-t-l}(\mathcal{E})c_t(Y)s_l(\mathcal{E}).
\end{aligned}
$$

Hence for every integer i with $0 \leq i \leq m$

$$
\begin{aligned}
(24)\quad & e_{2m-2-i}(X, L) \\
&= \sum_{t=0}^{i}\sum_{l=0}^{i-t}(-1)^{i-t}\binom{m-t-2}{i-t-l}c_{m-t-l}(\mathcal{E})c_t(Y)s_l(\mathcal{E}) + (m-i-1)c_m(Y).
\end{aligned}
$$

(B) Next we consider $e_{2m-1}(X, L)$ and $e_{2m}(X, L)$. Then by [2, Theorem 3.1 (3.1.1)] we have

$$e_{2m-1}(X, L) = mc_m(Y), \tag{25}$$

$$e_{2m}(X, L) = (m+1)c_m(Y). \tag{26}$$

(C) By (24), (25) and (26), we get the assertion of Claim 3.1. □

Here we set

$$\begin{aligned}&E_i(c_0(\mathcal{E}),\dots,c_i(\mathcal{E});s_{m-i}(\mathcal{E}),\dots,s_m(\mathcal{E}))\\&:=\sum_{t=0}^{i}\sum_{k=0}^{i-t}(-1)^{i-t}\binom{m-t-2}{i-t-k}c_k(\mathcal{E})\,c_t(Y)\,s_{m-k-t}(\mathcal{E}).\end{aligned}$$

Then by Proposition 2.1 we have

$$E_i(c_0(\mathcal{E}),\dots,c_i(\mathcal{E});s_{m-i}(\mathcal{E}),\dots,s_m(\mathcal{E}))=e_i(X,L). \tag{27}$$

Moreover by Claim 3.1 we have

$$\begin{aligned}&e_{2m-2-i}(X,L)\\&=E_i(s_0(\mathcal{E}),\dots,s_i(\mathcal{E});c_{m-i}(\mathcal{E}),\dots,c_m(\mathcal{E}))+(m-i-1)c_m(Y),\end{aligned} \tag{28}$$

$$\begin{aligned}&e_{2m-1-i}(X,L)\\&=E_{i-1}(s_0(\mathcal{E}),\dots,s_{i-1}(\mathcal{E});c_{m-i+1}(\mathcal{E}),\dots,c_m(\mathcal{E}))+(m-i)c_m(Y),\end{aligned} \tag{29}$$

$$\begin{aligned}&e_{2m-i}(X,L)\\&=E_{i-2}(s_0(\mathcal{E}),\dots,s_{i-2}(\mathcal{E});c_{m-i+2}(\mathcal{E}),\dots,c_m(\mathcal{E}))+(m-i+1)c_m(Y)\end{aligned} \tag{30}$$

for every integer i with $0\le i\le m$. By (27) and Definitions 1.1 and 2.2 we get

$$\begin{aligned}&F_i(s_1(\mathcal{E}),\dots,s_m(\mathcal{E}))\\&=\mathrm{cl}_i(X,L)\\&=(-1)^i\{E_i(c_0(\mathcal{E}),\dots,c_i(\mathcal{E});s_{m-i}(\mathcal{E}),\dots,s_m(\mathcal{E}))\\&\quad-2E_{i-1}(c_0(\mathcal{E}),\dots,c_{i-1}(\mathcal{E});s_{m-i+1}(\mathcal{E}),\dots,s_m(\mathcal{E}))\\&\quad+E_{i-2}(c_0(\mathcal{E}),\dots,c_{i-2}(\mathcal{E});s_{m-i+2}(\mathcal{E}),\dots,s_m(\mathcal{E}))\}\\&=(-1)^i\{E_i(t_0(s_0(\mathcal{E})),\dots,t_i(s_0(\mathcal{E}),\dots,s_i(\mathcal{E}));s_{m-i}(\mathcal{E}),\dots,s_m(\mathcal{E}))\\&\quad-2E_{i-1}(t_0(s_0(\mathcal{E})),\dots,t_{i-1}(s_0(\mathcal{E}),\dots,s_{i-1}(\mathcal{E}));s_{m-i+1}(\mathcal{E}),\dots,s_m(\mathcal{E}))\\&\quad+E_{i-2}(t_0(s_0(\mathcal{E})),\dots,t_{i-2}(s_0(\mathcal{E}),\dots,s_{i-2}(\mathcal{E}));s_{m-i+2}(\mathcal{E}),\dots,s_m(\mathcal{E}))\}\end{aligned} \tag{31}$$

for every integer i with $0\le i\le m$. Here $t_i(x_0,\dots,x_i)$ denotes the polynomial which was defined in Definition 3.1.

On the other hand we see from Proposition 3.1, (28), (29), (30) and (31) that for every integer i with $0\le i\le m$

$$\begin{aligned}&\mathrm{cl}_{2m-i}(X,L)\\&=(-1)^{2m-i}\{e_{2m-i}(X,L)-2e_{2m-i-1}(X,L)+e_{2m-i-2}(X,L)\}\end{aligned}$$

$$
\begin{aligned}
&= (-1)^i\{E_{i-2}(s_0(\mathcal{E}),\dots,s_{i-2}(\mathcal{E});c_{m-i+2}(\mathcal{E}),\dots,c_m(\mathcal{E}))\\
&\quad - 2E_{i-1}(s_0(\mathcal{E}),\dots,s_{i-1}(\mathcal{E});c_{m-i+1}(\mathcal{E}),\dots,c_m(\mathcal{E}))\\
&\quad + E_i(s_0(\mathcal{E}),\dots,s_i(\mathcal{E});c_{m-i}(\mathcal{E}),\dots,c_m(\mathcal{E}))\\
&\quad + (m-i+1)c_m(Y) - 2(m-i)c_m(Y) + (m-i-1)c_m(Y)\}\\
&= (-1)^i\{E_i(s_0(\mathcal{E}),\dots,s_i(\mathcal{E});c_{m-i}(\mathcal{E}),\dots,c_m(\mathcal{E}))\\
&\quad - 2E_{i-1}(s_0(\mathcal{E}),\dots,s_{i-1}(\mathcal{E});c_{m-i+1}(\mathcal{E}),\dots,c_m(\mathcal{E}))\\
&\quad + E_{i-2}(s_0(\mathcal{E}),\dots,s_{i-2}(\mathcal{E});c_{m-i+2}(\mathcal{E}),\dots,c_m(\mathcal{E}))\}\\
&= (-1)^i\{E_i(t_0(c_0(\mathcal{E})),\dots,t_i(c_0(\mathcal{E}),\dots,c_i(\mathcal{E}));c_{m-i}(\mathcal{E}),\dots,c_m(\mathcal{E}))\\
&\quad - 2E_{i-1}(t_0(c_0(\mathcal{E})),\dots,t_{i-1}(c_0(\mathcal{E}),\dots,c_{i-1}(\mathcal{E}));c_{m-i+1}(\mathcal{E}),\dots,c_m(\mathcal{E}))\\
&\quad + E_{i-2}(t_0(c_0(\mathcal{E})),\dots,t_{i-2}(c_0(\mathcal{E}),\dots,c_{i-2}(\mathcal{E}));c_{m-i+2}(\mathcal{E}),\dots,c_m(\mathcal{E}))\}\\
&= F_i(c_1(\mathcal{E}),\dots,c_m(\mathcal{E})).
\end{aligned}
$$

Therefore we get the assertion of Theorem 3.1. □

Finally we note the following.

PROPOSITION 3.2. *Let (X, L) be an n-dimensional classical scroll over a smooth projective variety Y of dimension m such that $n \geq 2m + 1$. Then* $\mathrm{cl}_i(X, L) = 0$ *for every integer i with $2m + 1 \leq i \leq n$.*

Proof. By [2, Theorem 3.1 (3.1.1)], we see that $e_j(X, L) = (j - m + 1)c_m(Y)$ for every integer j with $j \geq 2m - 1$. Hence

$$\mathrm{cl}_i(X, L) = (-1)^i(e_i(X, L) - 2e_{i-1}(X, L) + e_{i-2}(X, L)) = 0.$$

This completes the proof. □

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