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CHARACTERIZATIONS OF GRAPHS TO INDUCE PERIODIC GROVER WALK

By

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Abstract. Recently, a research on quantum walks has been developed in various areas. In this paper we focus on the periodicity of the Grover walk which is one of the quantum walks on the discrete graphs. Then we find some special graphs to induce a periodic Grover walk: At some time k, the quantum state φ_k returns its initial quantum state φ_0. Our purpose is to characterize graphs which induce a k-periodic Grover walk for a fixed integer k. We do it for $k = 2, 3, 4, 5$ and gain a necessary condition for odd k.

1. Introduction

1.1 Background and Notation

Quantum walks (QWs) were introduced as quantizations of random walks (RWs) [5]. Every QW is determined by a given graph, its induced Hilbert space $\mathcal{H}$ and a unitary time evolution operator on $\mathcal{H}$. The amplitude is obtained by this unitary iteration to a given initial state. Due to the unitarity of the time iteration, the norm of the amplitude is preserved, which implies that the distribution can be defined at each time step. However it is believed that there are not any trivial representations of the present distribution of QWs by that of past time like Markov chain [8]. QWs have been applied to various study fields, for example, a problem of searching marked elements on graphs [22], [2], [17], [18], fundamental physics [21], [4], the limit theorems for its statistical behaviors [12], [13], spectral analysis [3], [15], and photon synthesis [19]. In [22], the Szegedy walk was formulated as a natural quantization of thr reversible Markov chain, and Szegedy showed that in most cases the quantized walk hits the marked set within the square root of the classical hitting time. The Grover walk, which is a special case of the Szegedy walk, is a widely studied quantum walk model. The Grover walk is related to the analysis of the zeta function and the isomorphic problem

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between two cospectrum strongly regular graphs [9], [14]. In [7], Grover's search algorithm was introduced to search marked elements in a database.

In this paper we focus on the periodicity of the Grover walk on graphs. The periodicity means that the quantum state at some time returns to the initial state. Recently, periodicities of QWs have been studied. In [16], Konno, Takei and Shimizu study Hadamard walk on cycle graph C_n, and find that only C_2, C_4, and C_8 induce periodic Hadamard walks, whose periods are $2, 8$, and 24, respectively. In [11], Higuchi, Konno, Sato and Segawa resarch some QWs on several finite graphs and find the conditions of these graphs to induce periodic QWs. The results are as follows:

- (Complete graphs) The Szegedy walks induced by isotropic random walk with laziness l with $l \in (0,1] (l \in \mathbb{Q})$ on K_n are periodic if and only if $(n,l) = (2,0), (3,0), (n,1/n), (2,1/4)$, or $(n,(n+1)/(2n))$, whose periods are $2, 3, 4, 6$ and 6, respectively.
- (Complete bipartite graphs) The Szegedy walks induced by isotropic random walk with laziness l with $l \in (0,1] (l \in \mathbb{Q})$ on $K_{n,m}$ with $m, n > 0$ and $m + n \geq 3$ are periodic if and only if $l = 0$, or $1/2$, whose periods are 4 and 12, respectively.
- (Strongly regular graphs) The Grover walks on the strongly regular graphs $\mathrm{SRG}(n,k,\lambda,\mu)$ are periodic if and only if

$$(n,k,\lambda,\mu) = (2k,k,0,k), (3\lambda, 2\lambda, \lambda, 2\lambda), (5,2,0,1),$$

  whose periods are $4, 12$ and 5, respectively.
- (Cycle graphs) The Szegedy walks on C_n induced by non-isotropic random walk such that a walker jumps around clockwise with a non-reversible probability $p (\neq 1/2)$ and around counterclockwise with probability $1-p$ are periodic if and only if $p = (2-\sqrt{3})/4, (2-\sqrt{2})/4,\ 1/4$ for $n = 2$, whose periods are $6, 8$ and 12, respectively, or $p = (2-\sqrt{3})/4, (2-\sqrt{2})/4,\ 1/4$ for $n = 4$, whose periods are $12, 8$ and 12, respectively, or $p = (2-\sqrt{2})/4$ for $n = 8$, whose period is 24.

If the underlying graph gives a periodicity to the Grover walk, then the sequence of the distributions is periodic. So we can say that such graphs are special class of graphs from the viewpoint of QWs. Our purpose is to characterize such special classes of graphs. In the previous results, for fixed graphs, the conditions of graphs to induce periodic QWs are found. On the contrary, we fix an integer k and characterize graphs to induce k-periodic Grover walks.

For a given finite graph G, we denote $\mathbb{C}^{|D(G)|}$ by $\mathcal{H}$, and give a $|D(G)| \times |D(G)|$ unitary operator U, where $D(G)$ is the set of symmetric arcs of G, that is, $D(G) = \{(u,v),(v,u) | uv \in E(G)\}$. A walker of the Grover walk on G transfers on arcs. The motion of the quantum walker is interpreted as a dynamics of plane wave on the metric graphs [10].

First, we introduce the notations and QWs on graphs. All graphs considered in this paper are finite and simple graphs without loops and multiple edges. Let $V(G)$, $E(G)$ be a set of vertices and edges of G and set $n = |V(G)|, m = |E(G)|$, respectively. The matrix $T = (T_{u,v})\ (u, v \in V(G))$ is the $n \times n$ transition matrix of isotropic RWs, that is,

$$T_{u,v} = \begin{cases} 1/\deg(u) & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

For $uv \in E(G)$, let $e = (u, v)$ be an arc from u to v, and its inverse arc (v, u) is denoted by e^{-1}. The origin and terminus of e are denoted by $o(e), t(e)$, respectively. For a square matrix A, if we write

$$\mathrm{Spec}(A) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix},$$

it implies that the multiplicity of the eigenvalue λ_i of A is m_i for $1 \le i \le r$. In this paper we denote $\lambda \in \mathrm{Spec}(A)$, if λ is an eigenvalue of A.

Throughout this paper, a path graph and a complete graph with l vertices are denoted by P_l and K_l, respectively. The graph C_l is a cycle graph with l vertices and we call it an even-cycle if l is even, otherwise an odd-cycle. The tree graphs are defined as graphs without cycles. In this paper, we say that a graph G is a unicycle graph if it contains exactly one cycle C and the subgraph $G \backslash C$ is a non-empty forest graph, where the forest graph is a disjoint union of tree graphs. So we do not call cycle graphs unicycle graphs in this paper. In addition, we call a unicycle graph even-unicycle graph if the length of the cycle is even, otherwise odd-unicycle graph. In this paper the graphs in Figures 1, 2 are unicycle graphs, but the graph in Figure 3 is not a unicycle graph. Let $K_{r,s}$ be a complete bipartite graph with two partitions with r and s vertices. The girth of G is denoted by $g(G)$, which is the length of the minimum cycle in G.

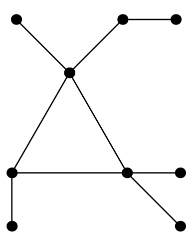

Figure 1

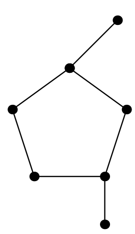

Figure 2

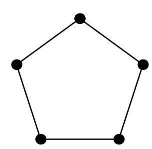

Figure 3

Furthermore, we assign vectors $\vec{x}_e$ to every $e \in D(G)$, where $\vec{x}_e$ s are the standard basis of the Hilbert space $\mathbb{C}^{2m}$, that is, $(\vec{x}_e)_f = \delta_{e,f}$ for every $e, f \in D(G)$. We construct the quantum state at time t, $\varphi_t \in \mathbb{C}^{2m}$ as

$$\varphi_t = \Sigma_{e \in D(G)} \alpha_e^t \vec{x}_e,$$

where $\alpha_e^t \in \mathbb{C}$, and set $\Sigma_{e \in D(G)} | \alpha_e^t |^2 = 1$. Then the finding probability of the walker on an arc e at time t is $|\alpha_e^t|^2$. Giving a $2m \times 2m$ unitary matrix U, we determine φ_{t+1} as

$$\varphi_{t+1} = U\varphi_t.$$

Hence, we can denote φ_t as

$$\varphi_t = U^t \varphi_0$$

using the initial state φ_0.

Next, we introduce the Grover walk. The evolution operator of the Grover walk is the following $2m \times 2m$ unitary matrix $U = U(G) = (U_{e,f})\ (e, f \in D(G))$:

$$U_{e,f} = \begin{cases} 2/\deg(t(f)) & \text{if } t(f) = o(e) \text{ and } f \neq e^{-1}, \\ 2/\deg(t(f)) - 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The quantum waves on an arc f transmits to an arc e with $t(f) = o(e)$ and $f \neq e$ with a rate of $2/\deg(t(f))$, and reflects to the arc f^{-1} with a rate of $2/\deg(t(f)) - 1$. This U is called the Grover transfer matrix. We shall give an example of the Grover walk and consider the periodicity. We will provide an example of graphs which induces periodic Grover walks.
Example: $G = K_{1,3}$.

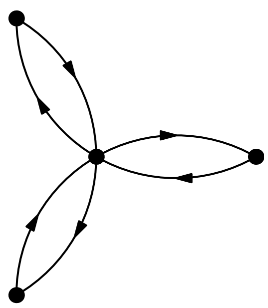

Figure 4 $K_{1,3}$

$$U = U(K_{1,3}) = \begin{pmatrix} 0 & 0 & 0 & -1/3 & 2/3 & 2/3 \\ 0 & 0 & 0 & 2/3 & -1/3 & 2/3 \\ 0 & 0 & 0 & 2/3 & 2/3 & -1/3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

In fact, this U satisfies $U^4 = I_6$, that is, $\varphi_4 = U^4\varphi_0 = \varphi_0$ for an arbitrary initial state φ_0. We say that $G = K_{1,3}$ is a graph to induce a 4-periodic Grover walk. The other examples are given in Figures 5, 6, 7.

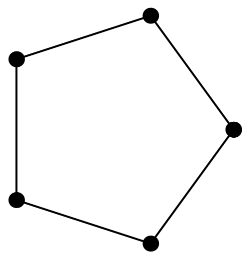

Figure 5

Figure 6

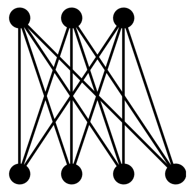

Figure 7

These graphs induce $5, 6, 4$-periodic Grover walks, respectively.

1.2 Main Results

For a positive integer k, inducing a k-periodic Grover walk implies that $U^k = I_{2m}$, while $U^j \neq I_{2m}$ for every j with $j < k$ for the Grover transfer matrix U. These conditions can be regarded as the following spectral problem.

PROPOSITION 1.1. *A graph G induces a k-periodic Grover walk if and only if $\lambda_U^k = 1$ for every $\lambda_U \in \mathrm{Spec}(U)$, and there exists $\lambda_U \in \mathrm{Spec}(U)$ such that $\lambda_U^j \neq 1$ for every j with $j < k$.*

So what we have to do is checking whether all the eigenvalues of U satisfy the condition of the above Proposition. In order to consider it, we need the following Theorem.

THEOREM 1.2. (Emms, Hancock, Severini and Wilson [3], [4]) *For the Grover transfer matrix U and the transition matrix T on a graph G, it holds*

$$det(\lambda_U I_{2m} - U) = (\lambda_U^2 - 1)^{m-n}((\lambda_U^2 + 1)I_n - 2\lambda_U T)$$

for every $\lambda_U \in \mathbb{C}$.

So U has $2n$ eigenvalues of the following form

$$\lambda_U = \lambda_T \pm i\sqrt{1-\lambda_T^2},$$

where $\lambda_T \in \mathrm{Spec}(T)$. The remaining $2(m-n)$ eigenvalues are $-1, 1$, which have the same multiplicities. Therefore if these two kinds of eigenvalues satisfy the condition of Proposition 1.1, then G induces a k-periodic Grover walk. We characterize such graphs for $k = 2, 3, 4, 5$ and obtain a necessary condition for odd k.

Theorem 1.3. *The graph P_2 is the only graph to induce the 2-periodic Grover walk.*

Theorem 1.4. *If G induces an odd-periodic Grover walk, then G is an odd-cycle or an odd-unicycle graph.*

Theorem 1.5. *The graphs C_3, C_5 are the only graphs to induce the 3, 5-periodic Grover walks, respectively.*

Theorem 1.6. *The graph $K_{r,s}$ is the only graph to induce the 4-periodic Grover walk for every $r, s \in \mathbb{N}$.*

This paper is organized as follows: In section 2, we mention the 2-periodic case and prove Theorem 1.3. In section 3, we first give a necessary condition for graphs to induce an odd-periodic Grover walk and prove Theorems 1.4, 1.5 with several Lemmas. In section 4, we prove Theorem 1.6 by using a property of bipartite graphs. At the end of this paper, we summarize our results and make some discussions in section 5.

2. Proof of Theorem 1.3

Here we explain the graph to induce the 2-periodic Grover walk, and prove Theorem 1.3 with some Lemmas.

Lemma 2.1. *For any $\lambda_T \in \mathrm{Spec}(T)$, it holds that $|\lambda_T| \leq 1$ and $1 \in \mathrm{Spec}(T)$.*

Lemma 2.2. (Perron-Frobenius) *If A is a non-negative matrix, that is, all entries are non-negative, then the eigenvector of the maximum eigenvalue of A is a non-negative vector and its multiplicity is 1.*

Proof of Theorem 1.3. Obviously P_2 induces the 2-periodic Grover walk. Hence, we prove that if G induces a 2-periodic Grover walk, then G is P_2. By Proposition

1.1, for any $\lambda_U \in \mathrm{Spec}(U)$, it should hold that $\lambda_U^2 = 1$. According to Theorem 1.2, U has eigenvalues of the form $\lambda_T \pm i\sqrt{1-\lambda_T^2}$, and the remaining $2(m-n)$ eigenvalues are ± 1. The latter values satisfy the condition of Proposition 1.1, then the former values should satisfy it, which implies $\lambda_T \pm i\sqrt{1-\lambda_T^2} = \pm 1$, that is, $\lambda_T = \pm 1$. Since T is a non-negative matrix and 1 is the maximum eigenvalue of T, then its multiplicity is 1 by Lemmas 2.1, 2.2. Moreover the multiplicity of -1 is also 1 because of $\mathrm{Tr}(T) = 0$. So we can obtain

$$\mathrm{Spec}(T) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{1}$$

Considering the connectivity of G and the summation of their multiplicities, we can obtain that P_2 is the only graph which leads (1) as a spectrum of its transition matrix. Indeed,

$$U(P_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Obviously it induces the 2-periodic Grover walk. Then such a graph is only P_2. □

3. Proofs of Theorems 1.4, 1.5

In this section we show that the graphs inducing an odd-periodic Grover walk should satisfy some conditions. Then we prove Theorems 1.4, 1.5.

3.1 Proof of Theorem 1.4

Lemma 3.1. *A graph G is a bipartite graph if and only if it holds that $(\lambda_T)_{min} = -(\lambda_T)_{max}$ for the eigenvalues of its transition matrix T.*

Furthermore using Lemmas 2.1, 2.2, we can gain the following corollary:

Corollary 3.2. *A graph G is a bipartite graph if and only if $-1 \in \mathrm{Spec}(T)$.*

Proof of Theorem 1.4. Let k be an odd integer. We assume that $m - n > 0$. For all of the eigenvalues of U, λ_U should satisfy $\lambda_U^k = 1$. If $m - n > 0$, then at least one -1 is an eigenvalue of U by Theorem 1.2. Since k is odd, -1 does not satisfy the condition. Thus, it should hold that $m - n \leq 0$. It follows that $m = n-1, m = n$ from the connectivity of G. Then such graphs must be trees, which satisfy $m = n - 1$, or cycles, unicycle graphs, which satisfy $m = n$. From Corollary 3.2 and Theorem 1.2, trees, even-cycles, and even-unicycle graphs are improper graphs since these are bipartite. Hence, an odd-cycle and an odd-unicycle graph can induce a k-periodic Grover walk for an odd k. □

3.2 Proof of Theorem 1.5

First, we introduce an easy Lemma and obtain a restriction for unicycle graphs which induce an odd-periodic Grover walk. We prove Theorem 1.5 with them.

Lemma 3.3. *The graph C_k induces a k-periodic Grover walk.*

Proof. Let A be an adjacent matrix of C_k, and λ_A be an eigenvalue of A. For every j with $0 \leq j \leq k$,

$$\lambda_A = 2\cos\frac{2\pi}{k}j.$$

Since C_k is 2-regular graph, it holds that

$$T = \frac{1}{2}A.$$

Then $\lambda_T = \cos(2\pi j/k)$ for $0 \leq j \leq k$. Hence, the eigenvalues of U are

$$\lambda_T \pm i\sqrt{1-\lambda_T^2} = e^{\pm\frac{2\pi i}{k}j}.$$

Thus, C_k induces a k-periodic Grover walk. □

Proposition 3.4. *Let G be an odd-unicycle graph. If G induces a k-periodic Grover walk, then it should hold that $g(G) \leq k-4$.*

Proof. Let k be an odd integer and $g(G) = t$. We assume that an odd-unicycle graph G induces a k-periodic Grover walk. Since G is a unicycle graph, G contains the graph on Figure 8 as its subgraph. Let the vertices of the cycle C_t be $v_1, \cdots, v_t$. Also we define arcs $e_i \in D(G)$ such as

$$e_i = \begin{cases} (v_i, v_{i+1}) & \text{if } 1 \leq i \leq t-1, \\ (v_t, v_1) & \text{if } i = t. \end{cases}$$

Let e be the arc (v, v_1). For the Grover transfer matrix U, $U^k_{e,f}$ can be written by

$$U^k_{e,f} = \sum U_{e,h_{k-1}} U_{h_{k-1},h_{k-2}} \cdots U_{h_2,h_1} U_{h_1,f},$$

where $h_1, h_2, \cdots, h_{k-1} \in D(G)$ run over the arcs which make the walk $f \to h_1 \to h_2 \to \cdots \to h_{k-1} \to e$ with the length k in G. From its periodicity, it should

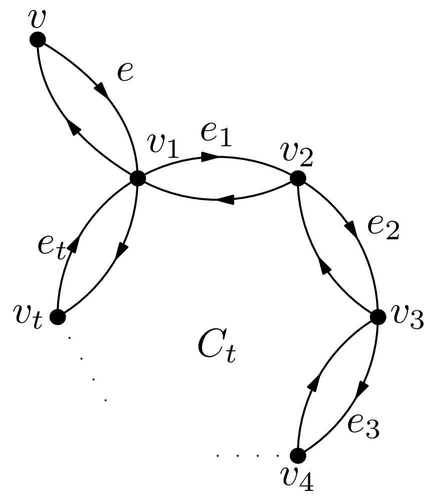

Figure 8

hold that $U^k_{e,e} = 1$. We consider two cases (i) $t \geq k$, (ii) $t = k-2$ and then we show $U^k_{e,e} \neq 1$ in the both of cases.

(i) $t \geq k$. In order to obtain $U^k_{e,e}$, we consider the walks with length k from e to e. Since the distance between e and e is even and G is a unicycle graph, we have to traverse C_t to go from e to e with odd steps. This walk has the length at least $t+2$. So there are no walks from e to e with length k since $t \geq k$. Therefore we can conclude that $U^k_{e,e} = 0$, and it gives us a contradiction.

(ii) $t = k-2$. From the observation of (i), the walks should run through C_t at once. Since $t = k-2$, there are only two walks such as

$$e \to e^{-1} \underbrace{\to e_1 \to e_2 \to \cdots \to e_t}_{k-2\,steps} \to e,$$
$$e \to e^{-1} \underbrace{\to e_t^{-1} \to \cdots \to e_2^{-1} \to e_1^{-1}}_{k-2\,steps} \to e.$$

Hence,

$$\begin{aligned} U^k_{e,e} &= U_{e,e_t} \cdots U_{e_2,e_1} U_{e_1,e^{-1}} U_{e^{-1},e} \\ &\quad + U_{e,e_1^{-1}} \cdots U_{e_{t-1}^{-1},e_t^{-1}} U_{e_t^{-1},e^{-1}} U_{e^{-1},e} \\ &= \frac{2}{\deg(v_1)} \cdots \frac{2}{\deg(v_2)} \frac{2}{\deg(v_1)} \left(\frac{2}{\deg(v)} - 1 \right) \\ &\quad + \frac{2}{\deg(v_1)} \cdots \frac{2}{\deg(v_t)} \frac{2}{\deg(v_1)} \left(\frac{2}{\deg(v)} - 1 \right) \\ &= \left(\frac{2}{\deg(v)} - 1 \right) \frac{8}{\{\deg(v_1)\}^2} \frac{2}{\deg(v_2)} \cdots \frac{2}{\deg(v_t)}. \end{aligned} \tag{2}$$

It follows $\frac{2}{\deg(v)} - 1 > 0$ and $\deg(v) = 1$ from $U^k_{e,e} = 1$. Therefore

$$U^k_{e,e} = \frac{8}{\{\deg(v_1)\}^2} \frac{2}{\deg(v_2)} \cdots \frac{2}{\deg(v_t)}.$$

However it holds that $\deg(v_1) \geq 3, \deg(v_2), \cdots, \deg(v_t) \geq 2$ from the choice of the subgraph. Thus, it follows that $U^k_{e,e} \neq 1$. □

Proof of Theorem 1.5. From the previous arguments, the graphs inducing the 3, 5-periodic Grover walks are C_3, C_5 or the other unicycle graphs, respectively. Proposition 3.4 leads the fact that the girth $g(G)$ of such unicycle graphs should be less than or equal to 1 for $k = 3$, and 5. However $g(G)$ is greater than 2 from the definition of unicycle graphs. Thus, no unicycle graphs induce 3, and 5-periodic Grover walks. Therefore C_3, C_5 are the only graphs to induce the 3, 5-periodic Grover walks, respectively. □

4. Proof of Theorem 1.6

In this section we prove Theorem 1.6 by using a property of the bipartite graphs. Similar to the previous sections, we introduce some Lemmas to prove Theorem 1.6.

Lemma 4.1. *A graph G induces a 4-periodic Grover walk if and only if the spectrum of its transition matrix T is of the form*

$$\mathrm{Spec}(T) = \begin{pmatrix} -1 & 0 & 1 \\ 1 & n-2 & 1 \end{pmatrix}. \tag{3}$$

Proof. First, we show its necessity. In other words, we show that if G induces 4-periodic Grover walk, then spectrum of T is of the form (3). Similar to Proof of Theorem 1.3, for any $\lambda_U \in \mathrm{Spec}(U)$, it should hold that $\lambda_U^4 = 1$. Then it follows that

$$\lambda_T \pm i\sqrt{1 - \lambda_T^2} = \pm 1, \pm i.$$

Thus, $\lambda_T = \pm 1, 0$. Therefore we can obtain

$$\mathrm{Spec}(T) = \begin{pmatrix} -1 & 0 & 1 \\ 1 & n-2 & 1 \end{pmatrix}.$$

Immediately, their multiplicities are determined by Lemma 2.1 and Lemma 2.2.

Next, we show its sufficiency. If the spectrum of T of G is of the form (3), then the spectrum of U is of the form

$$\mathrm{Spec}(U) = \begin{pmatrix} -1 & -i & i & 1 \\ m-n+2 & n-2 & n-2 & m-n+2 \end{pmatrix}.$$

From Proposition 1.1, G induces a 4-periodic Grover walk. □

In fact this spectrum induces the complete bipartite graphs immediately.

Lemma 4.2. *A graph G is a complete bipartite graph if and only if the spectrum of its transition matrix T is of the form*

$$\mathrm{Spec}(T) = \begin{pmatrix} -1 & 0 & 1 \\ 1 & n-2 & 1 \end{pmatrix}. \tag{4}$$

Proof. First, we show its necessity. If G is a complete bipartite graph, then its transition matrix T can be written by

$$T = \begin{pmatrix} O_{r,r} & \frac{1}{s}J_{r,s} \\ \frac{1}{r}J_{s,r} & O_{s,s} \end{pmatrix},$$

where $O_{i,j}$, $J_{i,j}$ denote the $i \times j$ matrix with all 0 entries, and the $i \times j$ matrix with all 1 entries, respectively. Then it follows that $\mathrm{rank}(T) = 2$, and $\dim(\ker(T)) = n-2$. It implies that the multiplicity of the eigenvalue 0 of T is $n-2$ since T is a diagonalizable matrix. Furthermore 1 is an eigenvalue of T with multiplicity 1, and so is -1 from Corollary 3.2. Therefore we can obtain (4) as the spectrum of the transition matrix of G.

Next, we show its sufficiency. If the spectrum of T is of the form (4), then -1 is an eigenvalue of T. Thus, G is a bipartite graph by Corollary 3.2. Then its transition matrix T can be represented by

$$T = \begin{pmatrix} O_{r,r} & V_{r,s} \\ W_{s,r} & O_{s,s} \end{pmatrix}$$

with some $r \times s$ matrix $V_{r,s}$, and $s \times r$ matrix $W_{s,r}$. These matrices are the transition matrices from a partition R to another partition S, and from S to R, respectively, where R, S are the bipartition of G with $r = |R|$, $s = |S|$. In addition each summations over the row of T are 1. We show that all of the entries of $V_{r,s}$, and $W_{s,r}$ are not 0. From the assumption of $\dim(\ker(T)) = n-2$, we can see that $\mathrm{rank}(T) = 2$. Therefore $V_{r,s}$ and $W_{s,r}$ can be represented by

$$V_{r,s} = \begin{pmatrix} c_1\vec{a} \\ \vdots \\ c_r\vec{a} \end{pmatrix},$$

$$W_{s,r} = \begin{pmatrix} d_1\vec{b} \\ \vdots \\ d_s\vec{b} \end{pmatrix},$$

where $\vec{a}$ is a $1 \times r$ rational vector and $\vec{b}$ is a $1 \times s$ rational vector, and c_i, d_j are rational numbers for $1 \leq i \leq r$, $1 \leq j \leq s$. We can obtain $c_i = 1$, $d_j = 1$ for $1 \leq i \leq r$, $1 \leq j \leq s$ since the summations over the row of T are 1. If there exists i such that the i-th entry of $\vec{a}$ is 0, then the i-th column vector of T is the zero vector. It contradicts the connectivity of G. Thus, $V_{r,s}$ does not contain 0 as its entries. Similarly $W_{s,r}$ also does not contain 0 as its entries. Therefore any vertices in R are adjacent to the vertices in S, and so are the vertices in S. We can conclude that G is a complete bipartite graph. □

Proof of Theorem 1.6. From Lemma 4.1, G induces a 4-periodic Grover walk if and only if the spectrum of its transition matrix is of the form (4). Having such a spectrum of T means that G is a complete bipartite graph by Lemma 4.2. Then we can show that the complete bipartite graph $K_{r,s}$ is the only graph to induce the 4-periodic Grover walk for $r, s \in \mathbb{N}$. □

5. Summary and Discussions

In this paper, we have given some characterizations of the graphs to induce a k-periodic Grover walk for $k = 2, 3, 4, 5$. We proved that $P_2, C_3, K_{r,s}, C_5$ are the only graphs to induce $2, 3, 4, 5$-periodic Grover walks, respectively. One of what we want to do is to determine such graphs for an integer k with $k \geq 6$. The main method used in this paper to characterize such graphs is to analyze the spectrum of its transition matrix, that is, to find graphs whose spectrum of its transition matrix is occupied by the real part of the k-th root of 1 and have an eigenvalue that is not the real part of j-th root of 1 for every j with $j < k$. Generally speaking, it is difficult to characterize graphs with any given spectrum. For the cases of $k \geq 6$, we might take another method to solve it.

Next, we will provide some examples to induce a k-periodic Grover walk for $k \geq 6$ and a special operator between graphs.

Lemma 5.1. *The graph P_k induces a $2(k-1)$-periodic Grover walk.*

Using this Lemma and Lemma 3.3, we can conclude that P_4 and C_6 induce 6-periodic Grover walks. In addition both of the graphs on Figure 9, 10 also induce 6-periodic Grover walks. These graphs are made by identifying two endpoints of some P_4 s. These graphs include P_4 and C_6. Furthermore both of the graphs

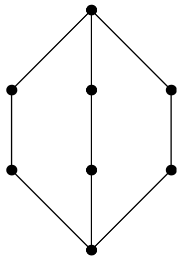

Figure 9

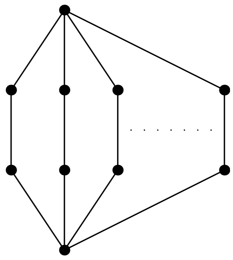

Figure 10

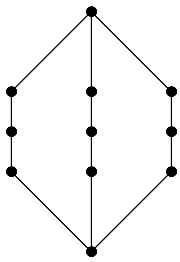

Figure 11

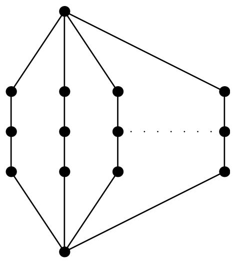

Figure 12

on Figures 11, 12 induce 8-periodic Grover walks. These graphs include P_5, C_8. Then the graphs to induce an even-periodic Grover walk contain these graphs, and we must find any other graphs. For an odd k, it is thought that C_k is the only graph which induce a k-periodic Grover walk. So we have to eliminate the possibility of the unicycle graphs. Moreover we got the following Proposition.

Proposition 5.2. *If G induces a k-periodic Grover walk then its subdivision graph $S(G)$ induces a $2k$-periodic Grover walk.*

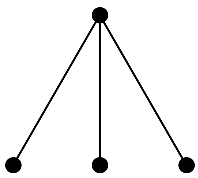

Figure 13 Graph G

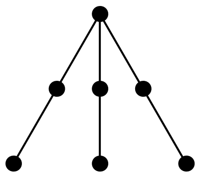

Figure 14 Graph $S(G)$

The graphs $G, S(G)$ induce $4, 8$-periodic Grover walks, respectively. We can regard the subdivision as an operator which conserves the periodicity of the Grover walk between graphs. To find such operators between graphs is also our interest. The Grover walk on the graphs are determined by the Grover transfer matrix. So we are also interesting in investigating the periodicity of another QWs on the graphs determined by a unitary matrix except the Grover transfer matrix.

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