

ON A PROPERTY OF THE SIMPLE RANDOM WALK ON \mathbb{Z}

By

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Abstract. The subject of this paper is the simple random walk on \mathbb{Z} . We give a very simple answer to the following problem: under the condition that the random walk has already spent A -percent of the traveling time on the positive side $\mathbb{Z}_{\geq 0}$, what is the probability that the random walk is now on the positive side?

All the possible walks which step $2n$ -times on \mathbb{Z} can be decomposed in the following two ways: (1) how many times the walk steps on the positive side, (2) whether the last step is on the positive side or on the negative side. To answer the above problem, we clarify the number of the walks classified by (1) and (2). It has been already known that the distribution of the number indicated by (1) makes the arcsine law. Combining with the decomposition with respect to (2), we obtain a decomposition of the arcsine law into the Marčenko-Pastur law.

1. Introduction

An invisible simple random walker on \mathbb{Z} suddenly telephoned to you at the time $t = 2n$ and informed you that she has spent 70 percent of the time on the positive side, from $t = 0$ to $t = 2n$.

Question: What is the probability that she is on the positive side **now** ($t = 2n$)?

Answer: 70 percent.

... But WHY?

In the present paper, we will explain WHY (and to show some remarkable corollary connecting classical probability theory and noncommutative probability). In Section 2 we briefly review basic definitions and known results on the simple random walk on \mathbb{Z} . We prove the main theorem (Theorem 5) and as a corollary we explain WHY (Theorem 6) in Section 3.

In the last section we show that the main theorem implies the new limit theorem connecting the simple random walk with positive ends and the Marčenko-Pastur law [2]. It plays a crucial role in random matrix theory and in free

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probability. For instance, it appears in the context of universality of $M \times N$ real rectangular random matrices with independent and identically distributed components whose average is zero and variance is σ . For such A , in the case that $M, N \rightarrow \infty$ and $\frac{M}{N} \rightarrow \alpha$, the distribution of the eigenvalues of the symmetric random matrices $A^T A$ tends to the distribution called the Marčenko-Pastur law with parameter α . The distribution also appears as the “free Poisson distribution” in the theory of free probability (see [3, Section 3.7]), which is a branch of noncommutative probability (quantum probability) and is deeply related to random matrix theory and combinatorics. The term “free Poisson” comes from the fact that its “free cumulants” are constant, while Poisson distribution has constant cumulants. In Theorem 7 we show that the percentage of the time which the Brownian motion with positive end has spent is described by Marčenko-Pastur law with parameter $\alpha = 1$. In other words, we obtain a decomposition of the arcsine law into the Marčenko-Pastur law.

2. Preliminaries on the simple random walk on \mathbb{Z}

First we introduce a notation frequently used throughout the paper:

Notation: Discrete time interval $[m, n] := \{m, m + 1, \dots, n\}$ **for integers satisfying** $m \leq n$.

In the following we define several basic notions and theorems for the simple random walk:

DEFINITION 1. $(\mathcal{P}_N, \mathcal{P}_N^0)$

We denote the set of all possible paths of simple random walk $X(t)$ on discrete time interval $[0, N] = \{0, 1, 2, 3, \dots, N\}$ by \mathcal{P}_N . The set of all possible paths of simple random walk $X(t)$ on time interval $[0, N]$ such that $X(N) = 0$ is denoted by \mathcal{P}_N^0 .

These sets can be written as

$$\begin{aligned} \mathcal{P}_N &= \{X: [0, N] \rightarrow \mathbb{Z} \mid X(0) = 0, |X(t) - X(t-1)| = 1 \ (t \in [1, N])\}, \\ \mathcal{P}_N^0 &= \{X \in \mathcal{P}_N \mid X(N) = 0\}. \end{aligned}$$

The number of the elements of \mathcal{P}_N is 2^N . The number of the elements of \mathcal{P}_N^0 can be written as follows:

THEOREM 1. (Feller [1], Chap III Sec 4 Theorem1-(a))

$$\#\mathcal{P}_N^0 = \begin{cases} \binom{2n}{n}, & N = 2n, \quad (n = 0, 1, 2, 3, \dots) \\ 0, & N \text{ is odd.} \end{cases}$$

The central notions in the present paper are defined as follows:

DEFINITION 2. (Positive side, Soujourn time) Let $X(t)$ be a simple random walk on \mathbb{Z} . For positive integer t , we say that “ $X(t)$ is on the positive side” if $X(t) \geq 0$ and $X(t-1) \geq 0$. In such a case, we also say that the path X is on the positive side at t .

The soujourn time on the positive side T_N is defined by

$$T_N := \# \{t \in [1, N] \mid X(t) \text{ is on the positive side.}\}.$$

DEFINITION 3. ($\mathcal{P}_{N,m}, \mathcal{P}_{N,m}^0, \mathcal{P}_N^+, \mathcal{P}_{N,m}^+$) We define $\mathcal{P}_{N,m}$ by

$$\mathcal{P}_{N,m} = \{p \in \mathcal{P}_N \mid p \text{ is a path for which } T_N = m \text{ is satisfied.}\}.$$

We also define $\mathcal{P}_{N,m}^0 := \mathcal{P}_{N,m} \cap \mathcal{P}_N^0$. We denote by \mathcal{P}_N^+ the set of all the random walks $X(t)$ defined for $t \in [0, N]$ which are on the positive side at N . and define $\mathcal{P}_{N,m}^+ := \mathcal{P}_{N,m} \cap \mathcal{P}_N^+$.

For $\#\mathcal{P}_{2n,2k}$ and $\#\mathcal{P}_{2n,2k}^0$, the following general formula is well-known.

THEOREM 2. (Arcsine law for simple random walk, Feller [1], Chap III Sec 5 Theorem 1)

$$\#\mathcal{P}_{2n,2k} = \binom{2k}{k} \binom{2n-2k}{n-k}.$$

THEOREM 3. (Uniform principle, Feller [1], Chap III Sec 2 Theorem 3)

$$\#\mathcal{P}_{2n,2k}^0 = \frac{1}{n+1} \binom{2n}{n}, \quad \text{for every } k \in [0, n].$$

Then how about $\#\mathcal{P}_{2n,2k}^+$? In the case that $k = n$, it is obvious that $\#\mathcal{P}_{2n,2n}^+ = \#\mathcal{P}_{2n,2n}$, so Theorem 2 gives the following:

PROPOSITION 4. (Feller [1], Chap III Sec 4 Theorem 1-(c))

$$\#\mathcal{P}_{2n,2n}^+ = \binom{2n}{n}.$$

Theorem 5 in the next section gives a general formula for $\#\mathcal{P}_{2n,2k}^+$. As a corollary (Theorem 6), we get an answer to the question “WHY” in the introduction.

3. Main Theorem

In this section, we present our main result (Theorem 5). To prove this, we prepare Lemma 1 and Lemma 2 as follows.

LEMMA 1. *Let n, k be positive integers and $1 \leq k \leq n - 1$. Then*

$$\#\mathcal{P}_{2n,2k}^+ = \sum_{l \in [1,k]} \frac{1}{n-l+1} \cdot \binom{2n-2l}{n-l} \cdot 2 \binom{2l-2}{l-1}.$$

Proof. Since $1 \leq k \leq n - 1$, it is obvious that for $X \in \mathcal{P}_{2n,2k}^+$

$$\{t \in [1, 2n - 1] \mid X(t) = 0\} \neq \emptyset.$$

Let $2\tau := \text{Max}\{t \in [1, 2n - 1] \mid x(t) = 0\}$. Then $\tau \in [n - k, n - 1]$. We can totally decompose the set $\mathcal{P}_{2n,2k}^+$ by the values of 2τ . It is easy to see that $X(t)$ is on the positive side for $2k - (2n - 2\tau)$ times in $[0, 2\tau]$ because $X(t)$ is always (i.e. for $2n - 2\tau$ times) on the positive side; and more strongly,

- $X(2\tau + 1) = 1$
- $X(t) \geq 1$ on $[2\tau + 1, 2n - 1]$
- $X(2n) = X(2n - 1) + 1$ or $X(2n - 1) - 1$.

It is easy to see that the number of paths defined on $[2\tau + 1, 2n]$ satisfying $X(2\tau + 1) = 1$ and $X(t) \geq 1$ for all $t \in [2\tau + 1, 2n - 1]$ is nothing but $2 \cdot \#\mathcal{P}_{2n-2\tau-2, 2n-2\tau-2}^+$. Hence, from the above argument and Theorem 3, we obtain

$$\begin{aligned} \#\mathcal{P}_{2n,2k}^+ &= \sum_{\tau \in [n-k, n-1]} \#P_{2\tau, 2k-(2n-2\tau)}^0 \cdot (2 \cdot \#\mathcal{P}_{2n-2\tau-2, 2n-2\tau-2}^+) \\ &= \sum_{\tau \in [n-k, n-1]} \frac{1}{\tau+1} \binom{2\tau}{\tau} \cdot 2 \binom{2(n-\tau)-2}{(n-\tau)-1} \\ &= \sum_{l \in [1,k]} \frac{1}{n-l+1} \binom{2n-2l}{n-l} \cdot 2 \binom{2l-2}{l-1}, \end{aligned}$$

where $l := n - \tau$. \square

LEMMA 2. *Let n, k, l be positive integers and $1 \leq k \leq n - 1$. Then the identity*

$$\sum_{l \in [1,k]} \frac{1}{n-l+1} \cdot \binom{2n-2l}{n-l} \cdot 2 \binom{2l-2}{l-1} = \frac{k}{n} \binom{2k}{k} \binom{2n-2k}{n-k}$$

holds.

Proof. We show the identity by induction with respect to k .

(i) The case that $k = 1$ is trivial.

(ii) Suppose that the lemma holds for $k = m < n - 1$. Then we have

$$\begin{aligned} & \sum_{l \in [1, m+1]} \frac{1}{n-l+1} \cdot \binom{2n-2l}{n-l} \cdot 2 \binom{2l-2}{l-1} \\ &= \sum_{l \in [1, m]} \frac{1}{n-l+1} \cdot \binom{2n-2l}{n-l} \cdot 2 \binom{2l-2}{l-1} + \frac{1}{n-m} \cdot \binom{2n-2m-2}{n-m-1} \cdot 2 \binom{2m}{m}. \end{aligned}$$

By the induction hypothesis, the above quantity is

$$\begin{aligned} & \frac{m}{n} \binom{2m}{m} \binom{2n-2m}{n-m} + \frac{1}{n-m} \cdot \binom{2n-2m-2}{n-m-1} \cdot 2 \binom{2m}{m} \\ &= \frac{m}{n} \binom{2m}{m} \frac{(2n-2m)(2n-2m-1)}{(n-m)^2} \binom{2n-2m-2}{n-m-1} \\ & \quad + \frac{1}{n-m} \cdot \binom{2n-2m-2}{n-m-1} \cdot 2 \binom{2m}{m} \\ &= \frac{2}{n-m} \binom{2m}{m} \binom{2n-2m-2}{n-m-1} \left\{ \frac{m}{n} (2n-2m-1) + 1 \right\} \\ &= \frac{2}{n-m} \binom{2m+2}{m+1} \frac{(m+1)^2}{(2m+2)(2m+1)} \binom{2n-2m-2}{n-m-1} \frac{(n-m)(2m+1)}{n} \\ &= \frac{m+1}{n} \binom{2m+2}{m+1} \binom{2n-2m-2}{n-m-1} \\ &= \frac{m+1}{n} \binom{2(m+1)}{m+1} \binom{2n-2(m+1)}{n-(m+1)}. \end{aligned}$$

This is nothing but the identity for $k = m + 1$.

By (i) and (ii) the identity is proved for any $k \in [1, n - 1]$. \square

The following is the main theorem of the present paper:

THEOREM 5.

$$\#\mathcal{P}_{2n, 2k}^+ = \frac{k}{n} \binom{2k}{k} \binom{2n-2k}{n-k}.$$

Proof. For $k = 0$, it is trivial. For $k = n$, it follows from Proposition 4.

The equation for the case of $0 < n < k$ directly follows from Lemma 1 and Lemma 2. \square

As a corollary, we obtain an explanation for WHY:

THEOREM 6. *For the simple random walk on \mathbb{Z} , the identity below holds.*

$$\mathbf{P}(X(2n) \text{ is on the positive side} \mid T_{2n} = 2k) = \frac{k}{n}.$$

Proof. By the definition of the conditional probability and from Theorem 2 and Theorem 5, we have

$$\mathbf{P}(X(2n) \text{ is on the positive side} \mid T_{2n} = 2k) = \frac{\#\mathcal{P}_{2n,2k}^+}{\#\mathcal{P}_{2n,2k}} = \frac{\frac{k}{n} \binom{2k}{k} \binom{2n-2k}{n-k}}{\binom{2k}{k} \binom{2n-2k}{n-k}} = \frac{k}{n}.$$

□

This is why

“the probability that we can now find on the positive side a simple random walker who has already spent 70 percent of time on the positive side is 70 percent.”

4. Marčhenko-Pastur Law

For the Brownian motion $B(\tau)$, it is well known that

$$\mathbf{P}(\mu(\tau \in [0, t] \mid B(\tau) \geq 0) \leq rt) = \frac{1}{\pi} \int_0^r \frac{dx}{\sqrt{x(1-x)}},$$

where $\mu(dx) = dx$ denotes the Lebesgue measure of the real line. The fact above is called the Arcsine law [1].

Based on the theorems in the previous section and on the standard transition from digital to analog, we easily obtain the arcsine law for the Brownian motion $x(\tau)$ with positive ends:

THEOREM 7. *We have*

$$\mathbf{P}(\mu(\tau \in [0, t] \mid x(\tau) \geq 0) \leq rt) = \frac{2}{\pi} \int_0^r \frac{xdx}{\sqrt{x(1-x)}} = \frac{2}{\pi} \int_0^r \sqrt{\frac{x}{1-x}} dx,$$

and dually,

$$\mathbf{P}(\mu(\tau \in [0, t] \mid x(\tau) \leq 0) \leq rt) = \frac{2}{\pi} \int_0^r \frac{(1-x)dx}{\sqrt{x(1-x)}} = \frac{2}{\pi} \int_0^r \sqrt{\frac{1-x}{x}} dx,$$

where $\mu(dx) = dx$ denotes the Lebesgue measure of the real line.

The probability law above is nothing but the Marčhenko-Pastur law, which plays fundamental and universal roles in the theory of random matrices and free probability. It strongly suggests that there is some hidden relationship between the classical probability, combinatorics, random matrices and quantum probability.

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