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U-STATISTICS AND U-PROCESSES BASED ON WEAKLY $\mathcal{M}$-DEPENDENT DATA

By

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Summary. The object of this paper is to study asymptotics of U-statistics and U-processes based on centered stationary sequences, weakly $\mathcal{M}$-dependent in L^p with rate function $\delta(\cdot)$. The notion of the weakly $\mathcal{M}$-dependence is recently introduced by Berkes et al (2011) and has many examples such as NED processes, augmented GARCH sequences, linear processes with dependent innovations, etc.

1. Introduction

1.1 Weakly $\mathcal{M}$-dependent in L^p with rate function $\delta(\cdot)$

Berkes et al. (2011) introduced one of the notion of weak dependence which is called " weakly $\mathcal{M}$-dependent in L^p with rate fuction $\delta(m)$" and proved a new type of strong approximation of partial sums of dependent processes.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $p \geq 1$ and for any random variable Y, let $\|Y\|_p = (E|Y|^p)^{\frac{1}{p}}$. For A and B ($\subset \{\cdots, -1, 0, 1, 2, \cdots\}$), we define

$$d(A, B) = \inf\{|a - b|; a \in A, b \in B\}.$$

We say that a real-valued process $\{\eta_i\}$ is weakly $\mathcal{M}$-dependent in L^p with rate function $\delta(\cdot)$ if $\{\eta_i\}$ satisfies the following conditions:

Condition I. (A) For any integer k and positive integer m one can find a random variable $\eta_i^{(m)}$ with finite p-th moment such that

$$\|\eta_k - \eta_k^{(m)}\|_p \leq \delta(m) \downarrow 0 \quad m \to \infty; \tag{1}$$

(B) For any disjoint subsets $I_1, \cdots, I_r$ of integers and any positive integers $m_1, \cdots, m_r$ the vectors $\{\eta_{j_1}^{(m_1)}, j_1 \in I_1\}, \cdots, \{\eta_{j_r}^{(m_r)}, j_r \in I_r\}$ are independent provided

$$d(I_k, I_l) > \max\{m_k, m_l\} \quad (1 \leq k < l \leq r).$$

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(1) implies that

$$\|\eta_i\|_p \le \|\eta_i^{(m)}\|_p + \|\eta_i - \eta_i^{(m)}\|_p < \infty. \tag{2}$$

Note that sequences satisfying conditions (A) and (B) are approximable by m-dependent processes for any fixed order m (≥ 1) with termwise approximation error $\delta(m)$.

Properties of weakly $\mathcal{M}$-dependent in L^p

(I) If $\{\eta_i\}$ is weakly $\mathcal{M}$-dependent in L^p with rate function $\delta(\cdot)$ and h is a Lipschitz α function ($0 < \alpha \le 1$) with Lipschitz constant K, then

$$\begin{aligned} &\|h(\eta_k) - h(\eta_k^{(m)})\|_p \\ &\le K\|\eta_k - \eta_k^{(m)}\|_{\alpha p}^{\alpha} \le K\|\eta_k - \eta_k^{(m)}\|_p^{\alpha} \le K\delta^{\alpha}(m) \end{aligned} \tag{3}$$

(II) The following theorem was proved in Berkes et al (2011):

Theorem A. *Let $p > 2$, $\kappa > 0$ and let $\{\eta_i\}$ be a centered stationary sequence, weakly $\mathcal{M}$-dependent in L^p and a rate function $\delta(\cdot)$ satisfying*

$$\delta(m) \le Cm^{-\alpha} \tag{4}$$

where

$$\alpha > \frac{p-2}{2\kappa}\left(1 - \frac{1+\kappa}{p}\right) \vee 1, \quad \frac{1+\kappa}{p} < \frac{1}{2}. \tag{5}$$

Then, we can redefine on a new probability space together with two standard Wiener processes $\{W_1(t); t \ge 0\}$ and $\{W_2(t); t \ge 0\}$ such that

$$\sum_{k=1}^{n} \eta_k = W_1(s_n) + W_2(t_n) + O\big(n^{\frac{1+\kappa}{p}}\big) \quad a.s. \tag{6}$$

where $\{s_n\}$ and $\{t_n\}$ are nondecreasing numerical sequences with

$$s_n \sim n, \quad t_n \sim C_1 n^{\rho}, \tag{7}$$

$0 < \rho < 1$ and $C_1 > 0$ are some constants.

1.2 A critrion of weak convergence

Let X_n and X be random elements of $\mathcal{D}[0,1]$. Let $\mathcal{T}_X$ be the set of all t in $[0,1]$ which contains 0 and 1, and $0 < t < 1$, and $t \in \mathcal{T}_X$ if and only if $P(X(t) \ne X(t-)) = 0$. The following theorem assures the weak convergence of X_n.

Theorem B. (Billingsley (1968) Theorem 15.6) *Suppose that*

$$(X_n(t_1), X_n(t_2), \cdots, X_n(t_k)) \xrightarrow{D} (X(t_1), X(t_2), \cdots, X(t_k))$$

holds whenever $t_1, t_2, \cdots, t_k$ *all lie in* $\mathcal{T}_X$ *and that*

$$P(|X_n(t) - X_n(t_1)| > \lambda, |X_n(t_2) - X_n(t)| > \lambda) \leq \frac{1}{\lambda^{2\kappa}}\{G(t_2) - G(t_1)\}^{2\tau}$$

for $t_1 \leq t \leq t_2$ *and* $n \geq 1$, *where* $\kappa \geq 0$, $\tau > \frac{1}{2}$ *and* G *is a nondecreasing continuous function on* $[0,1]$. *Then* $X_n \to^D X$.

1.3 U-statistics

Let $\{\eta_i\}$ be a centered stationary sequence, weakly $\mathcal{M}$-dependent in L^p with rate function $\delta(\cdot)$. Let F be the distribution function of η_1.

Let $\psi(y_1, y_2, \cdots, y_l) : \mathsf{R}^l \to \mathsf{R}$ be a measurable symmetric function, which will be called a kernel and define the U-statistic of degree l by

$$U_n = \binom{n}{l}^{-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_l \leq n} \psi(\eta_{i_1}, \eta_{i_2}, \cdots, \eta_{i_l}).$$

We assume that for some $r > 2$

$$\sup_{-\infty < i_1, i_2, \cdots, i_l < \infty} E|\psi(\eta_{i_1}, \eta_{i_2}, \cdots, \eta_{i_l})|^r < \infty. \tag{8}$$

Put

$$\theta(F) = \int\int \cdots \int \psi(y_1, y_2, \cdots, y_l) \prod_{j=1}^{l} dF(y_{i_j}). \tag{9}$$

To consider Hoeffding's H-decomposition of the U-statistic, we define $\psi_j(y_1, y_2, \cdots, y_j)$ $(j = 1, \cdots, l-1)$ recursively by

$$\psi_j(y_1, y_2, \cdots, y_j) = \int \psi_{j+1}(y_1, y_2, \cdots, y_j, y_{j+1}) dF(y_{j+1}).$$

Assume

$$\int \psi_1(y_1) dF(y_1) = \theta(F)$$

holds.

We introduce kernels of degrees $1, 2, \cdots, l$ which are also defined recursively by the equations

$$h^{(1)}(y_1) = \psi_1(y_1) - \theta(F)$$

and

$$h^{(k)}(y_1,\cdots,y_k) = \psi_k(y_1,\cdots y_k) - \sum_{j=1}^{k-1}\sum_{1\le i_1<\cdots<i_j\le k} h^{(j)}(y_{i_1},\cdots,y_{i_j}) - \theta(F) \tag{10}$$

for $k=2,3,\cdots,l$.

By the above definitions of $h^{(j)}$ we have that for $j=1,\cdots,k-1$ and $k=1,2,\cdots,l$

$$h_j^{(k)}(y_1,\cdots,y_j) = \int\cdots\int h^{(k)}(y_1,\cdots,y_j,y_{j+1},\cdots,y_k)\prod_{i=j+1}^{k} dF(y_i) = 0$$

and

$$\int\cdots\int h^{(k)}(y_1,\cdots,y_k)\prod_{j=1}^{k} dF(y_j) = 0. \tag{11}$$

Now, we have the following well known H-decomposition of U_n:

$$U_n = \binom{n}{l}^{-1}\sum_{1\le i_1<i_2<\cdots<i_l\le n}\psi(\eta_{i_1},\eta_{i_2},\cdots,\eta_{i_l}) = \theta(F) + \sum_{k=1}^{l}\binom{l}{k}U_n^{(k)} \tag{12}$$

where $U_n^{(k)}$ is the U-statistic of degree k based on kernel $h^{(k)}$.

Next, to estimate $E|\hat{U}_n^{(k)}|^2$ when $\hat{U}_n^{(k)}$ is constructed by an m-dependent sequence, we consider a kind of weak dependence conditions. Let $\{\xi_i\}$ be a stationary sequence of random variables and denote by M_a^b be the σ-algebra generated by $\xi_a,\cdots,\xi_b$ $(a\le b)$. We say that $\{\xi_i\}$ is ϕ-mixing if

$$\phi(n) = \sup_{A\in\mathsf{M}_{-\infty}^0, B\in\mathsf{M}_n^\infty}\left|\frac{P(AB)-P(A)P(B)}{P(A)}\right| \to 0 \quad (n\to\infty). \tag{13}$$

For any stationary sequence $\{\xi_i\}$ let

$$M(r) = \max\{\|\psi(X_{i_1},X_{i_2},\cdots,X_{i_l})\|_r, \sup_{1\le i_1<i_2<\cdots<i_l}\|\psi(\xi_{i_1},\xi_{i_2},\cdots,\xi_{i_l})\|_r\} \tag{14}$$

where $X_{i_1},X_{i_2},\cdots,X_{i_l}$ are i.i.d. random variables with the same distribution as that of ξ_1.

The following is known:

Lemma A. *Suppose $\{\xi_i\}$ is a stationary ϕ-mixing sequence satisfying $M(r) < \infty$ holds for some $r > 4$ and*

$$\sum_{n=1}^{\infty} n\phi^{\frac{1}{4}}(n) < \infty. \tag{15}$$

Then,

$$E|\hat{U}_n^{(2)}|^2 = O(n^{-2}), \tag{16}$$

$$E|\hat{U}_n^{(k)}|^2 = O(n^{-3}), \quad (k = 3, \cdots, l). \tag{17}$$

Furthermore, there exists a constant $\gamma > 0$ such that

$$E|\hat{U}_n^{(2)}|^4 \ = \ O(n^{-3-\gamma}). \tag{18}$$

(cf. Yoshihara (1976) and Yoshihara (1993) Lemma 3.2.4.)

It is obvious that if $\{\xi_i\}$ is an m-dependent sequence, then it satisfies the ϕ-mixing condition, and so we can use this lemma when $\{\eta_i^{(m)}\}$ is needed.

2. Main results

Let $\{\eta_i\}$ be a centered stationary sequence, weakly $\mathcal{M}$-dependent in L^p with rate function $\delta(\cdot)$ and $\{\eta_i^{(m)}\}$ (with fixed m) be a centered stationary sequence of random variables satisfying Conditions (A) and (B), so that $\{\eta_i^{(m)}\}$ is a centered stationary m-dependent sequence with $E|\eta_1|^p < \infty$.

Let F be the distribution function of η_1.

2.1 U-statistics based on $\{\eta_i\}$

Let $\psi(y_1, \cdots, y_l) : \mathsf{R}^l \to \mathsf{R}$ be a symmetric kernel function which has continuous bounded partial derivatives, i.e., for some positive constant K

$$\sup_{y_1,\cdots,y_l} \max_{1\le j\le l} \left| \frac{\partial \psi(y_1, y_2, \cdots, y_l)}{\partial y_j} \right| \le K. \tag{19}$$

We prove the following theorems.

Theorem 1. *Let $p > 4$. Let $\{\eta_i\}$ be a centered, stationary sequence, weakly $\mathcal{M}$-dependent in L^p with rate function $\delta(m)$ satisfying*

$$\delta(m) \le Cm^{-\beta} \tag{20}$$

where $\beta > 6$. Let $\psi(y_1, y_2, \cdots, y_l)$ be a measurable symmetric kernel satisfying (19).

Define U-statistic of degree l by

$$U_n = \binom{n}{l}^{-1} \sum_{1 \le i_1 < i_2 < \cdots < i_l \le n} \psi(\eta_{i_1}, \eta_{i_2}, \cdots, \eta_{i_l}). \tag{21}$$

and $\theta(F)$ by (9). Suppose $M(p)$, defined for $\{\eta_i\}$, is finite.

Then, the series in the next equation is absolutely convergent and

$$\sigma^2 = \mathrm{Var}(\psi_1(\eta_1) + 2\sum_{j=2}^{\infty} \mathrm{cov}(\psi_1(\eta_1), \psi_1(\eta_j)) < \infty \tag{22}$$

is well defined.

Furthermore, if $\sigma^2 > 0$, then

$$\left| \frac{\sqrt{n}}{2\sigma}(U_n - \theta(F)) - W_1 \right| = o(1) \quad a.s. \quad (n \to \infty), \tag{23}$$

where W_1 is an $\mathcal{N}(0,1)$ random variable.

Theorem 2. *Suppose conditions of Theorem 1 hold. Then*

$$U_n \ \rightarrow \ \theta(F) \quad a.s. \tag{24}$$

For the function $h^{(2)}(x_1, x_2)$ defined by (10) associated with the kernel $\psi(x_1, x_2, \cdots, x_l)$ $(l \ge 2)$, we define an operator A on the function space L^2 by

$$A\phi(x) = \int h^{(2)}(x, y)\phi(y)dF(y), \quad x \in \mathsf{R}, \phi \in L^2. \tag{25}$$

In connection with any such operator A, we define the associated eigenvalues $\lambda_1, \lambda_2 \cdots$ to be the real numbers λ (not necessarily distinct) corresponding to the distinct solutions $\phi_1, \phi_2, \cdots$ of the equation

$$A\phi - \lambda\phi = 0.$$

Then

$$\int \phi_j(x)\phi_k(x)dF(x) = \begin{cases} 1 & (j = k) \\ 0 & (j \ne k) \end{cases} \tag{26}$$

and

$$\lim_{N \to \infty} \int \int \left\{ h^{(2)}(x, y) - \sum_{k=1}^{N} \lambda_k \phi_k(x)\phi_k(y) \right\}^2 dF(x)dF(y) = 0.$$

Thus, we can write

$$h^{(2)}(x,y)=\sum_{q=1}^{\infty}\lambda_q\phi_q(x)\phi_q(y). \tag{27}$$

Let

$$S_{q,n}=\sum_{i=1}^{n}\phi_q(\eta_i).$$

Using facts $E\phi_k(\eta_i)=0$ and $\|\phi_k(\eta_i)\|_2=1$, by the usual method (cf. Lemma 3 (below)) we can prove that for some constant $C>0$ (independnt of $n\geq 1$ and K_0)

$$ES_{q,n}^2=E\bigg|\sum_{i=1}^{n}\phi_q(\eta_i)\bigg|^2\leq Cn.$$

and so put

$$\sigma_q^2=\lim_{n\to\infty}\frac{1}{n}ES_{q,n}^2=1+2\sum_{i=1}^{\infty}E\phi_q(\eta_1)\phi_q(\eta_{i+1}) \tag{28}$$

and

$$\begin{aligned}\sigma_{q,q'}&=\lim_{n\to\infty}\frac{1}{n}ES_{q,n}S_{q',n}\\&=E\phi_q(\eta_1)\phi_{q'}(\eta_1)+\sum_{i=1}^{\infty}\{E\phi_q(\eta_1)\phi_{q'}(\eta_{i+1})+E\phi_{q'}(\eta_1)\phi_q(\eta_{i+1})\}.\end{aligned} \tag{29}$$

Theorem 3. *Suppose conditions of Theorem 1 hold and*

$$E|h^{(1)}(\eta_1)|^2=0,\quad \textit{and}\quad \inf_{-\infty<i,j<\infty}E|h^{(2)}(\eta_i,\eta_j)|^2>0.$$

Suppose the above defined eigenvalues satisfy the conditions

$$|\lambda_1|\geq|\lambda_2|\geq|\lambda_3|\geq\cdots,\quad\text{and}\quad\sum_{l=1}^{\infty}|\lambda_l|<\infty, \tag{30}$$

and the above defined eigenfunctions satisfy the Lipschitz condition

$$\sup_{q\geq 1}|\phi_q(x+h)-\phi_q(x)|\leq K|h|. \tag{31}$$

Furthermore, assume

$$0<\inf_{q\geq 1}\sigma_q^2\leq\sup_{q\geq 1}\sigma_q^2<\infty. \tag{32}$$

Then

$$nU_n^{(2)} \xrightarrow{D} \mathbf{Y} = \sum_{k=1}^{\infty} \lambda_k (W_k^2 - 1), \tag{33}$$

where $U_n^{(2)}$ *is defined by* (12), $W_q \sim \mathcal{N}(0, \sigma_q^2)$ $(q \geq 1)$, $EW_qW_{q'} = \sigma_{q,q'}$ $(q \neq q' \geq 1)$ *and* $\mathbf{Y}$ *is defined in the sense of the limit in mean square.*

Consequently, from (16) *and* (17) *in Lemma A, we have*

$$n(U_n - \theta(F)) \xrightarrow{D} \frac{l(l-1)}{2} \mathbf{Y}. \tag{34}$$

2.2 Empirical processes based on $\{\eta_i\}$

Define the empirical distribution function by

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} I(\eta_i \in (-\infty, t]).$$

Let $\mathcal{U} = \{\{u_t; 0 \leq t \leq 1\}\}$ be the class of functions satisfying the conditions

$$\begin{cases} 0 \leq u_t(y) \leq 1, & 0 \leq t \leq 1, \\ u_s(y) \leq u_t(y), & 0 \leq s \leq t \leq 1 \quad \text{for all y.} \end{cases} \tag{35}$$

Put

$$G(t) = Eu_t(\eta_1) \quad \text{and} \quad g_t(y) = u_t(y) - G(t).$$

We assume that G is Lipschitz continuous on $[0, 1]$, i.e.

$$|G(t) - G(s)| \leq K|t - s|. \tag{36}$$

For $u_t \in \mathcal{U}$, define

$$\begin{aligned} W_n(t) &= \sqrt{n} \int u_t(s)(dF_n(s) - dF(s)) \\ &= \sqrt{n} \Big\{ \frac{1}{n} \sum_{i=1}^{n} u_t(\eta_i) - \int u_t(y) dF(y) \Big\}. \end{aligned} \tag{37}$$

Then, $\{W_n(t); 0 \leq t \leq 1\}$ is a random element of $\mathcal{D}[0, 1]$.

Theorem 4. *Let* $p > 4$. *Let* $\{\eta_i\}$ *be a centered, stationary sequence, weakly* $\mathcal{M}$*-dependent in* L^p *with rate function* $\delta(m)$ *satisfying*

$$\delta(m) \leq Cm^{-\beta} \tag{38}$$

where $\beta > 4$. Let $u_t \in \mathcal{U}$ and the function G satisfies (36).

Then, the empirical processes $\{W_n(t); 0 \le t \le 1\}$ converges weakly to the centered Gaussian process $\{W(t); 0 \le t \le 1\}$ with covariance structure

$$\begin{aligned}\operatorname{cov}(W(s), W(t)) &= \operatorname{cov}(u_s(\eta_1), u_t(\eta_1)) + \sum_{j=1}^{\infty} \operatorname{cov}(u_s(\eta_1), u_t(\eta_j + 1)) \\ &+ \sum_{j=1}^{\infty} \operatorname{cov}(u_s(\eta_{j+1}), u_t(\eta_1)).\end{aligned} \tag{39}$$

Moreover, the series on the right hand side of (39) *converges absolutely and the limit process W has continuous paths almost surely.*

2.3 U-processes of stochastic sequences

Let $\mathcal{H}$ be a class of kernel functions $h(x, y; t) : \mathsf{R}^2 \times \mathsf{R} \to \mathsf{R}$ such that

$$\begin{cases} 0 \le h(x, y; t) \le 1, \quad h(x, y; 0) = 0, \\ h(x, y; t) \text{ is increasing in } t, \text{ for fixed } x, y \in \mathsf{R}. \end{cases} \tag{40}$$

Define

$$\begin{aligned} U_n(t) &= \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} h(\eta_i, \eta_j; t), \\ U(t) &= \int\int h(x, y; t) dF(x) dF(y) \quad (t \in \mathsf{R}), \\ W_n(t) &= \sqrt{n}(U_n(t) - U(t)). \end{aligned}$$

Put

$$h^{(1)}(x; t) = \int h(x, y; t) dF(y) \quad (x \in \mathsf{R}).$$

Note that if $h \in \mathcal{H}$,

$$0 \le h^{(1)}(x; t) \le 1, \quad h^{(1)}(x; 0) = 0$$

and it is increasing in t.

We impose the following condition.

Condition II There exists a constant $K > 0$ such that

$$|U(t) - U(s)| \le K|t - s|, \tag{41}$$

$$|Eh(\eta_0, \eta_k; t) - Eh(\eta_0, \eta_k; s)| \le K|t - s|, \tag{42}$$

$$\left| \int\int h(y_0, y_k; t) dF(y_0) dF(y_k) - \int\int h(y_0, y_k; s) dF(y_0) dF(y_k) \right| \le K|t - s| \tag{43}$$

hold for all s, t ($\in [0, 1]$) and $k \ge 1$.

Theorem 5. *Let $p > 4$. Let $\{\eta_i\}$ be a centered, stationary sequence, weakly $\mathcal{M}$-dependent in L^p with rate function $\delta(m)$ satisfying*

$$\delta(m) \le Cm^{-\beta} \tag{44}$$

where $\beta > 4$. Suppose that $h(\cdot, \cdot; t) \in \mathcal{H}$ for all $t \in [0, 1]$ and Condition II holds.
Then

$$\{\sqrt{n}(U_n(t) - U(t); 0 \le t \le 1\} \xrightarrow{D} \{W(t); 0 \le t \le 1\} \quad \text{in } \mathcal{D}[0, 1] \tag{45}$$

where $\{W(t); 0 \le t \le 1\}$ is a centered Gaussian process with covariance structure

$$\begin{aligned} &\operatorname{cov}(W(s), W(t)) \\ &= 4\operatorname{cov}(h^{(1)}(\eta_1, s), h^{(1)}(\eta_1, t)) + 4\sum_{j=1}^{\infty} \operatorname{cov}(h^{(1)}(\eta_1, s), h^{(1)}(\eta_{j+1}, t)) \\ &\quad +4\sum_{j=1}^{\infty} \operatorname{cov}(h^{(1)}(\eta_{j+1}, s), h^{(1)}(\eta_1, t)), \end{aligned} \tag{46}$$

which converges absolutely and the limit process W has continuous paths on $[0, 1]$ almost surely.

3. Proofs

3.1 Proofs of Theorems 1-3

In the sequel, we use $\sum_{(n,l)}$ to denote the sum taken over all subsets $1 \le i_1 < \cdots < i_l \le n$ of $\{1, 2, \cdots, n\}$.

Let $p > 4$. Let $\{\eta_i\}$ be a centered, stationary sequence, weakly $\mathcal{M}$-dependent in L^p with rate function $\delta(\cdot)$. Then, corresponding to $\{\eta_i\}$, we can choose a centered, stationary m dependent sequence $\{\eta_j^{(m)}\}$ with $E|\eta_i^{(m)}|^p < \infty$ satisfying (1). For a while we assume that m is fixed. Let $\hat{F}(y_1)$ be the ditribution function of $\eta_1^{(m)}$.

Corresponding to U_n we consider the following U-statistic and its H-decomposition as follows:

$$\hat{U}_n = \binom{n}{l}^{-1} \sum_{(n,l)} \psi(\eta_{i_1}^{(m)}, \eta_{i_2}^{(m)}, \cdots, \eta_{i_l}^{(m)}) = \theta(F) + \sum_{k=1}^{l} \binom{l}{k} \hat{U}_n^{(k)} \tag{47}$$

where $\hat{U}_n^{(k)}$ is the U-statistic of degree k based on kernel $\hat{h}^{(k)}$ which is defind by $\{\eta_i^{(m)}\}$.

We note that by (19) and (3)

$$\|\psi(\eta_1, \cdots, \eta_l) - \psi(\eta_1^{(m)}, \cdots, \eta_l^{(m)})\|_p \le lK\delta(m)$$

which implies

$$\|U_n - \hat{U}_n\|_2 \le \|U_n - \hat{U}_n\|_p \le c\delta(m). \tag{48}$$

We prove the following lemma, which will be used in the proofs of Theorems.

Lemma 1. *Suppose conditions of Theorem 1 hold. Then, for some* $\gamma > 0$

$$E|U_n^{(2)}|^4 \le C_0 n^{-3-\gamma} \tag{49}$$

and

$$E|U_n^{(k)}|^2 \le C_0 n^{-3} \quad (k = 3, \cdots, l) \tag{50}$$

where C_0 *is a positive constant independent of* n.

Consequently, we have

$$\sqrt{n} U_n^{(k)} = O\big(n^{-\frac{\gamma}{2}}\big) \quad a.s. \quad (k = 2, \cdots, l). \tag{51}$$

Proof. By the Hölder inequality and (1)

$$|E(U_n^{(2)})^4 - E(\hat{U}_n^{(2)})^4| \le \|U_n^{(2)} - \hat{U}_n^{(2)}\|_4 \|(|U_n^{(2)}| + |\hat{U}_n^{(2)}|)((U_n^{(2)})^2 + (\hat{U}_n^{(2)})^2)\|_{\frac{4}{3}} \le c\delta(m),$$

and similarly

$$|E(U_n^{(k)})^2 - E(\hat{U}_n^{(k)})^2| \le c\delta(m) \quad (k = 3, \cdots, l).$$

Hence, by Lemma A we have

$$E|U_n^{(2)}|^4 \le c\{E|\hat{U}_n^{(2)}|^4 + |E(U_n^{(2)})^4 - E(\hat{U}_n^{(2)})^4|\} \tag{52}$$

$$\leq c\{n^{-3-\gamma} + \delta(m)\}$$

and similarly

(53)
$$\begin{aligned} E|U_n^{(k)}|^2 &\leq c\{E|\hat{U_n}^{(k)}|^2 + |E(U_n^{(k)})^2 - E(\hat{U_n}^{(k)})^2|\} \\ &\leq c\{n^{-3} + \delta(m)\} \qquad (k = 3, \cdots, l). \end{aligned}$$

Now, put $m = [n^{\frac{1}{2}}]$ and we have

$$\delta(m) \leq n^{\frac{-\beta}{2}} \quad (\beta > 6).$$

Thus, (49) and (50) follow from (52) and (53).

By the Markov inequality and (49)

$$P(\sqrt{n}|U_n^{(2)}| > n^{-\frac{\gamma}{8}}) \leq n^{\frac{\gamma}{2}} E|U_n^{(2)}|^4 \leq cn^{-1-\frac{\gamma}{2}}.$$

Thus, from the Borel-Cantelli lemma, (51) (with $k = 2$) is obtained.

Similarly, for $k = 3, \cdots, l$ we have

(54)
$$P(\sqrt{n}|U_n^{(k)}| \geq n^{-\frac{1}{3}}) \leq n^{\frac{2}{3}} n E|U_n^{(k)}|^2 \leq cn^{\frac{5}{3}} n^{-3} \leq cn^{-\frac{4}{3}},$$

which, via the Borel-Cantelli lemma, implies (51) (with $k = 3, \cdots, l$). □

Proof of Theorem 1. Since U_n defined by (21) may be written as (12), we have

$$\sqrt{n}(U_n - \theta(F)) = \frac{l}{\sqrt{n}} \sum_{i=1}^{n} h^{(1)}(\eta_i) + \sum_{k=2}^{l} \binom{l}{k} \sqrt{n} U_n^{(k)}.$$

Noting that $\{h^{(1)}(\eta_i)\}$ is a weakly, centered stationary sequence, weakly $\mathcal{M}$-dependent in L^p with rate function $\delta(m)$ satisfying $\delta(m) \leq m^{-\beta}$ $(\beta > 6)$, from Theorem A we obtain

(55)
$$\left| \frac{l}{\sqrt{n}\sigma} \sum_{i=1}^{n} h^{(1)}(\eta_i) - W_1 \right| = O(n^{-\frac{1}{4}}) \quad a.s.$$

where

$$\begin{aligned} \sigma^2 &= E(h^{(1)}(\eta_1))^2 + 2 \sum_{i=1}^{\infty} E h^{(1)}(\eta_1) h^{(1)}(\eta_{1+i}) \\ &= E(\psi_1(\eta_1) - \theta(F))^2 + 2 \sum_{i=1}^{\infty} E(\psi_1(\eta_1) - \theta(F))(\psi_1(\eta_{1+i}) - \theta(F)) > 0. \end{aligned}$$

Further, by (51)

$$\sum_{k=2}^{l} \binom{l}{k} \sqrt{n} U_n^{(k)} = O\big(n^{-\frac{\gamma}{2}}\big) \quad a.s. \quad (n \to \infty).$$

for some $\gamma > 0$. Combining these results we obtain (23). □

Proof of Theorem 2. We note that by Theorem A

$$\frac{1}{n}\sum_{j=1}^{n} h^{(1)}(\eta_j) = o\big(n^{-\frac{1}{2}}\big) \quad a.s. \quad (n \to \infty). \tag{56}$$

Hence, from (56) and Lemma 1 we obtain that for some $\gamma > 0$

$$U_n - \theta(F) = \frac{1}{n}\sum_{j=1}^{n} h^{(1)}(\eta_j) + \sum_{k=2}^{l}\binom{l}{k} U_n^{(k)} = o\big(n^{-\frac{1}{2}}\big) \quad a.s.$$

as $n \to \infty$ and the proof is completed. □

Lemma 2. *Let $p > 2$. Let $\{\eta_i\}$ be a centered, stationary sequence, weakly $\mathcal{M}$–dependent in L^p with rate function $\delta(m)$ satisfying*

$$D_p = \sum_{m=0}^{\infty} \delta(m) < \infty.$$

Let $g_l : \mathsf{R} \to \mathsf{R}$ $(l = 1, 2)$ be Lipschitz continuous functions such that

$$\begin{cases} Eg_l(\eta_1) = 0, \quad E|g_l(\eta_1)|^p < \infty \\ |g_l(y) - g_l(x)| \le K|y - x|. \end{cases} \tag{57}$$

Then

$$\begin{aligned} &\frac{1}{n}E\Big(\sum_{i=1}^{n} g_1(\eta_i)\sum_{j=1}^{n} g_2(\eta_j)\Big) \\ &\quad \to V(g_1, g_2) = Eg_1(\eta_1)g_2(\eta_1) + \sum_{i=1}^{\infty} Eg_1(\eta_1)g_2(\eta_{i+1}) \\ &\qquad\qquad + \sum_{i=1}^{\infty} Eg_2(\eta_1)g_1(\eta_{i+1}) \end{aligned} \tag{58}$$

as $n \to \infty$. The series in $V(g_1, g_2)$ is absolutely convergent.

Proof. We use the method of the proof in Berkes et al (2011).

We note first that both $\{g_1(\eta_k)\}$ and $\{g_2(\eta_k)\}$ are centered, stationary sequences, weakly $\mathcal{M}$-dependent in L^p with rate function $K\delta(m)$. We use below that

$$\begin{aligned} &\sup_{m\ge 0}\max\big\{\|g_1(\eta_k^{(m)})\|_2, \|g_2(\eta_k^{(m)})\|_2\big\} \\ &\le \max\big\{\|g_1(\eta_k)\|_2 + KD_p, \|g_2(\eta_k)\|_2 + KD_2\big\} \le C_1 \end{aligned}$$

Without loss of generality we assume that $Eg_1(\eta_k^{(m)}) = Eg_2(\eta_k^{(m)}) = 0$ for all $k \in \mathsf{Z}$ and $m \in \mathsf{N}$.

Since $g_1(\eta_k^{(j-1)})$ and $g_2(\eta_{k+j}^{(j-1)})$ are independent, by Condition I (B)

$$Eg_1(\eta_k^{(j-1)})g_2(\eta_{k+j}^{(j-1)}) = 0.$$

Hence, we have

$$\begin{aligned}(59)\qquad |Eg_1(\eta_k)g_2(\eta_{k+j})| &= \big|E\big\{(g_1(\eta_k) - g_1(\eta_k^{(j-1)}))g_2(\eta_{k+j}) \\ &\quad + g_1(\eta_k^{(j-1)})(g_2(\eta_{k+j}) - g_2(\eta_{k+j}^{(j-1)})) + g_1(\eta_k^{(j-1)})g_2(\eta_{k+j}^{(j-1)})\big\}\big| \\ &\le \big|E(g_1(\eta_k) - g_1(\eta_k^{(j-1)}))g_2(\eta_{k+j})\big| \\ &\quad + E\big|g_1(\eta_k^{(j-1)})(g_2(\eta_{k+j}) - g_2(\eta_{k+j}^{(j-1)}))\big| \\ &\le \|g_1(\eta_{k+j})\|_2\|g_1(\eta_k) - g_1(\eta_k^{(j-1)})\|_2 \\ &\quad + \|g_1(\eta_k^{(j-1)})\|_2\|g_2(\eta_{k+j}) - g_2(\eta_{k+j}^{(j-1)})\|_2 \\ &\le (\|g_2(\eta_{k+j})\|_2 + \|g_1(\eta_k^{(j-1)})\|_2)K\delta(j-1) \\ &\le K(\|g_1(\eta_1)\|_2 + \|g_2(\eta_1)\|_2 + KD_2)\delta(j-1) \le C\delta(j-1)\end{aligned}$$

and by the same method we obtain

$$\begin{aligned}(60)\qquad |Eg_1(\eta_k)g_2(\eta_{k+j})| &\le K(\|g_1(\eta_1)\|_2 + \|g_2(\eta_1)\|_2 + KD_2)\delta(j-1) \\ &\le C\delta(j-1).\end{aligned}$$

Here $C > 0$ is a constant independent of k and j. From relations (59) and (60) we obtain

$$\begin{aligned}&\left|V(g_1, g_2) - \frac{1}{n}E\left(\sum_{i=1}^n g_1(\eta_i)\sum_{j=1}^n g_2(\eta_j)\right)\right| \\ &= \Bigg|V(g_1, g_2) - \Bigg\{Eg_1(\eta_1)g_2(\eta_1) \\ &\quad + \sum_{j=1}^{n-1}\frac{n-j}{n}E\{g_1(\eta_1)g_2(\eta_{1+j}) + g_2(\eta_1)g_1(\eta_{1+j})\} \\ &\quad + \sum_{j=n}^{\infty}E\{(g_1(\eta_1)g_2(\eta_{1+j}) + g_2(\eta_1)g_1(\eta_{1+j})\}\Bigg\}\Bigg| \\ &\le \sum_{j=1}^{n-1}\frac{j}{n}\{|Eg_1(\eta_1)g_2(\eta_{1+j})| + |Eg_2(\eta_1)g_1(\eta_{1+j})|\} \\ &\quad + \sum_{j=n}^{\infty}\{|Eg_1(\eta_1)g_2(\eta_{1+j})| + |Eg_2(\eta_1)g_1(\eta_{1+j})|\}\end{aligned}$$

$$\leq \frac{c}{n}\sum_{j=1}^{n-1} j\delta(j) + c\sum_{j=n}^{\infty}\delta(j).$$

Since $D_p < \infty$, we have

$$\frac{1}{n}\sum_{j=1}^{n-1} j\delta(j) = o(1) \quad \text{and} \quad \sum_{j=n}^{\infty}\delta(j) = o(1)$$

which implies

$$\left| V(g_1, g_2) - \frac{1}{n}E\Bigg(\sum_{i=1}^{n} g_1(\eta_i)\sum_{j=1}^{n} g_2(\eta_j)\Bigg)\right| = o(1) \quad (n \to \infty).$$

□

Proof of Theorem 3. Let

(61) $$G_{q,n} = \binom{n}{2}^{-1}\sum_{(n,2)}\lambda_q\phi_q(\eta_{i_1})\phi_q(\eta_{i_2}) = (n-1)^{-1}\lambda_q\big(W_{q,n}^2 - Z_{q,n}\big)$$

where

(62) $$W_{q,n} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\phi_q(\eta_i) \quad \text{and} \quad Z_{q,n} = \frac{1}{n}\sum_{i=1}^{n}\phi_q^2(\eta_i),$$

and consider

(63) $$(n-1)V_n^{(N,L)} = \sum_{q=N+1}^{L} G_{q,n}.$$

Since ϕ_q's satisfy (57), from Lemma 3 we obtain

$$\lim_{n\to\infty} EW_{q,n,N}^2 = \lim_{n\to\infty}\frac{1}{n}E\Bigg(\sum_{i=1}^{n}\phi_q(\eta_i)\Bigg)^2 = \sigma_q^2$$

and

$$\lim_{n\to\infty} EW_{q,n,N}W_{q',n,N}$$
$$= \lim_{n\to\infty}\frac{1}{n}E\Bigg(\sum_{i=1}^{n}\phi_q(\eta_i)\Bigg)\Bigg(\sum_{i=1}^{n}\phi_{q'}(\eta_i)\Bigg) = \sigma_{q,q'}.$$

In addition, as $E\phi_q(\eta_1) = 0$, $E\phi_q^2(\eta_1) = 1$ and $E\phi_q^4(\eta_1) < \infty$, by Theorem A we can construct a sequence of Gaussian random variables $\{W_q; q \geq 1\}$ on the common probability space such that $\mathrm{Var}(W_q) = \sigma_q^2$, $\mathrm{cov}(W_q, W_{q'}) = \sigma_{q,q'}$ and

$$|W_{q,n} - W_q| \leq \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \phi_q(\eta_i) - W_q \right| = o\big(n^{-\frac{1}{4}}\big) \quad a.s. \tag{64}$$

and

$$\sqrt{n}|(Z_{q,n} - 1)| = \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\phi_q^2(\eta_i) - 1) \right| = o\big(n^{-\frac{1}{4}}\big) \quad a.s. \tag{65}$$

By the Schwarz inequality

$$\begin{aligned}
(n-1)^2 E|V_n^{(N,L)}|^2 &= E\bigg(\sum_{q=N+1}^{L} \lambda_q \big(W_{q,n}^2 - Z_{q,n}\big) \bigg)^2 \\
&\leq E\bigg(\sum_{q=N+1}^{L} |\lambda_q| |W_q^2 - 1 + o(n^{-\frac{1}{4}})| \bigg)^2 \\
&\leq cE\bigg(\sum_{q=N+1}^{L} |\lambda_q| |W_q^2 - 1| \bigg)^2 + cn^{-\frac{1}{2}} \bigg(\sum_{q=N+1}^{L} |\lambda_q| \bigg)^2 \\
&\leq c\bigg(\sum_{q=N+1}^{L} |\lambda_q| \bigg) E\bigg(\sum_{q=N+1}^{L} |\lambda_q| (W_q^2 - 1)^2 \bigg) + cn^{-\frac{1}{2}} \bigg(\sum_{q=N+1}^{L} |\lambda_q| \bigg)^2 \\
&\leq c\bigg(\sum_{q=N+1}^{L} |\lambda_q| \bigg)^2 (1 + n^{-\frac{1}{2}}).
\end{aligned}$$

Hence, as $n \to \infty$ first, and then $N, L \to \infty$, from (30) we obtain

$$(n-1)^2 E|V_n^{(N,L)}|^2 \to 0. \tag{66}$$

Now, put

$$T_{n,N} = (n-1) \sum_{q=1}^{N} G_{q,n}.$$

By (66) we have that for all n sufficiently large

$$E|T_{n,L} - T_{n,N}|^2 = (n-1)^2 E|V_n^{(N,L)}|^2 \quad \to 0 \quad (N, L \to \infty). \tag{67}$$

In terms of the representation (27), T_n may be expressed as

$$T_n = (n-1) \sum_{q=1}^{\infty} G_{q,n}$$

Hence, by Fatou's lemma

$$E|T_n - T_{n,N}|^2 \le \lim_{L\to\infty} E|T_{n,L} - T_{n,N}|^2 = 0 \tag{68}$$

for all n sufficiently large.

Next, let

$$\mathbf{Y}_N = \sum_{q=1}^{N} \lambda_q (W_q^2 - 1). \tag{69}$$

By (69), (64), (65) and (30) we have

$$\begin{aligned} E|T_{n,N} - \mathbf{Y}_N|^2 &= E\Big(\sum_{q=1}^{N} \lambda_q \big\{(W_{q,n}^2 - W_q^2) - (Z_{q,n} - 1)\big\}^2\Big)^2 \\ &\le E\Big(\sum_{q=1}^{N} \lambda_q \big(cn^{-\frac{1}{4}}\big)\Big)^2 \le cn^{-\frac{1}{2}}\Big(\sum_{q=1}^{N} |\lambda_q|\Big)^2 \to 0 \end{aligned} \tag{70}$$

as $n \to \infty$.

Further, by the Schwarz inequality we have

$$\begin{aligned} E|\mathbf{Y}_L - \mathbf{Y}_N|^2 &= E\Big|\sum_{q=N+1}^{L} \lambda_q (W_q^2 - 1)\Big|^2 \\ &\le E\Big\{\Big(\sum_{q=N+1}^{L} |\lambda_q|\Big)\Big(\sum_{q=N+1}^{L} |\lambda_q|(W_q^2 - 1)^2\Big)\Big\} \\ &\le c\Big(\sum_{q=N+1}^{L} |\lambda_q|\Big)\Big(\sum_{q=N+1}^{L} |\lambda_q| E(W_q^2 - 1)^2\Big) \\ &\le c\Big(\sum_{q=N+1}^{L} |\lambda_q|\Big)^2 \ \to\ 0 \quad (N, L \to \infty), \end{aligned} \tag{71}$$

which implies that $\{\mathbf{Y}_N\}$ is a Cauchy sequence in the L^2 space. Thus, we can define in the sense of the limit in mean square

$$\mathbf{Y} = \sum_{q=1}^{\infty} \lambda_q (W_q^2 - 1)$$

and hence

$$E|\mathbf{Y} - \mathbf{Y}_N|^2 \to 0 \quad (N \to \infty). \tag{72}$$

From (68),(70) and (72) we obtain that for an arbitrary positive ϵ there are positive constants N_0 and n_0 such that for all $N \geq N_0$ and $n \geq n_0$

$$\begin{aligned}&E|\mathbf{Y} - T_n|^2 \\ &\leq 3(E|\mathbf{Y} - \mathbf{Y}_N|^2 + E|\mathbf{Y}_N - T_{n,N}|^2 + E|T_{n,N} - T_n|^2) \leq 9\epsilon,\end{aligned}$$

which implies

$$\begin{aligned}&\big|E(\exp(it\mathbf{Y}) - E(\exp itT_n)| \\ &\leq |t|E|\mathbf{Y} - T_n| \leq |t|\|\mathbf{Y} - T_n\|_2 \leq 3\sqrt{\epsilon}.\end{aligned}$$

Hence, the distribution of T_n converges to that of $\mathbf{Y}$. □

4. Proofs of Theorems 4 and 5

4.1 Proof of Theorem 4

Lemma 3. *Let $p > 4$. Let $\{\eta_i\}$ be a centered stationary sequence, weakly $\mathcal{M}$-dependent in L^p with rate function $\delta(m)$. If $u_t(x)$ satisfies the conditions in Theorem 4, then for fixed t the $\{u_t(x)I(\eta_i < x)\}$ is weakly $\mathcal{M}$-dependent in L^1 with rate function $K\sqrt{\delta(m)}$ where $K > 0$ some constant.*

Proof. Let $\{\eta_i^{(m_i)}\}$ be the random variables which corresponding to $\{\eta_i\}$ satisfy Condition I (A) and (B). For brevity put $m = m_i$. Define

$$B = \big\{\omega : |\eta_i - \eta_i^{(m)}| \leq \sqrt{\delta(m)}\big\} \quad \text{and} \quad B^c = \Omega - B.$$

Since for fixed t $0 \leq u_t(x) \leq 1$,

$$\begin{aligned}&E|u_t(x)I(\eta_i < x) - u_t(x)I(\eta_i^{(m)} < x)| \\ &\leq E|I(\eta_i < x) - I(\eta_i^{(m)} < x)| \\ &\leq E\{E\{|I(\eta_i < x) - I(\eta_i^{(m)} < x)|B\}\} \\ &\quad + E\{E\{|I(\eta_i < x) - I(\eta_i^{(m)} < x)|B^c\}\}.\end{aligned}$$

Further, noting that on B

$$\eta_i - \sqrt{\delta(m)} \leq \eta_i^{(m)} \leq \eta_i + \sqrt{\delta(m)}$$

and so using the absolute continuity of F we have

$$E\{E|I(\eta_i < x) - I(\eta_i^{(m)} < x)|B\}\}$$

$$
\begin{aligned}
&\le \max\Big\{E(I(\eta_i < x) - I\big(\eta_i < x - \sqrt{\delta(m)}\big)\big), \\
&\qquad E\big(I\big(\eta_i < x + \sqrt{\delta(m)}\big) - I(\eta_i < x)\big)\Big\} \\
&= \max\Big\{F(x) - F\big(x - \sqrt{\delta(m)}\big), F\big(x + \sqrt{\delta(m)}\big) - F(x)\Big\} \\
&\le K\sqrt{\delta(m)}.
\end{aligned}
$$

On the other hand, by the Markov inequality

$$
\begin{aligned}
&E\{E\{|I(\eta_i < x) - I(\eta_i^{(m)} < x)|B^c\}\} \le 2P(B^c) \\
&\le 2\frac{1}{\sqrt{\delta(m)}^p}E|\eta_i - \eta_i^{(m)}|^p \le c\sqrt{\delta(m)}^p.
\end{aligned}
$$

Combining these results, we have

$$
\sup_{-\infty<x<\infty} E|u_t(x)I(\eta_i < x) - u_t(x)I(\eta_i^{(m)} < x)| \le c\sqrt{\delta(m)}.
$$

Thus, we have the desired conclusion. □

Lemma 4. *Suppose conditions of Theorem 4 are satisfied. Then*

$$
E\Big(\sum_{i=1}^n \eta_i\Big)^4 \le cn^2\|\eta_1\|_2^4. \tag{73}
$$

Proof. We note first that for fixed m $\{\eta_i^{(m)}\}$ is a stationary m-dependent sequence which satisfies a ϕ-mixing condition. Hence, we have

$$
E\Big|\sum_{i=1}^n \eta_i^{(m)}\Big|^4 \le cn^2\|\eta_i^{(m)}\|_2^4,
$$

(cf. Uteev (1984) and Doukhan(1994)). Consequently, we have

$$
E\Big|\sum_{i=1}^n \eta_i^{(m)}\Big|^4 \le cn^2\{\|\eta_i\|_2^4 + \|\eta_i - \eta_i^{(m)}\|_2^4\}. \tag{74}
$$

From (74) it follows that

$$
\begin{aligned}
E\Big(\sum_{i=1}^n \eta_i\Big)^4 &\le E\Big|\sum_{i=1}^n \eta_i^{(m)}\Big|^4 + E\Big|\sum_{i=1}^n(\eta_i - \eta_i^{(m)})\Big|^4 \\
&\le cn^2\big\{\|\eta_1\|_2^4 + \|\eta_1 - \eta_1^{(m)}\|_2^4\big\} + cn^4\|\eta_1 - \eta_1^{(m)}\|_2^4\big\} \\
&\le cn^2\big\{\|\eta_1\|_2^4 + \delta^4(m) + n^2\delta^4(m)\big\}.
\end{aligned}
$$

Put $m = [n^{\frac{1}{2}}]$. Then, $\delta^4(m) = o(n^{-2})$ and so $n^2\delta^4(m) \le \|\eta_1\|_2^4$. Thus, we have (73). □

Proof of Theorem 4. We show first that for fixed $t \in [0,1]$

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_t(\eta_i) \xrightarrow{D} \mathcal{N}(0, \sigma_t^2) \tag{75}$$

where

$$\sigma_t^2 = Eg_t^2(\eta_0) + 2\sum_{j=1}^{\infty} \mathrm{cov}(g_t(\eta_0), g_t(\eta_j)) > 0.$$

The random variables $g_t(\eta_i)$ are centered and bounded, and themselves again weakly $\mathcal{M}$-dependent in L^p with rate function $2\sqrt{\delta(m)}$. Hence, from Theorem A (75) is obtained.

Now, an application of the Cramér-Wold device (cf. Billingsley 1968) yields that for any $(t_1, t_2, \cdots, t_k)$ the vector $(W_n(t_1), W_n(t_2), \cdots, W_n(t_k))$ a k-dimensional centered normal distribution with covariance given by (46).

It remains to show that the sequence of processes $\{W_n(t); 0 \le t \le 1\}$ $(n \ge 1)$ is tight. Thightness follows if we can show that the condition of Theorem B is satisfied.

Consider the difference

$$W_n(t) - W_n(s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{g_t(\eta_i) - g_s(\eta_i)\}.$$

Note that if $t > s$, then $u_t(x) - u_s(x) \ge 0$ and $G(t) - G(s) \ge 0$ and hence

$$\begin{aligned} &|g_t(\eta_0) - g_s(\eta_0)| \\ &= |(u_t(\eta_0) - G(t)) - (u_s(\eta_0) - G(s))| \\ &= |(u_t(\eta_0) - u_s(\eta_0)) - (G(t) - G(s))| \le G(t) - G(s) \end{aligned} \tag{76}$$

and hence, for any $2 < r \le p$

$$\|g_t(\eta_0) - g_s(\eta_0)\|_p^r \le c|G(t) - G(s)|^r.$$

Applying Lemma 5 we have that for any $2 < r \le p$

$$\begin{aligned} E|W_n(t) - W_n(s)|^r &= E\left|\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{g_t(\eta_i) - g_s(\eta_i)\}\right|^r \\ &\le cn^{-\frac{r}{2}} n^{-\frac{r}{2}} \|g_t(\eta_0) - g_s(\eta_0)\|^{\frac{r}{2}} \le c|G(t) - G(s)|^{\frac{r}{2}} \end{aligned} \tag{77}$$

Thus, by the Markov inequality and the Schwarz inequality we have that for any $0 \le t_1 < t < t_2 \le 1$

$$P(|W_n(t_2) - W_n(t)| > \lambda, |W_n(t) - W_n(t_1)| > \lambda)$$

$$\le \frac{1}{\lambda^r} E\Big\{|W_n(t_2) - W_n(t)|^{\frac{r}{2}} |W_n(t) - W_n(t_1)|^{\frac{r}{2}}\Big\}$$
$$\le \frac{1}{\lambda^r} \Big\{E|W_n(t_2) - W_n(t)|^r\Big\}^{\frac{1}{2}} \Big\{E|W_n(t) - W_n(t_1)|^r\Big\}^{\frac{1}{2}}$$
$$\le \frac{c}{\lambda^2}(G(t) - G(t_1))^{\frac{r}{4}}(G(t_2) - G(t))^{\frac{r}{4}}$$
$$\le \frac{c}{\lambda^2}(G(t_2) - G(t_1))^{\frac{r}{2}}.$$

Since $r > 2$, we can use Theorem B with $2\tau = r$ and the tightness of $\{W_n(t)\}$ follows.

Hence, $\{W_n(t) : 0 \le t \le 1\}$ converges weakly to some Gaussian process $\{W(t) : 0 \le t \le 1\}$.

Since

$$EW_n(t)W_n(s) = \frac{1}{n} E\bigg(\sum_{i=1}^{n} g_t(\eta_i)\bigg)\bigg(\sum_{j=1}^{n} g_t(\eta_i)\bigg)$$

and

$$Eg_t(\eta_i)g_s(\eta_j) = \operatorname{cov}(u_t(\eta_i), u_s(\eta_i)) \quad (1 \le i, j \le n; s, t \in [0, 1]),$$

by Lemma 3 we see that (46) is the covariance structure of $\{W(t)\}$. □

4.2 Proof of Theorem 5

Proof of Theorem 5. Put

$$g(x, t) = h_1(x, t) - U(t)$$

and

$$J(x, y, t) = h(x, y, t) - h_1(x, t) - h_1(y, t) + U(t).$$

Define

$$V_n(t) = \frac{1}{n} g(\eta_i, t) \quad \text{and} \quad R_n(t) = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} J(\eta_i, \eta_j, t).$$

Then

$$U_n - U(t) = 2V_n(t) + R_n(t)$$

and by Lemma 1

$$\sup_{0 \le t \le 1} \sqrt{n} R_n(t) = O\big(n^{-\frac{\gamma}{2}}\big) \quad a.s. \tag{78}$$

for some $\gamma > 0$. Further, as $h_1(x,t)$ satisfies the conditions of Theorem 4, we have that

$$\{2\sqrt{n}V_n(t); 0 \le t \le 1\} \xrightarrow{D} \{W(t); 0 \le t \le 1\} \quad (n \to \infty). \tag{79}$$

Now, from (78) and (79) the conclusion of Theorem 5 follows. □

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