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OSCILLATION CRITERIA OF DIFFERENCE EQUATIONS WITH SEVERAL DEVIATING ARGUMENTS

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Abstract. New sufficient conditions for the oscillation of all solutions of difference equations with several deviating arguments are presented. Corresponding difference equations of both retarded and advanced type are studied. The significance of the conditions established are demonstrated by comparing with known oscillation conditions. Examples illustrating the results are also given.

1. Introduction

In this paper we study the oscillation of all solutions of the difference equation with several variable retarded arguments of the form

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, \quad n \in \mathbb{N}_0,\, m \in \mathbb{N} \tag{E$_R$}$$

and the (dual) difference equation with several variable advanced arguments of the form

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\sigma_i(n)) = 0, \quad n \in \mathbb{N},\, m \in \mathbb{N}, \tag{E$_A$}$$

where $(p_i(n))$, $1 \le i \le m$ are sequences of nonnegative real numbers, $(\tau_i(n))$, $1 \le i \le m$ are sequences of integers such that

$$\tau_i(n) \le n-1 \quad \forall n \in \mathbb{N}_0, \quad \text{and} \quad \lim_{n\to\infty} \tau_i(n) = \infty, \quad 1 \le i \le m \tag{1.1}$$

$(\sigma_i(n))$, $1 \le i \le m$ are sequences of integers such that

$$\sigma_i(n) \ge n+1 \quad \forall n \in \mathbb{N}, \quad 1 \le i \le m. \tag{1.2}$$

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Here, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ and ∇ denotes the backward difference operator $\nabla x(n) = x(n) - x(n-1)$.

If $\tau_i(n) = n - k_i$ and $\sigma_i(n) = n + k_i$ where $k_i \in \mathbb{N}$, $1 \leq i \leq m$, then equations (E_R) and (E_A) take the forms

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(n-k_i) = 0, \quad n \in \mathbb{N}_0 \tag{E'_R}$$

and

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(n+k_i) = 0, \quad n \in \mathbb{N}, \tag{E'_A}$$

respectively.

Set $w = -\min_{\substack{n\geq 0 \\ 1\leq i\leq m}} \tau_i(n)$. Clearly, w is a positive integer.

By a *solution* of (E_R), we mean a sequence of real numbers $(x(n))_{n\geq -w}$ which satisfies (E_R) for all $n \geq 0$. It is clear that, for each choice of real numbers $c_{-w}, c_{-w+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $(x(n))_{n\geq -w}$ of (E_R) which satisfies the initial conditions $x(-w) = c_{-w}$, $x(-w+1) = c_{-w+1}, \ldots, x(-1) = c_{-1}$, $x(0) = c_0$.

By a solution of (E_A), we mean a sequence of real numbers $(x(n))_{n\geq 0}$ which satisfies (E_A) for all $n \geq 1$.

A solution $(x(n))_{n\geq -w}$ (or $(x(n))_{n\geq 0}$) of (E_R) (or (E_A)) is called *oscillatory,* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory.*

In the last few decades, the asymptotic and oscillatory behavior of the solutions of difference equations has been extensively studied. See, for example, [2−9, 12−17] and the references cited therein. Most of these papers concern the special case of the equations (E'_R) and (E'_A) with $m = 1$, while a small number of these papers are dealing with the general case of the equations (E_R) and (E_A) with $m = 1$, in which the arguments $(n - \tau_i(n))_{n\geq 0}$, $(\sigma_i(n) - n)_{n\geq 1}$, $1 \leq i \leq m$ are variable. For the general theory of difference equations the reader is referred to the monographs [1, 10, 11].

In 1989 Erbe and Zhang [9], in 1999 Tang and Yu [14], and in 2001, Tang and Zhang [16] proved that either one of the following conditions

$$\sum_{i=1}^{m} \left(\liminf_{n\to\infty} p_i(n) \right) \frac{(k_i+1)^{k_i+1}}{(k_i)^{k_i}} > 1, \tag{1.3}$$

$$\liminf_{n\to\infty} \sum_{i=1}^{m} \left(\frac{k_i+1}{k_i} \right)^{k_i+1} \sum_{j=n+1}^{n+k_i} p_i(j) > 1, \tag{1.4}$$

or

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{n+k_i} p_i(j) > 1, \tag{1.5}$$

implies that all solutions of the equation (E_R') oscillate, while in 2002, Li and Zhu [12] proved that if

$$\liminf_{n\to\infty} \sum_{i=1}^{m} \left(\frac{k_i+1}{k_i}\right)^{k_i+1} \sum_{j=n-k_i}^{n-1} p_i(j) > 1. \tag{1.6}$$

then all solutions of the equation (E_A') oscillate.

In 2005, Yan, Meng and Yan [17], and in 2006, Berezansky and Braverman [2], proved that if

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(j) \left(\frac{n-\tau_i(j)+1}{n-\tau_i(j)}\right)^{n-\tau_i(j)+1} > 1, \tag{1.7}$$

or

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0 \qquad \text{and} \qquad \liminf_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n-1} p_i(j) > \frac{1}{e}, \tag{1.8}$$

where $\tau(n) = \max_{1\le i\le m} \tau_i(n)$, $\forall n \ge 0$, then all solutions of (E_R) oscillate.

Very recently, Chatzarakis et al. [6] established the following oscillation criteria for the equations (E_R) and (E_A).

Theorem 1.1. (See [6, Theorems 2.1 [3.1], 2.2 [3.2]) *Assume that the sequences* $(\tau_i(n))$ $[(\sigma_i(n))]$, $1 \le i \le m$ *are increasing,* (1.1) [(1.2)] *holds, and*

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) > 1, \qquad \left[\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) > 1,\right] \tag{1.9}$$

where $\tau(n) = \max_{1\le i\le m} \tau_i(n)$, $\forall n \ge 0$, $[\sigma(n) = \min_{1\le i\le m} \sigma_i(n)$, $\forall n \ge 1]$, *or*

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0 \text{ and } \liminf_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau_i(n)}^{n-1} p_i(j) > \frac{1}{e} \left[\liminf_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n+1}^{\sigma_i(n)} p_i(j) > \frac{1}{e}\right]. \tag{1.10}$$

Then all solutions of Eq. (E_R) [(E_A)] *oscillate.*

The authors study further (E_R) and (E_A) and derive new sufficient oscillation conditions when neither (1.9) nor (1.10) is satisfied. Examples illustrating the results are also given.

2. Oscillation criteria for Eq. (E_R)

In this section we establish sufficient conditions for the oscillation of all solutions of (E_R), when the conditions (1.9) and (1.10) are not satisfied. To that end, the following lemma provides a useful tool.

Lemma 2.1. *Assume that the sequences* $(\tau_i(n))$, $1 \le i \le m$ *are increasing,* (1.1) *holds and* $(x(n))_{n \ge -w}$ *is a nonoscillatory solution of* (E_R). *Set*

$$\alpha = \min\{\alpha_i : 1 \le i \le m\} \qquad \textit{where} \qquad \alpha_i = \liminf_{n\to\infty} \sum_{j=\tau_i(n)}^{n-1} p_i(j)\,. \tag{2.1}$$

If $0 < \alpha \le 1$, *then*

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \ge (1-\sqrt{1-\alpha})^2, \tag{2.2}$$

where

$$\tau(n) = \max_{1\le i\le m} \tau_i(n), \quad \forall n \ge 0. \tag{2.3}$$

If $0 < \alpha < \frac{3\sqrt{5}-5}{2}$ *and, in addition,*

$$p_i(n) \ge 1-\sqrt{1-\alpha} \quad \textit{for all large } n, \quad (1 \le i \le m)\,, \tag{2.4}$$

then

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \ge \alpha\left[\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1\right]. \tag{2.5}$$

Proof. Since the solution $(x(n))_{n \ge -w}$ of (E_R) is nonoscillatory, it is either eventually positive or eventually negative. As $(-x(n))_{n\ge -w}$ is also a solution of (E_R), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $\rho \ge -w$ be an integer such that $x(n) > 0$ for all $n \ge \rho$, and consider an integer $r \ge 0$ so that $\tau_i(n) \ge \rho$, $1 \le i \le m$ for $n \ge r$ (clearly, $r > \rho$). Then it follows immediately from (E_R) that $\Delta x(n) \le 0$ for every $n \ge r$, which means that the sequence $(x(n))_{n\ge r}$ is decreasing.

Assume that $0 < \alpha \le 1$, where α is defined by (2.1). Consider an arbitrary real number ε with $0 < \varepsilon < \alpha$. Then we can choose an integer $n_0 \ge r$ such that $\tau_i(n) \ge r$, $1 \le i \le m$ for $n \ge n_0$, and

$$\sum_{j=\tau_i(n)}^{n-1} p_i(j) \ge \alpha_i - \varepsilon \ge \alpha - \varepsilon \quad \text{for all } n \ge n_0, \quad (1 \le i \le m)\,. \tag{2.6}$$

Observe that $0 < 1-\sqrt{1-(\alpha-\varepsilon)} < \alpha - \varepsilon$. We will establish the following claim.

Claim. For each $n \geq n_0$, there exists an integer $n_i^* \geq n$ for each $i = 1, 2, \cdots, m$ such that $\tau_i(n_i^*) \leq n - 1$, and

$$\sum_{j=n}^{n_i^*} p_i(j) \geq 1 - \sqrt{1 - (\alpha - \varepsilon)}, \tag{2.7}$$

$$\sum_{j=\tau_i(n_i^*)}^{n-1} p_i(j) > (\alpha - \varepsilon) - \left(1 - \sqrt{1 - (\alpha - \varepsilon)}\right). \tag{2.8}$$

To prove this claim, let us consider an arbitrary integer $n \geq n_0$. Assume, first, that $p_i(n) \geq 1 - \sqrt{1 - (\alpha - \varepsilon)}$, and choose $n_i^* = n$. Then $\tau_i(n_i^*) = \tau_i(n) \leq n - 1$. Moreover, we have

$$\sum_{j=n}^{n_i^*} p_i(j) = \sum_{j=n}^{n} p_i(j) = p_i(n) \geq 1 - \sqrt{1 - (\alpha - \varepsilon)}$$

and, by (2.6),

$$\sum_{j=\tau_i(n_i^*)}^{n-1} p_i(j) = \sum_{j=\tau_i(n)}^{n-1} p_i(j) \geq (\alpha - \varepsilon) > (\alpha - \varepsilon) - \left(1 - \sqrt{1 - (\alpha - \varepsilon)}\right).$$

So, (2.7) and (2.8) are fulfilled. Next, we suppose that $p_i(n) < 1 - \sqrt{1 - (\alpha - \varepsilon)}$. It is not difficult to see that (2.6) guarantees that $\sum_{j=0}^{\infty} p_i(j) = \infty$. In particular, it holds

$$\sum_{j=n}^{\infty} p_i(j) = \infty.$$

Thus, as $p_i(n) < 1 - \sqrt{1 - (\alpha - \varepsilon)}$, there always exists an integer $n_i^* > n$ so that

$$\sum_{j=n}^{n_i^*-1} p_i(j) < 1 - \sqrt{1 - (\alpha - \varepsilon)} \tag{2.9}$$

and (2.7) holds. We assert that $\tau_i(n_i^*) \leq n - 1$. Otherwise, $\tau_i(n_i^*) \geq n$. We also have $\tau_i(n_i^*) \leq n_i^* - 1$. Hence, in view of (2.9), we get

$$\sum_{j=\tau_i(n_i^*)}^{n_i^*-1} p_i(j) \leq \sum_{j=n}^{n_i^*-1} p_i(j) < 1 - \sqrt{1 - (\alpha - \varepsilon)}.$$

On the other hand, (2.6) gives

$$\sum_{j=\tau_i(n_i^*)}^{n_i^*-1} p_i(j) \geq \alpha - \varepsilon > 1 - \sqrt{1 - (\alpha - \varepsilon)}.$$

We have arrived at a contradiction, which shows our assertion. Furthermore, by using (2.6) (for the integer n_i^*) as well as (2.9), we obtain

$$\sum_{j=\tau_i(n_i^*)}^{n-1} p_i(j) = \sum_{j=\tau_i(n_i^*)}^{n_i^*-1} p_i(j) - \sum_{j=n}^{n_i^*-1} p_i(j) > (\alpha-\varepsilon) - \left(1-\sqrt{1-(\alpha-\varepsilon)}\right)$$

and consequently (2.8) holds true. Our claim has been proved.

By the definition of τ, suppose that $\tau(n) = \tau_\ell(n)$ where $\ell \in [1, m] \cap \mathbb{N}$. Clearly, ℓ is a fixed natural number. Taking into account that (2.6), (2.7) and (2.8) hold for each $i = 1, 2, \ldots, m$, we conclude that

$$\sum_{j=\tau(n)}^{n-1} p_\ell(j) \ge \alpha - \varepsilon \quad \text{for } n \ge n_0,$$

$$\sum_{j=n}^{n^*} p_\ell(j) \ge 1 - \sqrt{1-(\alpha-\varepsilon)} \quad \text{for } n \ge n_0,$$

$$\sum_{j=\tau(n^*)}^{n-1} p_\ell(j) > (\alpha-\varepsilon) - \left(1-\sqrt{1-(\alpha-\varepsilon)}\right) \quad \text{for } n \ge n_0,$$

where $n^* = n_\ell^*$. Thus, for all $n \ge n_0$, the above three inequalities guarantee that

$$\sum_{i=1}^{m} \sum_{j=\tau(n)}^{n-1} p_i(j) \ge \sum_{j=\tau(n)}^{n-1} p_\ell(j) \ge \alpha - \varepsilon, \tag{2.6$'$}$$

$$\sum_{i=1}^{m} \sum_{j=n}^{n^*} p_i(j) \ge \sum_{j=n}^{n^*} p_\ell(j) \ge 1 - \sqrt{1-(\alpha-\varepsilon)} \tag{2.7$'$}$$

and

$$\sum_{i=1}^{m} \sum_{j=\tau(n^*)}^{n-1} p_i(j) \ge \sum_{j=\tau(n^*)}^{n-1} p_\ell(j) > (\alpha-\varepsilon) - \left(1-\sqrt{1-(\alpha-\varepsilon)}\right). \tag{2.8$'$}$$

Summing up (E_R) from n to n^*, and using the fact that the function x is decreasing and the function τ (as defined by (2.3)) is increasing, we have

$$x(n) = x(n^*+1) + \sum_{i=1}^{m} \sum_{j=n}^{n^*} p_i(j)\, x(\tau_i(j)) \ge x(n^*+1) + \sum_{i=1}^{m} \sum_{j=n}^{n^*} p_i(j)\, x(\tau(j)),$$

or

$$x(n) \geq x(n^* + 1) + x(\tau(n^*)) \sum_{i=1}^{m} \sum_{j=n}^{n^*} p_i\,(j)\,,$$

which, in view of (2.7′), gives

$$x(n) \geq x(n^* + 1) + \left(1 - \sqrt{1 - (\alpha - \varepsilon)}\right) x(\tau(n^*)). \tag{2.10}$$

Summing up (E_R) from $\tau(n^*)$ to $n - 1$, and using the same arguments, we have

$$x(\tau(n^*)) = x(n) + \sum_{i=1}^{m} \sum_{j=\tau(n^*)}^{n-1} p_i\,(j)\, x(\tau_i(j)) \geq x(n) + \sum_{i=1}^{m} \sum_{j=\tau(n^*)}^{n-1} p_i\,(j)\, x(\tau(j)),$$

or

$$x(\tau(n^*)) \geq x(n) + x(\tau(n-1)) \sum_{i=1}^{m} \sum_{j=\tau(n^*)}^{n-1} p_i\,(j)\,,$$

which, in view of (2.8′), gives

$$x(\tau(n^*)) > x(n) + \left[(\alpha - \varepsilon) - \left(1 - \sqrt{1 - (\alpha - \varepsilon)}\right)\right] x(\tau(n-1)). \tag{2.11}$$

Combining inequalities (2.10) and (2.11), we obtain

$$\begin{aligned} &x(n) > x(n^* + 1) \\ &+ \left(1 - \sqrt{1 - (\alpha - \varepsilon)}\right) \left\{x(n) + \left[(\alpha - \varepsilon) - \left(1 - \sqrt{1 - (\alpha - \varepsilon)}\right)\right] x(\tau(n-1))\right\}, \end{aligned}$$

or

$$\frac{x(n)}{x(\tau(n-1))} > \left(1 - \sqrt{1 - (\alpha - \varepsilon)}\right)^2$$

and, for large n, we have

$$\frac{x(n+1)}{x(\tau(n))} > \left(1 - \sqrt{1 - (\alpha - \varepsilon)}\right)^2. \tag{2.12}$$

Hence,

$$\liminf_{n \to \infty} \frac{x(n+1)}{x(\tau(n))} \geq \left(1 - \sqrt{1 - (\alpha - \varepsilon)}\right)^2,$$

which, for arbitrarily small values of ε, implies (2.2).

Next, we consider the particular case where (2.4) holds. Therefore

$$p_i(n) > 1 - \sqrt{1 - (\alpha - \varepsilon)}\,, \quad \text{for all large } n, \qquad (1 \leq i \leq m)\,.$$

Assume now that $0 < \alpha < \frac{3\sqrt{5}-5}{2}$. In view of (2.12), it is clear that

$$x(n+1) > \left(1 - \sqrt{1-(\alpha-\varepsilon)}\right)^2 x(\tau(n)).$$

Thus, from (E_R) we have

$$\begin{aligned} x(n) &= x(n+1) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) \geq x(n+1) + \sum_{i=1}^{m} p_i(n)x(\tau(n)) \\ &> \left(1 - \sqrt{1-(\alpha-\varepsilon)}\right)^2 x(\tau(n)) + \left(1 - \sqrt{1-(\alpha-\varepsilon)}\right) x(\tau(n)), \end{aligned}$$

or

$$x(n) > \left(1 - \sqrt{1-(\alpha-\varepsilon)}\right)\left(2 - \sqrt{1-(\alpha-\varepsilon)}\right) x(\tau(n)). \tag{2.13}$$

Summing up (E_R) from $\tau(n)$ to $n-1$, and using the same arguments, we have

$$\begin{aligned} x(\tau(n)) &= x(n) + \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n-1} p_i\,(j)\, x(\tau_i(j)) \geq x(n) + \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n-1} p_i\,(j)\, x(\tau(j)) \\ &\geq x(n) + x(\tau(n-1)) \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n-1} p_i\,(j)\,, \end{aligned}$$

which, in view of (2.6′), gives

$$x(\tau(n)) \geq x(n) + (\alpha-\varepsilon)\, x(\tau(n-1)). \tag{2.14}$$

Combining inequalities (2.13) and (2.14), we obtain

$$x(n) > \left(1 - \sqrt{1-(\alpha-\varepsilon)}\right)\left(2 - \sqrt{1-(\alpha-\varepsilon)}\right)\left[x(n) + (\alpha-\varepsilon)\, x(\tau(n-1))\right],$$

or, using the assumption that $\alpha - \varepsilon < \frac{3\sqrt{5}-5}{2}$, we have

$$\frac{x(n)}{x(\tau(n-1))} > (\alpha-\varepsilon)\left[\frac{1}{3\sqrt{1-(\alpha-\varepsilon)} + (\alpha-\varepsilon) - 2} - 1\right],$$

and, for large n, we have

$$\frac{x(n+1)}{x(\tau(n))} > (\alpha-\varepsilon)\left[\frac{1}{3\sqrt{1-(\alpha-\varepsilon)} + (\alpha-\varepsilon) - 2} - 1\right].$$

Hence,

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \geq (\alpha-\varepsilon)\left[\frac{1}{3\sqrt{1-(\alpha-\varepsilon)} + (\alpha-\varepsilon) - 2} - 1\right],$$

which, for arbitrarily small values of ε, implies (2.5).

The proof of the lemma is complete.

Theorem 2.1. *Assume that the sequences* $(\tau_i(n))$, $1 \le i \le m$ *are increasing,* (1.1) *holds, and define* α *by* (2.1).

If $0 < \alpha \le 1$, *and*

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) > 1 - \left(1 - \sqrt{1-\alpha}\right)^2, \tag{2.15}$$

where τ *is defined by* (2.3), *then all solutions of Eq.* (E_R) *oscillate.*

If $0 < \alpha < \frac{3\sqrt{5}-5}{2}$, (2.4) *holds, and*

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) > 1 - \alpha \left[\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1\right], \tag{2.16}$$

then all solutions of Eq. (E_R) *oscillate.*

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\ge -w}$ is a nonoscillatory solution of (E_R). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\ge -w}$ is also a solution of (E_R), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $\rho \ge -w$ be an integer such that $x(n) > 0$ for all $n \ge \rho$, and consider an integer $r \ge 0$ so that $\tau_i(n) \ge \rho$, $1 \le i \le m$ for $n \ge r$ (clearly, $r > \rho$). Then it follows immediately from (E_R) that $\Delta x(n) \le 0$ for every $n \ge r$, which means that the sequence $(x(n))_{n\ge r}$ is decreasing.

Now, we choose an integer $n_0 > r$ such that $\tau(n) \ge r$ for $n \ge n_0$. Furthermore, we consider an integer $N > n_0$ so that $\tau(n) \ge n_0$ for $n \ge N$. Then, as the sequence $(\tau(n))_{n\ge 0}$ is increasing and the sequence $(x(n))_{n\ge r}$ is decreasing, it follows from (E_R) that, for every $n \ge N$,

$$x(\tau(n)) = x(n+1) + \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j)\, x(\tau_i(j)) \ge x(n+1) + x(\tau(n)) \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j)\,.$$

Consequently,

$$\sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) \le 1 - \frac{x(n+1)}{x(\tau(n))} \quad \text{for all } n \ge N,$$

which gives

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) \le 1 - \liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))}. \tag{2.17}$$

First, assume that $0 < \alpha \leq 1$ and (2.15) holds. Then by Lemma 2.1, inequality (2.2) is fulfilled, and so (2.17) leads to

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) \leq 1 - \left(1-\sqrt{1-\alpha}\right)^2,$$

which contradicts condition (2.15).

Next, let us suppose that $0 < \alpha < \frac{3\sqrt{5}-5}{2}$ and (2.4), (2.16) hold. Then by Lemma 2.1, inequality (2.5) is fulfilled, and so (2.17) leads to

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) \leq 1 - \alpha\left[\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1\right],$$

which contradicts condition (2.16).

The proof of the theorem is complete.

3. Oscillation criteria for Eq. (E_A)

In this section we establish sufficient conditions for the oscillation of all solutions of (E_A), when the conditions (1.9) and (1.10) are not satisfied.

Lemma 3.1. *Assume that the sequences $(\sigma_i(n))$, $1 \leq i \leq m$ are increasing, (1.2) holds and $(x(n))_{n\geq 0}$ is a nonoscillatory solution of (E_A). Set*

$$\alpha = \min\{\alpha_i : 1 \leq i \leq m\} \qquad \textit{where} \qquad \alpha_i = \liminf_{n\to\infty} \sum_{j=n+1}^{\sigma_i(n)} p_i(j). \tag{3.1}$$

If $0 < \alpha \leq 1$, then

$$\liminf_{n\to\infty} \frac{x(n-1)}{x(\sigma(n))} \geq (1-\sqrt{1-\alpha})^2 \tag{3.2}$$

where

$$\sigma(n) = \min_{1\leq i\leq m} \sigma_i(n), \quad \forall n \geq 1. \tag{3.3}$$

If $0 < \alpha < \frac{3\sqrt{5}-5}{2}$ and, in addition,

$$p_i(n) \geq 1 - \sqrt{1-\alpha} \quad \textit{for all large } n, \qquad (1 \leq i \leq m), \tag{3.4}$$

then

$$\liminf_{n\to\infty} \frac{x(n-1)}{x(\sigma(n))} \geq \alpha\left[\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1\right]. \tag{3.5}$$

Proof. Since the solution $(x(n))_{n\geq 0}$ of (E_A) is nonoscillatory, it is either eventually positive or eventually negative. As $(-x(n))_{n\geq 0}$ is also a solution of (E_A), we may restrict ourselves only to the case where $x(n)>0$ for all large n. Let $r\geq 0$ be an integer such that $x(n-1)>0$ for all $n\geq r$. Then $x(n)>0$, $x(\sigma_i(n))>0$, $\forall n\geq r$, $1\leq i\leq m$. Then it follows immediately from (E_A) that $\Delta x(n)\geq 0$ for every $n\geq r$, which means that the sequence $(x(n))_{n\geq r}$ is increasing.

Assume that $0<\alpha\leq 1$, where α is defined by (3.1). Consider an arbitrary real number ε with $0<\varepsilon<\alpha$. Then we can choose an integer $n_0\geq r$ such that

$$\sum_{j=n+1}^{\sigma_i(n)} p_i\,(j)\geq \alpha_i-\varepsilon\geq \alpha-\varepsilon \quad \text{for all } n\geq n_0, \qquad (1\leq i\leq m)\,. \tag{3.6}$$

Consequently, (3.6) guarantees that

$$\sum_{j=1}^{\infty} p_i\,(j)=\infty, \qquad \text{for each } i=1,\ldots,m.$$

Therefore, it is clear that there exists $n_1\geq n_0$ such that

$$\sum_{j=1}^{n_1} p_i\,(j)\geq 1-\sqrt{1-(\alpha-\varepsilon)}, \qquad \text{for each } i=1,\ldots,m.$$

We will establish the following claim.

Claim. For each $n\geq n_1$, there exists an integer $n_i^*\leq n$ for each $i=1,2,\cdots,m$ such that $\sigma_i(n_i^*)\geq n+1$, and

$$\sum_{j=n_i^*}^{n} p_i\,(j)\geq 1-\sqrt{1-(\alpha-\varepsilon)}, \tag{3.7}$$

$$\sum_{j=n+1}^{\sigma_i(n_i^*)} p_i\,(j)>(\alpha-\varepsilon)-\left(1-\sqrt{1-(\alpha-\varepsilon)}\right). \tag{3.8}$$

To prove this claim, let us consider an arbitrary integer $n\geq n_1$. Assume, first, that $p_i(n)\geq 1-\sqrt{1-(\alpha-\varepsilon)}$, and choose $n_i^*=n$. Then $\sigma_i(n_i^*)=\sigma_i(n)\geq n+1$. Moreover, we have

$$\sum_{j=n_i^*}^{n} p_i\,(j)=\sum_{j=n}^{n} p_i\,(j)=p_i(n)\geq 1-\sqrt{1-(\alpha-\varepsilon)}$$

and, by (3.6),

$$\sum_{j=n+1}^{\sigma_i(n_i^*)} p_i(j) = \sum_{j=n+1}^{\sigma_i(n)} p_i(j) \geq (\alpha-\varepsilon) > (\alpha-\varepsilon) - \left(1-\sqrt{1-(\alpha-\varepsilon)}\right).$$

So, (3.7) and (3.8) are fulfilled. Next, we suppose that $p_i(n) < 1-\sqrt{1-(\alpha-\varepsilon)}$. Then there always exists an integer $n_i^* < n$ so that

$$\sum_{j=n_i^*+1}^{n} p_i(j) < 1-\sqrt{1-(\alpha-\varepsilon)} \tag{3.9}$$

and (3.7) holds. We assert that $\sigma_i(n_i^*) \geq n+1$. Otherwise, $\sigma_i(n_i^*) \leq n$. We also have $\sigma_i(n_i^*) \geq n_i^*+1$. Hence, in view of (3.9), we get

$$\sum_{j=n_i^*+1}^{\sigma_i(n_i^*)} p_i(j) \leq \sum_{j=n_i^*+1}^{n} p_i(j) < 1-\sqrt{1-(\alpha-\varepsilon)}.$$

On the other hand, (3.6) gives

$$\sum_{j=n_i^*+1}^{\sigma_i(n_i^*)} p_i(j) \geq \alpha-\varepsilon > 1-\sqrt{1-(\alpha-\varepsilon)}.$$

We have arrived at a contradiction, which shows our assertion. Furthermore, by using (3.6) (for the integer n_i^*) as well as (3.9), we obtain

$$\sum_{j=n+1}^{\sigma_i(n_i^*)} p_i(j) = \sum_{j=n_i^*+1}^{\sigma_i(n_i^*)} p_i(j) - \sum_{j=n_i^*+1}^{n} p_i(j) > (\alpha-\varepsilon) - \left(1-\sqrt{1-(\alpha-\varepsilon)}\right)$$

and consequently (3.8) holds true. Our claim has been proved.

By the definition of σ, suppose that $\sigma(n) = \sigma_\ell(n)$ where $\ell \in [1,m] \cap \mathbb{N}$. Clearly, ℓ is a fixed natural number. Taking into account that (3.6), (3.7) and (3.8) hold for each $i = 1, 2, \ldots, m$, we conclude that

$$\sum_{j=n+1}^{\sigma(n)} p_\ell(j) \geq \alpha-\varepsilon \quad \text{for } n \geq n_1,$$

$$\sum_{j=n^*}^{n} p_\ell(j) \geq 1-\sqrt{1-(\alpha-\varepsilon)} \quad \text{for } n \geq n_1,$$

$$\sum_{j=n+1}^{\sigma(n^*)} p_\ell(j) > (\alpha-\varepsilon) - \left(1-\sqrt{1-(\alpha-\varepsilon)}\right) \quad \text{for } n \geq n_1,$$

where $n^* = n^*_\ell$. Thus, for all $n \geq n_1$, the above three inequalities guarantee that

$$\sum_{i=1}^{m}\sum_{j=n+1}^{\sigma(n)} p_i(j) \geq \sum_{j=n+1}^{\sigma(n)} p_\ell(j) \geq \alpha-\varepsilon, \tag{3.6$'$}$$

$$\sum_{i=1}^{m}\sum_{j=n^*}^{n} p_i(j) \geq \sum_{j=n^*}^{n} p_\ell(j) \geq 1-\sqrt{1-(\alpha-\varepsilon)}, \tag{3.7$'$}$$

and

$$\sum_{i=1}^{m}\sum_{j=n+1}^{\sigma(n^*)} p_i(j) \geq \sum_{j=n+1}^{\sigma(n^*)} p_\ell(j) > (\alpha-\varepsilon) - \left(1-\sqrt{1-(\alpha-\varepsilon)}\right). \tag{3.8$'$}$$

Summing up (E_A) from n^* to n, and using the fact that the functions x and σ (as defined by (3.3)) are increasing, we have

$$x(n) = x(n^*-1) + \sum_{i=1}^{m}\sum_{j=n^*}^{n} p_i(j)\, x(\sigma_i(j)) \geq x(n^*-1) + \sum_{i=1}^{m}\sum_{j=n^*}^{n} p_i(j)\, x(\sigma(j))$$

which, in view of (3.7$'$), gives

$$x(n) \geq x(n^*-1) + \left(1-\sqrt{1-(\alpha-\varepsilon)}\right) x(\sigma(n^*)). \tag{3.10}$$

Summing up (E_A) from $n+1$ to $\sigma(n^*)$, and using the same arguments, we have

$$x(\sigma(n^*)) = x(n) + \sum_{i=1}^{m}\sum_{j=n+1}^{\sigma(n^*)} p_i(j)\, x(\sigma_i(j)) \geq x(n) + \sum_{i=1}^{m}\sum_{j=n+1}^{\sigma(n^*)} p_i(j)\, x(\sigma(j)),$$

which, in view of (3.8$'$), gives

$$x(\sigma(n^*)) > x(n) + \left[(\alpha-\varepsilon) - \left(1-\sqrt{1-(\alpha-\varepsilon)}\right)\right] x(\sigma(n+1)). \tag{3.11}$$

Combining inequalities (3.10) and (3.11), we obtain

$$\begin{aligned} &x(n) > x(n^*-1) \\ &+ \left(1-\sqrt{1-(\alpha-\varepsilon)}\right)\left\{x(n) + \left[(\alpha-\varepsilon) - \left(1-\sqrt{1-(\alpha-\varepsilon)}\right)\right] x(\sigma(n+1))\right\}, \end{aligned}$$

or

$$\frac{x(n)}{x(\sigma(n+1))} > \left(1-\sqrt{1-(\alpha-\varepsilon)}\right)^2$$

and, for large n, we have

$$\frac{x(n-1)}{x(\sigma(n))} > \left(1 - \sqrt{1-(\alpha-\varepsilon)}\right)^2.$$

Hence,

$$\liminf_{n\to\infty} \frac{x(n-1)}{x(\sigma(n))} \geq \left(1 - \sqrt{1-(\alpha-\varepsilon)}\right)^2,$$

which, for arbitrarily small values of ε, implies (3.2). The rest of the proof is similar to the corresponding of Lemma 2.1 in the case where (3.4) holds.

The proof of the lemma is complete.

Oscillation of all solutions of (E_A) is described by the theorem below. Note that the proof is an easy modification of the proof of Theorem 2.1 and hence is omitted.

Theorem 3.1. *Assume that the sequences $(\sigma_i(n))$, $1 \leq i \leq m$ are increasing,* (1.2) *holds, and define α by* (3.1).

If $0 < \alpha \leq 1$, and

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) > 1 - \left(1 - \sqrt{1-\alpha}\right)^2, \tag{3.12}$$

where σ is defined by (3.3)*, then all solutions of Eq.* (E_A) *oscillate.*

If $0 < \alpha < \frac{3\sqrt{5}-5}{2}$, (3.4) *holds, and*

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) > 1 - \alpha\left[\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1\right], \tag{3.13}$$

then all solutions of Eq. (E_A) *oscillate.*

4. Remarks and examples

Remark 4.1. It is easy to see that

$$\alpha\left[\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1\right] > \left(1-\sqrt{1-\alpha}\right)^2, \qquad \forall \alpha \in \left(0, \frac{3\sqrt{5}-5}{2}\right).$$

Hence, if (2.4) [(3.4)] holds, then the condition (2.16) [(3.13)] is weaker than the condition (2.15) [(3.12)].

Remark 4.2. When $\alpha \to 0$, then the conditions (2.15), (2.16) [(3.12), (3.13)] reduce to

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) > 1, \qquad \left[\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) > 1, \right]$$

that is, to the condition (1.9). However the improvement is clear when $\alpha \to \frac{1}{e}$. For illustrative purposes we give the values of the lower bound on the above conditions when $\alpha = \frac{1}{e} = 0.3678794$. The lower bound in (2.15) [(3.12)] is 0.957999647, while in (2.16) [(3.13)] is 0.879366516. That is, our conditions (2.15) [(3.12)] and (2.16) [(3.13)] essentially improve (1.9).

Example 4.1. Consider the retarded difference equation

$$\Delta x(n) + p_1(n)x(n-2) + p_2(n)x(n-3) = 0, \qquad n \in \mathbb{N}_0 \tag{4.1}$$

where

$$p_1(3n) = \frac{6}{100}, \quad p_1(3n+1) = \frac{8}{100}, \quad p_1(3n+2) = \frac{4}{10}, \qquad n \in \mathbb{N}_0$$

and

$$p_2(4n) = \frac{3}{100}, \; p_2(4n+1) = \frac{5}{100}, \; p_2(4n+2) = \frac{7}{100}, \; p_2(4n+3) = \frac{335}{1000}, n \in \mathbb{N}_0$$

Here $m = 2$, $\tau_1(n) = n-2$, $\tau_2(n) = n-3$ and $\tau(n) = n-2$.

It is easy to see that

$$\alpha_1 = \liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p_1(j) = \frac{6}{100} + \frac{8}{100} = 0.14$$

and

$$\alpha_2 = \liminf_{n\to\infty} \sum_{j=n-3}^{n-1} p_2(j) = \frac{3}{100} + \frac{5}{100} + \frac{7}{100} = 0.15.$$

Thus

$$\alpha = \min\{\alpha_i : 1 \le i \le 2\} = \min\{0.14, 0.15\} = 0.14 < \frac{1}{e}.$$

Also,

$$\begin{aligned}\limsup_{n\to\infty} \sum_{i=1}^{2} \sum_{j=\tau(n)}^{n} p_i(j) &= \limsup_{n\to\infty} \left[\sum_{j=n-2}^{n} p_1(j) + \sum_{j=n-2}^{n} p_2(j) \right] \\ &= 0.14 + \frac{4}{10} + \frac{5}{100} + \frac{7}{100} + \frac{335}{1000} = 0.995.\end{aligned}$$

Observe that

$$0.995 > 1 - \left(1 - \sqrt{1-\alpha}\right)^2 \simeq 0.994723699,$$

that is, condition (2.15) of Theorem 2.1 is satisfied and therefore all solutions of equation (4.1) oscillate. On the other hand,

$$0.995 < 1,$$

$$\liminf_{n\to\infty} \sum_{i=1}^{2} \sum_{j=\tau(n)}^{n-1} p_i(j) = \liminf_{n\to\infty} \left[\sum_{j=n-2}^{n-1} p_1(j) + \sum_{j=n-2}^{n-1} p_2(j) \right] = 0.14 + 0.08 < \frac{1}{e},$$

$$\liminf_{n\to\infty} \sum_{i=1}^{2} \sum_{j=\tau_i(n)}^{n-1} p_i(j) = \liminf_{n\to\infty} \left[\sum_{j=n-2}^{n-1} p_1(j) + \sum_{j=n-3}^{n-1} p_2(j) \right] = 0.14 + 0.15 < \frac{1}{e},$$

$$\begin{aligned}\liminf_{n\to\infty} \sum_{i=1}^{m} \left(\frac{k_i+1}{k_i}\right)^{k_i+1} \sum_{j=n+1}^{n+k_i} p_i(j) &= \left(\frac{3}{2}\right)^3 \cdot 0.14 + \left(\frac{4}{3}\right)^4 \cdot 0.15 \\ &= 0.946574074 < 1,\end{aligned}$$

$$\begin{aligned}\liminf_{n\to\infty} \sum_{j=n-2}^{n-1} \sum_{i=1}^{2} p_i(j) \left(\frac{n-\tau_i(j)+1}{n-\tau_i(j)}\right)^{n-\tau_i(j)+1} &= \left(\frac{3}{2}\right)^3 \cdot 0.14 + \left(\frac{4}{3}\right)^4 \cdot 0.08 \\ &= 0.7253 < 1,\end{aligned}$$

and therefore none of the conditions (1.9), (1.8), (1.10), (1.4) and (1.7) is satisfied.

Example 4.2. Consider the retarded difference equation

$$\Delta x(n) + p_1(n)x(n-2) + p_2(n)x(n-1) = 0, \qquad n \in \mathbb{N}_0 \tag{4.2}$$

where

$$p_1(3n) = p_1(3n+1) = \frac{1}{10}, \quad p_1(3n+2) = \frac{1}{2}, \qquad n \in \mathbb{N}_0$$

and

$$p_2(2n) = \frac{8}{100}, \quad p_2(2n+1) = \frac{317}{1000}, \qquad n \in \mathbb{N}_0.$$

Here $m = 2$, $\tau_1(n) = n-2$, $\tau_2(n) = n-1$ and $\tau(n) = n-1$.

It is easy to see that

$$\alpha_1 = \liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p_1(j) = 2 \cdot \frac{1}{10} = 0.2$$

and

$$\alpha_2 = \liminf_{n\to\infty} \sum_{j=n-1}^{n-1} p_2(j) = \frac{8}{100} = 0.08.$$

Thus

$$\alpha = \min\{\alpha_i : 1 \le i \le 2\} = \min\{0.2, 0.08\} = 0.08 < \frac{1}{e}.$$

Furthermore, it is clear that

$$p_i(n) > 1 - \sqrt{1-\alpha} \simeq 0.040833695 \qquad \text{for all large } n, \qquad (1 \le i \le 2).$$

Also,

$$\begin{aligned}\limsup_{n\to\infty} \sum_{i=1}^{2} \sum_{j=\tau(n)}^{n} p_i(j) &= \limsup_{n\to\infty} \left[\sum_{j=n-1}^{n} p_1(j) + \sum_{j=n-1}^{n} p_2(j) \right] \\ &= \frac{1}{10} + \frac{1}{2} + \frac{8}{100} + \frac{317}{1000} = 0.997.\end{aligned}$$

Observe that

$$0.997 > 1 - \alpha \left[\frac{1}{3\sqrt{1-\alpha} + \alpha - 2} - 1 \right] \simeq 0.996448991,$$

that is, conditions (2.4) and (2.16) of Theorem 2.1 are satisfied and therefore all solutions of equation (4.2) oscillate. On the other hand,

$$0.997 < 1,$$

$$0.997 < 1 - \left(1 - \sqrt{1-\alpha}\right)^2 \simeq 0.998332609,$$

$$\liminf_{n\to\infty} \sum_{i=1}^{2} \sum_{j=\tau(n)}^{n-1} p_i(j) = \liminf_{n\to\infty} \left[\sum_{j=n-1}^{n-1} p_1(j) + \sum_{j=n-1}^{n-1} p_2(j) \right] = 0.18 < \frac{1}{e},$$

$$\liminf_{n\to\infty} \sum_{i=1}^{2} \sum_{j=\tau_i(n)}^{n-1} p_i(j) = \liminf_{n\to\infty} \left[\sum_{j=n-2}^{n-1} p_1(j) + \sum_{j=n-1}^{n-1} p_2(j) \right] = 0.28 < \frac{1}{e},$$

$$\liminf_{n\to\infty} \sum_{i=1}^{m} \left(\frac{k_i+1}{k_i} \right)^{k_i+1} \sum_{j=n+1}^{n+k_i} p_i(j) = \left(\frac{3}{2}\right)^3 \cdot 0.2 + 4 \cdot 0.08 = 0.995 < 1,$$

$$\begin{aligned}\liminf_{n\to\infty} \sum_{j=n-1}^{n-1} \sum_{i=1}^{2} p_i(j) \left(\frac{n-\tau_i(j)+1}{n-\tau_i(j)} \right)^{n-\tau_i(j)+1} &= \left(\frac{3}{2}\right)^3 \cdot 0.1 + 4 \cdot 0.08 \\ &= 0.6575 < 1,\end{aligned}$$

and therefore none of the conditions (1.9), (2.15), (1.8), (1.10), (1.4) and (1.7) is satisfied.

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