Yokohama Mathematical
Journal Vol. 60, 2014

BEND-FORMULAS IN SOME SPHERE-SYSTEMS

By

Hiroshi Maehara

(Received November 22, 2012)

Abstract. Let β_1, β_2 be a pair of disjoint circles in the plane such that there is a cyclic sequence of circles $\alpha_1\alpha_2\dots\alpha_n$ in which each consecutive pair are tangent and each α_i is tangent to both β_1, β_2. It is proved that $b(\alpha_1) + \cdots + b(\alpha_n) = (n/2)\cot^2(\pi/n)\{b(\beta_1)+b(\beta_2)\}$ holds, where $b(\cdot)$ denote the bend (that is, a signed reciprocal of the radius). Similar formulas are derived for Soddy's hexlet and other sphere-systems.

1. Introduction

A finite family of spheres $\mathcal{F}$ in $\mathbb{R}^n$ is called *normal* if for each $\sigma \in \mathcal{F}$, all other member of $\mathcal{F}$ lie in the same side (inside or outside) of σ. For example, Figure 1(a) shows a normal family of circles in $\mathbb{R}^2$, but other two are not. In a normal family of spheres, the *bend* $b(\sigma)$ of a sphere σ is defined as follows:

$$b(\sigma) = \begin{cases} 1/r(\sigma) & \text{if all other members of the family lie outside } \sigma \\ -1/r(\sigma) & \text{otherwise,} \end{cases}$$

where $r(\sigma)$ denotes the radius of σ.

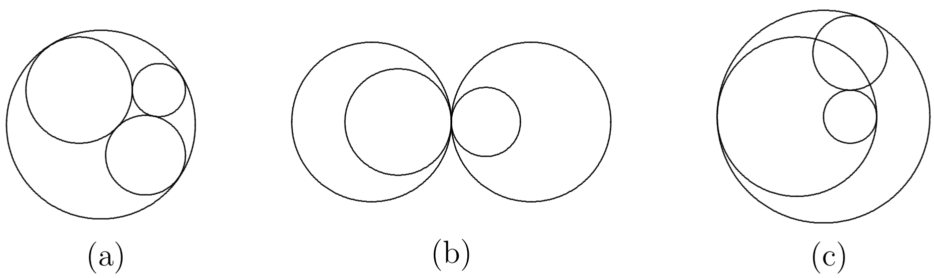


Figure 1 (a) is normal, (b) (c) are not

2010 Mathematics Subject Classification: Primary 52C99; Secondary 51N20
Key words and phrases: Steiner chain, Soddy's hexlet, K_2^{n+1}-system

For a normal family of mutually tangent circles $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ in $\mathbb{R}^2$,

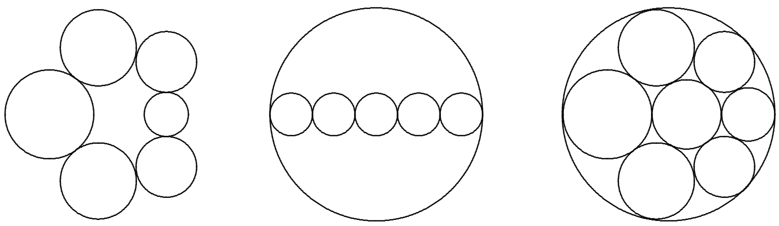
$$2\sum_{i=1}^{4} b(\alpha_i)^2 = \left(\sum_{i=1}^{4} b(\alpha_i)\right)^2$$

holds. This is known as *Descartes' circle formula*, see e.g. Coxeter [2]. This formula is also extended to higher dimensions: If $\{\sigma_1, \dots, \sigma_{n+2}\}$ is a normal family of mutually tangent $n+2$ spheres in $\mathbb{R}^n$, then

$$n\sum_{i=1}^{n+2} b(\sigma_i)^2 = \left(\sum_{i=1}^{n+2} b(\sigma_i)\right)^2$$

holds. This is called *Soddy's formula* (Soddy [7], see also [1,2,5]).

For $m \geq 3$, an *m-cycle of spheres* in $\mathbb{R}^n$ is a cyclic sequence of m spheres, in which each consecutive pair are tangent, and each non-consecutive pair are disjoint. Figure 2 (a), (b) are 6-cycles of circles.



(a) (b) (c)

Figure 2 Two 6-cycles of circles and a Steiner 6-cycle

Suppose that for an n-cycle $\alpha_1\alpha_2\dots\alpha_n$ of circles in $\mathbb{R}^2$, there is a pair of disjoint circles β_1, β_2 such that each α_i is tangent to both β_1, β_2. Such an n-cycle $\alpha_1\alpha_2\dots\alpha_n$ is called a *closed Steiner n-chain*, or simply, a *Steiner n-cycle* to the pair β_1, β_2, see Figure 2(c). Notice that in this case, the family $\{\alpha_1, \dots, \alpha_n, \beta_1, \beta_2\}$ is a normal family.

If a 4-cycle $\alpha_1\alpha_2\alpha_3\alpha_4$ is a Steiner 4-cycle to some pair of circles, then

$$b(\alpha_1) + b(\alpha_3) = b(\alpha_2) + b(\alpha_4)$$

holds. This formula, and the 3-dimensional case of Soddy's formula appeared in Japanese *Sangaku* tablet in 1822, see Rothman [6].

In this paper, we present similar formulas concerning bends for Steiner cycles and some other special families of spheres.

Theorem 1. *For $n \geq 3$, let $\alpha_1\alpha_2\dots\alpha_n$ be a Steiner n-cycle to a pair of disjoint circles β_1, β_2 in $\mathbb{R}^2$. If $n = km$ for some $k \geq 2$, then, for any j $(1 \leq j \leq m)$, the following holds:*

$$\sum_{i=0}^{k-1} b(\alpha_{j+im}) = \frac{k}{2}\cot^2(\pi/n)\,(b(\beta_1)+b(\beta_2))\,.$$

For example, if $\alpha_1\alpha_2\dots\alpha_{12}$ is a Steiner 12-cycle to a pair of disjoint circles β_1, β_2, and $k = 4$, then since $\cot^2(\pi/12) = 7 + 4\sqrt{3}$, we have

$$b(\alpha_j) + b(\alpha_{j+3}) + b(\alpha_{j+6}) + b(\alpha_{j+9}) = 2(7+4\sqrt{3})\{b(\beta_1)+b(\beta_2)\}$$

for each $j = 1, 2, 3$.

If $\tau_1\tau_2\tau_3$ is a 3-cycle of spheres in $\mathbb{R}^3$, and σ_1 is a sphere tangent to all τ_1, τ_2, τ_3, then there always exists a 6-cycle of spheres $\sigma_1\sigma_2\dots\sigma_6$ such that each σ_i is tangent to all τ_1, τ_2, τ_3. This 6-cycle $\sigma_1\sigma_2\dots\sigma_6$ is called **Soddy's hexlet** ([7], see also [3,4,5]) to the 3-cycle $\tau_1\tau_2\tau_3$. The family $\{\sigma_1, \dots, \sigma_6, \tau_1, \tau_2, \tau_3\}$ is a normal family.

Theorem 2. *If $\sigma_1\sigma_2\dots\sigma_6$ is a Soddy's hexlet to a 3-cycle $\tau_1\tau_2\tau_3$ of spheres in $\mathbb{R}^3$, then*

$$\begin{aligned} b(\sigma_1)+b(\sigma_4) = b(\sigma_2)+b(\sigma_5) = b(\sigma_3)+b(\sigma_6) = 2\{b(\tau_1)+b(\tau_2)+b(\tau_3)\},\\ b(\sigma_1)+b(\sigma_3)+b(\sigma_5) = b(\sigma_2)+b(\sigma_4)+b(\sigma_6) = 3\{b(\tau_1)+b(\tau_2)+b(\tau_3)\}. \end{aligned}$$

The *contact graph* of a normal family $\mathcal{F}$ of spheres is a graph whose vertices are in one-to-one correspondence with the members of $\mathcal{F}$, and two vertices are adjacent whenever the corresponding spheres are tangent. If the contact graph of a normal family $\mathcal{F}$ is isomorphic to a graph G, then the family is called a *G-system.* For example, if $\sigma_1\dots\sigma_6$ is a Soddy's hexlet to a 3-cycle $\tau_1\tau_2\tau_3$, then the family $\{\sigma_1, \dots, \sigma_6, \tau_1, \tau_2, \tau_3\}$ is a (C_6+K_3)-system in $\mathbb{R}^3$, where C_6+K_3 denotes the join of the 6-cycle C_6 and the complete graph K_3. (The *join* $G+H$ of two graphs G, H is the graph obtained from the disjoint union of G, H by adding every edges connecting each vertex of G to each vertex of H.) Let K_2^m denote the m-partite graph $K_{2,2,\dots,2}$. A K_2^4-system in $\mathbb{R}^3$ is called a *linked* 4-*pair* in [4].

For every $n \geq 2$, there is a K_2^{n+1}-system in $\mathbb{R}^n$. To see this, consider the n-dimensional cross polytope $\mathbb{R}^n$ with vertices

$$(\pm1, 0, \dots, 0), (0, \pm1, 0, \dots, 0), \dots, (0, \dots, 0, \pm1).$$

Its circum-radius is clearly equal to 1. Let $\sigma_1, \bar{\sigma}_1, \sigma_2, \bar{\sigma}_2, \dots, \sigma_n, \bar{\sigma}_n$ be the spheres of the same radius $\sqrt{2}/2$ centered at the vertices of this cross polytope, and let

$\sigma_0, \bar{\sigma}_0$ be the concentric spheres with center O and radii $1+\sqrt{2}/2, 1-\sqrt{2}/2$, respectively. Then, $\{\sigma_0, \bar{\sigma}_0, \sigma_1, \bar{\sigma}_1, \ldots, \sigma_n, \bar{\sigma}_n\}$ is a K_2^{n+1}-system in $\mathbb{R}^n$.

For a graph G, its complement is denoted by $\bar{G}$. For every $n \geq 2$, there is a $(\bar{K}_2 + K_{n+1})$-system in $\mathbb{R}^n$. To see this, consider a regular n-simplex of unit edge-length in $\mathbb{R}^n$ with barycenter at the origin O. The circum-radius r of this simplex is equal to $\sqrt{n/(2(n+1))}$. Let $\sigma_0, \sigma_1, \ldots, \sigma_n$ be the spheres of radius $1/2$ centered at the vertices of the simplex, and τ_1, τ_2 be the concentric spheres with center O and radii $r+1/2$, $r-1/2$, respectively. We have then a $(\bar{K}_2 + K_{n+1})$-system $\{\tau_1, \tau_2, \sigma_0, \sigma_1, \ldots, \sigma_n\}$.

Theorem 3. *In any $(\bar{K}_2 + K_{n+1})$-system in $\mathbb{R}^n$, we have*

$$\sum_{i=1}^{n+1} b(\sigma_i) = \frac{n-1}{2}\{b(\tau_1) + b(\tau_2)\},$$

where $\{\tau_1, \tau_2\}$ are the pair corresponding to the vertices of $\bar{K}_2$, and $\{\sigma_1, \ldots, \sigma_{n+1}\}$ are the spheres corresponding to the vertices of K_{n+1}.

Theorem 4. *In any K_2^{n+1}-system in $\mathbb{R}^n$ $(n \geq 2)$, we have*

$$\sum_{i=1}^{n}\{b(\sigma_i) + b(\bar{\sigma}_i)\} = n\{b(\sigma_0) + b(\bar{\sigma}_0)\},$$

where $\{\sigma_i, \bar{\sigma}_i\}$ denotes a pair of mutually non-tangent spheres, for each $i = 0, 1, 2, \ldots, n$. Therefore, $b(\sigma_0) + b(\bar{\sigma}_0) = b(\sigma_1) + b(\bar{\sigma}_1) = \cdots = b(\sigma_n) + b(\bar{\sigma}_n)$.

2. Inversions of $\mathbb{R}^n$

An inversion of $\mathbb{R}^n$ with center $\boldsymbol{p}$ and radius r is a transformation of $\mathbb{R}^n \setminus \{\boldsymbol{p}\}$ that sends $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{p}\}$ to a point $\boldsymbol{x}'$ lying on the ray $\overrightarrow{\boldsymbol{px}}$ and satisfying

$$|\boldsymbol{x} - \boldsymbol{p}| \cdot |\boldsymbol{x}' - \boldsymbol{p}| = r^2.$$

For a complete definition of an inversion, we need to add a point ∞ to $\mathbb{R}^n$ at the infinity and consider $\mathbb{R}^n \cup \{\infty\}$ instead of $\mathbb{R}^n$, see Coxeter [2] or Ogilvy [5] for details.

An inversion is an involution, that is, if we repeat the same inversion twice, then every points returns to the original position.

An important fact on an inversion of $\mathbb{R}^n$ is that it maps a sphere to a sphere, regarding a hyperplane as a sphere of infinite radius. More precisely, an inversion of $\mathbb{R}^n$ with center $\boldsymbol{p}$ satisfies the following:

1. It transforms a sphere not passing through $\boldsymbol{p}$ into a sphere not passing through $\boldsymbol{p}$.
2. It transforms a sphere passing through $\boldsymbol{p}$ into a hyperplane not passing through $\boldsymbol{p}$ and *vice versa*. In this case, the line through $\boldsymbol{p}$ and the center of the sphere is perpendicular to the hyperplane.
3. It transforms a hyperplane passing through $\boldsymbol{p}$ into itself.

The following fact is also used in this paper.

4. For any pair of disjoint spheres in $\mathbb{R}^n$, there is an inversion of $\mathbb{R}^n$ which transforms the pair into a pair of concentric spheres.

It is not difficult to check the following.

5. If a family $\mathcal{F}^*$ of spheres is obtained from a normal family $\mathcal{F}$ of spheres by applying an inversion, then $\mathcal{F}^*$ is also a normal family of spheres.

3. Proof of Theorem 1

For a point $\boldsymbol{p} \in \mathbb{R}^n$, let us denote by $f_{\boldsymbol{p}}$ the inversion of $\mathbb{R}^n$ with center $\boldsymbol{p}$ and *radius* 1.

Lemma 1. *Let $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be a normal family of k unit spheres in $\mathbb{R}^n$ with centers $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ such that $|\boldsymbol{x}_1| = \cdots = |\boldsymbol{x}_k| = d > 1$, $\boldsymbol{x}_1 + \boldsymbol{x}_2 + \cdots + \boldsymbol{x}_k = O$, the origin. For any $\boldsymbol{p} \in \mathbb{R}^n \setminus (\cup\sigma_i)$,*

$$\sum_{i=1}^{k} b(f_{\boldsymbol{p}}(\sigma_i)) = k(|\boldsymbol{p}|^2 + d^2 - 1).$$

Proof. Let $\sigma_i^* = f_{\boldsymbol{p}}(\sigma_i)$, $i = 1, 2, \ldots, k$. Let $\boldsymbol{a}_i, \boldsymbol{b}_i$ be the points where the line $\boldsymbol{p}\boldsymbol{x}_i$ intersects the sphere σ_i and $|\boldsymbol{a}_i - \boldsymbol{p}| \leq |\boldsymbol{b}_i - \boldsymbol{p}|$, see Figure 3. The diameter of σ_i^* is equal to $|f_{\boldsymbol{p}}(\boldsymbol{a}_i) - f_{\boldsymbol{p}}(\boldsymbol{b}_i)|$. If $\boldsymbol{p}$ lies outside σ_i, then since $|\boldsymbol{a}_i - \boldsymbol{p}| = |\boldsymbol{x}_i - \boldsymbol{p}| - 1$ and $|f_{\boldsymbol{p}}(\boldsymbol{a}_i) - \boldsymbol{p}| \cdot |\boldsymbol{a}_i - \boldsymbol{p}| = 1$, we have $|f_{\boldsymbol{p}}(\boldsymbol{a}_i) - \boldsymbol{p}| = 1/(|\boldsymbol{x}_i - \boldsymbol{p}| - 1)$. Similarly, we have $|f_{\boldsymbol{p}}(\boldsymbol{b}_i) - \boldsymbol{p}| = 1/(|\boldsymbol{x}_i - \boldsymbol{p}| + 1)$. Therefore,

$$\begin{aligned} 2r(\sigma_i^*) &= |f_{\boldsymbol{p}}(\boldsymbol{a}_i) - f_{\boldsymbol{p}}(\boldsymbol{b}_i)| = |f_{\boldsymbol{p}}(\boldsymbol{a}_i) - \boldsymbol{p}| - |f_{\boldsymbol{p}}(\boldsymbol{b}_i) - \boldsymbol{p}| \\ &= \frac{1}{|\boldsymbol{x}_i - \boldsymbol{p}| - 1} - \frac{1}{|\boldsymbol{x}_i - \boldsymbol{p}| + 1} = \frac{2}{|\boldsymbol{x}_i - \boldsymbol{p}|^2 - 1}, \end{aligned}$$

and hence $b(\sigma_i^*) = |\boldsymbol{x}_i - \boldsymbol{p}|^2 - 1$. If $\boldsymbol{p}$ lies inside σ_i, then since $|\boldsymbol{a}_i - \boldsymbol{p}| = 1 - |\boldsymbol{x}_i - \boldsymbol{p}|$ and $|f_{\boldsymbol{p}}(\boldsymbol{a}_i) - \boldsymbol{p}| = 1/|\boldsymbol{a}_i - \boldsymbol{p}|$, we have $|f_{\boldsymbol{p}}(\boldsymbol{a}_i) - \boldsymbol{p}| = 1/(1 - |\boldsymbol{x}_i - \boldsymbol{p}|) = -1/(|\boldsymbol{x}_i -$

$\boldsymbol{p}|-1)$. Therefore,

$$\begin{aligned}2r(\sigma_i^*) &= |f_{\boldsymbol{p}}(\boldsymbol{a}_i) - f_{\boldsymbol{p}}(\boldsymbol{b}_i)| = |f_{\boldsymbol{p}}(\boldsymbol{a}_i) - \boldsymbol{p}| + |f_{\boldsymbol{p}}(\boldsymbol{b}_i) - \boldsymbol{p}| \\ &= \frac{-1}{|\boldsymbol{x}_i - \boldsymbol{p}| - 1} + \frac{1}{|\boldsymbol{x}_i - \boldsymbol{p}| + 1} = \frac{-2}{|\boldsymbol{x}_i - \boldsymbol{p}|^2 - 1}.\end{aligned}$$

In this case, however, σ_i^* encloses all other spheres. Hence $b(\sigma_i^*) = -1/r(\sigma_i^*) = |\boldsymbol{x}_i - \boldsymbol{p}|^2 - 1$. Thus, we always have $b(\sigma_i^*) = |\boldsymbol{x}_i - \boldsymbol{p}|^2 - 1$. And hence

$$\begin{aligned}\sum_{i=1}^{k} b(\sigma_i^*) &= \sum_{i=1}^{k} |\boldsymbol{x}_i - \boldsymbol{p}|^2 - k = kd^2 + k|\boldsymbol{p}|^2 - k - 2\sum_{i=1}^{k} \boldsymbol{x}_i \cdot \boldsymbol{p} \\ &= k(|\boldsymbol{p}|^2 + d^2 - 1) - 2\big(\sum_{i=1}^{k} \boldsymbol{x}_i\big) \cdot \boldsymbol{p} = k(|\boldsymbol{p}|^2 + d^2 - 1).\end{aligned}$$

This proves Lemma 1. □

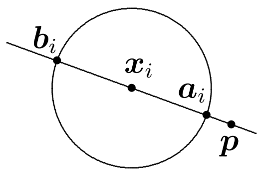


Figure 3 The diameter $|\boldsymbol{b}_i - \boldsymbol{a}_i|$ of σ_i

Lemma 2. *Let (τ_1, τ_2) be a pair of concentric spheres centered at the origin in $\mathbb{R}^n$ with radii $d+1, d-1\ (d > 1)$, respectively. For any point $\boldsymbol{p} \in \mathbb{R}^n \setminus (\cup \tau_i)$, we have*

$$b(f_{\boldsymbol{p}}(\tau_1)) + b(f_{\boldsymbol{p}}(\tau_2)) = \frac{2}{d^2-1}(|\boldsymbol{p}|^2 + d^2 - 1).$$

Proof. To make our argument clear, let us consider the case that $\boldsymbol{p}$ lies inside τ_2. (The other cases follows similarly.) The diameter of $f_{\boldsymbol{p}}(\tau_2)$ can be computed as

$$\frac{1}{d-1-|\boldsymbol{p}|} + \frac{1}{d-1+|\boldsymbol{p}|} = \frac{2(d-1)}{(d-1)^2 - |\boldsymbol{p}|^2}.$$

Similarly, the diameter of $f_{\boldsymbol{p}}(\tau_1)$ can be computed as $\frac{2(d+1)}{(d+1)^2-|\boldsymbol{p}|^2}$. Since $f_{\boldsymbol{p}}(\tau_2)$ encloses all other spheres, we have

$$\begin{aligned} b(f_{\boldsymbol{p}}(\tau_1)) + b(f_{\boldsymbol{p}}(\tau_2)) &= \frac{(d+1)^2 - |\boldsymbol{p}|^2}{d+1} - \frac{(d-1)^2 - |\boldsymbol{p}|^2}{d-1} \\ &= (d+1) - (d-1) + |\boldsymbol{p}|^2 \left(\frac{1}{d-1} - \frac{1}{d+1} \right) \\ &= \frac{2}{d^2-1}(|\boldsymbol{p}|^2 + d^2 - 1). \end{aligned}$$

□

Proof of Theorem 1. It is possible to transform the circles β_1, β_2 to a pair of concentric circles, namely, there is a point $\boldsymbol{p} \in \mathbb{R}^2$ such that $f_{\boldsymbol{p}}(\beta_1)$ and $f_{\boldsymbol{p}}(\beta_2)$ are concentric circles. In this case, all $f_{\boldsymbol{p}}(\alpha_i)$ must have the same radius. We may suppose that the center of $f_{\boldsymbol{p}}(\beta_1)$ (and $f_{\boldsymbol{p}}(\beta_2)$) is the origin. Since the formula in Theorem 1 is invariant under the similarity, we may suppose that all $f_{\boldsymbol{p}}(\alpha_i)$ have unit radius. Let d be the distance between the origin and the center of $f_{\boldsymbol{p}}(\alpha_i)$. We have

$$\sin(\pi/n) = \frac{1}{d}$$

and the radii of $f_{\boldsymbol{p}}(\beta_1)$, $f_{\boldsymbol{p}}(\beta_2)$ are $d+1, d-1$, respectively. If $\boldsymbol{x}_i$ denotes the center of $f_{\boldsymbol{p}}(\alpha_i)$, then for each $1 \le j \le m$, we have $\sum_{i=0}^{k-1} \boldsymbol{x}_{j+im} = O$. Hence, we can apply the 2-dimensional versions of Lemmas 1, 2, to the normal families of circles $\{f_{\boldsymbol{p}}(\alpha_1), \ldots, f_{\boldsymbol{p}}(\alpha_n)\}$ and $\{f_{\boldsymbol{p}}(\beta_1), f_{\boldsymbol{p}}(\beta_2)\}$. Noting that

$$f_{\boldsymbol{p}}(f_{\boldsymbol{p}}(\beta_j)) = \beta_j,\ f_{\boldsymbol{p}}(f_{\boldsymbol{p}}(\alpha_i)) = \alpha_i,$$

we have

$$\sum_{i=0}^{k-1} b(\alpha_{j+im}) = k(|\boldsymbol{p}|^2 + d^2 - 1)$$

$$|b(\beta_1) + b(\beta_2)| = \frac{2}{d^2-1}(|\boldsymbol{p}|^2 + d^2 - 1).$$

Since $d = 1/\sin(\pi/n)$, we have $d^2 - 1 = \cot^2(\pi/n)$. Hence we have the theorem. □

4. Soddy's hexlet and other sphere-systems

Lemma 3. *Let τ be the unit sphere centered at the origin in $\mathbb{R}^n$, and let ξ, η denote the hyperplanes $x_n = 1$ and $x_n = -1$, respectively. Let $\boldsymbol{p} = (p_1, p_2, \ldots, p_n) \in$*

$\mathbb{R}^n$ be a point not lying on $\tau \cup \xi \cup \eta$, and put $\tau^* = f_{\boldsymbol{p}}(\tau), \xi^* = f_{\boldsymbol{p}}(\xi), \eta^* = f_{\boldsymbol{p}}(\eta)$. The family $\{\tau^*, \xi^*, \eta^*\}$ is a normal family of spheres, and

$$b(\tau^*) + b(\xi^*) + b(\eta^*) = |\boldsymbol{p}|^2 + 3.$$

Proof. Let us consider the case that $|p_n| < 1$ and $\boldsymbol{p}$ lies outside of τ. (Other cases are similar.) The diameters of τ^*, ξ^*, η^* are calculated as

$$\begin{aligned} 2r(\tau^*) &= \frac{1}{|\boldsymbol{p}| - 1} - \frac{1}{|\boldsymbol{p}| + 1} = \frac{2}{|\boldsymbol{p}|^2 - 1}, \\ 2r(\xi^*) &= \frac{1}{1 - p_n}, \\ 2r(\eta^*) &= \frac{1}{1 + p_n}. \end{aligned}$$

Hence we have the lemma. □

Proof of Theorem 2. Let $\boldsymbol{p}$ be the contact point of τ_1 and τ_2. Put $\xi = f_{\boldsymbol{p}}(\tau_1)$, $\eta = f_{\boldsymbol{p}}(\tau_2)$ and $\tau^* = f_{\boldsymbol{p}}(\tau_3)$. The images ξ, η are two parallel planes, and τ^* is a sphere tangent to these two planes. Let $\sigma_i^* = f_{\boldsymbol{p}}(\sigma_i)$, $i = 1, 2, \ldots, 6$. The images σ_i^*, $i = 1, 2, \ldots, 6$, are spheres all tangent to ξ, η, τ^*. Since the formulas in Theorem 2 are invariant under the similarity, we may suppose that both σ_i^*, τ^* are unit spheres, and the center of τ^* is the origin of $\mathbb{R}^3$. Let $\boldsymbol{x}_i$ denote the center of σ_i^*. Then, $|\boldsymbol{x}_i| = 2$. Now, applying Lemma 1 with $d = 2$, we have

$$\sum_{i=0}^{2} b(f_{\boldsymbol{p}}(\sigma_{j+2i}^*)) = 3(|\boldsymbol{p}|^2 + 3) \qquad (k = 3)$$

$$\sum_{i=0}^{1} b(f_{\boldsymbol{p}}(\sigma_{j+3i}^*)) = 2(|\boldsymbol{p}|^2 + 3) \qquad (k = 2)$$

By Lemma 3, we have

$$b(f_{\boldsymbol{p}}(\tau^*)) + b(f_{\boldsymbol{p}}(\xi)) + b(f_{\boldsymbol{p}}(\eta)) = |\boldsymbol{p}|^2 + 3.$$

Hence we have the theorem.

Proof of Theorem 3. There is a point $\boldsymbol{p}$ such that $f_{\boldsymbol{p}}(\tau_1), f_{\boldsymbol{p}}(\tau_2)$ are concentric spheres. Put $\tau_1^* = f_{\boldsymbol{p}}(\tau_1), \tau_2^* = f_{\boldsymbol{p}}(\tau_2)$ and put $\sigma_i^* = f_{\boldsymbol{p}}(\sigma_i)$, $i = 1, 2, \ldots n + 1$. All σ_i^* are spheres of the same size. We may suppose that the center of τ_1^* is the origin, and all σ_i^* are unit spheres. Let $\boldsymbol{x}_i$ be the center of σ_i^*. Since $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+1}$ are the vertices of a regular simplex of edge length 2, its circum-radius is equal to $\sqrt{2n/(n+1)}$, that is, $|\boldsymbol{x}_i| = \sqrt{2n/(n+1)}$. Now, applying Lemmas 1 and 2 with $d = \sqrt{2n/(n+1)}$, we have

$$\sum_{i=1}^{n+1} b(f_{\boldsymbol{p}}(\sigma_i^*)) = (n+1)(|\boldsymbol{p}|^2 + d^2 - 1),$$

$$b(f_{\boldsymbol{p}}(\tau_1^*)) + b(f_{\boldsymbol{p}}(\tau_2^*)) = \frac{2}{d^2-1}(|\boldsymbol{p}|^2 + d^2 - 1).$$

and

$$\sum_{i=1}^{n+1} b(f_{\boldsymbol{p}}(\sigma_i^*)) = \frac{(d^2-1)(n+1)}{2}\{b(f_{\boldsymbol{p}}(\tau_1^*)) + b(f_{\boldsymbol{p}}(\tau_2^*))\}.$$

Since

$$d^2 - 1 = \frac{2n}{n+1} - 1 = \frac{n-1}{n+1},$$

we have the theorem.

Proof of Theorem 4. There is a point $\boldsymbol{p}$ such that $f_{\boldsymbol{p}}(\sigma_0), f_{\boldsymbol{p}}(\bar{\sigma}_0)$ are concentric spheres. Put $\sigma_0^* = f_{\boldsymbol{p}}(\sigma_0), \bar{\sigma}_0^* = f_{\boldsymbol{p}}(\bar{\sigma}_0)$ and $\sigma_i^* = f_{\boldsymbol{p}}(\sigma_i)$, $\bar{\sigma}_i^* = f_{\boldsymbol{p}}(\bar{\sigma}_i)$. Now, all σ_i^* and $\bar{\sigma}_i^*$, $i > 0$, are spheres of the same size. We may suppose that the center of σ_0^* is the origin, and all $\sigma_i^*, \bar{\sigma}_i^*$, $i > 0$, are unit spheres. Let $\boldsymbol{x}_i$ be the center of σ_i^* and $\bar{\boldsymbol{x}}_i$ be the center of $\bar{\sigma}_i^*$, $i > 0$. Since

$$|\boldsymbol{x}_i - \boldsymbol{x}_j| = |\bar{\boldsymbol{x}}_i - \boldsymbol{x}_j| = |\boldsymbol{x}_i - \bar{\boldsymbol{x}}_j| = |\bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}}_j| = 2$$

for all $0 < i, j\,(i \neq j)$, we can deduce that n vectors $\overrightarrow{\boldsymbol{x}_i\bar{\boldsymbol{x}}_i}$, $i = 1, 2, \ldots, n$ are mutually perpendicular, and the midpoint of $\boldsymbol{x}_i\bar{\boldsymbol{x}}_i$ is equidistant from all $2(n-1)$ points $\boldsymbol{x}_j$, $\bar{\boldsymbol{x}}_j$, $j \neq i$. Since these $2(n-1)$ points span a hyperplane, we can deduce that the midpoint of $\boldsymbol{x}_i\bar{\boldsymbol{x}}_i$ is the center of $\bar{\sigma}_0^*$, the origin. Hence $\sum_{i=1}^{n}(\boldsymbol{x}_i + \bar{\boldsymbol{x}}_i) = 0$, and $|\boldsymbol{x}_i| = |\bar{\boldsymbol{x}}_i| = \sqrt{2}$. Now, applying Lemmas 1 and 2 with $d = |\boldsymbol{x}_i| = |\bar{\boldsymbol{x}}_i| = \sqrt{2}$, we have

$$\sum_{i=1}^{n}\{b(f_{\boldsymbol{p}}(\sigma_i^*)) + b(f_{\boldsymbol{p}}(\bar{\sigma}_i))\} = 2n(|\boldsymbol{p}|^2 + d^2 - 1),$$

$$b(f_{\boldsymbol{p}}(\sigma_0^*)) + b(f_{\boldsymbol{p}}(\bar{\sigma}_0^*)) = \frac{2}{d^2-1}(|\boldsymbol{p}|^2 + d^2 - 1).$$

Hence

$$\sum_{i=1}^{n}\{b(f_{\boldsymbol{p}}(\sigma_i^*)) + b(f_{\boldsymbol{p}}(\bar{\sigma}_i))\} = n\{b(f_{\boldsymbol{p}}(\sigma_0^*)) + b(f_{\boldsymbol{p}}(\bar{\sigma}_0^*)\},$$

which proves the theorem.

5. Remarks

5.1. In a normal family of spheres, we may include hyperplanes as spheres with infinite radius. The formulas in Theorems 1, 2, 3, 4 are still true.

5.2. Let Q_n denote the graph of an n-dimensional cube in $\mathbb{R}^n$. For every $n > 0$, there also exists a $(\bar{K}_2+Q_n)$-system of spheres in $\mathbb{R}^n$. In this sphere-system, there is no bend-formula like one in a $(\bar{K}_2 + K_{n+1})$-system in $\mathbb{R}^n$ or in a K_2^{n+1}-system in $\mathbb{R}^n$.

To see this, first consider a 3-dimensional cube of edge length 2 centered at the origin in $\mathbb{R}^3$. Let σ_i, $i = 1, 2, \ldots, 8$ be the unit spheres with centers at the vertices of this cube. Let τ_1 be the sphere with center $O = (0, 0, 0)$, radius $1+\sqrt{3}$, and τ_2 be the sphere with center O, radius $\sqrt{3}-1$. The family $\{\sigma_1, \ldots, \sigma_8, \tau_1, \tau_2\}$ is a $(\bar{K}_2 + Q_3)$-system in $\mathbb{R}^3$. In this system, we have

$$\sum_{i=1}^{8} b(\sigma_i) = 8\{b(\tau_1) + b(\tau_2)\}.$$

Next, consider the sphere $x^2+y^2+z^2 = 11/4$ in $\mathbb{R}^3$. The sections of this sphere by the planes $z = \pm\sqrt{3}/2$ are circles with the same radius $\sqrt{2}$. We can inscribe a square $\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4$ in the upper circle on the plane $z = \sqrt{3}/2$, and inscribe a square $\boldsymbol{y}_1\boldsymbol{y}_2\boldsymbol{y}_3\boldsymbol{y}_4$ in the lower circle on the plane $z = -\sqrt{3}/2$, so that $|\boldsymbol{x}_i - \boldsymbol{y}_i| = 2$ for $i = 1, 2, 3, 4$. Let $\sigma'_1, \ldots, \sigma'_8$ be unit spheres with centers $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_4, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_4$ and let τ'_1 be the sphere with center O, radius $\sqrt{11/4}+1$ and τ'_2 be the sphere with center O, radius $\sqrt{11/4}-1$. The family $\{\sigma'_1, \ldots, \sigma'_8, \tau'_1, \tau'_2\}$ is a $(\bar{K}_2+Q_3)$-system. In this sphere-system, we have

$$\sum_{i=1}^{8} b(\sigma'_i) = 7\{b(\tau'_1) + b(\tau'_2)\}.$$

Thus, for a $(\bar{K}_2 + Q_4)$-system in $\mathbb{R}^n$, there is no general formula like one in K_2^{n+1}-system in $\mathbb{R}^n$.

References

[1] W. S. Brown, The kiss precise, *Amer. Math. Monthly*, **76** (1969) 661–663.
[2] H. S. M. Coxeter, *Introduction to Geometry*, John Wiley and Sons Inc., New York 1965.
[3] H. S. M. Coxeter, Interlocked ring of spheres, *Scrita Mathematica*, **18** (1952) 113–121.
[4] H. Maehara, A. Oshiro, On Soddy's hexlet and a linked 4-pair, Proc. JCDCG '98 (Ed. J. Akiyama et al.) Springer (2000) 188–193.
[5] C. S. Ogilvy, *Excursions in Geometry*, Dover, 1990.

[6] T. Rothman, Japanese temple geometry, *Sci. Amer.*, **278** (1998) 85–91.
[7] F. Soddy, The hexlet, *Nature*, **139** (1937) 154.

862-2 Yagi, Nakagusuku, Okinawa 901-2405 Japan
Email: hmaehara@edu.u-ryukyu.ac.jp