Yokohama Mathematical
Journal Vol. 59, 2013

DIFFERENCE EQUATIONS OF FUNCTIONS OF PROCESSES BASED ON WEAKLY DEPENDENT DATA

By

Ken-ichi Yoshihara

(Received February 12, 2013; Revised July 10, 2013)

Summary. Corresponding to the Itô formula we consider difference equations defined by some weakly dependent sequence of random variables and examine the asymptotic behavior of their solutions.

1. Introduction

Let $\{\xi_n\}$ be a strictly stationary sequence of zero mean random variables defined on a probability space $(\Omega, \mathcal{M}, P)$ and satisfies the strong mixing condition

$$\alpha(n) = \sup_{A \in \mathcal{M}_{-\infty}^0, B \in \mathcal{M}_n^\infty} |P(AB) - P(A)P(B)| \to 0 \quad (n \to \infty)$$

where $\mathcal{M}_a^b$ $(a < b)$ denotes the σ-algebra generated by $\xi_a, \cdots, \xi_b$.

Then, under the conditions on $\{\xi_i\}$ in Theorem 1 (below),

$$\rho^2 = E\xi_1^2 + 2\sum_{i=2}^{\infty} E\xi_1\xi_i < \infty \tag{1}$$

holds. We always assume $\rho > 0$.

Remark. In Yoshihara (2009) it was shown that under the conditions in Theorem 1 (below)

$$\frac{1}{n}\sum_{l=1}^{k}\left(\sum_{j=1}^{r}\left(\xi_{(l-1)r+j} - \frac{1}{n}\sum_{i=1}^{n}\xi_i\right)\right)^2 \to \rho^2 \quad a.s. \tag{2}$$

where $r = [n^\gamma]$ with $0 < \gamma < \frac{1}{8}$ and $k = [n/r]$. So, we can obtain an approximate value of ρ by simulation.

2010 Mathematics Subject Classification:
Key words and phrases:

Let $\{X(t); t \geq 0\}$ be a continuous process. Corresponding to the stochastic differential equation

$$dX(t) = X(t)\big(\mu dt + \sigma dW(t)\big), \tag{3}$$

where μ and $\sigma > 0$ are constants and $\{W(t) : t \geq 0\}$ is a standard Wiener process, Yoshihara (2013) considered the difference equation

$$\Delta X(t_k) = X(t_{k-1})\left(\mu\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\xi_k\right) \quad (k = 1, \cdots, n) \tag{4}$$

where $t_k = (kT)/n$ $(k = 1, \cdots, n)$, and obtained that the solution $X^{(n)}(T)$ of (4) converges almost surely to

$$X(T) = X(0)\left\{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\rho W(T)\right\}. \tag{5}$$

as $n \to \infty$.

In particular, if $\{\xi_n\}$ is a sequence of i.i.d. $\mathcal{N}(0, 1)$ random variables, then $\rho = 1$ and (5) becomes

$$X(T) = X(0)\left\{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma W(T)\right\}. \tag{6}$$

In this paper, we consider the more general type of difference equations, corresponding to differential equations of types

$$dX(t) = h(X(t), t)dt + v(X(t), t)dW(t). \tag{7}$$

Denote by $\mathcal{C}^a_*(\mathsf{A}^b)$ the set of functions $\mathsf{A}^b \to \mathsf{R}$ which possess continuous bounded partial derivatives up to order a.

For $F(x_1, x_2) \in \mathcal{C}^3_*(\mathsf{R}^2)$ write

$$F_{x_q}(x_1, x_2) = \frac{\partial F(x_1, x_2)}{\partial x_q}, \; F_{x_q, x_{q'}}(x_1, x_2) = \frac{\partial^2 F(x_1, x_2)}{\partial x_q \partial x_{q'}} \quad (q, q' = 1, 2).$$

In the following sections, we use "c" to denote some absolute constant which does not depend on i, j, k, n and may differ from line to line and write $\|\zeta\|_p = \{E|\zeta|^p\}^{1/p}$ for any random variable ζ.

2. Main results

Let $T > 0$ be fixed and put

$$s_i = \frac{iT}{n} \quad (1 \leq i \leq n), \quad s_0 = 0,$$

and for any continuous process $\{Z(t) : 0 \le t < \infty\}$ put

$$\Delta Z(s_i) = Z(s_i) - Z(s_{i-1}) \quad (1 \le i \le n).$$

We consider the difference equations of functions of $\{X(t) : 0 \le t < T\}$.

The following theorem corresponds to the Itô formula.

Theorem 1. *Let $\{X(t) : t \ge 0\}$ be a continuous process. Let $h(x,t)$ and $v(x,t) > 0$ be elements of $\mathcal{C}^3_*([0,\infty))$. Let $\{\xi_n\}$ be a strictly stationary strong mixing sequence of random variables such that $E\xi_1 = 0$, $E\xi_1^2 = 1$ and*

$$E|\xi_1|^{13} < \infty \quad \text{and} \quad \alpha(n) = O(n^{-40}). \tag{8}$$

Assume $\rho > 0$.

Suppose that for an arbitrarily fixed positive integer n

$$\Delta X(s_i) = h(X(s_{i-1}), s_{i-1})\frac{T}{n} + \sqrt{\frac{T}{n}}v(X(s_{i-1}), s_{i-1})\xi_i \qquad (1 \le i \le n). \tag{9}$$

For $F \in \mathcal{C}^3_*(\mathsf{R}^2)$ define the process $\{Z(t) = F(X(t),t) : 0 \le t \le T\}$. Then

$$\begin{aligned}\Delta Z(s_i) &= Z(s_i) - Z(s_{i-1}) \\ &= \frac{T}{n}\Big(F_x(X(s_{i-1}), s_{i-1})h(X(s_{i-1}), s_{i-1}) + F_t(X(s_{i-1}), s_{i-1})) \\ &\quad + \frac{1}{2}F_{x,x}(X(s_{i-1}), s_{i-1})v^2(X(s_{i-1}), s_{i-1})\xi_i^2\Big) \\ &\quad + \sqrt{\frac{T}{n}}F_x(X(s_{i-1}), s_{i-1})v(X(s_{i-1}), s_{i-1})\xi_i + R_i\end{aligned} \tag{10}$$

where $R_i = R(s_i)$ denotes the residual such that

$$|R_i| \le \frac{c}{n^{\frac{5}{4}}} \quad a.s. \quad (1 \le i \le n). \tag{11}$$

Denote the solution of (10) by $Z^{(n)}(t)$ with $Z^{(n)}(0) = Z(0) = z$ $(n \ge 1)$, i.e.,

$$\begin{aligned}Z^{(n)}(T) &= z + \frac{T}{n}\sum_{i=1}^{n} F_x(X(s_{i-1}), s_{i-1}), s_{i-1})h(X(s_{i-1}), s_{i-1}) \\ &\quad + \frac{T}{n}\sum_{i=1}^{n} F_t(X(s_{i-1}), s_{i-1}) \\ &\quad + \frac{T}{2n}\sum_{i=1}^{n} F_{x,x}(X(s_{i-1}), s_{i-1})v^2(X(s_{i-1}), s_{i-1})\xi_i^2 \\ &\quad + \sqrt{\frac{T}{n}}\sum_{i=1}^{n} F_x(X(s_{i-1}), s_{i-1})v(X(s_{i-1}), s_{i-1})\xi_i + \sum_{i=1}^{n} R_i.\end{aligned} \tag{12}$$

Then, $Z^{(n)}(T)$ converges almost surely to

$$Z(T) = z + \int_0^T \Big(F_x(X(t),t)h(X(t),t) + F_t(X(t),t) + \frac{1}{2}F_{x,x}(X(t),t)v^2(X(t),t)\Big)dt + \rho \int_0^T F_x(X(t),t)v(X(t),t)dW(t) \tag{13}$$

as $n \to \infty$, where $\{W(t) : 0 \le t \le T\}$ is a standard Wiener process.

As an application of Theorem 1, we prove the following theorem which corresponds to the Feynman-Kac theorem.

Theorem 2. *Suppose that the conditions of Theorem 1 are satisfied. Let $X^{(n)}(t)$ be the solution of (9). Suppose further that for some functions $f(x,t) \in \mathcal{C}^3_*(\mathsf{R}^2)$ and $h(x,t)$ the partial differential equation*

$$\frac{\partial f}{\partial t} + \frac{1}{2}v^2(x,t)\frac{\partial^2 f}{\partial x^2} + h(x,t)\frac{\partial f}{\partial x} + a(x,t)f(x,t) = 0, \quad f(X(0),0) = F(0) \tag{14}$$

holds. Then, as $n \to \infty$ $f(X^{(n)}(T),T)$ converges almost surely to

$$f(X(T),T) = F(0)e^{-\int_0^T a(X(u),u)du} + \rho e^{-\int_0^T a(X(u),u)du} \int_0^T e^{\int_0^s a(X(u),u)du} dW(s). \tag{15}$$

3. Preliminaries

To prove theorems we need the following inequality and the strong approximation theorems.

Lemma A. *Let η (ζ) be an $\mathcal{M}^a_{-\infty}$ ($\mathcal{M}^{\infty}_{a+n}$)- measurable random variable. Suppose there are positive numbers p and q with $p^{-1} + q^{-1} < 1$ such that $\|\eta\|_p < \infty$ and $\|\zeta\|_q < \infty$. Then*

$$|E\eta\zeta - E\eta E\zeta| \le 10\|\eta\|_p\|\zeta\|_q\alpha^{1-p^{-1}+q^{-1}}(n).$$

Specifically, if $E\eta E\zeta = 0$ and $\|\eta\|_8\|\zeta\|_8 < \infty$, then

$$|E\eta\zeta| \le c\|\eta\|_8\|\zeta\|_8\alpha^{\frac{3}{4}}(n). \tag{16}$$

Theorem A. *Let $\{\xi_n\}$ be a stationary strong mixing sequence of zero mean random variables and having $(2+\delta)$-th moments $(0<\delta\leq 1)$. Assume that for some $\tau>0$*

$$\alpha(n)\leq cn^{-(1+\tau)(1+\frac{2}{\delta})}.$$

Then, we can redefine the sequence $\{\xi_n\}$ on a new probability space together with a standard Wiener process $W(t)$ such that

$$\left|\sum_{n\leq t}\xi_n-\rho W(t)\right|=O\big(t^{\frac{1}{4}}\big)\quad a.s. \tag{17}$$

Remark. Precise explanations on strong approximations of sums are found in Berkes, *et al* (2011). In the i.i.d. case, the right hand side of (16) is of order $o(n^{\frac{1}{2}})$ if $E\xi_1=0$ and $E\xi_1^2=1$.

Theorem B. *Let $\{\xi_n\}$ be a stationary strong mixing sequence of centered random variables. Suppose $\{\xi_n\}$ satisfies the conditions of Theorem A. Then*

$$\limsup_{n\to\infty}(2n\rho\log\log\rho n)^{-\frac{1}{2}}\sum_{i=1}^{n}\xi_i=1\quad a.s. \tag{18}$$

4. Proof of Theorem 1

Firstly, we prove the following lemmas.

Lemma 1. *Suppose the conditions in Theorem 1. If h_0 and v_0 are elements of $\mathcal{C}_*^1(\mathsf{R}^2)$, then for fixed i $(1\leq i\leq n)$*

$$\begin{aligned}|\Delta X(s_i)|&\leq\left\{|h_0(X(s_{i-1}),s_{i-1})|\frac{T}{n}+\sqrt{\frac{T}{n}}|v_0(X(s_{i-1}),s_{i-1})\xi_i|\right\}\\&=O\big(n^{-\frac{5}{12}}\big)\quad a.s.\end{aligned} \tag{19}$$

Proof. We note that

$$\begin{aligned}&E\left|h_0(X(s_{i-1}),s_{i-1})\frac{T}{n}+\sqrt{\frac{T}{n}}v_0(X(s_{i-1}),s_{i-1})\xi_i\right|^{13}\\&\leq c\left\{E|h_0(X(s_{i-1}),s_{i-1})|^{13}\frac{T^{13}}{n^{13}}+\frac{T^{\frac{13}{2}}}{n^{\frac{13}{2}}}E\left|v_0(X(s_{i-1}),s_{i-1})\xi_i\right|^{13}\right\}\\&\leq\frac{c}{n^{\frac{13}{2}}},\end{aligned}$$

since $h_0(x,t)$ and $v_0(x,t)$ are continuous functions with bounded derivatives. Thus, for all n sufficiently large

$$
\begin{aligned}
&P\bigg(\bigg|h_0(X(s_{i-1}),s_{i-1})\frac{T}{n}+\sqrt{\frac{T}{n}}v_0(X(s_{i-1}),s_{i-1})\xi_i\bigg|\geq n^{-\frac{5}{12}}\bigg)\\
&\leq n^{\frac{65}{12}}E\bigg|h_0(X(s_{i-1}),s_{i-1})\frac{T}{n}+\sqrt{\frac{T}{n}}v_0(X(s_{i-1}),s_{i-1})\xi_{i-1}\bigg|^{13}\\
&\leq cn^{\frac{65}{12}}\frac{1}{n^{\frac{13}{2}}}\leq\frac{c}{n^{\frac{13}{12}}}.
\end{aligned}
$$

which implies that for all n sufficienly large

$$
\begin{aligned}
&\sum_{n=1}^{\infty}P\bigg(\bigg|h_0(X(s_{i-1}),s_{i-1})\frac{T}{n}\\
&+\sqrt{\frac{T}{n}}v_0(X(s_{i-1}),s_{i-1})\xi_i\bigg|\geq n^{-\frac{5}{12}}\bigg)<\infty\quad(1\leq i\leq n).
\end{aligned}
$$

Now, (19) follows from the Borel-Cantelli lemma. □

Lemma 2. *Suppose $\{\xi_n\}$ be a stationary sequence of zero mean random variables with mixing coefficient $\alpha(n)$. If $E|\xi_1|^8<\infty$, then for any $q\geq 1$*

$$
E\big|E\{\xi_0|\mathcal{M}_{-\infty}^{-r}\}\big|\leq c\alpha^{\frac{3}{8}}(r)\|\xi_0\|_8. \tag{20}
$$

Proof. Since $E\xi_0=0$ and $E\{\xi_0|\mathcal{M}_{-\infty}^{-r}\}$ is $\mathcal{M}_{-\infty}^{-r}$-measurable, by (16)

$$
\begin{aligned}
&E\big|E\{\xi_0|\mathcal{M}_{-\infty}^{-r}\}\big|^2\\
&=E\big\{E\{\xi_0|\mathcal{M}_{-\infty}^{-r}\}E\{\xi_0|\mathcal{M}_{-\infty}^{-r}\}\big\}=E\big\{\xi_0E\{\xi_0|\mathcal{M}_{-\infty}^{-r}\}\big\}\\
&\leq c\alpha^{\frac{3}{4}}(r)\|\xi_0\|_8\big\|E\{\xi_0|\mathcal{M}_{-\infty}^{-r}\}\big\|_8\leq c\alpha^{\frac{3}{4}}(r)\|\xi_0\|_8^2.
\end{aligned}
$$

Since

$$
E\big|E\{\xi_0|\mathcal{M}_{-\infty}^{-r}\}\big|\leq\big\|E\{\xi_0|\mathcal{M}_{-\infty}^{-r}\}\big\|_2,
$$

(20) is obtained. □

Proof of Theorem 1. By the Taylor theorem

$$
\begin{aligned}
(21)\qquad \Delta Z(s_i) &= Z(s_i) - Z(s_{i-1}) \\
&= F(X(s_i), s_i) - F(X(s_{i-1}), s_{i-1}) \\
&= \Big(F_x(X(s_{i-1}), s_{i-1})\Delta X(s_i) + F_t(X(s_{i-1}), s_{i-1})(s_i - s_{i-1}) \Big) \\
&\quad + \frac{1}{2}\Big(F_{x,x}(X(s_{i-1}), s_{i-1})(\Delta X(s_i))^2 \\
&\quad + 2F_{x,t}(X(s_{i-1}), s_{i-1})(\Delta X(s_i))(s_i - s_{i-1}) \\
&\quad + F_{t,t}(X(s_{i-1}), s_{i-1})(s_i - s_{i-1})^2 \Big) + R_i \quad (1 \le i \le n)
\end{aligned}
$$

and hence

$$
\begin{aligned}
(22)\qquad & Z(T) - Z(0) \\
&= \sum_{i=1}^{n} \Big\{ \Big(F_x(X(s_{i-1}), s_{i-1})\Delta X(s_i) + F_t(X(s_{i-1}), s_{i-1})\frac{T}{n} \Big) \\
&\quad + \frac{1}{2}\Big(F_{x,x}(X(s_{i-1}), s_{i-1})(\Delta X(s_i))^2 \\
&\quad + 2F_{x,t}(X(s_{i-1}), s_{i-1})(\Delta X(s_i))\frac{T}{n} \\
&\quad + F_{t,t}(X(s_{i-1}), s_{i-1})\Big(\frac{T}{n}\Big)^2 \Big) + R_i \Big\} \\
&= (U_{1,1}^{(n)} + U_{1,2}^{(n)}) + \frac{1}{2}(U_{2,1}^{(n)} + U_{2,2}^{(n)} + U_{2,3}^{(n)}) + U_3^{(n)} \quad \text{(say)},
\end{aligned}
$$

where $R_1, \cdots, R_n$ are residuals.

Firstly, we consider $U_3^{(n)}$. We note that for each $1 \le i \le n$ R_i is may be written by uniformly bounded random variables $A_{1,i}, A_{2,i}, A_{3,i}$ and $A_{4,i}$ as

$$
\begin{aligned}
R_i = A_{1,i}(\Delta X(s_i))^3 &+ A_{2,i}(\Delta X(s_i))^2(s_i - s_{i-1}) \\
&+ A_{3,i}(\Delta X(s_i))(s_i - s_{i-1})^2 + A_{4,i}(s_i - s_{i-1})^3.
\end{aligned}
$$

(Uniform boundedness of $A_{1,i}, \cdots A_{4,i}$ are obtained from the assumption that $F \in \mathcal{C}_*^3(\mathsf{R}^2)$.)

Noting that $s_i - s_{i-1} = T/n$, from Lemma 1 we have

$$
\begin{aligned}
&\max_{1\le i\le n} |A_{1,i}(\Delta X(s_i))^3| \le \frac{c}{n^{\frac{5}{4}}}, \\
&\max_{1\le i\le n} |A_{2,i}(\Delta X(s_i))^2(s_i - s_{i-1})| \le \frac{c}{n^{\frac{11}{6}}}, \\
&\max_{1\le i\le n} |A_{3,i}(\Delta X(s_i))(s_i - s_{i-1})^2| \le \frac{c}{n^{\frac{29}{12}}}, \\
&\max_{1\le i\le n} |A_{4,i}(s_i - s_{i-1})^3| \le \frac{c}{n^3}
\end{aligned}
$$

almost surely. Thus, we have

$$|U_3^{(n)}| = \left|\sum_{i=1}^{n} R_i\right| \le \frac{c}{n^{\frac{1}{4}}}. \tag{23}$$

It is obvious that

$$U_{1,2}^{(n)} \to \int_0^T F_t(S(t),t)dt \quad (n \to \infty). \tag{24}$$

Next, since $F_{x,t}$ and $F_{t,t}$ are uniformly bounded and (19) holds, there is a bound M such that

$$|F_{t,t}(X(s_{i-1}),s_{i-1})|(s_i - s_{i-1})^2 \le M\left(\frac{T}{n}\right)^2 \le \frac{c}{n^2}$$

$$|F_{x,t}(X(s_{i-1}),s_{i-1})(\Delta X(s_i))(s_i - s_{i-1})| \le M|\Delta X(s_i)|\frac{T}{n} \le \frac{c}{n^{\frac{17}{12}}}$$

almost surely. Hence

$$|U_{2,2}^{(n)}| = \sum_{i=1}^{n} |F_{t,t}(X(s_{i-1}),s_{i-1})|(s_i - s_{i-1})^2 \le \frac{c}{n}, \tag{25}$$

$$|U_{2,3}^{(n)}| = \left|\sum_{i=1}^{n} F_{x,t}(X(s_{i-1}),s_{i-1})(\Delta X(s_i))(s_i - s_{i-1})\right| \le \frac{c}{n^{\frac{5}{12}}} \tag{26}$$

hold almost surely.

Now, we decompose $U_{2,1}^{(n)}$ as

$$\begin{aligned}
U_{2,1}^{(n)} &= \sum_{i=1}^{n} F_{x,x}(X(s_{i-1}),s_{i-1})(\Delta X(s_i))^2 \\
&= \left(\frac{T}{n}\right)^2 \sum_{i=1}^{n} F_{x,x}(X(s_{i-1}),s_{i-1})h^2(X(s_{i-1}),s_{i-1}) \\
&\quad +2\left(\frac{T}{n}\right)^{\frac{3}{2}} \sum_{i=1}^{n} F_{x,x}(X(s_{i-1}),s_{i-1})h(X(s_{i-1}),s_{i-1})v(X(s_{i-1}),s_{i-1})\xi_i \\
&\quad +\frac{T}{n}\sum_{i=1}^{n} F_{x,x}(X(s_{i-1}),s_{i-1})v^2(X(s_{i-1}),s_{i-1})\xi_i^2 \\
&= U_{2,1,1}^{(n)} + U_{2,1,2}^{(n)} + U_{2,1,3}^{(n)}, \quad \text{(say)}.
\end{aligned}$$

Since $F_{x,x}$, h and v are uniformly bounded, from Lemma 1 we obtain that

$$|U_{2,1,1}^{(n)}| \le c\sum_{i=1}^{n}\left(\frac{T}{n}\right)^2 \le \frac{c}{n},$$

$$|U_{2,1,2}^{(n)}| \le c\sum_{i=1}^{n}\frac{T^{\frac{3}{2}}}{n}\frac{1}{n^{\frac{5}{12}}} \le \frac{c}{n^{\frac{5}{12}}}$$

almost surely.

Next, we show that

$$U_{2,1,3}^{(n)} \to \int_0^T F_{x,x}(X(t),t)v^2(X(t),t)dt \quad a.s. \tag{27}$$

and consequently

$$\begin{aligned} U_{2,1}^{(n)} &= \frac{1}{2}\sum_{i=1}^{n} F_{x,x}(X(s_{i-1}),s_{i-1})(\Delta X(s_i))^2 \\ &\to \frac{1}{2}\int_0^T F_{x,x}(X(t),t)v^2(X(t),t)dt \quad a.s. \end{aligned} \tag{28}$$

To do so, let

$$l_2 = \left[n^{\frac{3}{16}}\right] \quad \text{and} \quad m_2 = \left[\frac{n}{l_2}\right]$$

$$t_{k,j}^{(2)} = \frac{(k-1)T}{m_2} + \frac{Tj}{m_2 l_2} \quad (1 \le j \le l_2, 1 \le k \le m_2), \quad t_{0,0}^{(2)} = 0$$

and for brevity put

$$G(x,t) = F_{xx}(x,t)v^2(x,t).$$

Then, it is obvious that $G \in \mathcal{C}_*^1(\mathsf{R}^2)$. Let

$$M_2 = \sup_{(x,t)\in \mathsf{R}\times[0,\infty)} \max\{|G_x(x,t)|, |G_t(x,t)|\}.$$

Now, we can write

$$\begin{aligned} U_{2,1,3}^{(n)} &= \frac{T}{n}\sum_{i=1}^{n} G_x(X(s_{i-1}),s_{i-1})\xi_i^2 \\ &= \frac{T}{n}\sum_{k=1}^{m_2}\sum_{j=1}^{l_2} G(X(t_{k,j}^{(2)}),t_{k,j}^{(2)})\xi_{(k-1)l_2+j}^2 \\ &\quad + \frac{T}{n}\sum_{i=m_2 l_2+1}^{n} G(X(s_{i-1}),s_{i-1})\xi_i^2. \end{aligned}$$

Since $G \in \mathcal{C}_*^1(\mathsf{R}^2)$, by Lemma 1 and the definitions of l_2 and m_2, we have

$$\begin{aligned} &\frac{T}{n}\left|\sum_{i=m_2 l_2+1}^{n} G(X(s_{i-1}),s_{i-1})\xi_i^2\right| \\ &\le c\sum_{i=m_2 l_2+1}^{n} \frac{\xi_i^2}{n} \le c l_2 (n^{-\frac{5}{12}})^2 \le n^{-\frac{31}{48}} \quad a.s. \end{aligned} \tag{29}$$

By the Taylor theorem and Lemma 1 we see that for all $1 \le r \le l_2$

$$
\begin{aligned}
(30)\qquad & |G(X(t_{k,r}^{(2)}), t_{k,r}^{(2)}) - G(X(t_{k,0}^{(2)}), t_{k,0}^{(2)})| \\
&= \Big|\sum_{j=1}^{r} (G(X(t_{k,j}^{(2)}), t_{k,j}^{(2)}) - G(X(t_{k,j-1}^{(2)}), t_{k,j-1}^{(2)}))\Big| \\
&\le c \sum_{j=1}^{r} M_2\Big(|\Delta X(t_{k,j}^{(2)})| + \frac{T}{n}\Big) \le c \sum_{j=1}^{r} (n^{-\frac{5}{12}} + n^{-1}) \\
&\le c l_2 n^{-\frac{5}{12}} \le c n^{-\frac{11}{48}} \quad a.s.
\end{aligned}
$$

and so from Lemma 1 we obtain

$$
\begin{aligned}
(31)\qquad & \Big|\frac{T}{n}\sum_{k=1}^{m_2}\sum_{j=1}^{l_2} G(X(t_{k,j}^{(2)}), t_{k,j}^{(2)})\xi^2_{(k-1)l_2+j} \\
&\quad - \frac{T}{n}\sum_{k=1}^{m_2} G(X(t_{k,0}^{(2)}), t_{k,0}^{(2)}) \sum_{j=1}^{l_2} \xi^2_{(k-1)l_2+j}\Big| \\
&\le \frac{T}{n}\sum_{k=1}^{m_2}\sum_{j=1}^{l_2} \big|G(X(t_{k,j}^{(2)}), t_{k,j}^{(2)}) - G(X(t_{k,0}^{(2)}), t_{k,0}^{(2)})\big|\xi^2_{(k-1)l_2+j} \\
&\le c n^{-\frac{11}{48}} \frac{T}{m_2}\sum_{k=1}^{m_2}\sum_{j=1}^{l_2} \frac{\xi^2_{(k-1)l_2+j}}{n} \le c n^{-\frac{11}{48}} n (n^{-\frac{5}{12}})^2 \le c n^{-\frac{13}{48}} \quad a.s.
\end{aligned}
$$

Further, by the assumption $E\xi_1^2 = 1$ and Theorem B

$$
\begin{aligned}
& \Big|\frac{1}{l_2}\sum_{j=1}^{l_2} \xi^2_{(k-1)l_2+j} - 1\Big| \\
&\le \frac{1}{\sqrt{l_2}}\Big|\frac{1}{\sqrt{l_2}}\sum_{j=1}^{l_2} (\xi^2_{(k-1)l_2+j} - E\xi^2_{(k-1)l_2+j})\Big| \\
&\le c\sqrt{\frac{\log\log l_2}{l_2}} \le c n^{-\frac{1}{11}} \quad a.s.
\end{aligned}
$$

which implies

$$
\begin{aligned}
(32)\qquad & \Big|\frac{T}{n}\sum_{k=1}^{m_2} G(X(t_{k,0}^{(2)}), t_{k,0}^{(2)}) \sum_{j=1}^{l_2} (\xi^2_{(k-1)l_2+j} - 1)\Big| \\
&\le c\frac{1}{m_2}\sum_{k=1}^{m_2}\Big|\frac{1}{l_2}\sum_{j=1}^{l_2} \xi^2_{(k-1)l_2+j} - 1\Big| \le c n^{-\frac{1}{11}} \quad a.s.
\end{aligned}
$$

Now, noting that $l_2 m_2 \sim n$, from (31) and (32) we have

$$
\begin{aligned}
(33) \qquad U_{2,1,3} &= \Big(\frac{T}{n}\sum_{k=1}^{m_2}\sum_{j=1}^{l_2} G(X(s_{i-1}), s_{i-1})\xi^2_{(k-1)l_2+j} \\
&\quad -\frac{T}{n}\sum_{k=1}^{m_2} G(X(t^{(2)}_{k,0}), t^{(2)}_{k,0})\sum_{j=1}^{l_2}\xi^2_{(k-1)l_2+j}\Big) \\
&\quad +\frac{T}{n}\sum_{k=1}^{m_2} G(X(t^{(2)}_{k,0}), t^{(2)}_{k,0})\sum_{j=1}^{l_2}(\xi^2_{(k-1)l_2+j}-1) \\
&\quad +\frac{T}{n} l_2 \sum_{k=1}^{m_2} G(X(t^{(2)}_{k,0}), t^{(2)}_{k,0}) \\
&= \frac{T}{m_2}\sum_{k=1}^{m_2} G(X(t^{(2)}_{k,0}), t^{(2)}_{k,0}) + O(n^{-\epsilon}) \quad a.s.
\end{aligned}
$$

for some $\epsilon > 0$ and (27) follows.

Finally, we consider $U^{(n)}_{1,1}$. We write $U^{(n)}_{1,1}$ as

$$
\begin{aligned}
U^{(n)}_{1,1} &= \sum_{i=1}^{n} F_x(X(s_{i-1}), s_{i-1})\Delta X(s_i) \\
&= \frac{T}{n}\sum_{i=1}^{n} F_x(X(s_{i-1}), s_{i-1}) h(X(s_{i-1}), s_{i-1}) \\
&\quad +\sqrt{\frac{T}{n}}\sum_{i=1}^{n} F_x(X(s_{i-1}), s_{i-1}) v(X(s_{i-1}), s_{i-1})\xi_i \\
&\quad +\frac{T}{n}\sum_{i=1}^{n} F_t(X(s_{i-1}), s_{i-1}) \\
&= U^{(n)}_{1,1,1} + U^{(n)}_{1,1,2} + U^{(n)}_{1,1,3} \quad \text{(say)}.
\end{aligned}
$$

It is obvious that

$$
(34) \qquad U^{(n)}_{1,1,1} \to \int_0^T F_x(X(s), t) h(X(s), s) ds,
$$

and

$$
(35) \qquad U^{(n)}_{1,1,3} \to \int_0^T F_t(X(s), t) ds,
$$

hold almost surely.

For brevity, we put $J(x,t) = F_x(x,t)v(x,t)$ and $H(t) = J(X(t),t)$. It is obvious that $J \in \mathcal{C}_*^2(\mathsf{R}^2)$. Hence, noting $s_i - s_{i-1} = T/n$, by the Taylor theorem and Lemma 1 we have

$$
\begin{aligned}
(36) \qquad & |H(s_i) - H(s_{i-1})| \\
&= |H(X(s_i), s_i) - H(X(s_{i-1}), s_{i-1})| \\
&= c\left\{ |\Delta X(s_i)| + \frac{T}{n} + |\Delta X(s_i)|^2 + \frac{T}{n}|\Delta X(s_i)| + \left(\frac{T}{n}\right)^2 \right\} \\
&\le cn^{-\frac{5}{12}} \quad (1 \le i \le n) \quad a.s.
\end{aligned}
$$

To consider $U_{1,1,2}^{(n)}$ let $l_1 = [n^{13/16}]$ and $m_1 = [n/l_1]$ and

$$
t_{k,j}^{(1)} = \frac{(k-1)T}{m_1} + \frac{jT}{m_1 l_1} \quad (1 \le j \le l_1, 1 \le k \le m_1), \quad t_{0,0}^{(1)} = 0.
$$

Let $p = [n^{3/16}]$.

We write

$$
\begin{aligned}
U_{1,1,2}^{(n)} &= \sqrt{\frac{T}{n}} \sum_{i=1}^{m_1 l_1} H(s_i)\xi_i + \frac{T}{n} \sum_{i=m_1 l_1 + 1}^{n} H(s_i)\xi_i \\
&= \sqrt{\frac{T}{n}} \sum_{k=1}^{m_1} \sum_{j=1}^{l_1} H(t_{k,j})\xi_{(k-1)l_1 + j} + \sqrt{\frac{T}{n}} \sum_{i=m_1 l_1 + 1}^{n} H(s_i)\xi_i \\
&= V_1^{(n)} + V_2^{(n)} \quad \text{(say)}
\end{aligned}
$$

We decompose further $V_2^{(n)}$ as

$$
\begin{aligned}
V_2^{(n)} &= \sqrt{\frac{T}{n}} \sum_{i=m_1 l_1 + 1}^{n} \sum_{r=1}^{p} (H(s_i) - H(s_{i-r}))\xi_i \\
&\quad + \sqrt{\frac{T}{n}} \sum_{i=m_1 l_1 + 1}^{n} H(s_{i-p})\xi_i \\
&= V_{2,1}^{(n)} + V_{2,2}^{(n)} \quad \text{(say)}.
\end{aligned}
$$

Firstly, we consider $V_{2,1}^{(n)}$. Let $m_1 l_1 + 1 \le i \le n$ be arbitrarily fixed. Since

$H \in \mathcal{C}^2_*(\mathsf{R}^2)$ and $E\xi_i = 0$, by the method of the proof of Lemma 2 and (36)

$$\begin{aligned}
&E\Big|\sum_{r=1}^{p}(H(s_i) - H(s_{i-r}))\xi_i\Big| \\
&= E\Big\{E\Big\{\Big|\sum_{r=1}^{p}(H(s_i) - H(s_{i-r}))\xi_i\Big|\Big|\mathcal{M}^i_{-\infty}\Big\}\Big\} \\
&\le \sum_{r=1}^{p} E\{E\{|H(s_i) - H(s_{i-r})||\xi_i||\mathcal{M}^i_{-\infty}\}\} \\
&\le c\sum_{r=1}^{p} n^{-\frac{5}{12}} E\{E\{|\xi_i||\mathcal{M}^i_{-\infty}\}\} \le cpn^{-\frac{5}{12}} E\{E\{|\xi_i||\mathcal{M}^{i-p}_{-\infty}\}\} \\
&\le cpn^{-\frac{5}{12}}\alpha^{\frac{3}{8}}(p) \le cn^{-\frac{73}{24}}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
E|V^{(n)}_{2,1}| &\le \sqrt{\frac{T}{n}} E\Big|\sum_{i=m_1 l_1+1}^{n}\sum_{r=1}^{p}(H(s_i) - H(s_{i-r}))\xi_i\Big| \\
&\le cn^{-\frac{1}{2}} l_1 n^{-\frac{73}{24}} \le cn^{-\frac{131}{48}},
\end{aligned}$$

which, by the Markov inequality, implies

$$P(|V^{(n)}_{2,1}| \ge n^{-1}) \le nE|V^{(n)}_{2,1}| \le cn^{-\frac{83}{48}}.$$

Now, by the Borel-Cantelli lemma

(37) $$V^{(n)}_{2,1} = O(n^{-1}) \quad a.s.$$

Next, we consider $V^{(n)}_{2,2}$. Since $H(s_{i-p})$ is $\mathcal{M}^{i-p}_0$-measurable and $H \in \mathcal{C}^2_*(\mathsf{R} \times [0,\infty))$ and $E\xi_1 = 0$, by the above method we have

$$\begin{aligned}
E|H(s_{i-p})\xi_i| &= E\{E\{|H(s_{i-p})\xi_i||\mathcal{M}^{i-p}_{-\infty}\}\} \\
&\le cE\{E\{|\xi_i||\mathcal{M}^{i-p}_{-\infty}\}\} \le c\alpha^{\frac{3}{8}}(p) \le cn^{-\frac{45}{16}}
\end{aligned}$$

which implies

$$E\Big|\sum_{i=m_1 l_1+1}^{n} H(s_{i-p})\xi_i\Big| \le \sum_{i=m_1 l_1+1}^{n} E|H(s_{i-p})\xi_i| \le cl_1 n^{-\frac{45}{16}} \le cn^{-2}.$$

Hence, by the Markov inequality

$$P(|V^{(n)}_{2,2}| \ge n^{-1}) \le cn^{-\frac{1}{2}} nn^{-2} \le cn^{-\frac{3}{2}}.$$

Thus, from the Borel-Cantteli lemma we obtain

$$V_{2,2}^{(n)} = O\big(n^{-1}\big) \quad a.s. \tag{38}$$

Combining (37) and (38), we have

$$V_{2}^{(n)} = O(n^{-1}) \quad a.s. \tag{39}$$

To consider the limiting behavior of $V_1^{(n)}$, we write $V_1^{(n)}$ as

$$\begin{aligned} V_1^{(n)} &= \sqrt{\frac{T}{n}} \sum_{k=1}^{m_1} \sum_{j=1}^{l_1} H(t_{k,0}) \xi_{(k-1)l_1+j} \\ &\quad + \sqrt{\frac{T}{n}} \sum_{k=1}^{m_1} \sum_{j=1}^{l_1} \sum_{r=2p+1}^{l_1-j} (H(t_{k,j-r}) - H(t_{k,0})) \xi_{(k-1)l_1+j} \\ &\quad + \sqrt{\frac{T}{n}} \sum_{k=1}^{m_1} \sum_{j=1}^{l_1} \sum_{r=1}^{2p} (H(t_{k,j-r}) - H(t_{k,0})) \xi_{(k-1)l_1+j} \\ &= V_{1,1}^{(n)} + V_{1,2}^{(n)} + V_{1,3}^{(n)} \quad \text{(say)}. \end{aligned}$$

To prove that for some $\kappa > 0$

$$V_{1,2}^{(n)} = O(n^{-\kappa}) \quad a.s. \tag{40}$$

we show that for each $1 \le k \le m_1$ and $1 \le j \le l_1$

$$\begin{aligned} &\sum_{r=2p+1}^{l_1-j} (H(t_{k,j-r}) - H(t_{k,0})) \xi_{(k-1)l_1+j} \\ &= \xi_{(k-1)l_1+j} \sum_{r=2p+1}^{l_1-j} (H(t_{k,j-r}) - H(t_{k,0})) = O\big(n^{-\frac{1}{2}-\kappa}\big) \quad a.s. \end{aligned} \tag{41}$$

Since

$$\begin{aligned} &E\{\xi_{(k-1)l_1+j} | \mathcal{M}_{(k-1)l_1+j-p}^{(k-1)l_1+j}\} \in \mathcal{M}_{(k-1)l_1+j-p}^{(k-1)l_1+j} \quad \text{and} \\ &\sum_{r=2p+1}^{l_1-j} (H(t_{k,j-r}) - H(t_{k,0})) \in \mathcal{M}_{(k-2)l_1+j}^{(k-1)l_1+j-2p}, \end{aligned}$$

by Lemma A, (36) and Lemma 2

$$
\begin{aligned}
&E\Big|\Big(\sum_{r=2p+1}^{l_1-j}(H(t_{k,j-r})-H(t_{k,0}))\Big)\xi_{(k-1)l_1+j}\Big| \\
&= E\Big\{E\Big\{\Big|\sum_{r=2p+1}^{l_1-j}(H(t_{k,j-r})-H(t_{k,0}))\Big||\xi_{(k-1)l_1+j}|\Big|\mathcal{M}_{(k-2)l_1+j}^{(k-1)l_1+j}\Big\}\Big\} \\
&= E\Big\{\Big|\sum_{r=2p+1}^{l_1-j}(H(t_{k,j-r})-H(t_{k,0}))\Big|E\{|\xi_{(k-1)l_1+j}||\mathcal{M}_{(k-2)l_1+j}^{(k-1)l_1+j}\}\Big\} \\
&\le E\Big\{\Big|\sum_{r=2p+1}^{l_1-j}(H(t_{k,j-r})-H(t_{k,0}))\Big|E\{|\xi_{(k-1)l_1+j}||\mathcal{M}_{(k-1)l_1+j-p}^{(k-1)l_1+j}\}\Big\} \\
&\le E\Big|\sum_{r=2p+1}^{l_1-j}(H(t_{k,j-r})-H(t_{k,0}))\Big|E|E\{|\xi_{(k-1)l_1+j}||\mathcal{M}_{(k-1)l_1+j-p}^{(k-1)l_1+j}\}| \\
&\quad +c\alpha^{\frac{3}{4}}(p)\Big\|\sum_{r=2p+1}^{l_1-j}(H(t_{k,j-r})-H(t_{k,0}))\Big\|_8\|\xi_1\|_8 \\
&\le cl_1n^{-\frac{5}{12}}\alpha^{\frac{3}{8}}(p)+c\alpha^{\frac{3}{4}}(p)l_1n^{-\frac{5}{12}}\le cn^{-\frac{29}{12}}
\end{aligned}
$$

Hence, by the Markov inequality we have

$$
P\Big(\Big|\xi_{(k-1)l_1+j}\sum_{r=2p+1}^{l_1-j}(H(t_{k,j-r})-H(t_{k,0}))\Big|\ge n^{-\frac{7}{12}}\Big)\le cn^{-\frac{11}{6}}.
$$

and so, by the Borel-Cantelli lemma, (41) with $\kappa=(1/12)$ is obtained and consequently (40) follows.

Next, we show

$$
V_{1,3}^{(n)}=O(n^{-\kappa})\quad a.s. \tag{42}
$$

As before, it suffices to show that

$$
\sum_{r=1}^{j}(H(t_{k,j-r})-H(t_{k,j-r-1}))\xi_{(k-1)l_1+j}=O\big(n^{-\frac{1}{2}-\kappa}\big)\quad a.s. \tag{43}
$$

By (36) and Lemma 2

$$
\begin{aligned}
&E\left\{\left|\sum_{r=1}^{2p}(H(t_{k,j-r})-H(t_{k,j-r-1}))\xi_{(k-1)l_1+j}\right|\right\}\\
&=E\left\{E\left\{\left|\sum_{r=1}^{2p}(H(t_{k,j-r})-H(t_{k,j-r-1}))\xi_{(k-1)l_1+j}\right|\middle|\mathcal{M}_{(k-2)l_1}^{(k-1)l_1+j}\right\}\right\}\\
&\le cpn^{-\frac{5}{12}}E\{E\{|\xi_{(k-1)l_1+j}||\mathcal{M}_{(k-2)l_1}^{(k-1)l_1+j}\}\}\\
&\le cpn^{-\frac{5}{12}}E\{E\{|\xi_{(k-1)l_1+j}||\mathcal{M}_{(k-2)l_1}^{(k-1)l_1+j-2p}\}\}\\
&\le cpn^{-\frac{5}{12}}\alpha^{\frac{3}{8}}(p)\le cn^{-\frac{29}{12}}.
\end{aligned}
$$

and so

$$
P\left(\left|\sum_{r=1}^{2p}(H(t_{k,j-r})-H(t_{k,j-r-1}))\xi_{(k-1)l_1+j}\right|\ge n^{-\frac{3}{4}}\right)\le cn^{-\frac{4}{3}}
$$

which, via the Borel-Cantelli lemma, we have (43) with $\kappa = (1/4)$ and consequently (42).

Finally, we consider the limiting behavior of $V_{1,1}^{(n)}$. Since $m_1l_1/n = 1 + O(n^{-3/16})$ as $n \to \infty$, it suffices to consider the case $n = m_1l_1$: for some $\kappa > 0$

$$
V_{1,1}^{(m_1l_1)}=\frac{\sqrt{T}}{\sqrt{m_1}}\sum_{k=1}^{m_1}H(X(t_{k,0}^{(1)}),t_{k,0}^{(1)})\frac{1}{\sqrt{l_1}}\sum_{j=1}^{l_1}\xi_{(k-1)l_1+j}+O(n^{-\kappa})\quad a.s.
$$

Since by Theorem A

$$
\begin{aligned}
&\left|\frac{1}{\sqrt{l_1}}\sum_{j=1}^{l_1}\xi_{(k-1)l_1+j}-\frac{\rho}{\sqrt{l_1}}\{W(kl_1)-W((k-1)l_1)\}\right|\\
&=\left|\frac{1}{\sqrt{l_1}}\sum_{j=1}^{l_1}\xi_{(k-1)l_1+j}-\rho\{W(k)-W(k-1)\}\right|\\
&\le cl_1^{-\frac{1}{4}}=cn^{-\frac{11}{60}}\quad a.s.,
\end{aligned}
$$

we have

$$
\begin{aligned}
V_{1,1}^{(m_1l_1)}&=\frac{\rho\sqrt{T}}{\sqrt{m_1}}\sum_{k=1}^{m_1}H(X(t_{k,0}^{(1)}),t_{k,0}^{(1)})\{W(k)-W(k-1)\}+O\big(m_1^{\frac{1}{2}}n^{-\frac{11}{60}}\big)\\
&=\rho\sum_{k=1}^{m_1}H(X(t_{k,0}^{(1)}),t_{k,0}^{(1)})\left\{W\left(\frac{kT}{m_1}\right)-W\left(\frac{(k-1)T}{m_1}\right)\right\}\\
&\quad+O\big(n^{-\frac{1}{20}}\big)\quad a.s.
\end{aligned}
$$

and cosequently

$$V_{1,1}^{(n)} \to \rho \int_0^T H(X(s),s)dW(s) \quad a.s. \tag{44}$$

It is obvious that

$$U_{1,2}^{(n)} \to \int_0^T F_t(X(s),s)ds. \tag{45}$$

Hence, from (45), (41), (43) and (46) we have

$$\begin{aligned} &\sum_{i=1}^{n}\Big(F_x(X(s_{i-1}),s_{i-1})\Delta X(s_i) + F_t(X(s_{i-1}),s_{i-1})(s_i - s_{i-1})\Big) \\ &= U_{11}^{(n)} + U_{1,2}^{(n)} = (V_{1,1}^{(n)} + V_{1,2}^{(n)} + V_{1,3}^{(n)}) + U_{1,2}^{(n)} \\ &\to \int_0^T F_x(X(s),t)h(X(s),s)ds \\ &\quad + \rho\int_0^T F_x(X(s),s)v(X(s),s)dW(s) + \int_0^T F_t(X(s),s)ds \end{aligned} \tag{46}$$

almost surely as $n \to \infty$.

Thus, (13) follows from (46), (28), (24), (25) and (22) and the proof is completed. □

Example. As an example we consider the following case. Suppose the time-continuous process $\{X(t) : 0 \le t \le T\}$ satisfies the difference equation (4), i.e.,

$$\Delta X(s_i) = \mu X(X_{i-1})\frac{T}{n} + \sigma X(s_{i-1})\sqrt{\frac{T}{n}}\xi_i \quad (1 \le i \le n)$$

Let $F(x,t) = \log x$. Then,

$$F_t(x,t) = 0, \quad F_x(x,t) = \frac{1}{x}, \quad F_{x,x}(x,t) = -\frac{1}{x^2}$$

and so the solution $\{X^{(n)}(t); 0 \le t \le T\}$ of (4) satisfies

$$\begin{aligned} &F_t(X^{(n)}(t),t) + F_x(X^{(n)}(t),t)(\mu X^{(n)}(t)) \\ &\quad + \frac{1}{2}F_{x,x}(X^{(n)}(t),t)(\sigma X^{(n)}(t))^2 = \mu - \frac{\sigma^2}{2}, \\ &F_x(X^{(n)}(t),t)(\sigma X^{(n)}(t)) = \sigma. \end{aligned}$$

Hence, by Theorem 1 we have

$$\begin{aligned} &\log X(T) - \log X(0) \\ &= \log\int_0^T\Big(\mu - \frac{\sigma^2}{2}\Big)dt + \rho\int_0^T \sigma dW = \Big(\mu - \frac{\sigma^2}{2}\Big)T + \rho\sigma W(T), \end{aligned}$$

which coincides with (6).

As an easy application of Theorem 1, we can prove the following corollary.

Corollary. *Let $T > 0$ be fixed and $\{X(t) : 0 \le t \le T\}$ be a continuous random process. Suppose the conditions in Theorem 1 are fulfilled. If for any positive integer n*

$$\Delta X(s_i) = \alpha X(s_{i-1})\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\xi_i \quad (1 \le i \le n), \tag{47}$$

then the solution of the difference equation (47) with $X^{(n)}(0) = x$, i.e.,

$$X^{(n)}(T) = x + \alpha\sigma e^{\alpha T}\sum_{i=1}^{n}\exp(-\alpha s_{i-1})\sqrt{\frac{T}{n}}\xi_i + e^{\alpha T}\bar{R}_n \tag{48}$$

converges almost surely to

$$X(T) = x + \rho\alpha\sigma\int_0^T e^{\alpha(T-s)}dW(s), \tag{49}$$

as $n \to \infty$ where $\{W(t) : 0 \le t \le T\}$ is a standard Wiener process and $\bar{R}_n$'s are residuals such that

$$\bar{R}_n \to 0 \quad a.s. \quad (n \to \infty).$$

Proof. Let $F(x,t) = e^{-\alpha t}x$. Then, it is obvious that

$$F_x(x,t) = e^{-\alpha t},\ F_t(x,t) = -\alpha e^{-\alpha t}x,\ F_{x,x}(x,t) = 0.$$

Now, we put $Z(t) = e^{-\alpha t}X(t)$. Then, by Theorem 1, we have

$$\Delta Z(s_i) = e^{-\alpha s_{i-1}}\sqrt{\frac{T}{n}}\xi_i + R_i^{(n)} \quad (1 \le i \le n)$$

where the $R_i = R(s_i)$ are residuals such that

$$|R_i^{(n)}| \to o(n^{-1}) \quad a.s. \quad (n \to \infty). \tag{50}$$

Since $X^{(n)}(0) = x$, $Z^{(n)}(0) = x$, and we have

$$Z^{(n)}(T) - e^{-\alpha T}x = \sum_{i=1}^{n} e^{-\alpha s_{i-1}}\sqrt{\frac{T}{n}}\xi_i + \sum_{i=1}^{n} R_i^{(n)}$$

or equivalently

$$X^{(n)}(T) = x + e^{\alpha T}\sum_{i=1}^{n} e^{-\alpha s_{i-1}}\sqrt{\frac{T}{n}}\xi_i + e^{\alpha T}\sum_{i=1}^{n} R_i^{(n)} \tag{51}$$

Hence, (48) is obtained. (49) follows from (51). □

5. Proof of Theorem 2

We use the notations and results in the proof of Theorem 1. Let

$$
\begin{aligned}
(52)\qquad V_i^{(n)} &= \Delta\Bigg(\exp\Big(\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)\Big) f(X(s_i),s_i)\Bigg)\\
&= \exp\Big(\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)\Big)\Delta f(X(s_i),s_i)\\
&\quad +\Delta\Bigg(\exp\Big(\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)\Big)\Bigg) f(X(s_i),s_i)\\
&= V_{1,i}^{(n)} + V_{2,i}^{(n)} \quad \text{(say)}
\end{aligned}
$$

We note first that by the Taylor theorem

$$
\begin{aligned}
(53)\qquad &\Delta\Bigg(\exp\Big(\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)\Big)\Bigg)\\
&= \Bigg(\exp\Big(\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)\Big) - \Bigg(\exp\Big(\frac{T}{n}\sum_{r=1}^{i-1} a(X(s_r),s_r)\Big)\Bigg)\\
&= \frac{T}{n} a(X(s_i),s_i) e^{\frac{T}{n}\sum_{r=1}^{i-1} a(X(s_r),s_r)} + O\big(n^{-2}\big)\\
&= \frac{T}{n} a(X(s_{i-1}),s_{i-1}) e^{\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)} + O\big(n^{-2}\big) \quad a.s.
\end{aligned}
$$

since $a(x,t) \in \mathcal{C}_*^1(\mathsf{R}^2)$.

By (53) and the assumption $f \in \mathcal{C}_*^3(\mathsf{R}^2)$

$$
\begin{aligned}
(54)\qquad V_{2,i}^{(n)} &= \frac{T}{n} e^{\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)} a(X(s_{i-1}),s_{i-1}) f(X(s_{i-1}),s_{i-1})\\
&\quad + O\big(n^{-2}\big) \quad a.s.
\end{aligned}
$$

Furthermore, by (10), we have that for some $\epsilon > 0$

$$
\begin{aligned}
(55)\qquad V_{1,i}^{(n)} &= e^{\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)}\\
&\times\Bigg\{ f_x(X(s_{i-1}),s_{i-1}) h(X(s_{i-1}),s_{i-1}) + f_t(X(s_{i-1}),s_{i-1})\frac{T}{n}\\
&+\frac{1}{2}\Big(f_{x,x}(X(s_{i-1}),s_{i-1}) v^2(X(s_{i-1}),s_{i-1})\Big)\frac{T}{n}\\
&+\frac{1}{2} f_{x,x}(X(s_{i-1}),s_{i-1}) v^2(X(s_{i-1}),s_{i-1})(\xi_i^2 - 1)\frac{T}{n}\\
&+\sqrt{\frac{T}{n}} f_x(X(s_{i-1}),s_{i-1}) v(X(s_{i-1}),s_{i-1})\xi_i\Bigg\} + O\big(n^{-1-\epsilon}\big)
\end{aligned}
$$

holds almost surely. Thus, by (14) we have

$$
\begin{aligned}
(56) \qquad V_i^{(n)} &= V_{1,i}^{(n)} + V_{2,i}^{(n)} \\
&= \frac{T}{n} e^{\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)} a(X(s_{i-1}), s_{i-1}) f(X(s_{i-1}), s_{i-1}) \\
&\quad + e^{\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)} \\
&\quad \times \Big\{ f_x(X(s_{i-1}), s_{i-1}) h(X(s_{i-1}), s_{i-1}) + f_t(X(s_{i-1}), s_{i-1}) \frac{T}{n} \\
&\quad + \frac{1}{2}\Big(f_{x,x}(X(s_{i-1}), s_{i-1}) v^2(X(s_{i-1}), s_{i-1}) \Big) \frac{T}{n} \\
&\quad + \frac{1}{2} f_{x,x}(X(s_{i-1}), s_{i-1}) v^2(X(s_{i-1}), s_{i-1}) (\xi_i^2 - 1) \frac{T}{n} \\
&\quad + \sqrt{\frac{T}{n}} f_x(X(s_{i-1}), s_{i-1}) v(X(s_{i-1}), s_{i-1}) \xi_i \Big\} + O\big(n^{-1-\epsilon}\big) \\
&= e^{\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)} \\
&\quad \times \Big\{ \frac{1}{2} f_{x,x}(X(s_{i-1}), s_{i-1}) v^2(X(s_{i-1}), s_{i-1}) (\xi_i^2 - 1) \frac{T}{n} \\
&\quad + \sqrt{\frac{T}{n}} f_x(X(s_{i-1}), s_{i-1}) v(X(s_{i-1}), s_{i-1}) \xi_i \Big\} + O\big(n^{-1-\epsilon}\big)
\end{aligned}
$$

Summing i from 1 to n on both side of (56) and using Theorem B and (44)

$$
\begin{aligned}
(57) \qquad & e^{\frac{T}{n}\sum_{r=1}^{n} a(X(s_r),s_r)} f(X(T), T) - f(X(0), 0) = \sum_{i=1}^{n} V_i^{(n)} \\
&= \sqrt{\frac{T}{n}} \sum_{i=1}^{n} e^{\frac{T}{n}\sum_{r=1}^{i} a(X(s_r),s_r)} f_x(X(s_{i-1}), s_{i-1}) v(X(s_{i-1}), s_{i-1}) \xi_i \\
&\quad + O\big(n^{-\frac{1}{2}} \log\log n\big) \quad a.s.
\end{aligned}
$$

Thus, letting $n \to \infty$, (15) is obtained. □

Acknowledgements I would like to thank the referees for their careful readings of the manuscript.

References

[1] Berkes, I. Hörmann, S. Schauer, J. (2011). Slip invariance principles for stationary processes. *Ann. Probab.* **39** 2441-2473.

[2] Doukhan, P. (1995) *Mixing* Springer-Verlag, Berlin.

[3] Liu, W. and Lin, Z. (2009). Strong approximation for a class of stationary processes. *Stoch. Process. Appl.* **119** 249-280.

[4] Yoshihara, K. (1992). *Weakly dependent stochastic sequences and their applications, Vol. 1, Summation theory for weakly dependent sequences.* Sanseido, Tokyo.

[5] Yoshihara, K. (2009). Self-normalized U-statistics defined by absolutely regular sequences. *Yokohama Math. J.* **55** 1-19

[6] Yoshihara, K. (2013). Asymptotic behavior of solutions of Black-Scholes type equations based on weakly dependent random variables. (To be published).

[7] Yoshihara, K. and Kanagawa, S. (2013). Approximation to optimal prices when basic data are weakly dependent. (Preprint.)