

# RELATIONS OF SMOOTH Kervaire CLASSES OVER THE MOD 2 STEENROD ALGEBRA II

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**Abstract.** In smooth surgery theory, it is worthwhile to find relations holding among universal characteristic classes of surgery, because those relations give us information on the possible values of surgery obstructions. We present and prove a new series of relations between smooth Kervaire classes.

## 1. Main Result

Let  $M^n$  be a smooth closed manifold. In the surgery theory of differentiable manifolds, a normal map with the target manifold  $M$  is represented by a map  $f : M \rightarrow G/O$ , where  $G/O$  is the fiber of the stable  $J$ -map  $BJ : BSO \rightarrow BSG$ . There exist cohomology classes  $K_{2^i-2} \in H^{2^i-2}(G/O; \mathbb{Z}/2)$  with the property that the Kervaire obstruction  $c(f)$  is equal to

$$\langle V(M)^2 \sum_{i \geq 2} f^* K_{2^i-2}, [M]_2 \rangle,$$

where  $V(M)$  is the total Wu class of  $M$  and  $[M]_2$  is the mod 2 homology fundamental class of  $M$ . The cohomology classes  $K_{2^i-2}$  are called the *smooth Kervaire classes*.

In [3] we proved a series of relations

$$Sq^{2^r} Sq^{2^s} Sq^{2^t} K_{2^r-2} + Sq^{2^s} Sq^{2^t} K_{2^{r+1}-2} = 0,$$

where  $r, s$  and  $t$  satisfy  $r > s > t \geq 0$ . The purpose of this paper is to prove another series of relations:

**MAIN THEOREM.** *Among the smooth Kervaire classes, the following relation holds:*

$$Sq^{2^{r+1}} Sq^{2^r} Sq^{2^s} K_{2^r-2} + (Sq^{2^{r+1}} Sq^{2^s} + Sq^{2^{s+1}+2^s} Sq^{2^{r+1}-2^{s+1}}) K_{2^{r+1}-2} = 0,$$

where  $r$  and  $s$  are integers satisfying  $r > s > 0$ .

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## 2. Preliminaries

All the binomial coefficients in this paper are considered mod 2. In the proof of our main theorem, we shall encounter many binomial coefficients. The following is the most fundamental criterion for determining the modulo 2 value of the binomial coefficient ([8, I.2.6.LEMMA]).

**LEMMA 2.1.** *Let*

$$a = \sum_{i \geq 0} a_i 2^i, \quad b = \sum_{i \geq 0} b_i 2^i$$

*be 2-adic expansions of non-negative integers  $a$  and  $b$ , where  $a_i$  and  $b_i$ 's are 0 or 1. Then the binomial coefficient  $\binom{a}{b}$  is zero if and only if there exists an  $i$  such that  $a_i = 0$  and  $b_i = 1$ .*

For  $a, b$  not satisfying  $a \geq b \geq 0$ , we read the binomial coefficient  $\binom{a}{b}$  as zero by convention. By using Lemma 2.1 we have the following lemmas (see [3]) :

**LEMMA 2.2.** ([3, Lemma 2.6]) *Let  $a \geq 0$ . The binomial coefficient  $\binom{2^a-1+i}{2^i}$  is nonzero if and only if  $i = 2^a - 2^k$  where  $0 \leq k \leq a$ .*

**LEMMA 2.3.** ([3, Lemma 2.10]) *Let  $a \geq b \geq 0$ . The binomial coefficient  $\binom{2^a-2^b+i}{2^i}$  is nonzero if and only if  $i = (2^a - 2^k) + (2^b - 2^l)$  or  $i = 2^a - 2^b$  where  $b+1 \leq k \leq a$  and  $0 \leq l \leq b$ .*

**LEMMA 2.4.** ([3, Lemma 2.11]) *Let  $a \geq b \geq 0$ . The binomial coefficient  $\binom{2^a+2^b-1+i}{2^i}$  is nonzero if and only if  $i = (2^a - 2^k) + (2^b - 2^l)$ ,  $i = 2^a + (2^b - 2^l)$  or  $i = 2^a - 2^b$  where  $b+1 \leq k \leq a$  and  $0 \leq l \leq b$ .*

**LEMMA 2.5.** ([3, Lemma 2.13]) *Let  $a > b \geq 0$ . The binomial coefficient  $\binom{2^a-2^b-1+i}{2^i}$  is nonzero if and only if  $i = (2^a - 2^k) + (2^b - 2^l)$  or  $i = (2^a - 2^m) + 2^b$  where  $b+1 \leq k \leq a$ ,  $0 \leq l \leq b$  and  $b+2 \leq m \leq a$ .*

**LEMMA 2.6.** ([3, Lemma 2.16]) *Let  $a > b > c \geq 0$ . The binomial coefficient  $\binom{2^a+2^b-2^c-1+i}{2^i}$  is nonzero if and only if  $i = (2^a - 2^k) + (2^b - 2^l) + (2^c - 2^m)$ ,  $i = 2^a + (2^b - 2^l) + (2^c - 2^m)$ ,  $i = 2^a - 2^b + (2^c - 2^m)$ ,  $i = 2^a - 2^b + 2^c$ ,  $i = (2^a - 2^k) + (2^b - 2^q) + 2^c$  or  $i = 2^a + (2^b - 2^q) + 2^c$  where  $b+1 \leq k \leq a$ ,  $c+1 \leq l \leq b$ ,  $0 \leq m \leq c$  and  $c+2 \leq q \leq b$ .*

In addition to the above lemmas, we need the following two lemmas to prove the main theorem.

**LEMMA 2.7.** *Let  $a > b > c \geq 0$ . The binomial coefficient  $\binom{2^a-2^b+2^c+i}{2^{i+1}}$  is*

nonzero if and only if  $i = (2^a - 2^k) + (2^b - 2^l) + 2^c - 1$  or  $i = 2^a - 2^b + 2^c - 1$  where  $b + 1 \leq k \leq a$  and  $c + 1 \leq l \leq b$ .

*Proof.* We use induction on  $c$ . When  $c = 0$ ,  $\binom{2^a - 2^b + 1 + i}{2i + 1}$  is zero if  $i$  is odd. Hence without loss of generality, we may assume  $i = 2j$ . By Lemma 2.3,  $\binom{2^a - 2^b + 1 + 2j}{4j + 1} = \binom{2^{a-1} - 2^{b-1} + j}{2j}$  is nonzero if and only if

$$j = \begin{cases} (2^{a-1} - 2^\alpha) + (2^{b-1} - 2^\beta) \\ 2^{a-1} - 2^{b-1} \end{cases}$$

where  $b \leq \alpha \leq a - 1$  and  $0 \leq \beta \leq b - 1$ . That is

$$i = \begin{cases} (2^a - 2^{\alpha+1}) + (2^b - 2^{\beta+1}) \\ 2^a - 2^b \end{cases}$$

where  $b + 1 \leq \alpha + 1 \leq a$  and  $1 \leq \beta + 1 \leq b$ .

When  $c \geq 1$ , it is enough to consider the case  $i = 2j + 1$ . Then on the inductive assumption, we see that  $\binom{2^a - 2^b + 2^c + i}{2i + 1} = \binom{2^a - 2^b + 2^c + 2j + 1}{4j + 3} = \binom{2^{a-1} - 2^{b-1} + 2^{c-1} + j}{2j + 1}$  is nonzero if and only if

$$j = \begin{cases} (2^{a-1} - 2^\alpha) + (2^{b-1} - 2^\beta) + (2^{c-1} - 1) \\ 2^{a-1} - 2^{b-1} + 2^{c-1} - 1 \end{cases}$$

where  $b \leq \alpha \leq a - 1$  and  $c \leq \beta \leq b - 1$ . That is

$$i = \begin{cases} (2^a - 2^{\alpha+1}) + (2^b - 2^{\beta+1}) + 2^c - 1 \\ 2^a - 2^b + 2^c - 1 \end{cases}$$

where  $b + 1 \leq \alpha + 1 \leq a$  and  $c + 1 \leq \beta + 1 \leq b$ . □

**LEMMA 2.8.** *Let  $a > b > c \geq 0$ . The binomial coefficient  $\binom{2^a - 2^b + 2^c - 1 + i}{2i}$  is nonzero if and only if  $i = (2^a - 2^k) + (2^b - 2^l) + (2^c - 2^m)$ ,  $i = (2^a - 2^k) + 2^b - 2^c$  or  $i = 2^a - 2^b + 2^c - 2^m$  where  $b + 1 \leq k \leq a$ ,  $c + 1 \leq l \leq b$  and  $0 \leq m \leq c$ .*

*Proof.* We use induction on  $c$ . When  $c = 0$ , by Lemma 2.3,  $\binom{2^a - 2^b + i}{2i}$  is nonzero if and only if

$$i = \begin{cases} (2^a - 2^k) + (2^b - 2^l) \\ 2^a - 2^b \end{cases}$$

where  $b + 1 \leq k \leq a$  and  $0 \leq l \leq b$ .

When  $c \geq 1$  and  $i = 2j$ , on the inductive assumption we see that  $\binom{2^a - 2^b + 2^{c-1} + i}{2i} = \binom{2^a - 2^b + 2^{c-1} + 2j}{4j} = \binom{2^{a-1} - 2^{b-1} + 2^{c-1} - 1 + j}{2j}$  is nonzero if and only if

$$j = \begin{cases} (2^{a-1} - 2^\alpha) + (2^{b-1} - 2^\beta) + (2^{c-1} - 2^\gamma) \\ 2^{a-1} - 2^\alpha + 2^{b-1} - 2^{c-1} \\ 2^{a-1} - 2^{b-1} + (2^{c-1} - 2^\gamma) \end{cases}$$

where  $b \leq \alpha \leq a-1$ ,  $c \leq \beta \leq b-1$  and  $0 \leq \gamma \leq c-1$ . That is

$$i = \begin{cases} (2^a - 2^{\alpha+1}) + (2^b - 2^{\beta+1}) + 2^c - 2^{\gamma+1} \\ 2^a - 2^{\alpha+1} + 2^b - 2^c \\ 2^a - 2^b + 2^c - 2^{\gamma+1} \end{cases}$$

where  $b+1 \leq \alpha+1 \leq a$ ,  $c+1 \leq \beta+1 \leq b$  and  $1 \leq \gamma+1 \leq c$ .

When  $c \geq 1$  and  $i = 2j+1$ , by Lemma 2.7,  $\binom{2^a - 2^b + 2^{c-1} + i}{2i} = \binom{2^a - 2^b + 2^c + 2j}{4j+2} = \binom{2^{a-1} - 2^{b-1} + 2^{c-1} + j}{2j+1}$  is nonzero if and only if

$$j = \begin{cases} (2^{a-1} - 2^\alpha) + (2^{b-1} - 2^\beta) + 2^{c-1} - 1 \\ 2^{a-1} - 2^{b-1} + 2^{c-1} - 1 \end{cases}$$

where  $b \leq \alpha \leq a-1$  and  $c \leq \beta \leq b-1$ . That is

$$i = 2j+1 = \begin{cases} (2^a - 2^{\alpha+1}) + (2^b - 2^{\beta+1}) + 2^c - 1 \\ 2^a - 2^b + 2^c - 1 \end{cases}$$

where  $b+1 \leq \alpha+1 \leq a$  and  $c+1 \leq \beta+1 \leq b$ . □

Before we proceed to the details, we collect here notations and known facts which will be used in the proofs. We note that in this paper all homologies and cohomologies are in coefficients mod 2 and will be omitted from the notation.

The mod 2 homology and cohomology groups of the classifying space of surgery  $G/O$  were studied by Milgram ([6]) and others ([1], [4]). For a brief survey, see [5, chapter 6]. We do not know any explicit description of generators of the cohomology group  $H^*(G/O)$ . However, as to its dual  $H_*(G/O)$ , we know that it is a polynomial algebra and its generators are given as follows, using the Dyer-Lashof operations on  $G/O$ .

Let  $I = (i_1, i_2, \dots, i_n)$  be a finite sequence of non-negative integers. We shall write  $Q^I$  for the composite of mod 2 Dyer-Lashof homology operations  $Q^{i_1} Q^{i_2} \dots Q^{i_n}$ . We say that  $I$  or  $Q^I$  is allowable if  $i_j \leq 2i_{j+1}$  holds for all  $j$ ,  $1 \leq j \leq n-1$ . Define its length  $\ell(I) = n$  and its excess  $e(I)$  by

$$e(I) = \sum_{j=1}^{n-1} (i_j - 2i_{j+1}) + i_n = i_1 - i_2 - \dots - i_n.$$

The Pontrjagin ring of  $SG$  is known as follows ([6, Theorem C], [1, Chapter 6], [4, Theorem 4.10]).

**THEOREM.** *[Madsen-Milgram]*

$$H_*(SG) = E\{Q^i[1] * [-1] \mid i \geq 1\} \otimes P\{Q^i Q^i[1] * [-3] \mid i \geq 1\} \\ \otimes P\{Q^I[1] * [1 - 2^n] \mid I : \text{allowable}, \ell(I) = n \geq 2, e(I) \geq 1, i_n \geq 1\}.$$

Here the Dyer-Lashof operations  $Q^i$  are based on the infinite loop structure of  $\Omega^\infty S^\infty$  and  $H_*(SG)$  is considered as a subalgebra of  $H_*(\Omega^\infty S^\infty)$ .

The natural map  $SG \longrightarrow G/O$  in the sequence of fibrations

$$SO \longrightarrow SG \longrightarrow G/O \longrightarrow BSO \longrightarrow BSG$$

allows us to identify  $H_*(G/O)$  with the subalgebra of  $H_*(SG)$  :

$$H_*(G/O) = P\{Q^i Q^i[1] * [-3] \mid i \geq 1\} \\ \otimes P\{Q^I[1] * [1 - 2^n] \mid I : \text{allowable}, \ell(I) = n \geq 2, e(I) \geq 1, i_n \geq 1\}.$$

As to the homology operations, we have the Adem relation

$$Q^a Q^b = \sum_i \binom{i-b-1}{2i-a} Q^{a+b-i} Q^i \quad \text{for } a > 2b$$

and the Nishida relation ([7, Main Theorem])

$$Sq_*^a Q^b = \sum_i \binom{b-a}{a-2i} Q^{b-a+i} Sq_*^i,$$

where  $Sq_*^i$  denotes the dual of the Steenrod squaring operation  $Sq^i$ .

In what follows, given a sequence  $I = (i_1, i_2, \dots, i_n)$  (not necessarily allowable), we shall also write  $u(i_1, i_2, \dots, i_n)$  to represent the element  $Q^I[1] * [1 - 2^n]$  either in  $H_*(SG)$  or in  $H_*(G/O)$ .

As to the characterization of the smooth Kervaire classes, we have the following theorem which is a slight generalization of [1, Corollary 3.6] and [2, Proposition 2.8].

**LEMMA 2.9.** *For  $I = (i_1, i_2, \dots, i_n)$ , the Kronecker pairing*

$$\langle K_{2^{q+1}-2}, u(i_1, i_2, \dots, i_n) \rangle$$

*is nonzero if and only if  $n = 2$ ,  $i_1 + i_2 = 2^{q+1} - 2$  and  $i_2 = 2^m - 1$  for some  $m$ ,  $0 < m \leq q$ .*

*Proof.* Let  $I = (i_1, i_2, \dots, i_n)$  with  $i_n > 0$  and consider  $u_I = u(i_1, i_2, \dots, i_n)$ . When  $I$  is allowable, we know that the pairing  $\langle K_{2^{q+1}-2}, u_I \rangle$  is nonzero if and only if  $n = 2$  and  $i_1 = i_2 = 2^q - 1$ , by [1], [4] or [2, Proposition 2.8]. Suppose that  $I$  is not allowable. Then we can use the Adem relation and express  $u_I$  as a sum of allowable terms  $u_J$ . Since the Adem relation preserves the length of the indices,  $\langle K_{2^{q+1}-2}, u_I \rangle$  is zero if  $\ell(I) \neq 2$ . Let  $I = (i_1, i_2)$  be non-allowable. In this case, by the Adem relation we have

$$u(i_1, i_2) = \sum_j \binom{j - i_2 - 1}{2j - i_1} u(2^{q+1} - 2 - j, j).$$

The pairing  $\langle K_{2^{q+1}-2}, u(i_1, i_2) \rangle$  is nonzero if and only if the binomial coefficient  $\binom{j - i_2 - 1}{2j - i_1}$  is nonzero for  $j = 2^q - 1$ . Then we have  $\binom{j - i_2 - 1}{2j - i_1} = \binom{2^q - i_2 - 2}{i_2} = \binom{2^q - i_2 - 2}{2^q - 2i_2 - 2} = \binom{2^{q-1} - 1 + (2^{q-1} - i_2 - 1)}{2(2^{q-1} - i_2 - 1)}$ . By Lemma 2.2, this is nonzero if and only if  $2^{q-1} - i_2 - 1 = 2^{q-1} - 2^m$  for some  $m \leq q - 1$ , i.e.  $i_2 = 2^m - 1$ . This completes the proof.  $\square$

The elements of the form  $u(2^{q+1} - 2^m - 1, 2^m - 1) \in H^*(G/O)$  that appeared in the above proof will be referred to as Kervaire duals in the subsequent part of this paper.

### 3. Proof of the Main Theorem

The relation claimed in the main theorem

$$Sq^{2^{r+1}} Sq^{2^r} Sq^{2^s} K_{2^r-2} + (Sq^{2^{r+1}} Sq^{2^s} + Sq^{2^{s+1}+2^s} Sq^{2^{r+1}-2^{s+1}}) K_{2^{r+1}-2} = 0$$

is equivalent to the statement that

$$(3.1) \quad \begin{aligned} &\langle Sq^{2^{r+1}} Sq^{2^r} Sq^{2^s} K_{2^r-2}, u \rangle \\ &\quad + \langle Sq^{2^{r+1}} Sq^{2^s} K_{2^{r+1}-2}, u \rangle \\ &\quad + \langle Sq^{2^{s+1}+2^s} Sq^{2^{r+1}-2^{s+1}} K_{2^{r+1}-2}, u \rangle = 0 \end{aligned}$$

is true for all  $u \in H_*(G/O)$ . An element  $x \in H^*(G/O)$  is primitive, by definition, if it satisfies  $\mu^*(x) = x \otimes 1 + 1 \otimes x$ , where  $\mu : G/O \times G/O \rightarrow G/O$  is the H-space multiplication. The Kronecker pairing of a primitive class with a decomposable homology class always vanishes. It is known that the Kervaire classes  $K_{2^{i+1}-2}$  are primitive ([1, Theorem 3.3]). Since the Kervaire classes  $K_{2^{i+1}-2}$  are primitive and the Steenrod squaring operations  $Sq^j$  map primitive elements to primitive elements, the relation (3.1) holds when  $u$  is a decomposable element of  $H_*(G/O)$ .

Hence we may only consider the case where  $u$  is a polynomial generator having the form  $u = u(i_1, i_2, \dots, i_n) = Q^{i_1} Q^{i_2} \dots Q^{i_n} [1] * [1 - 2^n]$ . To prove (3.1) we have to show that

$$\begin{aligned} & \langle K_{2^r-2}, Sq_*^{2^s} Sq_*^{2^r} Sq_*^{2^{r+1}} u(i_1, i_2, \dots, i_n) \rangle \\ & + \langle K_{2^{r+1}-2}, Sq_*^{2^s} Sq_*^{2^{r+1}} u(i_1, i_2, \dots, i_n) \rangle \\ & + \langle K_{2^{r+1}-2}, Sq_*^{2^{r+1}-2^{s+1}} Sq_*^{2^{s+1}+2^s} u(i_1, i_2, \dots, i_n) \rangle = 0. \end{aligned}$$

By Lemma 2.9 and the Nishida relation, each term vanishes unless  $n = 2$  since  $Sq_*^j$  preserves the length of  $u$ . Therefore we may assume that  $u(i_1, i_2, \dots, i_n) = u(a, b)$ .

We put

$$\begin{aligned} T_1 &= \langle K_{2^r-2}, Sq_*^{2^s} Sq_*^{2^r} Sq_*^{2^{r+1}} u(a, b) \rangle, \\ T_2 &= \langle K_{2^{r+1}-2}, Sq_*^{2^s} Sq_*^{2^{r+1}} u(a, b) \rangle, \\ T_3 &= \langle K_{2^{r+1}-2}, Sq_*^{2^{r+1}-2^{s+1}} Sq_*^{2^{s+1}+2^s} u(a, b) \rangle. \end{aligned}$$

To prove the main theorem we have to show that  $T_1 + T_2 + T_3 = 0$ . In order to achieve this, we look for a condition for each  $T_i$ ,  $i = 1, 2, 3$ , to be nonzero.

Let  $r > s > 0$ . We have only to consider  $u(a, b)$ , where  $a$  and  $b$  satisfy  $a + b = 2^{r+2} + 2^s - 2$  and the allowability condition  $0 < b \leq a \leq 2b$ .

First, we calculate  $T_1$ . By repeated use of the Nishida relation, we have

$$Sq_*^{2^s} Sq_*^{2^r} Sq_*^{2^{r+1}} u(a, b) = \sum_{i,j,k} ABCDEF u(c, d)$$

where

$$\begin{aligned} A &= \binom{a - 2^{r+1}}{2^{r+1} - 2i} = \binom{2^{r+1} + 2^s - 2 - b}{2^{r+1} - 2i}, \\ B &= \binom{b - i}{i}, \\ C &= \binom{a - 2^{r+1} - 2^r + i}{2^r - 2j}, \\ D &= \binom{b - i - j}{j}, \\ E &= \binom{a - 2^{r+1} - 2^r - 2^s + i + j}{2^s - 2k}, \\ F &= \binom{b - i - j - k}{k}, \\ c &= a - 2^{r+1} - 2^r - 2^s + i + j + k, \quad d = b - i - j - k. \end{aligned}$$

We count the occurrences of Kervaire duals  $u(c, d)$ ,  $d = 2^m - 1$  ( $1 \leq m \leq r - 1$ ). Then we have

$$(3.2) \quad A = \binom{2^{r+1} + 2^s - 2^m - 1 - (i + j + k)}{2^{r+1} - 2i},$$

$$(3.3) \quad \begin{aligned} B &= \binom{2^m - 1 + j + k}{i}, \\ C &= \binom{2^r + 2^s - 2^m - 1 - (j + k)}{2^r - 2j}, \\ D &= \binom{2^m - 1 + k}{j}, \\ E &= \binom{2^r - 2^m - 1 - k}{2^s - 2k}, \\ F &= \binom{2^m - 1}{k}, \end{aligned}$$

$$(3.4) \quad b = 2^m - 1 + i + j + k, \quad a = 2^{r+2} + 2^s - 2 - b.$$

From the allowability condition  $3b \geq a + b = 2^{r+2} + 2^s - 2$ , we have  $3(2^m - 1 + i + j + k) \geq 2^{r+2} + 2^s - 2$ , that is

$$(3.5) \quad 3(i + j + k) + 2^{m+1} + 2^m \geq 2^{r+2} + 2^s + 1.$$

**LEMMA 3.1.** *The product  $BDF$  is zero if  $m \leq r - 3$ .*

*Proof.* Suppose that there exist  $i, j$  and  $k$  such that  $BDF$  is nonzero. Then we have  $0 \leq k \leq 2^m - 1$ ,  $0 \leq j \leq 2^m - 1 + k \leq 2(2^m - 1)$  and  $0 \leq i \leq 2^m - 1 + k + j \leq 4(2^m - 1)$ . Therefore from (3.5) and  $m \leq r - 3$ , we have

$$3 \cdot 2^r \geq 3 \cdot 7(2^m - 1) + 2^{m+1} + 2^m \geq 2^{r+2} + 2^s + 22.$$

This is clearly a contradiction. □

From this lemma, we have only to consider the case  $m = r - 2$  or  $r - 1$ .

**LEMMA 3.2.** *Let  $m = r - 1$  or  $r - 2$ . Then  $CDEF$  is nonzero if and only if  $s \leq m$ ,  $k = 2^\beta$ ,  $j = 2^m$  or  $s \leq m - 1$ ,  $k = 0$ ,  $j = 2^m - (2^s - 2^\beta)$  where  $\beta$  satisfies  $0 \leq \beta \leq s - 1$ .*

*Proof.* Case  $m = r - 1$ : We have

$$E = \binom{2^{r-1} - 1 - k}{2^s - 2k} = \binom{2^{r-1} - 2^{s-1} - 1 + (2^{s-1} - k)}{2(2^{s-1} - k)}.$$



By Lemma 2.5,  $E$  is nonzero if and only if  $k = 2^\beta$  where  $\beta$  satisfies  $0 \leq \beta \leq s-1$  or  $k = 0$ ,  $s \leq r-2$ .

*Subcase  $k = 2^\beta$  where  $\beta$  satisfies  $0 \leq \beta \leq s-1$ :* We have

$$C = \binom{2^r + 2^s - 2^{r-1} - 1 - (j + 2^\beta)}{2(2^{r-1} - j)},$$

$$D = \binom{2^{r-1} + 2^\beta - 1}{j}.$$

It is easy to see that  $D$  is nonzero if and only if  $0 \leq j \leq 2^\beta - 1$  or  $2^{r-2} \leq j \leq 2^{r-2} + 2^\beta - 1$ . From the form of  $C$  we see that  $2^r + 2^s - 2^{r-1} - 1 - (j + 2^\beta) \geq 2^r - 2j$ . But this inequality does not hold if  $0 \leq j \leq 2^\beta - 1$ . This shows that  $j \geq 2^{r-2} - (2^s - 1) + 2^\beta \geq 2^\beta$ . Therefore we have  $2^{r-1} \leq j \leq 2^{r-2} + 2^\beta - 1$ . By applying Lemmas 2.5 and 2.6 to  $C$  in the cases  $s = r-2$  and  $s \leq r-3$ , respectively, we see that  $C$  is nonzero if and only if  $j = 2^{r-2}$ . This contradicts  $s \leq r-1$ . Thus we have  $j = 2^{r-1} = 2^m$ .

*Subcase  $k = 0$  and  $s \leq r-2$ :* We have

$$C = \binom{2^{r-1} + 2^s - 1 - j}{2^r - 2j} = \binom{2^s - 1 + 2^{r-1} - j}{2(2^{r-1} - j)},$$

$$D = \binom{2^{r-1} - 1}{j}.$$

$D$  is nonzero if and only if  $0 \leq j \leq 2^{r-1} - 1$ . By Lemma 2.2,  $C$  is nonzero if and only if  $j = 2^{r-1} - (2^s - 2^\beta) = 2^m - (2^s - 2^\beta)$ ,  $0 \leq \beta \leq s$ . Then  $\beta \neq s$  because  $j \leq 2^{r-1} - 1$ . This completes the proof in the case  $m = r-1$ .

*Case  $m = r-2$  and  $s = r-1$ :* We have

$$E = \binom{2^{r-1} - 1 + (2^{s-1} - k)}{2(2^{s-1} - k)}.$$

By Lemma 2.2,  $E$  is nonzero if and only if  $k = 2^{r-2}$  or  $k = 0$ . On the other hand since  $F = \binom{2^{r-2}-1}{k} \neq 0$ , we have  $0 \leq k \leq 2^{r-2} - 1$ . Therefore we have  $k = 0$ . Then we have

$$C = \binom{2^r + 2^{r-1} - 2^{r-2} - 1 - j}{2^r - 2j} = \binom{2^r - 2^{r-2} - 1 + (2^{r-1} - j)}{2(2^{r-1} - j)},$$

$$D = \binom{2^{r-2} - 1}{j}.$$

$D$  is nonzero if and only if  $0 \leq j \leq 2^{r-2} - 1$ . By Lemma 2.5,  $C$  is nonzero if and only if  $j = 0$ . But this is contradictory to the allowability condition. Thus we have  $CDEF = 0$  in the case that  $m = r-2$ ,  $s = r-1$  and  $k = j = 0$ . This completes the proof in the case that  $m = r-2$  and  $s = r-1$ .

Case  $m = r - 2$  and  $s \leq r - 2$ : We have

$$E = \binom{2^r - 2^{r-2} - 1 - k}{2^s - 2k} = \binom{2^{r-1} + 2^{r-2} - 2^{s-1} - 1 + (2^{s-1} - k)}{2(2^{s-1} - k)}.$$

By Lemma 2.6,  $E$  is nonzero if and only if  $k = 2^\beta$  where  $\beta$  satisfies  $0 \leq \beta \leq s - 1$  or  $s \leq r - 3$ ,  $k = 0$ .

Subcase  $k = 2^\beta$  where  $\beta$  satisfies  $0 \leq \beta \leq s - 1$ : We have

$$C = \binom{2^r + 2^s - 2^{r-2} - 1 - (j + 2^\beta)}{2(2^{r-1} - j)} = \binom{2^{r-2} + 2^s - 2^\beta - 1 + (2^{r-1} - j)}{2(2^{r-1} - j)}.$$

It is easy to see that  $D$  is nonzero if and only if  $0 \leq j \leq 2^\beta - 1$  or  $j = 2^{r-2}$ . From the form of  $C$  we see that

$$2^r + 2^s - 2^{r-1} - 1 - (j + 2^\beta) \geq 2^r - 2j.$$

But this inequality is impossible if  $0 \leq j \leq 2^\beta - 1$ . Because if  $j \leq 2^\beta - 1$ , then inequalities  $2^r - 2^{r-1} - 2^s + 1 + 2^\beta \leq j \leq 2^\beta - 1$ , i.e.  $2^{r-1} + 2 \leq j \leq 2^s$ , hold. This contradicts  $s \leq r - 2$ . Thus  $j = 2^{r-2} = 2^m$ .

Subcase  $k = 0$ ,  $s \leq r - 3$ : We have

$$C = \binom{2^r + 2^s - 2^{r-2} - 1 - j}{2^r - 2j} = \binom{2^{r-2} + 2^s - 1 + (2^{r-1} - j)}{2(2^{r-1} - j)},$$

$$D = \binom{2^{r-2} - 1}{j}.$$

So we have  $0 \leq j \leq 2^{r-2} - 1$ . By Lemma 2.4, we can easily see that  $C$  is nonzero if and only if  $j = 2^{r-2}$ . This is impossible. This completes the proof in the case that  $m = r - 2$  and  $s \leq r - 2$ . Thus the proof of this lemma is completed.  $\square$

**LEMMA 3.3.** *If  $m \leq r - 2$ , then  $T_1 = 0$  holds.*

*Proof.* Suppose that  $T_1$  is nonzero. Then there exist  $i$ ,  $j$  and  $k$  such that  $ABCDEF$  is nonzero. But from Lemma 3.1,  $T_1$  is zero for  $m \leq r - 3$ . So we may assume that  $m = r - 2$ . From Lemma 3.2, it suffices to consider the following two cases. First we consider the case  $k = 2^\beta$  ( $0 \leq \beta \leq s - 1$ ),  $j = 2^{r-2}$ . We have

$$B = \binom{2^{r-1} + 2^\beta - 1}{i},$$

$$A = \binom{2^{r-1} + 2^s - 2^\beta - 1 - (2^r - i)}{2(2^r - i)}.$$

$B$  is nonzero if and only if  $0 \leq i \leq 2^\beta - 1$  or  $2^{r-1} \leq i \leq 2^{r-1} + 2^\beta - 1$ . By the form of  $A$ , the following inequality holds:

$$(3.6) \quad 2^{r-1} - 2^s + 2^\beta + 1 \leq i.$$

If  $0 \leq i \leq 2^\beta - 1$ , then  $2^{r-1} - 2^s + 2^\beta + 1 \leq i \leq 2^\beta - 1$ , i.e.  $2^{r-1} + 2 \leq i \leq 2^s$ . But this contradicts the condition  $s \leq m = r - 2$  of Lemma 3.1. Thus this case does not occur. If  $2^{r-1} \leq i \leq 2^{r-1} + 2^\beta - 1$ , then from Lemma 2.6 and (3.6) we see that there does not exist such an integer  $i$ .

Next we consider the second case  $k = 0$ ,  $j = 2^{r-2} - (2^s - 2^\beta)$ ,  $s \leq r - 3$ ,  $0 \leq \beta \leq s - 1$ . We have

$$B = \binom{2^{r-1} - 2^s + 2^\beta - 1}{i},$$

$$A = \binom{2^{r-1} + 2^{s+1} - 2^\beta - 1 + (2^r - i)}{2(2^r - i)}.$$

From the form of  $A$ , the following inequalities hold:

$$2^{r-1} - 2^{s+1} + 2^\beta + 1 \leq i \quad \text{and} \quad 0 \leq i \leq 2^r.$$

If  $0 \leq i \leq 2^{r-1}$ , then the allowability condition (3.5) is not satisfied. Therefore  $2^{r-1} + 1 \leq i \leq 2^r$ . From the form of  $B$  in (3.3), we have the inequality  $i \leq 2^{r-1} - 2^s + 2^\beta - 1$ . Hence we have

$$2^{r-1} + 1 \leq 2^{r-1} - 2^s + 2^\beta - 1,$$

i.e.  $2^s + 2 \leq 2^\beta$ . This contradicts  $0 \leq \beta \leq s - 1$ . Therefore there does not exist such an integer  $i$ . This proves that  $T_1 = 0$  if  $m = r - 2$ .  $\square$

**LEMMA 3.4.**  $T_1$  is nonzero if and only if

$$(a, b) = \begin{cases} (2^{r+1} + 2^s - 2^\beta - 1, 2^{r+1} + 2^\beta - 1) \\ (2^{r+1} + 2^{s+1} + 2^s - 2^\beta - 1, 2^{r+1} - 2^{s+1} + 2^\beta - 1) \end{cases}$$

where  $\beta$  satisfies  $0 \leq \beta \leq s - 1$ .

*Proof.* From Lemma 3.3, we may assume that  $m = r - 1$ . By Lemma 3.2, it is enough to consider the following two cases. We consider the first case:  $k = 2^\beta$ ,  $j = 2^{r-1}$  where  $\beta$  satisfies  $0 \leq \beta \leq s - 1$ .

$$B = \binom{2^r + 2^\beta - 1}{i}$$

is nonzero if and only if  $0 \leq i \leq 2^\beta - 1$  or  $2^r \leq i \leq 2^r + 2^\beta - 1$ . If  $0 \leq i \leq 2^\beta - 1$ , then from the allowability condition (3.5), we have

$$2^{r+2} + 2^s + 1 - 2^r - 2^{r-1} \leq 3(i + j + k) \leq 3(2^{s-1} - 1 + 2^{r-1} + 2^{s-1}).$$

From this we have  $2^r + 4 \leq 2^{s+1}$ . But this is impossible because  $s + 1 \leq r$ . Thus we have  $2^r \leq i \leq 2^r + 2^\beta - 1$ . From the form of  $A$  in (3.2), we have  $i \leq 2^r$ , and hence  $i = 2^r$ . Thus  $b = 2^{r+1} + 2^\beta - 1$  and  $a = 2^{r+1} + 2^s - 2^\beta - 1$ .

We consider the second case:  $k = 0$ ,  $j = 2^{r-1} - (2^s - 2^\beta)$ ,  $s \leq r - 2$  where  $\beta$  satisfies  $0 \leq \beta \leq s - 1$ .

$$B = \binom{2^r - 2^s + 2^\beta - 1}{i}$$

is nonzero if and only if  $i = 2^s i_1 + i_2$  where  $0 \leq i_1 \leq 2^{r-s}$  and  $0 \leq i_2 \leq 2^\beta - 1$ . By Lemma 2.5,

$$A = \binom{2^{s+1} - 2^\beta - 1 + (2^r - i)}{2(2^r - i)}$$

is nonzero if and only if  $i = 2^r - 2^s$ . Thus we have  $b = 2^{r+1} - 2^{s+1} + 2^\beta - 1$  from (3.4) and  $a = 2^{r+1} + 2^{s+1} + 2^s - 2^\beta - 1$ .  $\square$

Next, we calculate  $T_2$ . By repeated use of the Nishida relation, we have

$$Sq_*^{2^s} Sq_*^{2^{r+1}} u(a, b) = \sum_{i,j} A' B' C' D' u(c, d)$$

where

$$\begin{aligned} A' &= \binom{2^{r+1} + 2^s - 2 - b}{2^{r+1} - 2i}, \\ B' &= \binom{b - i}{i}, \\ C' &= \binom{2^{r+1} - 2 - (b - i)}{2^s - 2j}, \\ D' &= \binom{b - i - j}{j}, \\ c &= a - 2^{r+1} - 2^s + i + j, \quad d = b - i - j. \end{aligned}$$

We count the occurrences of Kervaire duals  $u(c, d)$  with  $d = 2^m - 1$  ( $1 \leq m \leq r$ ).

Then we have

$$\begin{aligned}
 A' &= \binom{2^{r+1} - 2^m + 2^s - 1 - (i + j)}{2^{r+1} - 2i}, \\
 B' &= \binom{2^m - 1 + j}{i}, \\
 C' &= \binom{2^{r+1} - 2^m - 1 - j}{2^s - 2j}, \\
 D' &= \binom{2^m - 1}{j} \\
 (3.7) \quad b &= 2^m - 1 + i + j, \quad a = 2^{r+2} + 2^s - 2 - b
 \end{aligned}$$

The allowability condition  $2b \geq a$  can be expressed as

$$(3.8) \quad 3(2^m - 1 + i + j) \geq 2^{r+2} + 2^s - 2.$$

**LEMMA 3.5.** *If  $m \leq r - 1$ , then  $T_2 = 0$  holds.*

*Proof.* Suppose that  $A'B'C'D'$  is nonzero for some  $i, j$ . From the form of  $B'$ , we have  $i \leq 2^m - 1 + j$ . Combining this with (3.8), we have

$$(3.9) \quad 6(2^m + j) \geq 2^{r+2} + 2^s + 4.$$

From the form of  $C'$ , we may assume that  $j \leq 2^{s-1}$ . Suppose that  $m \leq r - 1$ , then from (3.9), we have  $3(2^r + 2^s) \geq 2^{r+2} + 2^s + 4$ . It follows that  $2^{s+1} \geq 2^r + 4$ . But this is a contradiction because  $s \leq r - 1$ . This completes the proof.  $\square$

**LEMMA 3.6.**  *$T_2$  is nonzero if and only if*

$$(a, b) = \begin{cases} (2^{r+1} + 2^s - 2^\beta - 1, 2^{r+1} + 2^\beta - 1) \\ (2^{r+1} + 2^{s+1} - 2^\beta - 1, 2^{r+1} - 2^s + 2^\beta - 1) \end{cases}$$

where  $0 \leq \beta \leq s - 1$ .

*Proof.* If  $T_2$  is nonzero, then from Lemma 3.5, we have  $m = r$  and there exist  $i, j$  such that  $A'B'C'D'$  is nonzero. By Lemma 2.5,

$$C' = \binom{2^r - 1 - j}{2^s - 2j} = \binom{2^r - 2^{s-1} - 1 + (2^{s-1} - j)}{2(2^{s-1} - j)}.$$

is nonzero if and only if  $j = 2^\beta$  where  $0 \leq \beta \leq s - 1$  or  $j = 0$ .

Case  $j = 2^\beta$  where  $0 \leq \beta \leq s - 1$ : We have

$$\begin{aligned}
 A' &= \binom{2^r + 2^s - 2^\beta - 1 - i}{2^{r+1} - 2i} = \binom{2^s - 2^\beta - 1 + (2^r - i)}{2(2^r - i)}, \\
 B' &= \binom{2^r + 2^\beta - 1}{i}.
 \end{aligned}$$

$B'$  is nonzero if and only if  $0 \leq i \leq 2^\beta - 1$  or  $i = 2^r$ . But if  $0 \leq i \leq 2^\beta - 1$  then we have  $i \leq 2^{s-1} - 1$  and  $j \leq 2^{s-1}$ . This does not satisfy the allowability condition (3.8). Hence  $i = 2^r$ . Therefore by (3.7), we have  $b = 2^r - 1 + i + j = 2^{r+1} + 2^\beta - 1$  and  $a = 2^{r+2} + 2^s - 2 - b = 2^{r+1} + 2^s - 2^\beta - 1$ .

*Case  $j = 0$ :* We have  $B' = \binom{2^r-1}{i}$  and

$$A' = \binom{2^r + 2^s - 1 - i}{2^{r+1} - 2i} = \binom{2^s - 1 + (2^r - i)}{2(2^r - i)}.$$

$B'$  is nonzero if and only if  $0 \leq i \leq 2^r - 1$ . By Lemma 2.2,  $A'$  is nonzero if and only if  $i = 2^r - (2^s - 2^\beta)$ , ( $0 \leq \beta \leq s - 1$ ). Then  $C'$  is nonzero. Hence  $b = 2^r - 1 + i + j = 2^{r+1} - 2^s + 2^\beta - 1$  and  $a = 2^{r+2} + 2^s - 2 - b = 2^{r+1} + 2^{s+1} - 2^\beta - 1$ .  $\square$

Finally, we deal with  $T_3$ . By repeated use of the Nishida relation, we have

$$Sq_*^{2^{r+1}-2^{s+1}} Sq_*^{2^{s+1}+2^s} u(a, b) = \sum_{i,j} A'' B'' C'' D'' u(c, d)$$

where

$$\begin{aligned} A'' &= \binom{a - 2^{s+1} - 2^s}{2^{s+1} + 2^s - 2i}, \\ B'' &= \binom{b - i}{i}, \\ C'' &= \binom{a - 2^{r+1} - 2^s + i}{2^{r+1} - 2^{s+1} - 2j}, \\ D'' &= \binom{b - i - j}{j}, \\ c &= a - 2^{r+1} - 2^s + i + j, \quad d = b - i - j. \end{aligned}$$

We count the occurrences of  $u(c, d)$  with  $d = 2^m - 1$  ( $1 \leq m \leq r$ ). Then we have

$$\begin{aligned} A'' &= \binom{2^{r+2} - 2^m - 2^{s+1} - 1 - (i + j)}{2^{s+1} + 2^s - 2i}, \\ B'' &= \binom{2^m - 1 + j}{i}, \\ C'' &= \binom{2^{r+1} - 2^m - 1 - j}{2^{r+1} - 2^{s+1} - 2j}, \\ D'' &= \binom{2^m - 1}{j} \end{aligned}$$

$$(3.10) \quad b = 2^m - 1 + i + j, \quad a = 2^{r+2} + 2^s - 2 - b,$$

and the allowability condition

$$(3.11) \quad 3(2^m - 1 + i + j) \geq 2^{r+2} + 2^s - 2.$$

**LEMMA 3.7.** *If  $m \leq r - 1$ , then  $T_3 = 0$  holds.*

*Proof.* Let  $m \leq r - 1$  and suppose that there exist  $i$  and  $j$  such that  $A''B''C''D''$  is nonzero.

Case  $m \leq r - 2$  : From the forms of  $B''$  and  $D''$ , we have  $i \leq 2^m - 1 + j$ ,  $j \leq 2^m - 1$ . From (3.11), we have  $3 \cdot 2^{m+2} \geq 2^{r+2} + 2^s + 10$ , but this is false if  $m \leq r - 2$ .

Case  $m = r - 1$  :  $D''$  is nonzero if and only if  $0 \leq j \leq 2^{r-1} - 1$ . We have

$$C'' = \binom{2^r + 2^{r-1} - 1 - j}{2^{r+1} - 2^{s+1} - 2j} = \binom{2^{r-1} + 2^s - 1 + (2^r - 2^s - j)}{2(2^r - 2^s - j)}.$$

By Lemma 2.4  $C''$  is nonzero if and only if  $j = 2^{r-1} - 2^{s+1} + 2^\beta$  where  $0 \leq \beta \leq s$ . From the form  $A''$ , we have  $0 \leq i \leq 2^s + 2^{s-1}$ . Therefore if  $A''B''$  is nonzero, then  $i + j \leq 2^{r-1} + 2^{s-1}$  and from the allowability condition (3.11), we have  $3(2^{r-1} - 1 + 2^{r-1} + 2^{s-1}) \geq 2^{r+2} + 2^s - 2$ . From this we have  $2^{s-1} \geq 2^r + 1$  and this is a contradiction.  $\square$

**LEMMA 3.8.**  *$T_3$  is nonzero if and only if*

$$(a, b) = \begin{cases} (2^{r+1} + 2^{s+1} - 2^\beta - 1, 2^{r+1} - 2^s + 2^\beta - 1) \\ (2^{r+1} + 2^{s+1} + 2^s - 2^\beta - 1, 2^{r+1} - 2^{s+1} + 2^\beta - 1) \end{cases}$$

where  $0 \leq \beta \leq s - 1$ .

*Proof.* Suppose that  $A''B''C''D''$  is nonzero for some  $i, j$ . Then from Lemma 3.7, we have  $m = r$ .  $D''$  is nonzero if and only if  $0 \leq j \leq 2^r - 1$ . By Lemma 2.6,  $C''$  is nonzero if and only if  $j = 2^r - 2^{s+1} + 2^l$  ( $0 \leq l \leq s$ ). From the form of  $A''$ , we see that  $0 \leq i \leq 2^s + 2^{s-1}$ . Therefore

$$B'' = \binom{2^{r+1} - 2^{s+1} + 2^l - 1}{i}$$

is nonzero if and only if  $0 \leq i \leq 2^l - 1$ .

Case  $l = s$ : We have

$$\begin{aligned} A'' &= \binom{2^{r+1} - 2^{s+1} - 2^{s-1} - 1 - (2^s + 2^{s-1} - i)}{2(2^s + 2^{s-1} - i)} \\ &= \binom{2^{s+1} - 2^{s-1} - 1 - (2^s + 2^{s-1} - i)}{2(2^s + 2^{s-1} - i)}. \end{aligned}$$

By Lemma 2.5,  $A''$  is nonzero if and only if  $i = 2^\beta$ ,  $0 \leq \beta \leq s - 1$ .

Case  $l = s - 1$ : We have

$$A'' = \binom{2^{r+1} - 2^{s+1} - 1 - (2^s + 2^{s-1} - i)}{2(2^s + 2^{s-1} - i)}.$$

By Lemma 2.5,  $A''$  is nonzero if and only if  $i = 0$ .

Case  $l \leq s - 2$ : We have

$$\begin{aligned} A'' &= \binom{2^{r+1} - 2^s - 2^{s-1} - 1 - 2^l - (2^s + 2^{s-1} - i)}{2(2^s + 2^{s-1} - i)} \\ &= \binom{2^{s+1} + 2^{s-1} - 2^l - 1 - (2^s + 2^{s-1} - i)}{2(2^s + 2^{s-1} - i)}. \end{aligned}$$

By Lemma 2.6,  $A''$  is nonzero if and only if  $i = 0$ .

Therefore, if  $l = s$ , then  $j = 2^r - 2^s$  and  $i = 2^\beta$ . So we have  $b = 2^{r+1} - 2^s + 2^\beta - 1$  and  $a = 2^{r+1} + 2^{s+1} - 2^\beta - 1$ . If  $l \leq s - 1$ , then  $i = 0$  and  $j = 2^r - 2^{s+1} + 2^\beta$ . So we have  $b = 2^{r+1} - 2^{s+1} + 2^\beta - 1$  and  $a = 2^{r+1} + 2^{s+1} + 2^s - 2^\beta - 1$ .  $\square$

*Proof of Main Theorem .* After a rather tedious examination, we conclude that each  $T_i$ ,  $i = 1, 2, 3$ , is nonzero exactly in the following cases where  $\beta$  is an integer with  $0 \leq \beta \leq s - 1$ :

For  $T_1$  by Lemma 3.4,

$$\begin{aligned} (a, b) &= (2^{r+1} + 2^s - 2^\beta - 1, 2^{r+1} + 2^\beta - 1) \\ (a, b) &= (2^{r+1} + 2^{s+1} + 2^s - 2^\beta - 1, 2^{r+1} - 2^{s+1} + 2^\beta - 1). \end{aligned}$$

For  $T_2$  by Lemma 3.6,

$$\begin{aligned} (a, b) &= (2^{r+1} + 2^s - 2^\beta - 1, 2^{r+1} + 2^\beta - 1) \\ (a, b) &= (2^{r+1} + 2^{s+1} - 2^\beta - 1, 2^{r+1} - 2^s + 2^\beta - 1). \end{aligned}$$

For  $T_3$  by Lemma 3.8,

$$\begin{aligned} (a, b) &= (2^{r+1} + 2^{s+1} - 2^\beta - 1, 2^{r+1} - 2^s + 2^\beta - 1) \\ (a, b) &= (2^{r+1} + 2^{s+1} + 2^s - 2^\beta - 1, 2^{r+1} - 2^{s+1} + 2^\beta - 1). \end{aligned}$$

This result shows that for each allowable  $u(a, b)$ , the sum  $T_1 + T_2 + T_3$  always vanishes and this completes the proof of the main theorem.  $\square$

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