Yokohama Mathematical
Journal Vol. 59, 2013

WEAK CONVERGENCE OF COMPLEX-VALUED MEASURE FOR BI-PRODUCT PATH SPACE INDUCED BY QUANTUM WALK

By

Norio Konno and Etsuo Segawa

(Received August 30, 2012)

Abstract. In this paper, a complex-valued measure of bi-product path space induced by quantum walk is presented. In particular, we consider three types of conditional return paths in a power set of the bi-product path space (1) $\Lambda \times \Lambda$, (2) $\Lambda \times \Lambda'$ and (3) $\Lambda' \times \Lambda'$, where Λ is the set of all $2n$-length ($n \in \mathbb{N}$) return paths and $\Lambda'(\subseteq \Lambda)$ is the set of all $2n$-length return paths going through nx ($x \in [-1, 1]$) at time n. We obtain asymptotic behaviors of the complex-valued measures for the situations (1)-(3) which imply two kinds of weak convergence theorems (Theorems 1 and 2). One of them suggests a weak limit of weak values.

1. Introduction

Let the set of all the n-truncated paths be $\Omega_n = \{-1, 1\}^n$. Denote the coin space $\mathcal{H}_C$ spanned by choice of direction at each time step, that is, $\boldsymbol{e}_{-1} = {}^T[1, 0]$ and $\boldsymbol{e}_1 = {}^T[0, 1]$. Let quantum coin on $\mathcal{H}_C$ be

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{U}(2)$$

with $abcd \neq 0$, where $\mathrm{U}(2)$ is the set of two-dimensional unitary matrices. Define weight of passage as $W : \Omega_n \to M_2(\mathbb{C})$ such that for $\xi = (\xi_n, \ldots, \xi_1) \in \Omega_n$,

$$W(\xi) = P_{\xi_n} \cdots P_{\xi_1} \tag{1.1}$$

with $P_j = \Pi_j U$, where Π_j is projection onto $\boldsymbol{e}_j$, that is,

$$P_{-1} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \text{ and } P_1 = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}.$$

Here $M_2(\mathbb{C})$ is the set of all the complex-valued 2×2 matrices. In this paper, we consider bi-product n-truncated path space $\Omega_n^2 = \Omega_n \times \Omega_n$. The algebra of

2010 Mathematics Subject Classification: 81Q99
Key words and phrases: quantum walk, bi-product path space, quantum measure

subsets of Ω_n^2 is denoted by $\mathcal{F}_n = 2^{\Omega_n^2}$. For fixed $\boldsymbol{\phi} \in \mathcal{H}_C$ with $||\boldsymbol{\phi}|| = 1$ called initial coin state, we define $\varphi_{\boldsymbol{\phi},n} : \mathcal{F}_n \to \mathbb{C}$ by for any $A \in \mathcal{F}_n$,

$$\varphi_{\boldsymbol{\phi},n}(A) = \left\langle \boldsymbol{\phi}, \sum_{(\xi,\eta)\in A} W(\xi)^\dagger \cdot W(\eta)\boldsymbol{\phi} \right\rangle . \tag{1.2}$$

If $A = \emptyset$, then $\varphi_{\boldsymbol{\phi},n}(A) \equiv 0$ for the convenience. We should remark that the map $\varphi_{\phi,n}$ expresses $\mathbb{C}$-valued measure on $\mathcal{F}_n$ in the following sense: for every $\boldsymbol{\phi} \in \mathcal{H}_C$ with $||\boldsymbol{\phi}|| = 1$,

Property of $\varphi_{\phi,n}$

(i) For $A \in \mathcal{F}_n$, $\varphi_{\boldsymbol{\phi},n}(A) \in \mathbb{C}$. Furthermore, $\varphi_{\phi,n}(\Omega_n^2) = 1$,
(ii) For any $A_1, \ldots, A_m \in \mathcal{F}_n$ with $A_i \cap A_j = \emptyset$ $(i \neq j)$,

$$\varphi_{\boldsymbol{\phi},n}\left(\bigcup_{i=1}^m A_i\right) = \sum_{i=1}^m \varphi_{\boldsymbol{\phi},n}(A).$$

In particular, for $\xi, \eta \in \Omega_n$,

$$(D)_{\xi,\eta} \equiv \varphi_{\boldsymbol{\phi},n}(\{(\xi,\eta)\})$$

is called the decoherence matrix starting from the initial coin state $\boldsymbol{\phi}$ which has been studied by [1, 2, 3]. Moreover for any $A_0 \in 2^{\Omega_n}$, $\nu_n(A_0) \equiv \varphi_{\phi,n}(A_0 \times A_0)$ is called q-measure on 2^{Ω_n} [1, 2].

Let $\Omega^2 = \Omega_n^2 \times \{-1,1\}^2 \times \{-1,1\}^2 \times \cdots = (\{-1,1\}^2)^{\mathbb{N}}$. A subset $A \subset \Omega^2$ is a cylinder set if and only if there exist $n \in \{1,2,\ldots\}$ and $B \in \mathcal{F}_n$ such that $A = B \times \{-1,1\}^2 \times \{-1,1\}^2 \times \cdots$. Denote $\mathcal{C}(\Omega^2)$ as the collection of all cylinder sets. From the unitarity of $U = P_1 + P_{-1}$, we see that for $A \in \mathcal{F}_n$,

$$\begin{aligned}&\varphi_{\boldsymbol{\phi},n+1}(A \times \{-1,1\}^2)\\ &= \left\langle \boldsymbol{\phi}, \sum_{(\xi,\eta)\in A} \{(P_1 + P_{-1})W(\xi)\}^\dagger \cdot \{(P_1 + P_{-1})W(\eta)\}\, \boldsymbol{\phi} \right\rangle = \varphi_{\boldsymbol{\phi},n}(A).\end{aligned}$$

Thus if $A \in \mathcal{F}_n$, then

$$\varphi_{\boldsymbol{\phi},n+m}(A \times \{-1,1\}^{2m}) = \varphi_{\boldsymbol{\phi},n}(A), \tag{1.3}$$

for any $m \geq 1$. Define $\varphi_{\boldsymbol{\phi}} : \mathcal{C}(\Omega^2) \to \mathbb{C}$ such that for any $A \in \mathcal{C}(\Omega^2)$ expressed by $A = B \times \{-1,1\}^2 \times \{-1,1\}^2 \times \cdots$ with $B \in \mathcal{F}_n$,

$$\varphi_{\boldsymbol{\phi}}(A) = \varphi_{\boldsymbol{\phi},n}(B).$$

Equation (1.3) implies that if $B = B_1 \times \{-1,1\}^{2(n-m)}$ with $B_1 \in \mathcal{F}_m$ and $m \le n$, then $\varphi_{\boldsymbol{\phi}}(A) = \varphi_{\boldsymbol{\phi},n}(B) = \varphi_{\boldsymbol{\phi},m}(B_1)$. So $\varphi_{\boldsymbol{\phi}}$ is well defined. Moreover, we easily find that $\varphi_{\boldsymbol{\phi}}$ satisfies both properties (i) and (ii).

Now we will connect the above statements to the quantum walk on $\mathbb{Z}$ originated by S. Gudder (1988) [4]. For $j \in \mathbb{Z}$, and $n \in \mathbb{N}$, define $T_n^{(j)} \in \mathcal{C}(\Omega^2)$ as $T_n^{(j)} = \{(\xi,\eta) \in \Omega^2 : \xi_1 + \cdots + \xi_n = \eta_1 + \cdots + \eta_n = j\}$, where $\Omega = \{-1,1\}^{\mathbb{N}}$. Indeed, we can check that $\varphi_{\boldsymbol{\phi}}(T_n^{(j)}) \ge 0$, and $\sum_{j\in\mathbb{Z}} \varphi_{\boldsymbol{\phi}}(T_n^{(j)}) = 1$. The property (i) implies that

$$\varphi_{\boldsymbol{\phi}}\left(\Omega_n^2 \setminus \bigcup_{j=-n}^{n} T_n^{(j)}\right) = 0.$$

Anyway, under the subalgebla $2^{\bigcup_{j=-n}^{n} T_n^{(j)}} \subset 2^{\Omega_n^2}$, the quantum walk at time n is denoted by a *random* variable $X_n^{(\boldsymbol{\phi})} : \bigcup_{j=-n}^{n} T_n^{(j)} \to \{-n, -(n-1), \ldots, n-1, n\}$. Here $X_n^{(\boldsymbol{\phi})}(\xi,\eta) = \xi_1 + \cdots + \xi_n = \eta_1 + \cdots + \eta_n$ has the following distribution:

$$P(X_n^{(\boldsymbol{\phi})} = j) \equiv P\left(\left\{(\xi,\eta) \in \bigcup_{j=-n}^{n} T_n^{(j)} : X_n^{(\boldsymbol{\phi})}(\xi,\eta) = j\right\}\right) = \varphi_{\boldsymbol{\phi}}(T_n^{(j)}).$$

This is an equivalent expression for the definition of the usual quantum walk on $\mathbb{Z}$ which has been intensively studied by many researchers. Now using $\varphi_{\boldsymbol{\phi}}$, we can measure various kinds of cylinder sets including $T_n^{(j)}$ corresponding to the usual quantum walk. In the next section, we choose three kinds of n-truncated cylinder sets in $\mathcal{C}(\Omega^2)$ by using our measure $\varphi_{\boldsymbol{\phi}}$ and find their asymptotics for large n.

By the way, it is possible to extend our $\mathbb{C}$-valued measure $\varphi_{n,\phi}$ to $\widetilde{\varphi}_{\phi,n}^{(i,j)}$ ($i, j \in \{\pm 1\}$) as follows: $\widetilde{\varphi}_{\phi,n}^{(i,j)} : \mathcal{F}_n \to \mathbb{C}$ such that for $A \in \mathcal{F}_n$,

$$\widetilde{\varphi}_{\phi,n}^{(i,j)}(A) = \sum_{(\xi,\eta)\in A} \langle W(\xi)\phi, \boldsymbol{e}_i\rangle\langle \boldsymbol{e}_j, W(\eta)\phi\rangle.$$

It is hold that

$$\varphi_{\phi,n} = \widetilde{\varphi}_{\phi,n}^{(1,1)} + \widetilde{\varphi}_{\phi,n}^{(-1,-1)}.$$

In particular, when we take $A \in \bigcup_{k=-n}^{n} T_n^{(k)}$, then $\varphi_{\phi,n}^{(i,j)}(A)$ becomes the argument proposed by [5]. The author [5] gives the weak convergence theorem of $\widetilde{\varphi}_{\phi,n}^{(i,j)}$, that is, $\sum_{k:k<nx} \widetilde{\varphi}_{\phi,n}^{(i,j)}(T_n(k))$ in the limit of n, for $i \neq j$ which is called interference term in his paper. So considering $\widetilde{\varphi}_{\phi,n}^{(i,j)}$ can be one of the candidates of the interesting future's problem.

2. Results

For $x, y \in \mathbb{R}$, define the set of all paths which go through the positions nx and ny at time n and $2n$, respectively as follows:

$$\Theta_{x|y}^{(n)} = \left\{ (\xi_1, \xi_2, \dots) \in \Omega : \frac{\xi_1 + \cdots + \xi_n}{n} = x, \; \frac{\xi_1 + \cdots + \xi_{2n}}{n} = y \right\}.$$

Now for simplicity, we concentrate on $y = 0$, and the following three cases with respect to the pair of $\Theta_x^{(n)} \times \Theta_y^{(n)} \in \mathcal{C}(\Omega^2)$, where $\Theta_x^{(n)} \equiv \Theta_{x|0}^{(n)}$:

(1) $A_1^{(n)} \equiv \bigcup_{x\in\mathbb{R}} \bigcup_{y\in\mathbb{R}} \Theta_x^{(n)} \times \Theta_y^{(n)}$ case
(2) $A_2^{(n)}(y) \equiv \bigcup_{x\in\mathbb{R}} \Theta_x^{(n)} \times \Theta_y^{(n)}$ with fixed $y \in \mathbb{R}$ case
(3) $A_3^{(n)}(y) \equiv \Theta_y^{(n)} \times \Theta_y^{(n)}$ case

Note that $A_1^{(n)} \supseteq A_2^{(n)}(y) \supseteq A_3^{(n)}(y)$. To explain the situations of $A_j^{(n)}$'s, we prepare two quantum walkers, walker 1 and walker 2, who produce the weight of path W. The measurement value is obtained by inner product of their weight of paths with an initial coin state (see Eq. (1.2)). Both walkers in $A_1^{(n)}$ give weight of all the paths returning back to the origin at time $2n$. Walkers 1 and 2 in $A_3^{(n)}(y)$ produce weight of every return path with length $2n$ restricted to passing the position ny at time n. In $A_2^{(n)}(y)$, despite of $A_1^{(n)}$ and $A_3^{(n)}(y)$, the classes of return paths for two walkers are different: walker 1 is in the situation (1) while walker 2 is in the situation (3).

The following theorem gives asymptotics of measurement value for each situation (1)-(3) by using $\varphi_{\boldsymbol{\phi},n}$. Define

$$\mathcal{D}_\kappa = e^{i\kappa}\Pi_{-1} + \Pi_1 \;\text{ with } \kappa = \arg(a) + \arg(c) - \det(U). \tag{2.4}$$

We use notation $a_n \sim b_n$ as $\lim_{n\to\infty} a_n/b_n = 1$.

Lemma 1. *Denote the Konno function $f_K(x;r)$ $(0 < r < 1)$ [10, 11] by*

$$f_K(x;r) = \frac{\sqrt{1-r^2}}{\pi(1-x^2)\sqrt{r^2-x^2}} \mathbf{1}_{\{|x|<r\}}(x),$$

where $\mathbf{1}_A(x)$ is the indicator function, that is, $\mathbf{1}_A(x) = 1$, $(x \in A)$, $= 0$, $(x \notin A)$. Let the initial coin state be $\boldsymbol{\phi}_0$, and $\boldsymbol{\phi}_\kappa \equiv \mathcal{D}_\kappa \boldsymbol{\phi}_0$. Then we have for large n,

(1) Case (1)

$$\varphi_{\boldsymbol{\phi}_0}(A_1^{(n)}) \sim \frac{f_K(0;|a|)}{n} = \frac{|c|}{\pi|a|n}. \tag{2.5}$$

(2) Case (2)

$$\sum_{j:j\le ny} \varphi_{\phi_0}\left(A_2^{(n)}(j/n)\right) \sim \mathbf{1}_{\{y>0\}}(y)\frac{f_K(0;|a|)}{n} = \mathbf{1}_{\{y>0\}}(y)\frac{|c|}{\pi|a|n}. \tag{2.6}$$

(3) Case (3)

$$\sum_{j\le ny} \varphi_{\phi_0}\left(A_3^{(n)}(j/n)\right) \sim \frac{|c|^2}{|a|^2 n}\int_{-\infty}^{y}(1+\langle \phi_\kappa, C_0\phi_\kappa\rangle x)\,\frac{\mathbf{1}_{\{|x|<|a|\}}(x)}{\pi^2(1-x^2)^2}dx, \tag{2.7}$$

where

$$C_0 = \begin{bmatrix} 1 & -|c|/|a| \\ -|c|/|a| & -1 \end{bmatrix}.$$

Now we present a distribution function with respect to q-measure [1, 2]. To do so, put

$$F_{n,\phi_0}(x|y) \equiv \frac{\sum_{j\le nx}\varphi_{\phi_0}\left(\Theta^{(n)}_{(j/n)|y}\times\Theta^{(n)}_{(j/n)|y}\right)}{\sum_{j\le n}\varphi_{\phi_0}\left(\Theta^{(n)}_{(j/n)|y}\times\Theta^{(n)}_{(j/n)|y}\right)}.$$

We can easily check that for fixed y, $F_{n,\phi_0}(x|y)$ becomes a distribution function, that is,

(a) $\lim_{x\to\infty} F_{n,\phi_0}(x|y)=1$, $\lim_{x\to-\infty} F_{n,\phi_0}(x|y)=0$,
(b) for any $x\le y$, $0\le F_{n,\phi_0}(x|z)\le F_{n,\phi_0}(y|z)\le 1$.

The function $F_{n,\phi_0}(x|y)$ corresponds to a *normalized* q-measure [1, 2] restricted to the event $\bigcup_x \Theta^{(n)}_{x|y}$. Part (3) in Lemma 1 leads the following theorem for $y=0$ case:

Theorem 1. *Assume that the initial coin state is $\phi_0 = {}^T[\alpha,\beta]$. We consider the sequence $\{F_{n,\phi_0}(x|0)\}_{n\ge 0}$. Let Y_n be a random variable whose distribution function is $F_{n,\phi_0}(x|0)$, that is, $P(Y_n\le x)=F_{n,\phi_0}(x|0)$. Then we have*

$$Y_n \Rightarrow Z, \ \ (n\to\infty) \tag{2.8}$$

where Z has the following density:

$$\nu_{\phi_0}(x|0) = \frac{|c|^2}{|a|+|c|^2\log\sqrt{\frac{1+|a|}{1-|a|}}}\left[1-\left\{(|\alpha|^2-|\beta|^2)+\frac{a\alpha\overline{b\beta}+\overline{a\alpha}b\beta}{|a|^2}\right\}x\right]\frac{\mathbf{1}_{\{|x|<|a|\}}(x)}{\pi^2(1-x^2)^2}$$

Here "$\Rightarrow$" means the weak convergence.

Next, define $W_x^{(n)} = \bigcup_y \Theta_{y|x}^{(n)}$ and

$$\widehat{G}_{n,\boldsymbol{\phi}_0}(y|x) = \frac{\sum_{j\le ny} \varphi_{\boldsymbol{\phi}_0}\left(W_x^{(n)} \times \Theta_{(j/n)|x}^{(n)}\right)}{\sum_{j\le n} \varphi_{\boldsymbol{\phi}_0}\left(W_x^{(n)} \times \Theta_{(j/n)|x}^{(n)}\right)}.$$

The value $\widehat{G}_{n,\boldsymbol{\phi}_0}(y|x)$ satisfies the above condition (a), but the condition (b) is not ensured, that is,

$$\lim_{y\to\infty} \widehat{G}_{n,\boldsymbol{\phi}_0}(y|x) = 1, \text{ and } \lim_{y\to-\infty} \widehat{G}_{n,\boldsymbol{\phi}_0}(y|x) = 0,$$

while $\widehat{G}_{n,\boldsymbol{\phi}_0}(y|x) \in \mathbb{C}$ for $|y| < \infty$ in general. From Parts (1) and (2) in Theorem 1, we obtain an asymptotic behavior of the value $\widehat{G}_{n,\boldsymbol{\phi}_0}(x|0)$ which is deeply related to the weak value [7, 8] as follows.

Before we show the result, here we briefly give the definition of the weak value. We can see more detailed explanations and its interesting related works in [9] and its references. Let $\mathcal{H}$ be a Hilbert space and $U(t_2, t_1)$ be an evolution from time t_1 to t_2 on $\mathcal{H}$. For an observable A and normalized states $\boldsymbol{\phi}_i, \boldsymbol{\phi}_f \in \mathcal{H}$, the weak value ${}_{\phi_f}\langle A\rangle_{\phi_i}^w$ is defined by

$$ {}_{\phi_f}\langle A\rangle_{\phi_i}^w = \frac{\langle \boldsymbol{\phi}_f, U(t_f,t)AU(t,t_i)\boldsymbol{\phi}_i\rangle}{\langle \boldsymbol{\phi}_f, U(t_f,t_i)\boldsymbol{\phi}_i\rangle}. \tag{2.9}$$

Here $\boldsymbol{\phi}_i$ and $\boldsymbol{\phi}_f$ are called pre-selected state and post-selected state, respectively.

From now on, we take the Hilbert space $\mathcal{H}$ as $\bigoplus_{x\in\mathbb{Z}} \mathcal{H}_x$, where $\mathcal{H}_x$ is the two-dimensional Hilbert space spanned by left and right chiralities $\{\boldsymbol{e}_L, \boldsymbol{e}_R\}$. Let the canonical basis of $\mathcal{H}$ be denoted by $\{\boldsymbol{\delta}_x\otimes\boldsymbol{e}_L, \boldsymbol{\delta}_x\otimes\boldsymbol{e}_R; x\in\mathbb{Z}\}$. Put a permutation operator S on $\mathcal{H}$ such that for $\boldsymbol{\delta}_x\otimes\boldsymbol{e}_J$ ($J\in\{L,R\}$),

$$S(\boldsymbol{\delta}_x\otimes\boldsymbol{e}_J) = \begin{cases} \boldsymbol{\delta}_{x+1}\otimes\boldsymbol{e}_R, & (J=R), \\ \boldsymbol{\delta}_{x-1}\otimes\boldsymbol{e}_L, & (J=L). \end{cases}$$

Define $E = SC$ be a unitary operator on $\mathcal{H}$, where $C = \sum_x \oplus U$. (Recall that U is the two-dimensional unitary operator.) We consider the iteration of E from the initial state $\boldsymbol{\Phi}_0 = \boldsymbol{\delta}_0\otimes\boldsymbol{\phi}$ with $||\boldsymbol{\phi}||^2 = 1$:

$$\boldsymbol{\Phi}_0 \overset{E}{\mapsto} \boldsymbol{\Phi}_1 \overset{E}{\mapsto} \boldsymbol{\Phi}_2 \overset{E}{\mapsto} \cdots .$$

This is another equivalent expression for the quantum walk on $\mathbb{Z}$ with initial state $\boldsymbol{\Phi}_0$. Indeed,

$$\left|\left|\Pi_j E^n \boldsymbol{\Phi}_0\right|\right|^2 = \varphi_{\boldsymbol{\phi}}(T_n^{(j)}),$$

where Π_j is the projection onto $\mathcal{H}_j$.

In particular, when we take for $t_1, t_2 \in \mathbb{N}$, $E^{t_2-t_1}$ as $U(t_2, t_1)$ and Π_j as the observable A, moreover $\mathbf{\Phi}_0$ as the pre-selected state and $\Pi_0 \overline{E}^{t_f} \mathbf{\Phi}_0$ as the post-selected state in Eq. (2.9) with $t_i = 0$, $t = n$ and $t_f = 2n$, then we have in this setting

$$\sum_{j \leq ny} {}_{\phi_f}\langle A \rangle^w_{\phi_i} = \widehat{G}_{n,\phi_0}(y|0). \tag{2.10}$$

This is a connection between our complex-valued measure and weak value. We find that the weak value weakly converges to the delta measure as follows.

Theorem 2. *It is hold that for large n,*

$$\lim_{n\to\infty} \widehat{G}_{n,\phi_0}(y|0) = \mathbf{1}_{\{y>0\}}(y). \tag{2.11}$$

The physical meaning of Theorem 2 remains as an interesting open problem.

3. Proof of Lemma 1

Let $\Xi_n(j) = \sum_{\xi:\xi_n+\cdots+\xi_1=j} W(\xi)$ be weight of all the n-truncated passages arriving at j. Our proof is based on the stationary phase method:

Lemma 2. *Let $f(x)$ denote an $\mathbb{R}$-valued function on $[a,b]$ satisfying that there exists a unique $c \in [a,b]$ such that $f'(c) = 0$ with $f''(c) \neq 0$. Then for continuous function $g(x)$ on $[a,b]$,*

$$\frac{1}{n} \sum_{j:\, an<j<bn} g(j/n) e^{inf(j/n)} \sim e^{i\mathrm{sgn}(f''(c))\pi/4} \sqrt{\frac{2\pi}{|f''(c)|n}} g(c) e^{inf(c)} + o\left(1/\sqrt{n}\right), \tag{3.12}$$

for large n, where $\mathrm{sgn}(y) = 1,\ (y>0),\ = 0,\ (y=0),\ = -1,\ (y<0)$.

At first we give the following key lemma whose proof is described in Appendix by using the stationary phase method:

Lemma 3. *Put $\mathbb{R}$-valued functions $k(x)$ and $\psi(x)$ $(x \in [-|a|, |a|])$ as*

$$e^{ik(x)} = \frac{1}{|a|}\sqrt{\frac{|a|^2 - x^2}{1-x^2}} + i\frac{|c|}{|a|}\frac{x}{\sqrt{1-x^2}}, \tag{3.13}$$

$$e^{i\psi(x)} = \sqrt{\frac{|a|^2 - x^2}{1-x^2}} + i\frac{|c|x^2}{\sqrt{1-x^2}}. \tag{3.14}$$

For any $j \in \mathbb{Z}$ with $j = nx$ ($x \in [-1,1]$), we obtain

$$\Xi_n(j) = \frac{1+(-1)^{n+j}}{2} e^{in\delta/2} \sqrt{\frac{2f_K(x;|a|)}{n}} \times \mathcal{D}_\kappa^\dagger \left(e^{i\pi/4} e^{in(\psi(x)-xk(x))} \Pi(x) + e^{-i\pi/4} e^{-in(\psi(x)-xk(x))} \overline{\Pi(x)} \right) \mathcal{D}_\kappa + o(1/\sqrt{n}), \tag{3.15}$$

where

$$\Pi(x) = \begin{bmatrix} |a|(1-x) & |c|x + i\sqrt{|a|^2 - x^2} \\ |c|x - i\sqrt{|a|^2-x^2} & |a|(1+x) \end{bmatrix}.$$

Here for $M \in M_2(\mathbb{C})$, $\left(\overline{M}\right)_{i,j} = \overline{(M)_{i,j}}$ for any $i,j \in \{1,2\}$.

Before the proof of Lemma 1, we can confirm a consistency of the statement of the above lemma as follows. Recall that $X_n^{(\phi)}$ is a random variable determined by $P(X_n^{(\phi)} = j) = ||\Xi_n(j)\phi||^2$ with the initial coin state $\phi = [\alpha, \beta]$ so called usual quantum walk. Then Lemma 3 and the Riemann-Lebesgue lemma imply the following corollary with respect to $X_n^{(\phi)}$:

Corollary 3.

$$\lim_{n\to\infty} P(X_n^{(\phi)}/n \le x) = \int_{-\infty}^{x} \left\{ 1 - \left(|\alpha|^2 - |\beta|^2 + \frac{2\mathrm{Re}[a\alpha\overline{b\beta}]}{|a|^2} \right) x \right\} f_K(x;|a|)dx.$$

This is consistent with results of [10, 11]. Now we give the proof of Lemma 1 in the following:

(1) *Proof of Case (1).* Put $g(x) = \psi(x) - xk(x)$. We should remark that $L_n(x) \equiv \sum_{\xi \in \Theta_x^{(n)}} W(\xi) = \Xi_n(-nx)\Xi_n(nx)$. Note that $\sum_{j=-n}^{n} L_n(j/n) = \Xi_{2n}(0)$. Lemma 3 reduces to

$$e^{-in\delta} D_\kappa \Xi_{2n}(0) D_\kappa^\dagger \sim \sqrt{\frac{f_K(0;|a|)}{n}} \left\{ e^{i\frac{\pi}{4}} e^{2ing(0)} \Pi(0) + e^{-i\frac{\pi}{4}} e^{-2ing(0)} \overline{\Pi(0)} \right\}. \tag{3.16}$$

By using the fact that for every $x \in \mathbb{R}$,

$$\Pi^2(x) = \Pi(x), \quad \Pi(x)\overline{\Pi(-x)} = 0, \tag{3.17}$$

and Eq. (3.16), we obtain

$$\varphi_{\phi_0}(A_1^{(n)}) = \sum_{i=-n}^{n} \sum_{j=-n}^{n} \langle L_n(i/n)\phi_0, L_n(j/n)\phi_0 \rangle \tag{3.18}$$

$$= \langle \Xi_{2n}(0)\phi_0, \Xi_{2n}(0)\phi_0 \rangle \tag{3.19}$$

$$\sim \frac{f_K(0;|a|)}{n} \left\langle \phi_0, \left\{ \Pi(0) + \overline{\Pi(0)} \right\} \phi_0 \right\rangle = \frac{f_K(0;|a|)}{n}. \tag{3.20}$$

Then we complete the proof of case (1). It is consistent with the result of [12] which treats the Hadamard walk. □

(2) *Proof of Case (2).* Using Eq. (3.17), Lemma 3 implies that

$$D_\kappa \Xi_n(-j)\Xi_n(j)D_\kappa^\dagger e^{-in\delta} \sim i\frac{1+(-1)^{n+j}}{2} \times \frac{2f_K(x;|a|)}{n} \times \left\{ e^{2ing(x)}\Pi(-x)\Pi(x) - e^{-2ing(x)}\overline{\Pi(-x)\Pi(x)} \right\}, \quad (3.21)$$

By Eq. (3.16),

$$e^{-in\delta}\sum_{j<ny} D_\kappa \Xi_n(-j)\Xi_n(j)D_\kappa^\dagger \sim \frac{2i}{n}\sum_{j<ny} f_K(j/n;|a|) \times \left\{ e^{2ing(j/n)}\Pi(-j/n)\Pi(j/n) - e^{-2ing(j/n)}\overline{\Pi(-j/n)\Pi(j/n)} \right\} \quad (3.22)$$

Now we consider the solution for $g'(x) = \psi'(x) - k(x) - xk'(x) = 0$. Equations (A.30)-(A.33) in Appendix imply that $\psi(x) = \theta(k(x))$ and $k(x)$ is the unique solution for

$$h(k) = \partial\theta(k)/\partial k = x \quad (3.23)$$

on $k \in [-\pi/2, \pi/2]$, where $\cos\theta(k) = |a|\cos k$ with $\sin\theta(k) \geq 0$. So we have

$$\frac{\partial\psi(x)}{\partial x} = \frac{\partial\theta(k(x))}{\partial x} = xk'(x).$$

Then we obtain

$$g'(x) = -k(x). \quad (3.24)$$

On the other hand, differentiating both sides of Eq. (3.23) with respect to x implies

$$\frac{\partial}{\partial x}\left(\frac{\partial\theta(k)}{\partial k}\right) = \frac{\partial k}{\partial x}\left(\frac{\partial^2\theta(k)}{\partial k^2}\right) = 1.$$

Then Eq. (3.24) gives

$$k'(x) = \left.\frac{1}{\partial^2\theta(k)/\partial k^2}\right|_{k=k(x)} = \pi f_K(x;|a|). \quad (3.25)$$

Thus, $g'(x) = 0$ if and only if $k(x) = 0$, which implies $e^{ik(x)} = 1$. Therefore by definition of $k(x)$ (see Eq. (3.13)), $x = 0$ is the unique solution for $g'(x) = 0$. Moreover Eqs. (3.24) and (3.25) give $g''(x) = -k'(x) =$

$-\pi f_K(x;|a|)$, which implies $g''(0) = -\pi f_K(0;|a|)$. So applying the stationary phase method described in Lemma 2 to Eq. (3.22), we obtain

$$e^{-in\delta} \sum_{j<ny} D_\kappa \Xi_n(-j)\Xi_n(j)D_\kappa^\dagger \sim i\mathbf{1}_{\{y>0\}}(y) \left(e^{2in\psi(0)}e^{-i\pi/4}\Pi(0) - e^{-2in\psi(0)}e^{i\pi/4}\overline{\Pi(0)}\right)\sqrt{\frac{f_K(0;|a|)}{n}}. \tag{3.26}$$

Combining Eq. (3.26) with Eq. (3.16), we arrive at

$$\varphi_{\boldsymbol{\phi}_0}(A_2^{(n)}(y)) = \sum_{j:j<ny} \langle \Xi_{2n}(0)\boldsymbol{\phi}_0, \Xi_n(-j)\Xi_n(j)\boldsymbol{\phi}_0\rangle \sim \mathbf{1}_{\{y>0\}}(y)\frac{f_K(0;|a|)}{n}. \tag{3.27}$$

So we complete the proof. □

(3) *Proof of Case (3).* Remark that

$$\sum_{j\le ny} \varphi_{\boldsymbol{\phi}_0}(A_3^{(n)}(j/n)) = \sum_{j\le ny} \langle L_n(j/n)\boldsymbol{\phi}_0, L_n(j/n)\boldsymbol{\phi}_0\rangle. \tag{3.28}$$

On the other hand, using the relations of $\Pi(x)$ described by Eq. (3.17), Eq. (3.21) gives

$$L_n^\dagger(x)\cdot L_n(x) \sim \frac{|c|^2}{n^2|a|^2}(I + C_0 x)\frac{\mathbf{1}_{\{|x|<|a|\}}(x)}{\pi^2(1-x^2)^2} \tag{3.29}$$

which leads the desired conclusion of case (3). □

Acknowledgments. NK acknowledges financial support of the Grant-in-Aid for Scientific Research (C) of Japan Society for the Promotion of Science (Grant No. 21540116).

References

[1] S. Gudder, Quantum measure theory, *Math. Slovaca*, **60**, 681–700 (2010).
[2] S. Gudder and R. D. Sorkin, Two-site quantum random walk, *Gen. Relativ. Gravit.*, **43**, 3451–3475 (2011).
[3] N. Konno and E. Segawa, On the entropy of decoherence matrix for quantum walks, *Yokohama Mathematical Journal*, in press, arXiv:1202.1376. (2012).
[4] S. Gudder, Quantum Probability, Academic Press Inc., CA (1988).

[5] T. Machida, Limit theorems for the interference terms of discrete-time quantum walks on the line, *Quantum Inf. Comp.*, **13**, 661–671 (2013).
[6] Y. Aharonov, L. Davidovich, and N. Zagury, Quantum random walks, *Phys. Rev. A*, **48**, 1687–1690 (1993).
[7] Y. Aharonov, D. Z. Albert, and L. Vaidman, How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100, *Phys. Rev. Lett.*, **60**, 1351 (1988).
[8] Y. Shikano and A. Hosoya, Weak values with decoherence, *J. Phys. A: Math. Theor.*, **43**, 025304 (2010).
[9] Y. Shikano, Theory of "weak value" and quantum mechanical measurements, "Measurements in Quantum Mechanics", edited by M. R. Pahlavani (InTech, 2012) 75–100 (2012).
[10] N. Konno, Quantum random walks in one dimension, *Quantum Inf. Proc.*, **1**, 345–354 (2002).
[11] N. Konno, A new type of limit theorems for the one-dimensional quantum random walk, *J. Math. Soc. Jpn.*, **57**, 1179–1195 (2005).
[12] N. Konno, Quantum walks and elliptic integrals, *Mathematical Structures in Computer Science*, **20**, 1091–1098 (2010).

Appendix

A. Proof of Lemma 3

We take the spatial Fourier transform of the weight of path $\Xi_n(j)$ such that

$$\widehat{\Xi}_n(k) = \sum_{j\in\mathbb{Z}} \Xi_n(j)e^{ijk}.$$

From the recurrence relation $\Xi_{n+1}(j) = Q\Xi_n(j-1) + P\Xi_n(j+1)$, we obtain

$$\widehat{\Xi}_n(k) = \left(e^{ik}Q + e^{-ik}P\right)^n.$$

The eigenvalues and their corresponding normalized eigenvectors are expressed by $\lambda_m(k+\tau)$ and $\boldsymbol{v}_m(k+\tau)$, $(m \in \{0,1\})$, where

$$\lambda_m(k) = e^{i\delta/2} \cdot e^{i(-1)^m\theta(k)}, \tag{A.30}$$

$$\boldsymbol{v}_m(k) = \frac{1}{\sqrt{2\{1-|a|\cos[(-1)^m\theta(k)-k]\}}} D_\kappa^\dagger \begin{bmatrix} |c| \\ |a| - e^{i((-1)^m\theta(k)-k)} \end{bmatrix}, \tag{A.31}$$

where $\tau = \delta/2 - \arg(a)$ and D_κ is defined in Eq. (2.4). Here $\cos\theta(k) = |a|\cos k$ with $\sin\theta(k) \geq 0$ and $\delta = \arg(\det(U))$. By the Fourier inversion theorem, we obtain for any $\gamma \in \mathbb{R}$,

$$\Xi_n(j) = \int_{\gamma}^{2\pi+\gamma} \widehat{\Xi}_n(k) e^{-ijk} \frac{dk}{2\pi},$$
$$= e^{in\delta/2} \sum_{m\in\{0,1\}} \int_{\gamma+\tau}^{2\pi+\gamma+\tau} e^{in((-1)^m\theta(k)-xk)} \boldsymbol{v}_m(k)\boldsymbol{v}_m(k)^\dagger \frac{dk}{2\pi}, \tag{A.32}$$

where $x = j/n$. We choose an arbitrary parameter γ as $-\tau - \pi/2$. From now on we apply the stationary phase method in Lemma 2 to Eq. (A.32). Put $f_m(k) = (-1)^m\theta(k) - xk$, $(m \in \{0,1\})$ as $\mathbb{R}$-valued function on $[-\pi/2, 3\pi/2)$. The solution for $\partial f_m(k)/\partial k = 0$ is given by

$$(-1)^m h(k) = x, \tag{A.33}$$

where $h(k) = \partial\theta(k)/\partial k$. In the following, we consider $m = 0$ case. The definition of $\theta(k)$ gives

$$h(k) = \frac{|a|\sin k}{\sqrt{1-|a|^2\cos^2 k}}.$$

The solutions for $h'(k) = 0$ in $[-\pi/2, 3\pi/2)$ are $\pm\pi/2$. We denote $h_\pm(k)$ with $h(k) = h_+(k) + h_-(k)$ so that $h'_+(k) > 0$ and $h'_-(k) \le 0$, as the function on $K_+ = [-\pi/2, \pi/2)$ and $K_- = [\pi/2, 3\pi/2)$, respectively. To apply the stationary phase method, we divide the integral in Eq. (A.32) into the four parts as follows:

$$e^{-in\delta/2} D_\kappa \Xi_n(j) D_\kappa^\dagger = \sum_{m\in\{0,1\}} \sum_{\epsilon\in\{-,+\}} \int_{k\in K_\epsilon} e^{in((-1)^m\theta(k)-xk)} \boldsymbol{v}_m(k)\boldsymbol{v}_m(k)^\dagger \frac{dk}{2\pi}. \tag{A.34}$$

An explicit expression for the solutions $k_\pm(x)$ for $h_\pm(k) = x$, respectively, are obtained as follows:

$$\cos k_\pm(x) = \pm\frac{1}{|a|}\sqrt{\frac{|a|^2 - x^2}{1-x^2}}, \tag{A.35}$$

$$\sin k_\pm(x) = \frac{|c|}{|a|}\frac{x}{\sqrt{1-x^2}}. \tag{A.36}$$

Thus we have

$$\left|\frac{1}{\partial^2 f_0(k)/\partial k^2}\right|_{k=k_\pm(x)} = \left|\frac{1}{\partial h(k)/\partial k}\right|_{k=k_\pm(x)} = \pi f_K(x;|a|). \tag{A.37}$$

Moreover some algebraic computations give

$$\boldsymbol{v}_0(k)\boldsymbol{v}_0(k)^\dagger|_{k=k_+(x)} = D_\kappa^\dagger \Pi(x) D_\kappa, \quad \boldsymbol{v}_0(k)\boldsymbol{v}_0(k)^\dagger|_{k=k_-(x)} = D_\kappa^\dagger \overline{\Pi(x)} D_\kappa,$$
$$\boldsymbol{v}_1(k)\boldsymbol{v}_1(k)^\dagger|_{k=k_+(x)} = D_\kappa^\dagger \overline{\Pi(-x)} D_\kappa, \quad \boldsymbol{v}_1(k)\boldsymbol{v}_1(k)^\dagger|_{k=k_-(x)} = D_\kappa^\dagger \Pi(-x) D_\kappa. \tag{A.38}$$

For the solutions of Eq. (A.33) in $m = 1$ case, we replace the parameter x in the result on $m = 0$ case given by the above discussion with $-x$. By putting $\psi(x)$ as $\psi(x) = \theta(k(x))$ with $k(x) \equiv k_+(x)$, note that $\theta(k_+(-x)) = \psi(x)$, $\theta(k_-(x)) = \pi - \psi(x)$, and $k_+(-x) = -k(x)$, $k_-(x) = -k(x) - \pi$. Inserting these relations and Eqs. (A.37) and $(A.38)$ into the formula in Lemma 2 for each term $(\epsilon, m) \in \{(+,0),(+,1),(-,0),(-,1)\}$ in Eq. (A.34), we have the desired conclusion. $\square$

Norio Konno
Department of Applied Mathematics,
Faculty of Engineering,
Yokohama National University
Hodogaya, Yokohama 240-8501,
Japan
E-mail: konno@ynu.ac.jp

Etsuo Segawa
Graduate School of Information Sciences,
Tohoku University,
Aoba, Sendai, 980-8579,
Japan
E-mail: e-segawa@m.tohoku.ac.jp