Yokohama Mathematical
Journal Vol. 58, 2012

SOME REMARKS ON ORBIT EQUIVALENCE OF TOPOLOGICAL MARKOV SHIFTS AND CUNTZ-KRIEGER ALGEBRAS

By

Kengo Matsumoto

(Received July 5, 2011; Revised March 12, 2012)

Abstract. Let A, B be square irredusible matrices with entries in $\{0, 1\}$. Assume that the sizes of A, B are both less than or equal to three. We will then show that the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) for A and B are continuously orbit equivalent if and only if the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic. The if part (and hence the only if part) is characterized by certain matrix relations between A and B.

1. Introduction

Measure theoretic studies of orbit equivalence of ergodic transformations have been initiated by H. Dye ([9], [10]). W. Krieger [17] has proved that two ergodic non-singular transformations are orbit equivalent if and only if the associated von Neumann crossed produtcs are isomorphic (cf.[5]). In topological setting, Giordano-Putnam-Skau [13], [14] (cf.[15],[24], etc.) have proved that two Cantor minimal systems are strongly orbit equivalent if and only if the associated C^*-crossed products are isomorphic. J. Tomiyama [26] (cf. [4], [27]) has studied a relationship between orbit equivalence and C^*-crossed products for topological free homeomorphisms on compact Hausdorff spaces. The class of one-sided topological Markov shifts is an important class of topological dynamical systems on Cantor sets with continuous surjections that are not homeomorphisms. The associated C^*-algebras to the topological Markov shifts are known to be the Cuntz-Krieger algebras. In a recent paper [21], the author has shown that the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) for irreducible matrices A and B with entries in $\{0, 1\}$ are continuously orbit equivalent if and only if there exists an isomorphism between the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ preserving their commutative C^*-subalgebras $C(X_A)$ and $C(X_B)$.

From the view point of the above Giordano-Putnam-Skau's works, we would

2010 Mathematics Subject Classification: Primary 46L55; Secondary 46L35, 37B10
Key words and phrases: Topological Markov shifts, orbit equivalence, Cuntz-Krieger algebras

expect that the isomorphism class of the C^*-algebras completely determines an orbit equivalence class of the underlying topological dynamical systems. Cuntz-Krieger algebras have been classified in terms of the underlying matrices and also in terms of its K-theory data by Enomoto-Fujii-Watatani [11] if the sizes of the matrices are three (for a general case, see Rørdam's work [25]). In this short note, by using the Enomoto-Fujii-Watatani's result, we will show the following theorem.

Theorem 1.1. *Let A, B be irreducible matrices with entries in $\{0,1\}$ satisfying condition (I). Suppose that the sizes of A, B are both less than or equal to three. Then the one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent if and only if the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic.*

Therefore the algebraic types of the Cuntz-Krieger algebras for irreducible matrices with entries in $\{0,1\}$, whose sizes are less than or equal to three, are completely classified by the continuous orbit equivalence classes of the underlying topological Markov shifts. We may also present a relationship between the associated directed graphs G_A and G_B to the matrices A and B under the condition that $\mathcal{O}_A$ and $\mathcal{O}_B$ are isomorphic.

2. Preliminaries

Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\{0,1\}$, where $1 < N \in \mathbb{N}$. Throughout the paper, we assume that A has both no zero columns and no zero rows. We denote by X_A the shift space

$$X_A = \{(x_n)_{n\in\mathbb{N}} \in \{1,\dots,N\}^{\mathbb{N}} \mid A(x_n, x_{n+1}) = 1 \text{ for all } n \in \mathbb{N}\}$$

of the right one-sided topological Markov shift for A. It is a compact Hausdorff space in natural product topology. The shift transformation σ_A on X_A defined by $\sigma_A((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$ is a continuous surjective map on X_A. The topological dynamical system (X_A, σ_A) is called the (right one-sided) topological Markov shift for A. We henceforth assume that A satisfies condition (I) in the sense of Cuntz-Krieger [8]. The condition (I) for A is equivalent to the condition that X_A is homeomorphic to a Cantor discontinuum.

A word $\mu = \mu_1 \cdots \mu_k$ for $\mu_i \in \{1,\dots,N\}$ is said to be admissible for X_A if μ appears in somewhere in some element x in X_A. The length of μ is k and denoted by $|\mu|$. We denote by $B_k(X_A)$ the set of all admissible words of length $k \in \mathbb{N}$. For $k = 0$ we denote by $B_0(X_A)$ the empty word $\emptyset$. We set $B_*(X_A) = \cup_{k=0}^{\infty} B_k(X_A)$ the set of admissible words of X_A.

The Cuntz-Krieger algebra $\mathcal{O}_A$ for the matrix A has been defined in [8] as the universal C^*-algebra generated by N partial isometries $S_1, \dots, S_N$ subject to the relations:

$$\sum_{j=1}^{N} S_j S_j^* = 1, \qquad S_i^* S_i = \sum_{j=1}^{N} A(i,j) S_j S_j^*, \quad i = 1, \dots, N. \tag{2.1}$$

The algebra $\mathcal{O}_A$ is the unique C^*-algebra subject to the above relations under the condition (I) for A. For a word $\mu = \mu_1 \cdots \mu_k$ with $\mu_i \in \{1, \dots, N\}$, we denote $S_{\mu_1} \cdots S_{\mu_k}$ by S_μ. Then $S_\mu \neq 0$ if and only if $\mu \in B_*(X_A)$. Let $C^*(S_\mu S_\mu^*; \mu \in B_*(X_A))$ be the C^*-subalgebra of $\mathcal{O}_A$ generated by the projections of the form $S_\mu S_\mu^*, \mu \in B_*(X_A)$, which we will denote by $\mathfrak{D}_A$. It is isomorphic to the commutative C^*-algebra $C(X_A)$ of all complex valued continuous functions on X_A through the correspondence $S_\mu S_\mu^* \in \mathfrak{D}_A \longleftrightarrow \chi_\mu \in C(X_A)$ where χ_μ denotes the characteristic function on X_A for the cylinder set $U_\mu = \{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = \mu_1, \dots, x_k = \mu_k\}$ for $\mu = \mu_1 \cdots \mu_k \in B_k(X_A)$. We identify $C(X_A)$ with the subalgebra $\mathfrak{D}_A$ of $\mathcal{O}_A$. Then it is well-known that the algebra $\mathfrak{D}_A$ is maximal abelian in $\mathcal{O}_A$ ([8, Remark 2.18], cf.[19, Proposition 3.3]).

For $x = (x_n)_{n \in \mathbb{N}} \in X_A$, the orbit $orb_{\sigma_A}(x)$ of x under σ_A is defined by

$$orb_{\sigma_A}(x) = \cup_{k=0}^{\infty} \cup_{l=0}^{\infty} \sigma_A^{-k}(\sigma_A^l(x)) \subset X_A.$$

Let (X_A, σ_A) and (X_B, σ_B) be topological Markov shifts. If there exists a homeomorphism $h : X_A \to X_B$ such that $h(orb_{\sigma_A}(x)) = orb_{\sigma_B}(h(x))$ for $x \in X_A$, then (X_A, σ_A) and (X_B, σ_B) are said to be topologically orbit equivalent. In this case, we have $h(\sigma_A(x)) \in orb_{\sigma_B}(h(x))$ for $x \in X_A$, so that $h(\sigma_A(x))$ belongs to $\cup_{k=0}^{\infty} \cup_{l=0}^{\infty} \sigma_B^{-k} \sigma_B^l(h(x))$. Hence there exist $k_1, l_1 : X_A \to \mathbb{Z}_+$ such that $\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x))$. Similarly there exist $k_2, l_2 : X_B \to \mathbb{Z}_+$ such that $\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y)))$ $= \sigma_A^{l_2(y)}(h^{-1}(y))$. If we may take $k_1, l_1 : X_A \longrightarrow \mathbb{Z}_+$ and $k_2, l_2 : X_B \longrightarrow \mathbb{Z}_+$ as continuous functions, the topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be *continuously orbit equivalent.* In [21], the following has been proved

Proposition 2.1. ([21, Theorem 5.7]) *Let A, B be irreducible matrices with entries in $\{0, 1\}$ satisfying condition (I). There exists an isomorphism $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\Psi(\mathfrak{D}_A) = \mathfrak{D}_B$ if and only if (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*

3. Primitive equivalence

In [11], a notion called primitive equivalence in square matrices with entries in $\{0, 1\}$ has been introduced. The equivalence relation completely classifies the isomorphism classes of the Cuntz-Krieger algebras defined by 3×3 matrices with entries in $\{0, 1\}$. By using the result [11, Theorem 4.1], we will show that the continuous orbit equivalence classes of the one-sided topological Markov shifts (X_A, σ_A) are completely classified by the isomorphism classes of the Cuntz-Krieger algebras $\mathcal{O}_A$ if the sizes of the matrices are three.

Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\{0, 1\}$. Following [11], for $i = 1, \dots, N$ let A_i be the i-th row vector $[A(i,j)]_{j=1}^N$ of A and E_i the row vector of size N whose i-th entry is one, other entries are zeros. Suppose that

$$A_p = E_{k(1)} + \cdots + E_{k(r)} + A_{m(1)} + \cdots + A_{m(s)} \tag{3. 1}$$

for some $k(1), \dots, k(r), m(1), \dots, m(s)$ which are mutually different and satisfy $p \notin \{m(1), \dots, m(s)\}$. Then the $N \times N$ matrix $B = [B(i,j)]_{i,j=1}^N$ is defined by setting for $i, j = 1, \dots, N$

$$B(i,j) = \begin{cases} A(i,j) & \text{for } i \neq p, \\ 1 & \text{for } i = p \text{ and } j \in \{k(1), \dots, k(r), m(1), \dots, m(s)\}, \\ 0 & \text{else.} \end{cases} \tag{3. 2}$$

We say that B is primitively transfered from A ([11]). Two $N \times N$ matrices C and D with entries in $\{0, 1\}$ are said to be primitively equivalent to each other if there exists a finite sequence of $N \times N$ matrices $M_1, \dots, M_n$ with entries in $\{0, 1\}$ such that $M_1 = C, M_n = D$ and M_i is primitively transfered from M_{i+1}, or M_{i+1} is primitively transfered from M_i. Enomoto-Fujii-Watatani have proved that for two 3×3 matrices A, B with entries in $\{0, 1\}$, A is primitively equivalent to B if and only if $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_B$ ([11, Theorem 4.1]).

Let A, B be $N \times N$ matrices with entries in $\{0, 1\}$. Assume that B is primitively transfered from A. Suppose that A_p satisfies (3. 1) and B is obtained from (3. 2). We further assume that $p = 1$. Let $S_1, \dots, S_N$ be the generating partial isometries of $\mathcal{O}_A$ satisfying the relations (2.1). We put

$$\begin{aligned} T_1 &= S_1(S_{k(1)}S_{k(1)}^* + \cdots + S_{k(r)}S_{k(r)}^* + S_{m(1)}^* + \cdots + S_{m(s)}^*), \\ T_i &= S_i \qquad \text{for } i \neq 1. \end{aligned}$$

In the proof of [11, Theorem 3.7], it was shown that the partial isometries $T_1, \dots, T_N$ generate $\mathcal{O}_A$ and satisfy the relations

$$\sum_{j=1}^N T_j T_j^* = 1, \qquad T_i^* T_i = \sum_{j=1}^N B(i,j) T_j T_j^*, \quad i = 1, \dots, N$$

so that $\mathcal{O}_A = \mathcal{O}_B$. We may regard $\mathfrak{D}_B$ as a C^*-subalgebra of $\mathcal{O}_A$ generated by the projections $T_\mu T_\mu^*, \mu \in B_*(X_B)$.

Lemma 3.1. *Keep the above notations. We have*

(i) $T_\mu T_\mu^* \in \mathfrak{D}_A$ *for* $\mu \in B_*(X_B)$ *implies* $T_i T_\mu T_\mu^* T_i^* \in \mathfrak{D}_A, i = 1, \dots, N$.
(ii) $S_\nu S_\nu^* \in \mathfrak{D}_B$ *for* $\nu \in B_*(X_A)$ *implies* $S_i S_\nu S_\nu^* S_i^* \in \mathfrak{D}_B, i = 1, \dots, N$.

Proof. (i) Suppose that $T_\mu T_\mu^* \in \mathfrak{D}_A$. It suffices to show that $T_1 T_\mu T_\mu^* T_1^* \in \mathfrak{D}_A$. As $k(1), \dots, k(r), m(1), \dots, m(s)$ are mutually different, one has $S_{k(i)} S_{k(i)}^* T_1^* = S_{k(i)} S_{k(i)}^* S_1^*$ and $S_{m(n)}^* T_1^* = S_{m(n)}^* S_{m(n)} S_1^*$. Since $T_\mu T_\mu^* \in \mathfrak{D}_A$, it then follows that

$$
\begin{aligned}
T_1 T_\mu T_\mu^* T_1^* &= \sum_{i=1}^{r} S_1 T_\mu T_\mu^* S_{k(i)} S_{k(i)}^* T_1^* + \sum_{n=1}^{s} S_1 S_{m(n)}^* T_\mu T_\mu^* S_{m(n)} S_{m(n)}^* T_1^* \\
&= \sum_{i=1}^{r} S_1 T_\mu T_\mu^* S_{k(i)} S_{k(i)}^* S_1^* + \sum_{n=1}^{s} S_1 S_{m(n)}^* T_\mu T_\mu^* S_{m(n)} S_1^*
\end{aligned}
$$

so that $T_1 T_\mu T_\mu^* T_1^* \in \mathfrak{D}_A$.

(ii) Suppose that $S_\nu S_\nu^* \in \mathfrak{D}_B$. It suffices to show that $S_1 S_\nu S_\nu^* S_1^* \in \mathfrak{D}_B$. As in the proof of [11, Theorem 3.7], one sees that

$$
S_1 = T_1 (T_{k(1)} T_{k(1)}^* + \cdots + T_{k(r)} T_{k(r)}^* + T_{m(1)} + \cdots + T_{m(s)}). \tag{3.3}
$$

We note that $T_{k(i)}^* T_{k(j)} = 0$ for $i \neq j$ and $T_{m(n)}^* T_{m(n)} = S_{m(n)}^* S_{m(n)}$ for $n = 1, \dots, s$. As $T_1 T_1^* = S_1 S_1^*$, one has $T_{k(i)} T_{k(i)}^* = S_{k(i)} S_{k(i)}^*$ for $i = 1, \dots, r$ so that

$$
T_{k(i)} T_{k(i)}^* S_{m(n)}^* S_{m(n)} = S_{k(i)} S_{k(i)}^* S_{m(n)}^* S_{m(n)} = 0
$$

because of (3.1). As $m(n) \neq 1$, we have $T_{k(i)} T_{k(i)}^* T_{m(n)}^* T_{m(n)} = 0$ for $i = 1, \dots, r,\ n = 1, \dots, s$ so that

$$
T_{k(i)} T_{k(i)}^* S_1^* = T_{k(i)} T_{k(i)}^* T_1^*, \qquad T_{m(n)}^* T_{m(n)} S_1^* = T_{m(n)}^* T_1^*
$$

by (3.3). It follows that

$$
T_{k(i)} T_{k(i)}^* S_\nu S_\nu^* S_1^* = S_\nu S_\nu^* T_{k(i)} T_{k(i)}^* S_1^* = S_\nu S_\nu^* T_{k(i)} T_{k(i)}^* T_1^*
$$

and

$$
T_{m(n)} S_\nu S_\nu^* S_1^* = T_{m(n)} S_\nu S_\nu^* T_{m(n)}^* T_{m(n)} S_1^* = T_{m(n)} S_\nu S_\nu^* T_{m(n)}^* T_1^*.
$$

Hence we have

$$
\begin{aligned}
S_1 S_\nu S_\nu^* S_1^* &= \sum_{i=1}^{r} T_1 T_{k(i)} T_{k(i)}^* S_\nu S_\nu^* S_1^* + \sum_{n=1}^{s} T_1 T_{m(n)} S_\nu S_\nu^* S_1^* \\
&= \sum_{i=1}^{r} T_1 S_\nu S_\nu^* T_{k(i)} T_{k(i)}^* T_1^* + \sum_{n=1}^{s} T_1 T_{m(n)} S_\nu S_\nu^* T_{m(n)}^* T_1^*
\end{aligned}
$$

so that $S_1 S_\nu S_\nu^* S_1^* \in \mathfrak{D}_B$. $\square$

As $S_jS_j^* = T_jT_j^*$ for $j = 1, \dots, N$, we have the following lemma.

Lemma 3.2. $C^*(S_\nu S_\nu^*; \nu \in B_*(X_A)) = C^*(T_\mu T_\mu^*; \mu \in B_*(X_B))$.

Therefore we have

Proposition 3.3. *Let A, B be irreducible matrices with entries in $\{0,1\}$ satisfying condition (I). Suppose that A is primitively equivalent to B. Then there exists an isomorphism $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\Psi(\mathfrak{D}_A) = \mathfrak{D}_B$.*

By Proposition 2.1 we obtain

Theorem 3.4. *Let A, B be irreducible 3×3 matrices with entries in $\{0,1\}$ satisfying condition (I). Then the following are equivalent:*

(i) *$\mathcal{O}_A$ is isomorphic to $\mathcal{O}_B$.*
(ii) *(X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*
(iii) *A is primitively equivalent to B.*

Proof. The implications (i) $\Longleftrightarrow$ (iii) come from [11, Theorem 4.1].

The implication (ii) $\Longrightarrow$ (i) comes from Proposition 2.1.

The implication (iii) $\Longrightarrow$ (ii) comes from Proposition 3.3 with Proposition 2.1. $\square$

4. Out-splitting and out-amalgamation

Primitive equivalence preserves the size of matrices. We need a weaker equivalence relation than primitive equivalence in the matrices to describe continuous orbit equivalence classes in the associated one-sided topological Markov shifts.

Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\{0,1\}$. We consider the associated directed graph $G_A = (V_A, E_A)$ with vertex set V_A consisting of N vertices labeled $\{1, \dots, N\}$. In this section, we use a technique from theory of symbolic dynamical systems so that the notations I, J, etc. for vertices of a graph will be used in stead of i, j, etc. following a common usage in symbolic dynamical systems. A directed edge $e \in E_A$ is defined if $A(I,J) = 1$ as its source vertex $s(e) = I$ and its terminal vertex $t(e) = J$. For a vertex $I \in V_A$, denote by $\mathcal{E}_I$ the set of edges in E_A starting at I. For each $I \in V_A$, partition $\mathcal{E}_I$ into disjoint sets $\mathcal{E}_I^1, \dots, \mathcal{E}_I^{m(I)}$ with $m(I) \geq 1$. Let $\mathcal{P}$ denote the resulting partition of E_A. Then as in [18, Definition 2.4.3], one may construct the state split graph $G_A^{[\mathcal{P}]} = (V_{A^{[\mathcal{P}]}}, E_{A^{[\mathcal{P}]}})$ formed from G_A using $\mathcal{P}$, where the vertex set $V_{A^{[\mathcal{P}]}}$ consists of $\{I^i \mid i = 1, \dots, m(I),\ I \in V_A\}$. For $e \in \mathcal{E}_I^i$ with $s(e) =$

$I, t(e) = J$, an edge $e^j \in E_{A^{[\mathcal{P}]}}$ is defined as $s(e^j) = I^i, t(e^j) = J^j$ for $j = 1, \dots, m(J)$. Denote by $A^{[\mathcal{P}]}$ the adjacency matrix of the directed graph $G_A^{[\mathcal{P}]}$. We say that the matrix $A^{[\mathcal{P}]}$ is an out-splitting matrix from A by using $\mathcal{P}$. Conversely A is called the out-amalgamation matrix from $A^{[\mathcal{P}]}$ by using $\mathcal{P}$. Then A is obtained from $A^{[\mathcal{P}]}$ by iteratively deleting multiple copies of repeated columns and adding corresponding rows. We will show that $\mathcal{O}_A = \mathcal{O}_{A^{[\mathcal{P}]}}$ and $\mathfrak{D}_A = \mathfrak{D}_{A^{[\mathcal{P}]}}$ as subalgebras of $\mathcal{O}_A (= \mathcal{O}_{A^{[\mathcal{P}]}})$. By [28], the one-sided topological Markov shifts (X_A, σ_A) and $(X_{A^{[\mathcal{P}]}}, \sigma_{A^{[\mathcal{P}]}})$ are topologically conjugate so that they are continuously orbit equivalent to each other. As a consequence, we know that there exists an isomorphism from $\mathcal{O}_A$ onto $\mathcal{O}_{A^{[\mathcal{P}]}}$ preserving their subalgebras $\mathfrak{D}_A$ and $\mathfrak{D}_{A^{[\mathcal{P}]}}$ by Proposition 2.1. We will here directly show the following proposition without using both [28] and Proposition 2.1. Let $S_I, I \in V_A$ be the canonical generating partial isometries of $\mathcal{O}_A$ satisfying the relations (2.1) for the matrix A. The vertex set $V_{A^{[\mathcal{P}]}}$ of $G_A^{[\mathcal{P}]}$ is written as $\{I^i \mid i = 1, \dots, m(I),\ I \in V_A\}$. Put

$$T_{I^i} = S_I \sum_{J \in t(\mathcal{E}_I^i)} S_J S_J^*, \qquad I^i \in V_{A^{[\mathcal{P}]}}$$

where $t(\mathcal{E}_I^i)$ denotes the set of teminal vertices $\{t(e) \in V_A \mid e \in \mathcal{E}_I^i\}$ of edges $\mathcal{E}_I^i$.

Proposition 4.1. *Keep the above notations. We have*

(i) $T_{I^i}, I^i \in V_{A^{[\mathcal{P}]}}$ *are partial isometries satisfying the relations*

$$\sum_{J^j \in V_{A^{[\mathcal{P}]}}} T_{J^j} T_{J^j}^* = 1, \qquad T_{I^i}^* T_{I^i} = \sum_{J^j \in V_{A^{[\mathcal{P}]}}} A^{[\mathcal{P}]}(I^i, J^j) T_{J^j} T_{J^j}^*, \quad I^i \in V_{A^{[\mathcal{P}]}}.$$

(ii) $T_{I^i}, I^i \in V_{A^{[\mathcal{P}]}}$ *generate* $\mathcal{O}_A$.

(iii) $C^*(S_\nu S_\nu^*; \nu \in B_*(X_A)) = C^*(T_\mu T_\mu^*; \mu \in B_*(X_{A^{[\mathcal{P}]}}))$.

Hence there exists an isomorphism between the C^*-algebras $\mathcal{O}_A$ and $\mathcal{O}_{A^{[\mathcal{P}]}}$ preserving their subalgebras $\mathfrak{D}_A$ and $\mathfrak{D}_{A^{[\mathcal{P}]}}$.

Proof. (i) Since $S_I^* S_I$ commutes with $S_J S_J^*, I, J \in V_A$, the operator T_{I^i} is a partial isometry. Put $P_J = S_J S_J^*, J \in V_A$. As $\sum_{j=1}^{m(J)} \sum_{K \in t(\mathcal{E}_J^j)} P_K = \sum_{K \in V_A} A(J, K) P_K = S_J^* S_J$, one has

$$\sum_{j=1}^{m(J)} T_{J^j} T_{J^j}^* = S_J (\sum_{j=1}^{m(J)} \sum_{K \in t(\mathcal{E}_J^j)} P_K) S_J^* = S_J S_J^*.$$

Since $V_{A^{[\mathcal{P}]}} = \{J^j \mid j = 1, \dots, m(J), J \in V_A\}$, one sees that

$$\sum_{J^j \in V_{A^{[\mathcal{P}]}}} T_{J^j} T_{J^j}^* = \sum_{J \in V_A} \sum_{j=1}^{m(J)} T_{J^j} T_{J^j}^* = \sum_{J \in V_A} S_J S_J^* = 1.$$

By the inequality $S_I^* S_I \ge \sum_{J\in t(\mathcal{E}_I^i)} P_J$ for $i = 1,\dots,m(I)$, one has

$$T_{I^i}^* T_{I^i} = (\sum_{J\in t(\mathcal{E}_I^i)} P_J) S_I^* S_I (\sum_{J\in t(\mathcal{E}_I^i)} P_J) = \sum_{J\in t(\mathcal{E}_I^i)} P_J.$$

We note that $A^{[\mathcal{P}]}(I^i, J^j) = 1$ if and only if $J \in t(\mathcal{E}_I^i)$ for $j = 1,\dots,m(J)$. It then follows that

$$\begin{aligned}
\sum_{J^j\in V_{A^{[\mathcal{P}]}}} A^{[\mathcal{P}]}(I^i, J^j) T_{J^j} T_{J^j}^* &= \sum_{J\in V_A} \sum_{j=1}^{m(J)} A^{[\mathcal{P}]}(I^i, J^j) T_{J^j} T_{J^j}^* \\
&= \sum_{J\in t(\mathcal{E}_I^i)} \sum_{j=1}^{m(J)} S_J (\sum_{K\in t(\mathcal{E}_J^j)} P_K) S_J^* \\
&= \sum_{J\in t(\mathcal{E}_I^i)} S_J (\sum_{K\in V_A} A(J,K) P_K) S_J^* \\
&= \sum_{J\in t(\mathcal{E}_I^i)} S_J S_J^* S_J S_J^* = T_{I^i}^* T_{I^i}.
\end{aligned}$$

(ii) For $I \in V_A$, we have

$$S_I = S_I S_I^* S_I = S_I (\sum_{i=1}^{m(I)} \sum_{J\in t(\mathcal{E}_I^i)} P_J) = \sum_{i=1}^{m(I)} T_{I^i}$$

so that the algebra $\mathcal{O}_A$ is generated by $T_{I^i}, I^i \in V_{A^{[\mathcal{P}]}}$.

(iii) The projection $T_{I^i} T_{I^i}^* = S_I (\sum_{J\in t(\mathcal{E}_I^i)} P_J) S_I^*$ belongs to $\mathfrak{D}_A$. For $I_1^{i_1}, I_2^{i_2}, \dots, I_n^{i_n}$ it is straightforward to see that

$$\begin{aligned}
&T_{I_1^{i_1}} T_{I_2^{i_2}} \cdots T_{I_n^{i_n}} T_{I_n^{i_n}}^* \cdots T_{I_2^{i_2}}^* T_{I_1^{i_1}}^* \\
=&\begin{cases} S_{I_1} S_{I_2} \cdots S_{I_n} (\sum_{J\in t(\mathcal{E}_{I_n}^{i_n})} P_J) S_{I_n}^* \cdots S_{I_2}^* S_{I_1}^* & \text{if } I_2 \in t(\mathcal{E}_{I_1}^{i_1}), \dots, I_n \in t(\mathcal{E}_{I_{n-1}}^{i_{n-1}}), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

On the other hand, as $t(\mathcal{E}_I^j) \cap t(\mathcal{E}_I^k) = \emptyset$ for $j \ne k$, we know

$$T_{I^j} T_{I^k}^* = S_I (\sum_{J\in t(\mathcal{E}_I^j)} \sum_{K\in t(\mathcal{E}_I^k)} P_J P_K) S_I^* = 0$$

so that

$$S_{I_1} S_{I_2} \cdots S_{I_n} S_{I_n}^* \cdots S_{I_2}^* S_{I_1}^* = \sum_{i_1=1}^{m(I_1)} \sum_{i_2=1}^{m(I_2)} \cdots \sum_{i_n=1}^{m(I_n)} T_{I_1^{i_1}} T_{I_2^{i_2}} \cdots T_{I_n^{i_n}} T_{I_n^{i_n}}^* \cdots T_{I_2^{i_2}}^* T_{I_1^{i_1}}^*.$$

Therefore we conclude that

$$C^*(S_\nu S_\nu^*; \nu \in B_*(X_A)) = C^*(T_\mu T_\mu^*; \mu \in B_*(X_{A^{[\mathcal{P}]}})). \qquad \square$$

5. Primitively amalgamated equivalence

We introduce an equivalence relation called primitively amalgamated equivalence (p.a. equivalence for short) in the set of square matrices with entries in $\{0,1\}$. For two square matrices A, B with entries in $\{0,1\}$, A is said to be primitively amalgamated equivalent to B if there exists a finite chain $C_0, C_1, \dots, C_K$ of square matrices with entries in $\{0,1\}$ such that $C_0 = A, C_K = B$ and C_{i-1}, C_i satisfy one of the following three conditions for $i = 1, \dots, K$:

(a) C_{i-1} is primitively equivalent to C_i,
(b) C_{i-1} is an out-splitting matrix from C_i,
(c) C_{i-1} is an out-amalgamation matrix from C_i.

We write this situation as $A \underset{p.a.}{\sim} B$. It is not difficult to see that both the properties of irreducibility and condition (I) are preserved under primitively amalgamated equivalence. Then we have

Theorem 5.1. *Let A, B be irreducible square matrices with entries in $\{0,1\}$ satisfying condition (I). Consider the following three conditions:*

(i) *$\mathcal{O}_A$ is isomorphic to $\mathcal{O}_B$.*
(ii) *(X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*
(iii) *$A \underset{p.a.}{\sim} B$.*

Then we have

$$\text{(iii)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(i)}.$$

If in particular, the sizes of the matrices A and B are both less than or equal to three, we have

$$\text{(i)} \Longrightarrow \text{(iii)},$$

so that all the three conditions above are mutually equivalent.

Proof. The implication (iii) $\Longrightarrow$ (ii) comes from Proposition 3.3 and Proposition 4.1 with Proposition 2.1. The implication (ii) $\Longrightarrow$ (i) comes from Proposition 2.1. It suffices to show the implication (i) $\Longrightarrow$ (iii) for the matrices A, B whose sizes are both less than or equal to three. Suppose that (i) holds. If A and B are both 3×3 matrices, (iii) holds by Theorem 3.4. If A and B are both 2×2 matrices, they are $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, because of its irreducibility with condition (I). Since the C^*-algebras $\mathcal{O}_{\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]}$ and $\mathcal{O}_{\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]}$ are both isomorphic to $\mathcal{O}_2$, and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is primitively equivalent to $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, the implication (i) $\Longrightarrow$ (iii) holds in this case. We may finally assume that A is a 3×3 matrix and B is a

2×2 matrix. Since $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underset{p.a.}{\sim} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, we may assume that $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. By the hypothesis that $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_B$, the 3×3 matrix A is one of the 13 matrices in the classification table [11, p.450], whose representative is $\mathcal{O}_2$. The 2×2 matrix B is an out-amalgamation matrix from $\left[\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{smallmatrix}\right]$, which is one of the 13 matrices. Since any two matrices of the 13 matrices are primitively equivalent to each other, we conclude $A \underset{p.a.}{\sim} B$. Therefore the implication (i) $\Longrightarrow$ (iii) holds. $\square$

6. A counter example

We will finally present an example of a pair of matrices A and B one of whose sizes is 4 such that the implication (i) $\Longrightarrow$ (iii) in Theorem 5.1 does not hold.

Lemma 6.1. *For a square matrix A with entries in $\{0,1\}$, $\det(1-A)$ is invariant under primitively amalgamated equivalence.*

Proof. $\det(1-A)$ is invariant under primitive equivalence by [11, Theorem 8.4]. An out-amalgamation yields a topological conjugacy on the associated two-sided topological Markov shifts. Hence it gives rise to a flow equivalence between them so that $\det(1-A)$ is invariant under out-amalgamation by [1]. $\square$

For an $N \times N$ square matrix $A = [A(i,j)]_{i,j=1}^N$ with entries in $\{0,1\}$, J. Cuntz in [7] has introduced an $(N+2)\times(N+2)$ matrix A_- defined by setting

$$A_- = \begin{bmatrix} A(1,1) & \dots & A(1,N) & & \\ \vdots & & \vdots & & 0 \\ A(N,1) & \dots & A(N,N) & 1 & \\ & & 1 & 1 & 1 \\ 0 & & & 1 & 1 \end{bmatrix}$$

In [25], it has shown that the Cuntz algebra $\mathcal{O}_A$ for the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is isomorphic to $\mathcal{O}_{A_-}$. Therefore we have

Proposition 6.2. *$\mathcal{O}_A$ is isomorphic to $\mathcal{O}_{A_-}$, however A is not primitively amalgamated equivalent to A_-. Therefore if one of the sizes of the matrices A and B is greater than or equal to four, the implication* (i) $\Longrightarrow$ (iii) *in Theorem 5.1 does not necessarily hold.*

Proof. As in [25], $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_{A_-}) \cong 0$ so that $\mathcal{O}_A$ is isomorphic to $\mathcal{O}_{A_-}$.

Since $\det(1 - A) = -1 \neq 1 = \det(1 - A_-)$, one sees that A is not primitively amalgamated equivalent to A_-. $\square$

A related result to Theorem 1.1 has been obtained in a recent paper [22].

Acknowledgment: This work was supported by JSPS Grant-in-Aid for Scientific Reserch ((C), No 20540215, No 23540237).

References

[1] R. Bowen and J. Franks, Homology for zero-dimensional nonwandering sets, *Ann. Math.* **106** (1977), 73–92.

[2] M. Boyle, *Topological orbit equivalence and factor maps in symbolic dynamics*, Ph. D. Thesis, University of Washington, 1983.

[3] M. Boyle and D. Handelman, Orbit equivalence, flow equivalence and ordered cohomology, *Israel J. Math.*, **95** (1996), 169–210.

[4] M. Boyle and J. Tomiyama, Bounded continuous orbit equivalence and C^*-algebras, *J. Math. Soc. Japan*, **50** (1998), 317–329.

[5] A. Connes and W. Krieger, Measure space automorphisms, the normalizers of their full groups, and approximate finiteness, *J. Funct. Anal.*, **18** (1975), 318–327.

[6] J. Cuntz, Simple C^*-algebras generated by isometries, *Comm. Math. Phys.*, **57** (1977), 173–185.

[7] J. Cuntz, The classification problem for the C^*-algebra $\mathcal{O}_A$, *Geometric methods in operator algebras, Pitman Reserch Notes in Mathematics Series*, **123** (1986), 145–151.

[8] J. Cuntz and W. Krieger, A class of C^*-algebras and topological Markov chains, *Invent. Math.* **56** (1980), 251–268.

[9] H. Dye, On groups of measure preserving transformations, *American J. Math.*, **81** (1959), 119–159.

[10] H. Dye, On groups of measure preserving transformations II, *American J. Math.*, **85** (1963), 551–576.

[11] M. Enomoto, M. Fujii and Y. Watatani, K_0-groups and classifications of Cuntz-Krieger algebras, *Math. Japon.*, **26** (1981), 443–460.

[12] J. Franks, Flow equivalence of subshifts of finite type, *Ergodic Theory Dynam. Systems*, **4** (1984), 53–66.

[13] T. Giordano, I. F. Putnam and C. F. Skau, Topological orbit equivalence and C^*-crossed products, *J. reine angew. Math.*, **469** (1995), 51–111.

[14] T. Giordano, I. F. Putnam and C. F. Skau, Full groups of Cantor minimal systems, *Isr. J. Math.*, **111** (1999), 285–320.

[15] T. Giordano, H. Matui, I. F. Putnam and C. F. Skau, Orbit equivalemce for Cantor minimal $\mathbb{Z}^2$-systems, *J. Amer. Math. Soc.*, **21** (2008), 863–892.

[16] B. P. Kitchens, *Symbolic dynamics*, Springer-Verlag, Berlin, Heidelberg and New York (1998).

[17] W. Krieger, On ergodic flows and isomorphisms of factors, *Math. Ann.*, **223** (1976), 19–70.

[18] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, (1995).

[19] K. Matsumoto, On automorphisms of C^*-algebras associated with subshifts, *J. Operator Theory*, **44** (2000), 91–112.

[20] K. Matsumoto, Orbit equivalence in C^*-algebras defined by actions of symbolic dynamical systems, *Contemporary Math.*, **503** (2009), 121–140.

[21] K. Matsumoto, Orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras, *Pacific J. Math.*, **246** (2010), 199–225.

[22] K. Matsumoto, Classification of Cuntz–Krieger algebras by orbit equivalence of topological Markov shifts, to appear in *Proc. Amer. Math. Soc.*.

[23] H. Matui, *Homology and topological full groups of étale groupoids on totally disconnected spaces*, *Proc. Lond. Math. Soc.*, **104** (2012), 27–52.

[24] I. F. Putnam, The C^*-algebras associated with minimal homeomorphisms of the Cantor set, *Pacific. J. Math.*, **136** (1989), 329–353.

[25] M. Rørdam, Classification of Cunzt-Krieger algebras, *K-theory*, **9** (1995), 31–58.

[26] J. Tomiyama, Topological full groups and structure of normalizers in transformation group C^*-algebras, *Pacific. J. Math.*, **173** (1996), 571–583.

[27] J. Tomiyama, Representation of topological dynamical systems and C^*-algebras, *Contemporary Math.*, **228** (1998), 351–364.

[28] R. F. Williams, Classification of subshifts of finite type, *Ann. Math.*, **98** (1973), 120–153. *erratum, Ann. Math.*, **99** (1974), 380–381.

Department of Mathematics,
Joetsu University of Education,
Joetsu, 943-8512, JAPAN
E-mail: `kengo@juen.ac.jp`