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A STONE-TYPE DECOMPOSITION OF THE RENEWAL MEASURE WITH EXACT ASYMPTOTICS OF THE SUMMANDS

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Abstract. A Stone-type decomposition $U = U_1 + U_2$ of the renewal measure U is established via Banach-algebraic approach. Banach algebras of measures are used with given functional $\mathscr{L}$ describing a certain asymptotic property of their elements. The values of $\mathscr{L}$ at U_1 and U_2 are given in terms of the value of $\mathscr{L}$ at the probability distribution generating the renewal measure U.

If ν and κ are two nonnegative measures defined on the σ-algebra $\mathscr{B}$ of all Borel subsets of the real line $\mathbb{R}$, then the measure

$$\nu * \kappa(A) := \iint_{\{x+y\in A\}} \nu(dx)\,\kappa(dy) = \int_{\mathbb{R}} \nu(A-y)\,\kappa(dy), \qquad A \in \mathscr{B},$$

is called the convolution of ν and κ; here $A - y := \{x \in \mathbb{R} \colon x + y \in A\}$.

Let F be a probability distribution on $\mathbb{R}$ with positive mean $\mu = \int_{\mathbb{R}} x\,F(dx)$, and let $U = \sum_{n=0}^{\infty} F^{n*}$ be the corresponding renewal measure; here $F^{1*} := F$, $F^{(n+1)*} := F * F^{n*}$, $n \geq 1$, and $F^{0*} := \delta_0$, the atomic measure of unit mass at the origin. Suppose that, for some $m \geq 1$, F^{m*} has a nonzero absolutely continuous component. Stone [14] showed that then there exists a decomposition $U = U_1 + U_2$, where U_2 is a finite measure and U_1 is absolutely continuous with bounded continuous density $h(x)$ such that

$$\lim_{x\to\infty} h(x) = \frac{1}{\mu}, \qquad \lim_{x\to-\infty} h(x) = 0. \tag{1}$$

A similar decomposition $U = U_1 + U_2$ was established in [13] via Banach-algebraic techniques which allowed us to extract rather detailed supplementary information about the asymptotic properties of the summands U_1 and U_2 depending on the corresponding properties of the underlying distribution F. Notice that in [14] and [13] the measures U_1 and U_2 were constructed differently. Nevertheless, in both cases the measure U_1 was absolutely continuous with bounded continuous density $h(x)$ satisfying the relations (1).

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In the present paper, we develop further the approach used in [13] by employing this time Banach algebras of measures with a functional $\mathscr{L}$ which characterizes a certain asymptotic property of their elements. For instance, $\mathscr{L}$ may characterize the asymptotic behaviour of the tails $\nu((x,\infty))$ or that of $\nu((x,x+h])$, or the asymptotics of the densities $f(x)$ of measures ν as $x \to \infty$ (see [10], [7], [9]). We evaluate $\mathscr{L}$ at the summands of the decomposition $U = U_1 + U_2$ depending on the value of $\mathscr{L}$ at the underlying distribution F and give exact convergence rates in (1).

Definition 1. A function $\varphi(x)$, $x \in \mathbb{R}$, is called *submultiplicative* if $\varphi(x)$ is a finite, positive, Borel-measurable function with the following properties:

$$\varphi(0) = 1, \qquad \varphi(x+y) \le \varphi(x)\varphi(y) \qquad \text{for all } x, y \in \mathbb{R}.$$

It is well known [5, Section 7.6] that

$$-\infty < r_1 := \lim_{x\to-\infty} \frac{\log\varphi(x)}{x} = \sup_{x<0} \frac{\log\varphi(x)}{x} \le \inf_{x>0} \frac{\log\varphi(x)}{x} = \lim_{x\to\infty} \frac{\log\varphi(x)}{x} =: r_2 < \infty. \quad (2)$$

Let $S(\varphi)$ be the collection of all complex-valued measures κ such that $\|\kappa\|_\varphi := \int_{\mathbb{R}} \varphi(x)\,|\kappa|(dx) < \infty$; here $|\kappa|$ stands for the total variation of κ. The collection $S(\varphi)$ is a Banach algebra with norm $\|\cdot\|_\varphi$ by the usual operations of addition and scalar multiplication of measures, the product of two elements ν and κ of $S(\varphi)$ is defined as their convolution $\nu * \kappa$ [5, Section 4.16]. The unit element of $S(\varphi)$ is the measure δ_0. Define the Laplace transform of a measure κ as $\hat{\kappa}(s) := \int_{\mathbb{R}} \exp(sx)\,\kappa(dx)$. Then relation (2) implies that the Laplace transform of any $\kappa \in S(\varphi)$ converges absolutely with respect to $|\kappa|$ for all s in the strip $\Pi(r_1, r_2) := \{s \in \mathbb{C} : r_1 \le \Re s \le r_2\}$. Put $S(r_1, r_2) := S(\varphi)$ where $\varphi(x) = \max(e^{r_1 x}, e^{r_2 x})$, $r_1 \le 0 \le r_2$.

Let ν be a finite complex-valued measure. Denote by $T\nu$ the σ-finite measure with the density $v(x;\nu) := \nu((x,\infty))$ for $x \ge 0$ and $v(x;\nu) := -\nu((-\infty, x])$ for $x < 0$. If $\int_{\mathbb{R}} |x|\,|\nu|(dx) < \infty$, then $T\nu$ is a finite measure whose Laplace transform is given by $(T\nu)^\wedge(s) = [\hat{\nu}(s) - \hat{\nu}(0)]/s$, $\Re s = 0$, the value $(T\nu)^\wedge(0)$ being defined by continuity as $\int_{\mathbb{R}} x\,\nu(dx)$.

The absolutely continuous part of any distribution F with respect to Lebesgue measure will be denoted by F_c, and its singular component by F_s, i.e., $F_s := F - F_c$; thus, here by the singular component of F we mean the sum of its *ordinary singular* component and its discrete component.

Let $\mathscr{A}$ be a Banach algebra of measures such that (i) $\mathscr{A} \subset S(r_1, r_2)$ and (ii) each homomorphism $\mathscr{A} \mapsto \mathbb{C}$ is the restriction to $\mathscr{A}$ of some homomorphism

$S(r_1, r_2) \mapsto \mathbb{C}$. Property (ii) can be restated as follows: Each maximal ideal M of $\mathscr{A}$ is of the form $M_1 \cap \mathscr{A}$, where M_1 is a maximal ideal of $S(r_1, r_2)$. It follows from the general theory of Banach algebras that if $\nu \in \mathscr{A}$ is invertible in $S(r_1, r_2)$, then $\nu^{-1} \in \mathscr{A}$.

In what follows, F will denote a probability distribution with finite mean $\mu > 0$ such that $F \in S(r_1, r_2)$, $r_1 \leq 0 \leq r_2$; $(F^{m*})_s^{\wedge}(r_i) < 1$, $i = 1, 2$, for some integer $m \geq 1$; and $\hat{F}(s) \neq 1$ for all $s \in \Pi(r_1, r_2) \setminus \{0\}$.

Let $\mathscr{A}_{\mathscr{L}}$ be a Banach algebra having properties (i) and (ii) and let there exist a continuous linear functional $\mathscr{L} : \mathscr{A}_{\mathscr{L}} \to \mathbb{C}$ such that $\mathscr{L}(\delta_0) = 0$ and

$$\mathscr{L}(\kappa * \theta) = \mathscr{L}(\kappa)\hat{\theta}(r_2) + \hat{\kappa}(r_2)\mathscr{L}(\theta) \tag{3}$$

for all $\kappa, \theta \in \mathscr{A}_{\mathscr{L}}$. Moreover, if $\kappa, T\kappa \in \mathscr{A}_{\mathscr{L}}$, then $\mathscr{L}(\kappa) = r_2\mathscr{L}(T\kappa)$. Concrete examples of such algebras $\mathscr{A}_{\mathscr{L}}$ may be found in [7], [9], [10] and others. Let L be the restriction of Lebesgue measure to $[0, \infty)$.

Theorem 1. *Let $\mathscr{A}_{\mathscr{L}}$ be a Banach algebra with the stated properties. Suppose F, $TF \in \mathscr{A}_{\mathscr{L}}$. Then the renewal measure $U = \sum_{n=0}^{\infty} F^{n*}$ admits a Stone-type decomposition $U = U_1 + U_2$, where $U_2 \in \mathscr{A}_{\mathscr{L}}$ and the measure $U_1 = L/\mu - rTU_2$ for some $r > r_2$ is absolutely continuous with bounded continuous density $h(x)$. Moreover,*

$$\mathscr{L}(U_2) = \begin{cases} \dfrac{r_2\mathscr{L}(F)}{(r_2 - r)[1 - \hat{F}(r_2)]^2} & \text{if } r_2 > 0, \\ -\dfrac{\mathscr{L}(TF)}{r\mu^2} & \text{if } r_2 = 0. \end{cases} \tag{4}$$

If, in addition, $T^2F \in \mathscr{A}_{\mathscr{L}}$, then $U_1 - L/\mu \in \mathscr{A}_{\mathscr{L}}$ and

$$\mathscr{L}\left(U_1 - \frac{L}{\mu}\right) = \begin{cases} \dfrac{r\mathscr{L}(F)}{(r - r_2)[1 - \hat{F}(r_2)]^2} & \text{if } r_2 > 0, \\ \dfrac{\mathscr{L}(T^2F)}{\mu^2} & \text{if } r_2 = 0. \end{cases} \tag{5}$$

Proof. The existence of such decomposition $U = U_1 + U_2$ follows from Theorem 3.1 in [13] for a Banach algebra of type $\mathscr{A}$ and, in particular, for $\mathscr{A}_{\mathscr{L}}$. It remains to evaluate the functional $\mathscr{L}$ at U_2 and $U_1 - L/\mu$. We have $U_2 = V^{-1} \in \mathscr{A}_{\mathscr{L}}$, where $V := \delta_0 - F + rTF \in \mathscr{A}_{\mathscr{L}}$ for some $r > r_2$ and

$$\hat{V}(s) := \frac{(s - r)[1 - \hat{F}(s)]}{s}, \qquad s \in \Pi(r_1, r_2),$$

the value $\hat{V}(0)$ being defined by continuity as $r\mu$ (see the proof of Theorem 3.1 in [13]). Notice that $\hat{U}_2(s) = 1/\hat{V}(s)$. Since $V * V^{-1} = \delta_0$, we have

$$\mathscr{L}(V * V^{-1}) = \mathscr{L}(V)(V^{-1})^{\wedge}(r_2) + \hat{V}(r_2)\mathscr{L}(V^{-1}) = \mathscr{L}(\delta_0) = 0.$$

If $r_2 > 0$, then

$$\begin{aligned}\mathscr{L}(U_2) &= \mathscr{L}(V^{-1}) = -\frac{\mathscr{L}(V)}{[\hat{V}(r_2)]^2} \\ &= -\frac{[-\mathscr{L}(F) + r\mathscr{L}(TF)]r_2^2}{(r_2 - r)^2[1 - \hat{F}(r_2)]^2} = -\frac{[-r_2\mathscr{L}(TF) + r\mathscr{L}(TF)]r_2^2}{(r_2 - r)^2[1 - \hat{F}(r_2)]^2} \\ &= \frac{\mathscr{L}(TF)r_2^2}{(r_2 - r)[1 - \hat{F}(r_2)]^2} = \frac{\mathscr{L}(F)r_2}{(r_2 - r)[1 - \hat{F}(r_2)]^2}.\end{aligned}$$

If $r_2 = 0$, then

$$\mathscr{L}(U_2) = -\frac{\mathscr{L}(V)}{(r\mu)^2} = -\frac{r\mathscr{L}(TF)}{(r\mu)^2} = -\frac{\mathscr{L}(TF)}{r\mu^2}.$$

Now let also $T^2F \in \mathscr{A}_{\mathscr{L}}$. The measure U_1 was defined as $U_1 := L/\mu - rTU_2$ (see the proof of Theorem 3.1 in [13]). Hence $U_1 - L/\mu = -rTU_2$. The hypothesis $T^2F \in \mathscr{A}_{\mathscr{L}}$ implies that T^2F is a finite measure, hence the integrals $\int_{\mathbb{R}} |x|\,|TF|(dx)$ and $\int_{\mathbb{R}} x^2\,F(dx)$ are finite. Put $\varphi(x) := 1 + |x|$, $x \in \mathbb{R}$. Then $S(\varphi)$ is a Banach algebra of type $\mathscr{A}$ with $r_1 = r_2 = 0$ and $V \in S(\varphi)$, whence $U_2 = V^{-1} \in S(\varphi)$, i.e. $\int_{\mathbb{R}} |x|\,|U_2|(dx) < \infty$ (see the proof of Theorem 3.1 in [13]). It follows that TU_2 is a finite measure. We have

$$\begin{aligned}(TU_2)^\wedge(s) &= \frac{\hat{U}_2(s) - \hat{U}_2(0)}{s} = -\frac{\hat{V}(s) - \hat{V}(0)}{s} \cdot \frac{1}{\hat{V}(s)\hat{V}(0)} \\ &= -\frac{\hat{U}_2(s)(TV)^\wedge(s)}{\hat{V}(0)} = \frac{\hat{U}_2(s)[(TF)^\wedge(s) - r(T^2F)^\wedge(s)]}{r\mu}.\end{aligned}$$

Thus,

$$U_1 - \frac{L}{\mu} = U_2 * \frac{rT^2F - TF}{\mu}. \tag{6}$$

If $r_2 > 0$, then

$$\mathscr{L}\left(U_1 - \frac{L}{\mu}\right) = -r\mathscr{L}(TU_2) = -\frac{r\mathscr{L}(U_2)}{r_2} = \frac{r\mathscr{L}(F)}{(r - r_2)[1 - \hat{F}(r_2)]^2}.$$

If $r_2 = 0$, then, taking into account $\mathscr{L}(TF) = 0 \cdot \mathscr{L}(T^2F)$, we have $\mathscr{L}(U_2) = 0$, and, by (3) and (6),

$$\mathscr{L}\left(U_1 - \frac{L}{\mu}\right) = \hat{U}_2(0)\frac{r\mathscr{L}(T^2F) - \mathscr{L}(TF)}{\mu} = \frac{r\mathscr{L}(T^2F)}{\hat{V}(0)\mu} = \frac{\mathscr{L}(T^2F)}{\mu^2}.$$

■

Corollary 2. *Suppose F, TF, $T^2F \in \mathscr{A}_{\mathscr{L}}$. Then*

$$\mathscr{L}\Big(U - \frac{L}{\mu}\Big) = \begin{cases} \dfrac{\mathscr{L}(F)}{[1-\hat{F}(r_2)]^2} & \textit{if } r_2 > 0, \\ \dfrac{\mathscr{L}(T^2F)}{\mu^2} & \textit{if } r_2 = 0. \end{cases}$$

Proof. We have $\mathscr{L}(U - L/\mu) = \mathscr{L}(U_2) + \mathscr{L}(U_1 - L/\mu)$. If $r_2 > 0$, then, by (4) and (5),

$$\mathscr{L}\Big(U - \frac{L}{\mu}\Big) = \frac{r_2\mathscr{L}(F)}{(r_2 - r)[1-\hat{F}(r_2)]^2} + \frac{r\mathscr{L}(F)}{(r - r_2)[1-\hat{F}(r_2)]^2} = \frac{\mathscr{L}(F)}{[1-\hat{F}(r_2)]^2}.$$

If $r_2 = 0$, then $\mathscr{L}(U_2) = 0$ and $\mathscr{L}(U - L/\mu) = \mathscr{L}(U_1 - L/\mu)$. ■

Remark 3. Let us dwell in more detail on the conditions $(F^{m*})_s^{\wedge}(r_i) < 1$, $i = 1, 2$, for some integer $m \geq 1$. These inequalities, along with the conditions $\hat{F}(s) \neq 1$ for all $s \in \Pi(r_1, r_2) \setminus \{0\}$ and $\mu > 0$, ensure the invertibility of the element V in $S(r_1, r_2)$ and hence in $\mathscr{A}_{\mathscr{L}}$ (see the proof of Theorem 3.1 in [13] and [12, Lemma 2]). Let $\mathscr{M}$ be the space of maximal ideals of the Banach algebra $S(r_1, r_2)$. The following facts are well known from the theory of Banach algebras. Each maximal ideal $M \in \mathscr{M}$ induces a homomorphism of the Banach algebra $S(r_1, r_2)$ onto the field of complex numbers $\mathbb{C}$; moreover, M is the kernel of this homomorphism. Denote by $\nu(M)$ the value of this homomorphism at $\nu \in S(r_1, r_2)$. An element $\nu \in S(r_1, r_2)$ has an inverse if and only if ν does not belong to any maximal ideal $M \in \mathscr{M}$. In other words, ν is invertible if and only if $\nu(M) \neq 0$ for all $M \in \mathscr{M}$. The space $\mathscr{M}$ is split into two sets: $\mathscr{M}_1$ is the set of those maximal ideals which do not contain the collection $L(r_1, r_2)$ of all absolutely continuous measures from $S(r_1, r_2)$ and $\mathscr{M}_2 = \mathscr{M} \setminus \mathscr{M}_1$. If $M \in \mathscr{M}_1$, then the homomorphism $S(r_1, r_2) \to \mathbb{C}$ induced by M is of the form $\nu(M) = \hat{\nu}(s_0)$, $\nu \in S(r_1, r_2)$, where s_0 is some complex number such that $r_1 \leq \Re s_0 \leq r_2$. In this case, $M = \{\mu \in S(r_1, r_2) : \hat{\mu}(s_0) = 0\}$ [5, Chapter IV, Section 4]. If $M \in \mathscr{M}_2$, then $\nu(M) = 0$ for all $\nu \in L(r_1, r_2)$. Due to the conditions $(F^{m*})_s^{\wedge}(r_i) < 1$, $i = 1, 2$, for $m \geq 1$, $V(M) = 1 - F(M) \neq 0$ for all $M \in \mathscr{M}_2$ [12, Lemma 2]. For each $M \in \mathscr{M}_1$ different from $M_0 = \{\nu \in S(r_1, r_2) : \hat{\nu}(0) = 0\}$, we have $V(M) = (s - r)[1 - \hat{F}(s)]/s \neq 0$, $s \in \Pi(r_1, r_2) \setminus \{0\}$, which is equivalent to $F(M) = \hat{F}(s) \neq 1$. As far as the maximal ideal M_0 is concerned, we have $V(M) = \hat{V}(0) = r\mu \neq 0$. Thus, if we replace the conditions $(F^{m*})_s^{\wedge}(r_i) < 1$, $i = 1, 2$, for $m \geq 1$ and $\hat{F}(s) \neq 1$ for all $s \in \Pi(r_1, r_2) \setminus \{0\}$ by the requirement that $F(M) \neq 1$ for all $M \in \mathscr{M} \setminus \{M_0\}$, then the proof of Theorem 1 remains valid, since in this case V is also invertible in $S(r_1, r_2)$.

Remark 4. For the sake of definiteness, one can put in Theorem 1 $r := 2r_2$ if $r_2 > 0$ and $r := 1$ if $r_2 = 0$. However, comparing Theorem 1 with Corollary 2, we see that the larger r is, the better the summand U_1 "approximates" the renewal measure U.

Let us apply Theorem 1 to some specific Banach algebras of measures of type $\mathscr{A}_{\mathscr{L}}$ and obtain exact rates of convergence in relations (1).

Definition 2. A probability distribution G concentrated on $[0,\infty)$ belongs to the class $\mathscr{S}(\gamma)$, $\gamma \geq 0$, if

(a) $\displaystyle\lim_{x\to\infty} \frac{G((x+y,\infty))}{G((x,\infty))} = \mathrm{e}^{-\gamma y}$ for all $y \in \mathbb{R}$,

(b) $\displaystyle\lim_{x\to\infty} \frac{G * G((x,\infty))}{G((x,\infty))} = c \in (0,\infty)$.

It follows that the constant c is equal to $2\int_0^\infty \mathrm{e}^{\gamma x}\, G(dx)$ (see corresponding discussions in [1, 2, 3, 4, 8, 6]).

Let γ be a fixed positive number. Fix also $G \in \mathscr{S}(\gamma)$. Put $\tau(x) := G((x,\infty))$, $x \geq 0$, and

$$Q(\nu) := \sup_{x\geq 0} \frac{|\nu|((x,\infty))}{\tau(x)}.$$

Consider the following collection of complex-valued σ-finite measures [10]:

$$\mathfrak{SL}(\tau) := \left\{ \nu \in S(\gamma',\gamma) : Q(\nu) < \infty, \text{ there exists } \lim_{x\to\infty} \frac{\nu((x,\infty))}{\tau(x)} =: \mathfrak{L}(\nu) \in \mathbb{C} \right\}.$$

The collection $\mathfrak{SL}(\tau)$ is a Banach algebra with a norm $\|\cdot\|'$, equivalent to the norm $\|\nu\| + Q(\nu)$. If ν, $\kappa \in \mathfrak{SL}(\tau)$, then

$$\mathfrak{L}(\nu * \kappa) = \mathfrak{L}(\nu)\hat{\kappa}(\gamma) + \hat{\nu}(\gamma)\mathfrak{L}(\kappa).$$

By Lemma 1 from [11], if $\gamma > 0$ and $\kappa \in \mathfrak{SL}(\tau)$, then $T\kappa \in \mathfrak{SL}(\tau)$ and $\mathfrak{L}(T\kappa) = \mathfrak{L}(\kappa)/\gamma$, and if $\gamma = 0$ and $T\kappa \in \mathfrak{SL}(\tau)$, then $\kappa \in \mathfrak{SL}(\tau)$ and $\mathfrak{L}(\kappa) = 0$. Thus, the Banach algebra $\mathfrak{SL}(\tau)$ is an algebra of type $\mathscr{A}_{\mathscr{L}}$ with $r_1 = 0$, $r_2 = \gamma$ and $\mathscr{L} = \mathfrak{L}$. Applying Theorem 1, we obtain the following assertions. Recall that $f(x) \sim cg(x)$ as $x \to \infty$ means that $\lim_{x\to\infty} f(x)/g(x) = c$.

Corollary 5. *Let $G \in \mathscr{S}(\gamma)$, $\gamma > 0$, and $\tau(x) = G((x,\infty))$, $x \geq 0$. Suppose $F \in \mathfrak{SL}(\tau)$, $(F^{m*})_s(\mathbb{R}) < 1$ and $(F^{m*})_s^\wedge(\gamma) < 1$ for some integer $m \geq 1$. Then the renewal measure $U = \sum_{n=0}^\infty F^{n*}$ admits a Stone-type decomposition $U = U_1 + U_2$, where $U_2 \in \mathfrak{SL}(\tau)$ and the measure $U_1 = L/\mu - rTU_2$ for some $r > \gamma$ has a bounded continuous density $h(x)$ and*

$$\mathfrak{L}(U_2) = \frac{\gamma\mathfrak{L}(F)}{(\gamma - r)[1-\hat{F}(\gamma)]^2}. \tag{7}$$

Moreover,

$$h(x) - \frac{1}{\mu} \sim \frac{\gamma r \mathfrak{L}(F)}{(r-\gamma)[1-\hat{F}(\gamma)]^2}\,\tau(x) \qquad \text{as } x \to \infty. \tag{8}$$

If, in addition, $\int_{\mathbb{R}} x^2\, F(dx) < \infty$, *then* $U_1 - L/\mu \in \mathfrak{S}\mathfrak{L}(\tau)$; *moreover,*

$$\mathfrak{L}\left(U_1 - \frac{L}{\mu}\right) = \frac{r\mathfrak{L}(F)}{(r-\gamma)[1-\hat{F}(\gamma)]^2}.$$

Corollary 6. *Let* $G \in \mathscr{S}(\gamma)$, $\gamma = 0$, *and* $\tau(x) = G((x,\infty))$, $x \ge 0$. *Suppose* $TF \in \mathfrak{S}\mathfrak{L}(\tau)$ *and* $(F^{m*})_s(\mathbb{R}) < 1$ *for some integer* $m \ge 1$. *Then the renewal measure* $U = \sum_{n=0}^{\infty} F^{n*}$ *admits a Stone-type decomposition* $U = U_1 + U_2$, *where* $U_2 \in \mathfrak{S}\mathfrak{L}(\tau)$ *and the measure* $U_1 = L/\mu - rTU_2$ *for some* $r > \gamma$ *has a bounded continuous density* $h(x)$ *and*

$$\mathfrak{L}(U_2) = -\frac{\mathfrak{L}(TF)}{r\mu^2}. \tag{9}$$

Moreover,

$$h(x) - \frac{1}{\mu} \sim \frac{\mathfrak{L}(TF)}{\mu^2}\,\tau(x) \qquad \text{as } x \to \infty. \tag{10}$$

If, in addition, $T^2F \in \mathfrak{S}\mathfrak{L}(\tau)$, *then* $U_1 - L/\mu \in \mathfrak{S}\mathfrak{L}(\tau)$; *moreover,*

$$\mathfrak{L}\left(U_1 - \frac{L}{\mu}\right) = \frac{\mathfrak{L}(T^2F)}{\mu^2}.$$

Proof of Corollaries 5 and 6. We need to prove only (8) and (10). We have

$$\frac{1}{\tau(x)}\left[h(x) - \frac{1}{\mu}\right] = -r\,\frac{U_2((x,\infty))}{\tau(x)}. \tag{11}$$

Let $\gamma > 0$. Relation (8) follows from (7) and (11). Let $\gamma = 0$. Relation (10) follows from (9) and (11). ■

Remark 7. Let $\mathscr{A}_{\mathscr{K}}$ be a Banach algebra of measures ν having properties (i) and (ii) and let there exist a continuous linear functional $\mathscr{K} : \mathscr{A}_{\mathscr{K}} \to \mathbb{C}$ such that $\mathscr{K}(\delta_0) = 0$ and

$$\mathscr{K}(\kappa * \theta) = \mathscr{K}(\kappa)\hat{\theta}(r_1) + \hat{\kappa}(r_1)\mathscr{K}(\theta)$$

for all $\kappa, \theta \in \mathscr{A}_{\mathscr{K}}$. If $\kappa, T\kappa \in \mathscr{A}_{\mathscr{K}}$, then $\mathscr{K}(\kappa) = r_1\mathscr{K}(T\kappa)$.

Analogues of Theorem 1 and Corollary 2 are also valid for Banach algebras of type $\mathscr{A}_{\mathscr{K}}$. We only need to replace in their statements and proofs $\mathscr{L}$ by $\mathscr{K}$, r_2 by r_1 and choose $r < r_1$.

By means of Banach algebras of type $\mathscr{A}_{\mathscr{L}}$ we investigate the asymptotic behaviour of the summands U_1 and U_2 and that of the density $h(x)$ of the measure U_1 at $+\infty$, whereas applying Banach algebras of type $\mathscr{A}_{\mathscr{K}}$ yields the asymptotic behaviour of U_1, U_2 and $h(x)$ at $-\infty$.

Relations (8) and (10) give exact rate of convergence in the first of the relations (1). In order to estimate the rate of convergence in $h(x) \to 0$ as $x \to -\infty$, it suffices to apply "symmetric" Banach algebras $\mathfrak{S}\mathfrak{K}(\tau)$ which are defined as follows. For an arbitrary measure ν, put $\nu_-(A) := \nu(-A)$, $A \in \mathscr{B}(\mathbb{R})$, where $-A := \{x \in \mathbb{R} : -x \in A\}$. Consider the Banach algebra $\mathfrak{S}\mathfrak{K}(\tau) := \{\nu \in S(-\gamma, 0) : \nu_- \in \mathfrak{S}\mathfrak{L}(\tau)\}$, where the norm of an element $\nu \in \mathfrak{S}\mathfrak{K}(\tau)$ is set to be equal to the norm of the corresponding element $\nu_- \in \mathfrak{S}\mathfrak{L}(\tau)$; moreover, we put $\mathfrak{K}(\nu) := \mathfrak{L}(\nu_-)$. The Banach algebra $\mathfrak{S}\mathfrak{K}(\tau)$ is a Banach algebra of type $\mathscr{A}_{\mathscr{K}}$ with $r_1 = -\gamma$, $r_2 = 0$, $\mathscr{K} = \mathfrak{K}$. As pointed out earlier, an analogue of Theorem 1 is valid for the Banach algebras of type $\mathscr{A}_{\mathscr{K}}$, whence corollaries for the algebra $\mathscr{A}_{\mathscr{K}} = \mathfrak{S}\mathfrak{K}(\tau)$ easily follow, similar to Corollaries 5 and 6. We restrict ourselves to the asymptotic behaviour of the density $h(x)$ of the measure U_1 as $x \to -\infty$. Recall that here $r < -\gamma$.

Corollary 8. *Let $G \in \mathscr{S}(\gamma)$, $\gamma \geq 0$, and $\tau(x) = G((x,\infty))$, $x \geq 0$. If $\gamma > 0$, $F \in \mathfrak{S}\mathfrak{K}(\tau)$, $(F^{m*})_s(\mathbb{R}) < 1$ and $(F^{m*})_s^\wedge(-\gamma) < 1$ for some integer $m \geq 1$, then*

$$h(x) \sim \frac{\gamma r \mathfrak{K}(F)}{(r+\gamma)[1-\hat{F}(-\gamma)]^2}\,\tau(-x) \qquad \text{as } x \to -\infty.$$

If $\gamma = 0$, $TF \in \mathfrak{S}\mathfrak{K}(\tau)$ and $(F^{m*})_s(\mathbb{R}) < 1$ for some integer $m \geq 1$, then

$$h(x) \sim \frac{\mathfrak{K}(TF)}{\mu^2}\,\tau(-x) \qquad \text{as } x \to -\infty.$$

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