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COMPLETELY INTEGRABLE IMPLICIT ORDINARY DIFFERENTIAL EQUATIONS

By

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Abstract. For smooth implicit ordinary differential equations, existence and uniqueness for solutions with initial condition do not hold in general. In this paper, we give a necessary and sufficient condition for existence of an immersive n-parameter family of geometric solutions, so-called a complete solution on the equation hypersurface in the smooth category. Moreover, we give a sufficient condition for existence of an immersive $(n-1)$-parameter family of geometric solutions on the contact singular set.

1. Introduction

For a smooth explicit ordinary differential equation

$$\frac{d^n y}{dx^n}(x) = f\left(x, y(x), \frac{dy}{dx}(x), \cdots, \frac{d^{n-1}y}{dx^{n-1}}(x)\right), \tag{1}$$

it is well-known that there exists a unique smooth local solution with an initial condition for (1), where f is a smooth function (for instance, see [1, 2, 4]). It follows that there exists an n-parameter family of smooth solutions at least locally.

On the other hand, for a smooth implicit ordinary differential equation (briefly, an implicit ODE)

$$F(x, y, p_1, \ldots, p_n) = 0, \tag{2}$$

existence for a local solution with initial condition does not hold in general, where F is a smooth function of the independent variable x, the function y and its i-th derivatives $p_i = d^i y/dx^i, i = 1, \ldots, n$.

A natural question is what conditions guarantee existence and uniqueness of a local solution and a family of solutions around a point for (2). In this paper we shall discuss a qualitative theory for implicit ODEs and establish basic notions.

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It is natural to consider (2) as being defined on a subset in the space of n-jets of smooth functions of one variable, $F : \mathcal{O} \to \mathbb{R}$ where $\mathcal{O}$ is an open subset in $J^n(\mathbb{R}, \mathbb{R})$. Throughout this paper, we assume that 0 is a regular value of F. It follows that the set $F^{-1}(0)$ is a hypersurface in $J^n(\mathbb{R}, \mathbb{R})$. We call $F^{-1}(0)$ the *equation hypersurface*. Let $(x, y, p_1, \dots, p_n)$ be a local coordinate on $J^n(\mathbb{R}, \mathbb{R})$ and $\xi \subset TJ^n(\mathbb{R}, \mathbb{R})$ be the canonical contact system on $J^n(\mathbb{R}, \mathbb{R})$ described by the vanishing of the 1-forms

$$\begin{cases} \alpha_1 &= dy - p_1 dx, \\ \alpha_2 &= dp_1 - p_2 dx, \\ & \vdots \\ \alpha_n &= dp_{n-1} - p_n dx. \end{cases}$$

We now define the notion of solutions. A *smooth solution* (or, a *classical solution*) of $F = 0$ passing through a point z_0 is a smooth function germ $y = f(x)$ at a point t_0 such that $(t_0, f(t_0), f'(t_0), \dots, f^{(n)}(t_0)) = z_0$ and $F(x, f(x), f'(x), \dots, f^{(n)}(x)) = 0$, where $f^{(i)}(x) = (d^i f/dx^i)(x)$. In other words, there exists a smooth function germ $f : (\mathbb{R}, t_0) \to \mathbb{R}$ such that the image of the n-jet extension, $j^n f : (\mathbb{R}, t_0) \to (J^n(\mathbb{R}, \mathbb{R}), z_0)$; $j^n f(x) = (x, f(x), f'(x), \dots, f^{(n)}(x))$, is contained in the equation hypersurface. It is easy to see that the map $j^n f$ is an immersion germ with $(j^n f)^* \alpha_i = 0$ for $i = 1, \dots, n$.

More generally, a *geometric solution* of $F = 0$ passing through a point z_0 is an integral immersion germ $\gamma : (\mathbb{R}, t_0) \to (J^n(\mathbb{R}, \mathbb{R}), z_0)$ such that the image of γ is contained in the equation hypersurface, namely, $\gamma' \neq 0$, $\gamma^* \alpha_i = 0$ for $i = 1, \dots, n$ and $F(\gamma(t)) = 0$ for each $t \in (\mathbb{R}, t_0)$.

By the definitions, a smooth solution is also a geometric solution. Conversely, it is easy to see that if $\gamma(t) = (x(t), y(t), p_1(t), \dots, p_n(t))$ is a geometric solution of $F = 0$ and $x'(t_0) \neq 0$, then we can reparametrize $\gamma(t)$ as a smooth solution.

The following notions are basic in this paper (cf. [3, 15, 17]). By the definition of parametrized version for smoothness of the solutions (*i.e.*, smooth solutions), a *smooth complete solution on* $F^{-1}(0)$ *at* z_0 is defined to be an n-parameter family of smooth function germs $y = f(t, \boldsymbol{c}) = f(t, c_1, \dots, c_n)$ such that

$$F\left(t, f(t, \boldsymbol{c}), \frac{\partial f}{\partial t}(t, \boldsymbol{c}), \dots, \frac{\partial^n f}{\partial t^n}(t, \boldsymbol{c})\right) = 0$$

and the map germ $j_1^n f : (\mathbb{R} \times \mathbb{R}^n, (t_0, \boldsymbol{c}_0)) \to (F^{-1}(0), z_0)$ defined by

$$j_1^n f(t, \boldsymbol{c}) = \left(t, f(t, \boldsymbol{c}), \frac{\partial f}{\partial t}(t, \boldsymbol{c}), \dots, \frac{\partial^n f}{\partial t^n}(t, \boldsymbol{c})\right)$$

is an immersion. It follows that the equation hypersurface is foliated by an n-parameter family of smooth solutions.

On the other hand, we consider the corresponding definition of parametrized version for geometric solutions. Let $\Gamma : (\mathbb{R} \times \mathbb{R}^n, (t_0, \boldsymbol{c}_0)) \to (F^{-1}(0), z_0)$ be an n-parameter family of geometric solutions, *i.e.*, $\Gamma(\cdot, \boldsymbol{c})$ is a geometric solution of $F = 0$ for each $\boldsymbol{c} \in (\mathbb{R}^n, \boldsymbol{c}_0)$.

We call Γ a *complete solution on* $F^{-1}(0)$ *at* z_0 if Γ is an immersion germ, namely,

$$\operatorname{rank} \begin{pmatrix} \partial x/\partial t & \partial y/\partial t & \partial p_1/\partial t & \cdots & \partial p_n/\partial t \\ \partial x/\partial \boldsymbol{c} & \partial y/\partial \boldsymbol{c} & \partial p_1/\partial \boldsymbol{c} & \cdots & \partial p_n/\partial \boldsymbol{c} \end{pmatrix} (t_0, \boldsymbol{c}_0) = n + 1,$$

where $\Gamma(t, \boldsymbol{c}) = (x(t, \boldsymbol{c}), y(t, \boldsymbol{c}), p_1(t, \boldsymbol{c}), \ldots, p_n(t, \boldsymbol{c}))$. It follows that the equation hypersurface is foliated by an n-parameter family of geometric solutions.

We say that an equation $F = 0$ is *smooth completely integrable* (respectively, *completely integrable*) at z_0 if there exists a smooth complete solution (respectively, a complete solution) on $F^{-1}(0)$ at z_0.

In the study of implicit ODEs from the view point of singularity theory, there are a lot of researches. For example, generic singularities and properties were given in [5, 6, 8, 16, 14] for the case of first order, in [12, 13] for the case of second order and in [7] for the case of any order etc. This paper is focused on the theory of completely integrable implicit ODEs.

In §2, we give a necessary and sufficient condition for existence of a smooth complete solution and a complete solution on the equation hypersurface at a point. We show that $F = 0$ is completely integrable at z_0 if and only if $F = 0$ is either of Clairaut type or of reduced type at z_0 in theorem 2.2. This result guarantees existence for a geometric solution in proposition 3.1. In §3, we give a sufficient condition for existence and uniqueness of a geometric solution with initial condition. In §4, we give a sufficient condition for existence of $(n-1)$-parameter family of geometric solutions of $F = 0$.

All map germs and manifolds considered here are differential of class C^∞.

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2. Existence and uniqueness for complete solutions

In this section, we consider existence and uniqueness conditions for a complete solution and a smooth complete solution on equation hypersurfaces. We denote a map $F_x + p_1 F_y + p_2 F_{p_1} + \cdots + p_n F_{p_{n-1}}$ by F_X. Here F_x (respectively, F_y, F_{p_i}) is the partial derivative of F with respect to x (respectively, with respect to y, p_i). We refer to the following lemma. See in case of first order in [9, 16], and of second order in [3, Lemma 3.1].

Lemma 2.1. *Let $F=0$ be an implicit ODE at z_0. $F=0$ is completely integrable at z_0 if and only if there exist function germs $\alpha, \beta : (F^{-1}(0), z_0) \to \mathbb{R}$, which do not vanish simultaneously, such that*

$$\alpha \cdot F_X|_{F^{-1}(0)} + \beta \cdot F_{p_n}|_{F^{-1}(0)} \equiv 0.$$

Proof. Suppose that $F=0$ is completely integrable at z_0 and let

$$\Gamma : (\mathbb{R} \times \mathbb{R}^n, (t_0, \boldsymbol{c}_0)) \to (F^{-1}(0), z_0)$$

be a complete solution on $F^{-1}(0)$ at z_0. Then differentiating Γ with respect to t yields a vector field $Z : (F^{-1}(0), z_0) \to TF^{-1}(0)$ given by $Z(\Gamma(t, \boldsymbol{c})) = \Gamma_t(t, \boldsymbol{c})$. Since $Z(z)$ lies in the contact plane ξ_z for each $z \in (F^{-1}(0), z_0)$, it has the form $Z = (\alpha, p_1\alpha, \dots, p_n\alpha, \beta)$ for some function germs $\alpha, \beta : (F^{-1}(0), z_0) \to \mathbb{R}$ which do not vanish simultaneously. Besides $Z(z)$ also lies in $T_zF^{-1}(0)$. It follows that the identity

$$\alpha \cdot F_X|_{F^{-1}(0)} + \beta \cdot F_{p_n}|_{F^{-1}(0)} \equiv 0$$

holds. Reversing the argument yields the converse. □

We say that an equation $F=0$ is of *(n-th order) Clairaut type* (for short, *type C*) at z_0 if there exists a function germ $\alpha_1 : (F^{-1}(0), z_0) \to \mathbb{R}$ such that

$$F_X|_{F^{-1}(0)} = \alpha_1 \cdot F_{p_n}|_{F^{-1}(0)},$$

and of *reduced type* (for short, *type R*) at z_0 if there exists a function germ $\beta_1 : (F^{-1}(0), z_0) \to \mathbb{R}$ such that

$$F_{p_n}|_{F^{-1}(0)} = \beta_1 \cdot F_X|_{F^{-1}(0)}.$$

We give a necessary and sufficient condition for existence of a smooth complete solution and a complete solution on equation hypersurfaces.

Theorem 2.2. *Let $F=0$ be an implicit ODE at z_0.*

(1) *$F=0$ is smooth completely integrable at z_0 if and only if $F=0$ is of type C at z_0.*

(2) *$F=0$ is completely integrable at z_0 if and only if $F=0$ is either of type C, or of type R at z_0.*

Proof. (1) The proof follows from a direct analogy of the proof for Theorem 2.2 in [10] or Theorem 3.1 in [15], so that we omit it.

(2) The result is a consequence of lemma 2.1. □

The uniqueness of the complete solution is the following.

Proposition 2.3. *Let* $\Gamma_1 : (\mathbb{R}\times\mathbb{R}^n, (t_1, \boldsymbol{c}_1)) \to (F^{-1}(0), z_0)$ *and* $\Gamma_2 : (\mathbb{R}\times\mathbb{R}^n, (t_2, \boldsymbol{c}_2)) \to (F^{-1}(0), z_0)$ *be complete solutions on* $F^{-1}(0)$ *at* z_0*. Then there exists a diffeomorphism germ* $\Phi : (\mathbb{R}\times\mathbb{R}^n, (t_2, \boldsymbol{c}_2)) \to (\mathbb{R}\times\mathbb{R}^n, (t_1, \boldsymbol{c}_1))$ *of the form* $\Phi(t, \boldsymbol{c}) = (\phi_1(t, \boldsymbol{c}), \phi_2(\boldsymbol{c}))$ *such that* $\Gamma_1 \circ \Phi = \Gamma_2$*.*

Proof. Suppose that the assertion does not hold. Since the complete solution is an immersive n-parameter family of curves in $F^{-1}(0)$, then there exists a point $z_1 \in (F^{-1}(0), z_0)$ such that $\Gamma_1(\cdot, \boldsymbol{c}_1)$ and $\Gamma_2(\cdot, \boldsymbol{c}_2)$ are transversal near the point z_1. Then we can construct a map germ $\Gamma : (\mathbb{R}\times\mathbb{R}^n, 0) \to (F^{-1}(0), z_1)$ such that (at least) $\Gamma(t, \cdot, \cdot, c_3, \dots, c_n)$ is an immersion germ,

$$\frac{\partial y}{\partial c_1}(t, \boldsymbol{c}) = p_1(t, \boldsymbol{c})\frac{\partial x}{\partial c_1}(t, \boldsymbol{c}), \dots, \frac{\partial p_{n-1}}{\partial c_1}(t, \boldsymbol{c}) = p_n(t, \boldsymbol{c})\frac{\partial x}{\partial c_1}(t, \boldsymbol{c}) \tag{3}$$

and

$$\frac{\partial y}{\partial c_2}(t, \boldsymbol{c}) = p_1(t, \boldsymbol{c})\frac{\partial x}{\partial c_2}(t, \boldsymbol{c}), \dots, \frac{\partial p_{n-1}}{\partial c_2}(t, \boldsymbol{c}) = p_n(t, \boldsymbol{c})\frac{\partial x}{\partial c_2}(t, \boldsymbol{c}), \tag{4}$$

where $\Gamma(t, \boldsymbol{c}) = (x(t, \boldsymbol{c}), y(t, \boldsymbol{c}), p_1(t, \boldsymbol{c}), \dots, p_n(t, \boldsymbol{c}))$. If we calculate second order partial derivatives of the last equality for (3) with respect to c_2 and for (4) with respect to c_1, we get

$$\frac{\partial^2 p_{n-1}}{\partial c_2 \partial c_1} = \frac{\partial p_n}{\partial c_2}\cdot\frac{\partial x}{\partial c_1} + p_n \cdot \frac{\partial^2 x}{\partial c_2 \partial c_1} \text{ and } \frac{\partial^2 p_{n-1}}{\partial c_1 \partial c_2} = \frac{\partial p_n}{\partial c_1}\cdot\frac{\partial x}{\partial c_2} + p_n \cdot \frac{\partial^2 x}{\partial c_1 \partial c_2}.$$

Therefore we obtain the equality $(\partial p_n/\partial c_2)\cdot(\partial x/\partial c_1) = (\partial p_n/\partial c_1)\cdot(\partial x/\partial c_2)$. This contradicts the fact that $\Gamma(t, \cdot, \cdot, c_3, \dots, c_n)$ is an immersion germ. □

3. Existence and uniqueness for geometric solutions

In this section, we give an existence and uniqueness condition for a geometric solution with initial condition.

Let $F = 0$ be an implicit ODE at z_0. Consider a point $z \in F^{-1}(0)$ such that the contact plane ξ_z intersects $T_z F^{-1}(0)$ transversally. Then it is easy to see that a complete solution exists at z by integrating the line field $\xi \cap TF^{-1}(0)$ (see, lemma 2.1). We call points where transversality fails *contact singular points* and denote the set of such points by $\Sigma_c(F)$. We call $\Sigma_c(F)$ the *contact singular set.* It is easy to check that the contact singular set is given by

$$\Sigma_c(F) = \{z \in F^{-1}(0) |\ F_X(z) = 0, F_{p_n}(z) = 0\}.$$

We say that a geometric solution $\gamma : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0)$ is a *singular solution* of $F = 0$ passing through z_0 if for any representative $\widetilde{\gamma} : I \to F^{-1}(0)$ of

γ and any open subinterval $(a,b) \subset I$ at t_0, $\widetilde{\gamma}|_{(a,b)}$ is never contained in a leaf of a complete solution (cf. [3, 7, 9, 11]). If a completely integrable implicit ODE has a singular solution passing through z_0, then uniqueness for geometric solutions does not hold.

Proposition 3.1. *Let $F = 0$ be an implicit ODE at z_0. If $z_0 \notin \Sigma_c(F)$, then there exists a unique geometric solution passing through z_0.*

If $z_0 \notin \Sigma_c(F)$, either $F_X \neq 0$ or $F_{p_n} \neq 0$ at z_0. The latter case, the consequence of proposition 3.1 follows from the classical results for existence and uniqueness of a smooth solution of smooth explicit equations. Thus in order to prove proposition 3.1, it is enough to show the former case. Here we give a proof by an elementary argument like as explicit ODEs. This method is useful to prove a result in [18].

Lemma 3.2. *Let $F = 0$ be an implicit ODE at z_0. If $F_X(z_0) \neq 0$, then there exists a unique geometric solution passing through z_0.*

Proof. It follows from $F_X(z_0) \neq 0$ that $F = 0$ is of reduced type at z_0. By theorem 2.2, there exists a complete solution on $F^{-1}(0)$ at z_0 and hence there exists a geometric solution passing through z_0. We may assume that $F_X \equiv 1$ on the equation hypersurface, if necessary, we consider $F/F_X = 0$ as $F = 0$. Moreover if $\gamma : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0); \gamma(t) = (x(t), y(t), p_1(t), \ldots, p_n(t))$ is a geometric solution passing through z_0, then $p_n'(t_0) \neq 0$. Hence we can reparametrize $\gamma(t)$ as $(x(t), y(t), p_1(t), \ldots, p_{n-1}(t), t)$. Let $\gamma(t) = (x(t), y(t), p_1(t), \ldots, p_{n-1}(t), t)$ and $\widetilde{\gamma}(t) = (\widetilde{x}(t), \widetilde{y}(t), \widetilde{p}_1(t), \ldots, \widetilde{p}_{n-1}(t), t)$ be geometric solutions passing through z_0, that is, $\gamma(t_0) = \widetilde{\gamma}(t_0) = z_0$. It is enough to show that $\gamma(t) = \widetilde{\gamma}(t)$ for $t_0 \leq t \leq t_0 + \varepsilon$, where ε is a small positive real number. Differentiate the equality $F(\gamma(t)) = F(x(t), y(t), p_1(t), \ldots, p_{n-1}(t), t) = 0$ with respect to t, then we get $x'(t) = -F_{p_n}(\gamma(t))$. By integrating this equality,

$$x(t) = x(t_0) - \int_{t_0}^{t} F_{p_n}(\gamma(t))dt.$$

Since $\gamma(t)$ is a geometric solution, namely, $y'(t) = p_1(t)x'(t)$, $p_i'(t) = p_{i+1}(t)x'(t)$ $(i = 1, \ldots, n-2)$ and $p_{n-1}'(t) = tx'(t)$, we have

$$y(t) = y(t_0) + \int_{t_0}^{t} p_1(t)x'(t)dt,\ p_i(t) = p_i(t_0) + \int_{t_0}^{t} p_{i+1}(t)x'(t)dt$$

$(i = 1, \ldots, n-2)$ and

$$p_{n-1}(t) = p_{n-1}(t_0) + \int_{t_0}^{t} tx'(t)dt.$$

It follows that

$$x(t)-\widetilde{x}(t)=\int_{t_0}^{t}\left(-F_{p_n}(\gamma(t))+F_{p_n}(\widetilde{\gamma}(t))\right)dt, \tag{5}$$

$$\begin{aligned} y(t)-\widetilde{y}(t) &= \int_{t_0}^{t}\left(p_1(t)x'(t)-\widetilde{p}_1(t)\widetilde{x}'(t)\right)dt \\ &= \int_{t_0}^{t} p_1(t)\left(x'(t)-\widetilde{x}'(t)\right)dt+\int_{t_0}^{t}\widetilde{x}'(t)\left(p_1(t)-\widetilde{p}_1(t)\right)dt \end{aligned} \tag{6}$$

$$\begin{aligned} p_i(t)-\widetilde{p}_i(t) &= \int_{t_0}^{t}\left(p_{i+1}(t)x'(t)-\widetilde{p}_{i+1}(t)\widetilde{x}'(t)\right)dt \\ &= \int_{t_0}^{t} p_{i+1}(t)\left(x'(t)-\widetilde{x}'(t)\right)dt+\int_{t_0}^{t}\widetilde{x}'(t)\left(p_{i+1}(t)-\widetilde{p}_{i+1}(t)\right)dt \end{aligned} \tag{7}$$

$(i=1,\dots,n-2)$ and

$$\begin{aligned} p_{n-1}(t)-\widetilde{p}_{n-1}(t) &= \int_{t_0}^{t} t\left(x'(t)-\widetilde{x}'(t)\right)dt. \\ &= \int_{t_0}^{t} t\left(-F_{p_n}(\gamma(t))+F_{p_n}(\widetilde{\gamma}(t))\right)dt. \end{aligned} \tag{8}$$

Since F is a smooth mapping, there exists some number K such that

$$|-F_{p_n}(\gamma(t))+F_{p_n}(\widetilde{\gamma}(t))|\leq K|\gamma(t)-\widetilde{\gamma}(t)|\leq K\alpha(t),$$

where $t_0\leq t\leq t_0+\varepsilon$ and

$$\alpha(t)=|x(t)-\widetilde{x}(t)|+|y(t)-\widetilde{y}(t)|+|p_1(t)-\widetilde{p}_1(t)|+\cdots+|p_{n-1}(t)-\widetilde{p}_{n-1}(t)|.$$

Moreover, since γ and $\widetilde{\gamma}'$ are smooth mappings, we put

$$a_i=\max_{t_0\leq t\leq t_0+\varepsilon}\{|p_i(t)|\}\ (i=1,\dots,n-1),\ a_n=|t_0+\varepsilon|,\ b=\max_{t_0\leq t\leq t_0+\varepsilon}\{|\widetilde{x}'(t)|\}.$$

Denote an integration repeated i-times $\int_{t_0}^{t}\big(\cdots\big(\int_{t_0}^{t}\alpha(t)dt\big)\cdots\big)dt$ by $\big(\int_{t_0}^{t}\big)^{i}\alpha(t)(dt)^{i}$. It follows from (5), (6), (7) and (8) that

$$\begin{aligned} \alpha(t) &\leq (1+a_1+\cdots+a_n)K\int_{t_0}^{t}\alpha(t)dt+(a_2+\cdots+a_n)bK\left(\int_{t_0}^{t}\right)^{2}\alpha(t)(dt)^2+ \\ &\cdots+(a_i+\cdots+a_n)b^{i-1}K\left(\int_{t_0}^{t}\right)^{i}\alpha(t)(dt)^i+\cdots+a_nb^{n-1}K\left(\int_{t_0}^{t}\right)^{n}\alpha(t)(dt)^n. \end{aligned} \tag{9}$$

Moreover we put

$$L = \max\{1 + a_1 + \cdots + a_n, (a_2 + \cdots + a_n)b, \cdots, a_n b^{n-1}\}, \ M = \max_{t_0 \le t \le t_0+\varepsilon}\{\alpha(t)\}.$$

By (9), we have

$$M \le LKM \sum_{i=1}^{n} \frac{1}{i!}(t - t_0)^i \le LKM \sum_{i=1}^{n} \frac{1}{i!}\varepsilon^i \tag{10}$$

We now consider a function $f(x) = \sum_{i=1}^{n}(1/i!)x^i - 1/LK$. If $x = 0$, then $f(x)$ is negative, and if x is sufficient large, then $f(x)$ is positive. By the mean value theorem, there exists a positive real number t_n such that $f(x) < 0$ on $0 < x < t_n$. If we take a small real number $\varepsilon > 0$ which satisfies $0 < \varepsilon < t_n$, then $f(\varepsilon)M \ge 0$ by (10). It follows $M = 0$ and concludes that $\gamma(t) = \widetilde{\gamma}(t)$ for $t_0 \le t \le t_0 + \varepsilon$. This completes the proof of lemma 3.2. □

By proposition 3.1, a geometric solution $\gamma : (\mathbb{R}, t_0) \to (F^{-1}(0), z_0)$ is a singular solution only if it is contained in $\Sigma_c(F)$.

Remark 3.3. For first order ODEs, the converse of proposition 3.1 also hold (cf. [9, 16]). However, higher order cases, the converse does not hold, see in [17, 18].

4. Complete solutions on the contact singular sets

We consider the cases for $z_0 \in \Sigma_c(F)$. It is easy to show the following result (cf. [17]).

Proposition 4.1. *Let $F = 0$ be an implicit ODE at $z_0 \in \Sigma_c(F)$.*

(1) *Suppose that 0 is a regular value of $F_{p_n}|_{F^{-1}(0)}$. Then, $F = 0$ is of type C at z_0 if and only if $\Sigma_c(F)$ is an n-dimensional manifold around z_0.*

(2) *Suppose that 0 is a regular value of $F_X|_{F^{-1}(0)}$. Then, $F = 0$ is of type R at z_0 if and only if $\Sigma_c(F)$ is an n-dimensional manifold around z_0.*

Now suppose that $F = 0$ is completely integrable at z_0 and $\Sigma_c(F)$ is an n-dimensional manifold around z_0. By proposition 4.1, the condition that $\Sigma_c(F)$ is an n-dimensional manifold around z_0 is generic for completely integrable ODEs.

We call a map germ

$$\Phi : (\mathbb{R} \times \mathbb{R}^{n-1}, (t_0, \boldsymbol{b}_0)) \to (\Sigma_c(F), z_0)$$

a *complete solution on $\Sigma_c(F)$ at z_0* if Φ is an immersion germ and $\Phi(\cdot, \boldsymbol{b})$ is a geometric solution for each $\boldsymbol{b} \in (\mathbb{R}^{n-1}, \boldsymbol{b}_0)$. Moreover, we call Φ a *complete* *singular solution on* $\Sigma_c(F)$ *at* z_0 if $\Phi(\cdot, \boldsymbol{b})$ is a singular solution for each $\boldsymbol{b} \in (\mathbb{R}^{n-1}, \boldsymbol{b}_0)$.

If ξ_z intersects $T_z\Sigma_c(F)$ transversally in $T_zF^{-1}(0)$, then integrating the line field $\xi \cap T\Sigma_c(F)$ yields a complete solution on $\Sigma_c(F)$. We call a point where transversality does not hold a *second order contact singular point* and denote the set of such points by $\Sigma_{cc}(F)$ (or, $\Sigma_{c^2}(F)$) (cf. [3, 17]). Inductively, if $\Sigma_{cc}(F)$ is an $(n-1)$-dimensional manifold around z_0, then we can define a *complete solution on* $\Sigma_{cc}(F)$ *at* z_0, a *complete singular solution on* $\Sigma_{cc}(F)$ *at* z_0 and third order contact singular set $\Sigma_{ccc}(F)$ (or, $\Sigma_{c^3}(F)$) etc. Therefore we have the following sequence when $\Sigma_{c^i}(F)$ are $(n-i+1)$-dimensional submanifolds, $i = 1, \dots, n$ (cf. [7]):

$$\Sigma_{c^n}(F) \subset \Sigma_{c^{n-1}}(F) \subset \cdots \subset \Sigma_{c^2}(F) \subset \Sigma_c(F) \subset F^{-1}(0).$$

We say that $F = 0$ is of *type CC* at z_0 if it is of type C at z_0 and there exists a function germ $\alpha_2 : (\Sigma_c(F), z_0) \to \mathbb{R}$ such that $(F_{p_n})_X = \alpha_2 \cdot (F_{p_n})_{p_n}$ on $\Sigma_c(F)$, is of *type CR* at z_0 if it is of type C at z_0 and there exists a function germ $\beta_2 : (\Sigma_c(F), z_0) \to \mathbb{R}$ such that $(F_{p_n})_{p_n} = \beta_2 \cdot (F_{p_n})_X$ on $\Sigma_c(F)$, is of *type RC* at z_0 if it is of type R at z_0 and there exists a function germ $\alpha_2 : (\Sigma_c(F), z_0) \to \mathbb{R}$ such that $(F_X)_X = \alpha_2 \cdot (F_X)_{p_n}$ on $\Sigma_c(F)$, and is *of type RR* at z_0 if it is of type R at z_0 and there exists a function germ $\beta_2 : (\Sigma_c(F), z_0) \to \mathbb{R}$ such that $(F_X)_{p_n} = \beta_2 \cdot (F_X)_X$ on $\Sigma_c(F)$.

Moreover, we can define $F = 0$ is of type CCC, CCR, CRC, CRR etc., when $\Sigma_c^i(F)$ are submanifolds. We give a sufficient condition for existence of a complete solution on the contact singular set.

Theorem 4.2. *Let* $F = 0$ *be an implicit ODE at* $z_0 \in \Sigma_c(F)$.

(1) *Suppose that* 0 *is a regular value of* $F_{p_n}|_{F^{-1}(0)}$ *and* $F = 0$ *is of type* C *at* z_0. *There exists a complete solution on* $\Sigma_c(F)$ *at* z_0 *if and only if* $F = 0$ *is either of type* CC *or of type* CR *at* z_0. *Moreover, if* $F = 0$ *is of type* CR *at* z_0 *with* $\beta_2(z_0) = 0$, *then there exists a complete singular solution on* $\Sigma_c(F)$ *at* z_0.

(2) *Suppose that* 0 *is a regular value of* $F_X|_{F^{-1}(0)}$ *and* $F = 0$ *is of type* R *at* z_0. *There exists a complete solution on* $\Sigma_c(F)$ *at* z_0 *if and only if* $F = 0$ *is either of type* RC *or of type* RR *at* z_0. *Moreover, if* $F = 0$ *is of type* RC *at* z_0 *with* $\alpha_2(z_0) = 0$, *then there exists a complete singular solution on* $\Sigma_c(F)$ *at* z_0.

Proof. (1) Suppose that $\Phi : (\mathbb{R} \times \mathbb{R}^{n-1}, 0) \to (\Sigma_c(F), z_0), (t, \boldsymbol{b}) \mapsto \Phi(t, \boldsymbol{b})$ is a complete solution on $\Sigma_c(F)$ at z_0. By proposition 4.1, $\Sigma_c(F)$ is an n-dimensional manifold around z_0. Since $F = 0$ is regular and $z_0 \in \Sigma_c(F)$, one of $F_y, F_{p_i} (i = 1, \dots, n-1)$ does not vanish at z_0.

Now assume that $F_y(z_0) \neq 0$. By the implicit function theorem, there exists a function $f : U \to \mathbb{R}$, where U is an open subset in $\mathbb{R}^{n+1}$, such that in a neighborhood of z_0, $(x, y, p_1, \ldots, p_n) \in F^{-1}(0)$ if and only if $-y + f(x, p_1, \ldots, p_n) = 0$. Thus, without loss of generality, we may assume that

$$F(x, y, p_1, \ldots, p_n) = -y + f(x, p_1, \ldots, p_n).$$

Define a diffeomorphism $\phi : U \to F^{-1}(0)$ given by

$$\phi(x, p_1, \ldots, p_n) = (x, f(x, p_1, \ldots, p_n), p_1, \ldots, p_n).$$

Differentiating $\phi^{-1} \circ \Phi$ with respect to t yields a vector field $Y : \phi^{-1}(\Sigma_c(F)) \to T\phi^{-1}(\Sigma_c(F))$ given by $Y(\phi^{-1} \circ \Phi(t, \boldsymbol{b})) = (\phi^{-1} \circ \Phi)_t(t, \boldsymbol{b})$. By definitions of a complete solution on $\Sigma_c(F)$ and of the diffeomorphism ϕ, Y has the form $(\alpha, p_2\alpha, \ldots, p_n\alpha, \beta)$ for some function germs $\alpha, \beta : (\phi^{-1}(\Sigma_c(F)), \phi^{-1}(z_0)) \to \mathbb{R}$ which do not vanish simultaneously. Since $F = 0$ is of type C, $\phi^{-1}(\Sigma_c(F))$ is given by $f_{p_n}^{-1}(0)$. It follows that there exist function germs $\widetilde{\alpha}, \widetilde{\beta} : (\Sigma_c(F), z_0) \to \mathbb{R}$ which do not vanish simultaneously, such that the identity

$$\widetilde{\alpha} \cdot (F_{p_n})_X + \widetilde{\beta} \cdot (F_{p_n})_{p_n} \equiv 0$$

on $\Sigma_c(F)$ holds. Then $F = 0$ is either of type CC or of type CR at z_0.

Assume that $F_{p_i}(z_0) \neq 0$, where $i = 1, \ldots, n-2$, or $n-1$. By the same arguments as above, there exists a function $f_i : U_i \to \mathbb{R}$, where U_i is an open subset in $\mathbb{R}^{n+1}$, such that in a neighborhood of z_0, we may assume that

$$F(x, y, p_1, \ldots, p_n) = -p_i + f_i(x, y, p_1, \ldots, \hat{p_i}, \ldots, p_n).$$

Here $\hat{p_i}$ means removing the component p_i. Define a diffeomorphism $\phi_i : U_i \to F^{-1}(0)$ given by

$$\phi_i(x, y, p_1, \ldots, \hat{p_i}, \ldots, p_n) = (x, y, p_1, \ldots, f_i(x, y, p_1, \ldots, \hat{p_i}, \ldots, p_n), \ldots, p_n).$$

Differentiating $\phi_i^{-1} \circ \Phi$ with respect to t yields a vector field $Y_i : \phi_i^{-1}(\Sigma_c(F)) \to T\phi_i^{-1}(\Sigma_c(F))$ given by the form $Y_i = (\alpha, p_1\alpha, \ldots, p_{i-1}\alpha, f_i\alpha, p_{i+2}\alpha, \ldots, p_n\alpha, \beta)$ for some function germs $\alpha, \beta : (\phi^{-1}(\Sigma_c(F)), \phi^{-1}(z_0)) \to \mathbb{R}$ which do not vanish simultaneously. Since $F = 0$ is of type C, $\phi^{-1}(\Sigma_c(F))$ is given by $(f_i)_{p_n}^{-1}(0)$. It follows that there exist function germs $\widetilde{\alpha}, \widetilde{\beta} : (\Sigma_c(F), z_0) \to \mathbb{R}$ which do not vanish simultaneously, such that the identity

$$\widetilde{\alpha} \cdot (F_{p_n})_X + \widetilde{\beta} \cdot (F_{p_n})_{p_n} \equiv 0$$

on $\Sigma_c(F)$ holds. Then $F = 0$ is either of type CC or of type CR at z_0. In either case, reversing the argument yields the converse.

Finally, suppose that $F = 0$ is of type CR at z_0 with $\beta_2(z_0) = 0$. By the forms of the vector fields of Z in the proof of lemma 2.1, Y and Y_i, each leaf of a complete solution on $\Sigma_c(F)$ at z_0 is not contained in the leaf of a complete solution on $F^{-1}(0)$ at z_0. Hence the complete solution on $\Sigma_c(F)$ is a complete singular solution on $\Sigma_c(F)$ at z_0.

(2) The argument is similar to that in case (1). □

Remark 4.3. For the implicit second order ODEs, we can give existence conditions for complete singular solution on the contact singular set in more detail ([18]).

Theorem 4.4. *Let $F = 0$ be an implicit ODE at $z_0 \in \Sigma_c(F)$.*
(1) *Let 0 be a regular value of $F_{p_n}|_{F^{-1}(0)}$ and $F = 0$ is of type C at z_0.*

(i) *Suppose that 0 is a regular value of $(F_{p_n})_{p_n}|_{\Sigma_c(F)}$ and $z_0 \in \Sigma_{cc}(F)$. Then, $F = 0$ is of type CC at z_0 if and only if $\Sigma_{cc}(F)$ is an $(n-1)$-dimensional manifold around z_0.*

(ii) *Suppose that 0 is a regular value of $(F_{p_n})_X|_{\Sigma_c(F)}$ and $z_0 \in \Sigma_{cc}(F)$. Then, $F = 0$ is of type CR at z_0 if and only if $\Sigma_{cc}(F)$ is an $(n-1)$-dimensional manifold around z_0.*

(2) *Let 0 be a regular value of $F_X|_{F^{-1}(0)}$ and $F = 0$ is of type R at z_0.*

(i) *Suppose that 0 is a regular value of $(F_X)_{p_n}|_{\Sigma_c(F)}$ and $z_0 \in \Sigma_{cc}(F)$. Then, $F = 0$ is of type RC at z_0 if and only if $\Sigma_{cc}(F)$ is an $(n-1)$-dimensional manifold around z_0.*

(ii) *Suppose that 0 is a regular value of $(F_X)_X|_{\Sigma_c(F)}$ and $z_0 \in \Sigma_{cc}(F)$. Then, $F = 0$ is of type RR at z_0 if and only if $\Sigma_{cc}(F)$ is an $(n-1)$-dimensional manifold around z_0.*

Proof. (1)(i) Since the contact singular set $\Sigma_c(F)$ is an n-dimensional manifold and given by $F^{-1}(0) \cap F_{p_n}^{-1}(0)$, the second order singular set $\Sigma_{cc}(F)$ is given by $\{z \in \Sigma_c(F) \mid (F_{p_n})_X = (F_{p_n})_{p_n} = 0\}$. It follows from the regularity condition that $F = 0$ is of type CC at z_0 if and only if $\Sigma_{cc}(F)$ is an $(n-1)$-dimensional manifold around z_0.

The other cases can be proved by using the same arguments in case (1)(i). □

By theorems 4.2 and 4.4, there exist immersive n and $(n-1)$-parameter families of geometric solutions of $F = 0$ if and only if $\Sigma_c(F)$ and $\Sigma_{cc}(F)$ are n and $(n-1)$-dimensional manifolds respectively under the regularity conditions.

Remark 4.5. Generally we can observe that there exists a complete solution on $\Sigma_{c^i}(F)$ at z_0, namely, there exists an immersive $(n-i)$-parameter family of geometric solutions of $F = 0$ if and only if $z_0 \notin \Sigma_{c^{i+1}}(F)$ or $\Sigma_{c^{i+1}}(F)$ is an $(n-i)$-dimensional manifold around z_0 under some regularity conditions.

As a conclusion, it is possible that there exist not only an n-parameter family of geometric solutions but also $(n-i)$-parameter $(i = 1, \dots, n)$ family of geometric solutions for implicit ODEs.

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