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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF BLACK-SCHOLES TYPE EQUATIONS BASED ON WEAKLY DEPENDENT RANDOM VARIABLES

By

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Summary. In this paper, we show the asymptotic normality of $(1/T)\log(X_T/X_0)$ where the sequence $\{X_t\}$ is defined as a solution of some stochastic difference equation based on weakly dependent random variables. As a result we obtain the asymptotic Black-Scholes formula.

1. Known results

Let $\{\xi_j\}$ be a strictly stationary stochastic process satisfying the strong mixing condition

$$\alpha(t) = \sup_{A\in\mathcal{M}_{-\infty}^{0}, B\in\mathcal{M}_t^{\infty}} |P(AB) - P(A)P(B)| \to 0 \quad (t\to\infty).$$

Lemma A. *Let $\{\xi_i\}$ be a strong mixing sequence of zero mean random variables with coefficient $\alpha(n)$. If*

$$\exists\delta > 0 : \quad \sup_{i\ge 1} E|\xi_i|^{2+\delta} < \infty,$$

then

$$|\mathrm{cov}(\xi_i, \xi_j)| \le c\|\xi_i\|_{2+\delta}\|\xi_j\|_{2+\delta}\alpha^{\frac{\delta}{2+\delta}}(|j-i|)$$

where $c > 0$ is some positive constant and

$$\|\xi\|_p = \{E|\xi|^p\}^{\frac{1}{p}} \quad (p > 0).$$

Next, for the sequence $\{\xi_i\}$ with mean θ, put

$$\sigma^2(\xi) = E(\xi_0-\theta)^2 + 2\sum_{i=1}^{\infty} E(\xi_0-\theta)(\xi_i-\theta)$$

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if exists. It is known that the series in the above definition of $\sigma^2(\xi)$ is absolutely convergent if (1) (below) holds.

Theorem A. *Let $\{\xi_i\}$ be a strictly stationary strong mixing sequence of zero mean random variables with coefficient $\alpha(n)$. Suppose there exists a $\delta > 0$ such that $\|\xi_1\|_{2+\delta} < \infty$ and*

$$\sum_{n=1}^{\infty} \alpha^{\frac{\delta}{2+\delta}}(n) < \infty. \tag{1}$$

If $\sigma(\xi) > 0$, then the weak invariance principle holds, that is, for any $T > 0$

$$\left\{ \frac{1}{\sqrt{n}\sigma(\xi)} \sum_{i=1}^{[nt]} \xi_i : 0 \le t \le T \right\} \xrightarrow{D} \{W(t) : 0 \le t \le T\}, \tag{2}$$

as $n \to \infty$ where $W(\cdot)$ denotes a standard Wiener process and "$\to^D$" means weak convergence in $D[0,T]$.

Furthermore, if the condition on $\alpha(n)$ is strengthened as

$$\exists\ \epsilon > 0 : \quad \sum_{i=1}^{\infty} \alpha^{(1+\epsilon)(1+(2/\delta))}(n) < \infty, \tag{3}$$

then, the strong invariance theorem holds, that is, for all t sufficiently large

$$\exists 0 \le \lambda < \frac{1}{2} : \quad | \sum_{1 \le i \le n \le t} \xi_i - \sigma(\xi) W(t)| = O(t^{\frac{1}{2}-\lambda}) \quad a.s. \tag{4}$$

Next, we consider the following self-normalizer introduced in Yoshihara (2009): For $n > 1$ let r and k be integer-valued functions of n such that

$$r = r(n) = o(n^{\frac{1}{4}-\gamma}) \text{ and } k = k(n) = \left[\frac{n}{r}\right] \tag{5}$$

where $0 < \gamma < 1/8$ and define the self-normalizer by

$$C_n^2 = \sum_{j=1}^{k} \left(\sum_{i=1}^{r} \left(\xi_{(j-1)r+i} - n^{-1} \sum_{l=1}^{n} \xi_l \right) \right)^2.$$

It is easy to see that if $\|\xi_1\|_{2+\delta} < \infty$ ($\delta > 0$), then

$$\lim_{n\to\infty} \frac{1}{n} C_n^2 = \sigma^2(\xi) \quad a.s.$$

(See, Yoshihara (2009).) Furthermore, let

$$Z_n = \frac{1}{C_n} \sum_{l=1}^{n} (\xi_l - \theta).$$

The following is Remark to Theorem 6 in Yoshihara (2009).

Theorem B. *Let $\{\xi_i\}$ be a strictly stationary strong mixing sequence of mean θ random variables with mixing coefficient $\alpha(n)$. Suppose there exists a $0 < \delta \le 1$ such that $\|\xi_1\|_{2+\delta} < \infty$ and*

$$\sum_{i=1}^{\infty} \alpha^{\frac{\delta}{16+\delta}}(i) < \infty. \tag{6}$$

hold. If $\sigma(\xi) > 0$, then

$$\{Z_n(t) = Z_{[nt]}; 0 \le t \le 1\} \xrightarrow{D} \{W(t) : 0 \le t \le 1\}. \tag{7}$$

In the sequel, we write "Z_n is $AN(\mu_n, \sigma_n^2)$" if

$$\frac{Z_n - \mu_n}{\sigma_n} \xrightarrow{D} N(0,1). \tag{8}$$

holds.

2. Time series

Firstly, we consider the sequence $\{X_k; k \ge 1\}$ satisfying the stochastic difference equation

$$\Delta X_k = (\nu + \sigma \xi_{k-1}) X_{k-1} \quad (k \ge 1) \tag{9}$$

where $\Delta X_k = X_k - X_{k-1}$, $X_0 \ne 0$ and $\{\xi_k : k \ge 1\}$ is a strictly stationary stochastic sequence and ν and $\sigma > 0$ are some absolute constants.

We prove the following theorem.

Theorem 1. *Let $\{X_k; k \ge 1\}$ be a solution of the stochastic difference equation (9) with $\nu = 0$ and $\sigma = 1$. Let $\{\xi_k\}$ be a strictly stationary sequence of strong mixing random variables with mixing coefficient $\alpha(n)$ satisfying the following conditions: (i)*

$$\xi_1 > -1 \quad a.s.; \tag{10}$$

(ii) for some $\delta > 0$ (1) and

$$E|\log(1 + \xi_k)|^{2+\delta} < \infty, \tag{11}$$

hold. Then,

$$\frac{1}{n} \log \frac{X_n}{X_0} \text{ is } AN\left(\mu, \frac{\rho^2}{n}\right) \tag{12}$$

where

$$\mu = E\log(\xi_1+1),$$
$$\rho^2 = \mathrm{Var}(\log(\xi_1+1)) + 2\sum_{k=2}^{\infty} E(\log(\xi_1+1)-\mu)(\log(\xi_k+1)-\mu) > 0.$$

If, instead of (1), the mixing coefficient satisfies (3), then

$$X_n = X_0 \exp\{\sqrt{n}\mu + \rho W(n) + o(n^{-\lambda})\} \quad a.s. \tag{13}$$

where $W(\cdot)$ denotes a standard Wiener process and $0 < \lambda < 1/2$.

Proof. We note that

$$\frac{X_k}{X_{k-1}} = \xi_k + 1 \quad (k \geq 1).$$

Hence, we can write as

$$\log\frac{X_n}{X_0} = \sum_{k=1}^{n} \log\frac{X_k}{X_{k-1}} = \sum_{k=1}^{n} \log(\xi_k+1).$$

Since $\{(\log(\xi_k+1)-\mu\}$ is a strictly stationary strong mixing sequence of mean zero random variables and satisfies (1) and $\rho > 0$, (12) follows from (2). On the other hand, (13) follows from (4). □

The following corollary is easily obtained by the proof of Theorem 1.

Corollary. *Let $\{X_k; k \geq 1\}$ be a solution of the difference equation (9). Let $\{\xi_k\}$ be a strictly stationary sequence of strong mixing random variables with mixing coefficient $\alpha(n)$ satisfying the following conditions: (i)*

$$\nu + \sigma\xi_1 > -1 \quad a.s.; \tag{14}$$

(ii) for some $\delta > 0$ (1) and

$$E|\log(1+\nu+\sigma\xi_k)|^{2+\delta} < \infty \tag{15}$$

hold. Then,

$$\frac{1}{n}\log\frac{X_n}{X_0} \text{ is } AN\left(\tilde{\mu}, \frac{\tilde{\rho}^2}{n}\right), \tag{16}$$

where

$$\tilde{\mu} = E\log(\nu + \sigma\xi_1 + 1),$$

$$\tilde{\rho}^2 = \mathrm{Var}(\nu + \sigma\xi_1 + 1) \\ +2\sum_{k=2}^{\infty} E(\log(\nu + \sigma\xi_1 + 1) - \tilde{\mu})(\log(\nu + \sigma\xi_k + 1) - \tilde{\mu}) > 0.$$

In addition, if $\{\xi_k\}$ satisfies (3), then for some $\lambda > 0$,

$$X_n = X_0 \exp\{\sqrt{n}\tilde{\mu} + \sigma\tilde{\rho}W(n) + o(n^{-\lambda})\} \quad a.s. \tag{17}$$

Usually, ρ^2 is not known. Hence, in practice, it is preferable to use the self-normalizer in Theorem 1 without using ρ^2. The following theorem gives a method.

THEOREM 2. *Let $\{X_k\}$ be the sequence defined in Theorem 1. Put $r = [n^{(1/4)-\gamma}]$ and $k = [n/k]$, where $0 < \gamma < 1/8$. Let*

$$\eta_i = \log(\xi_i + 1) \quad \text{and} \quad \bar{\eta}_n = \frac{1}{n}\sum_{i=1}^{n} \eta_i. \tag{18}$$

Define the self-normalizers by

$$C_n^2 = \sum_{j=1}^{k}\left(\sum_{i=1}^{r}(\eta_i - \bar{\eta}_n)\right)^2.$$

If (5), (6) and (11) hold and $\rho > 0$, then,

$$\frac{1}{C_n}\left(\log\frac{X_n}{X_0} - n\nu\right) \xrightarrow{D} N(0,1). \tag{19}$$

Proof. The conclusion follows from Theorem B. □

3. Stochastic Difference Equations

In this section, we assume that $\{X(t) : [0,\infty)\}$ be a time-continuous stochastic process. Let m be an arbitrary positive integer. Let $t_k = k/m$ $(k \geq 0)$ and put

$$\Delta X(t_k) = X(t_k) - X(t_{k-1}), \quad (k \geq 1).$$

We consider the difference equation where for all m

$$\Delta X(t_k) = X(t_{k-1})\int_{t_{k-1}}^{t_k}(\nu + \sigma\xi_s)ds,$$

which may be rewritten as

$$\frac{X(t_k)}{X(t_{k-1})} = 1 + \int_{t_{k-1}}^{t_k} (\nu + \sigma\xi_s)ds. \tag{20}$$

Here, $\{\xi_t\}$ is some strictly stationary stochatic process and ν and $\sigma > 0$ are some absolute positive constants.

In the sequel, c, with or without subscript, denotes an absolute constant.

We prove the following theorem.

THEOREM 3. *Suppose the process $\{\xi_t\}$ is strictly stationary with $E\xi_s = 0$ and satisfies the strong mixing condition with mixing coefficient $\alpha(t)$. Furthermore, suppose that for some $\delta > 0$*

$$E|\xi_s|^{4+2\delta} < \infty$$

and

$$\int_0^\infty \alpha^{\frac{2\delta}{2+\delta}}(t)dt < \infty. \tag{21}$$

Then, for the process defined by (20)

$$X(T) = X(0)\exp\{T\nu + \sigma\rho W(T)\} \quad a.s. \tag{22}$$

holds for all $T > 0$ where

$$\rho^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T\int_0^T \operatorname{cov}(\xi_s, \xi_{s'})dsds' > 0.$$

Remark. As for the definition of the time-continuous stationary mixing process $\{\xi_t\}$, see Ibragimov and Linnik (1971, p.362).

To prove Theorem 3 we need the following simple lemma.

LEMMA 1. *Let $t_k = km^{-1}$. Suppose $\{\xi_s\}$ is a stationary process such that $E|\xi_s|^p < \infty$ for $p \geq 2$. Then, for any $k \geq 1$*

$$E\left|\int_{t_{k-1}}^{t_k} (\nu + \sigma\xi_s)ds\right|^p = E\left|\int_0^{t_1} (\nu + \sigma\xi_s)ds\right|^p = O\left(\frac{1}{m^p}\right). \tag{23}$$

Proof. By the Jensen inequality we have

$$\begin{aligned}
&E\left|\int_0^{t_1} (\nu + \sigma\xi_s)ds\right|^p \\
&= t_1^p E\left(\frac{1}{t_1}\int_0^{t_1} |\nu + \sigma\xi_s|ds\right)^p \leq t_1^p E\left(\frac{1}{t_1}\int_0^{t_1} |\nu + \sigma\xi_s|^p ds\right) \\
&= t_1^p \left(\frac{1}{t_1}\int_0^{t_1} E|\nu + \sigma\xi_s|^p ds\right) \leq c t_1^p = c\frac{1}{m^p}. \quad \square
\end{aligned}$$

Proof of Theorem 3. We note firstly that for all m sufficiently large and any $k \geq 1$

$$\Big|\int_{t_{k-1}}^{t_k} (\nu + \sigma\xi_s) ds\Big| \leq m^{-\frac{1}{2}} \quad a.s. \tag{24}$$

In fact, since by Lemma 1 (with $p = 3$) and the Markov inequality, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} P\bigg(\Big|\int_{t_{k-1}}^{t_k} (\nu + \sigma\xi_s) ds\Big| > m^{-\frac{1}{2}}\bigg) \\
&\leq \sum_{m=1}^{\infty} m^{\frac{3}{2}} E\Big|\int_{t_{k-1}}^{t_k} (\nu + \sigma\xi_s) ds\Big|^3 \leq \sum_{m=1}^{\infty} m^{\frac{3}{2}} c m^{-3} = c \sum_{m=1}^{\infty} m^{-\frac{3}{2}} < \infty.
\end{aligned}$$

and, via the Borel-Cantelli Lemma, (24) is obtained.

Accordingly, for all m sufficiently large and all k we can define

$$U_{m,k} = \log\bigg(1 + \int_{t_{k-1}}^{t_k} (\nu + \sigma\xi_s) ds\bigg) \quad (k \geq 1).$$

We note that for each m fixed $\{U_{m,k} : k \geq 1\}$ is a strictly stationary strong mixing sequence with mixing coefficient $\alpha(n)$.

Now, we consider

$$\begin{aligned}
\log \frac{X(T)}{X(0)} &= \sum_{k=1}^{[mT]} \log \frac{X(t_k)}{X(t_{k-1})} + \log \frac{X(T)}{X([mT]m^{-1})} \\
&= \sum_{k=1}^{[mT]} EU_{m,k} + \sum_{k=1}^{[mT]} (U_{m,k} - EU_{m,k}) + \log \frac{X(T)}{X([mT]m^{-1})}
\end{aligned} \tag{25}$$

Firstly, we note that the last term in the above equation is negligible, which means

$$\log \frac{X(T)}{X([mT]m^{-1})} \ = \ O\big(m^{-\frac{1}{2}}\big) \quad a.s. \tag{26}$$

Since $\{\nu + \sigma\xi_s\}$ is strictly stationary, using the elementary inequality $\log(1 + x) < x$ $(x > 0)$ and Lemma 1 (with $p = 3$), we have

$$\begin{aligned}
E\Big|\log \frac{X(T)}{X([mT]m^{-1})}\Big|^3 &= E\Big|\log\bigg(1 + \int_{[mT]/m}^{T} (\nu + \sigma\xi_s) ds\bigg)\Big|^3 \\
&\leq E\Big|\log\bigg(1 + \int_{[mT]/m}^{([mT]+1)/m} |\nu + \sigma\xi_s| ds\bigg)\Big|^3
\end{aligned}$$

$$= E\Big|\log\Big(1+\int_0^{t_1}|\nu+\sigma\xi_s|ds\Big)\Big|^3$$
$$\le E\Big|\int_0^{t_1}|\nu+\sigma\xi_s|ds\Big|^3 \le \frac{c}{m^3},$$

Thus, by the Borel-Cantelli lemma we have (26).

Next, by the Taylor theorem we may write $U_{m,k}$ as

$$U_{m,k} = \log\Big(1+\int_{t_{k-1}}^{t_k}(\nu+\sigma\xi_s)ds\Big) = \int_{t_{k-1}}^{t_k}(\nu+\sigma\xi_s)ds + R_{m,k,2},$$

where $R_{m,k,2}$ denotes the residual, that is,

$$R_{m,k,2} = -\frac{1}{2}\Big(1+\theta_k(\omega)\int_{t_{k-1}}^{t_k}(\nu+\sigma\xi_s)ds\Big)^{-2}\Big(\int_{t_{k-1}}^{t_k}(\nu+\sigma\xi_s)ds\Big)^2$$

where $\theta_k(\omega)$ is a random variable such that $|\theta_k(\omega)| \le 1$. We note that by (24)

$$|R_{m,k,2}| \le \frac{1}{2}(1-2^{-1})^{-2}\Big(\int_{t_{k-1}}^{t_k}(\nu+\sigma\xi_s)ds\Big)^2 = 2\Big(\int_{t_{k-1}}^{t_k}(\nu+\sigma\xi_s)ds\Big)^2 \quad a.s.$$

for all m sufficiently large. Thus, by the stationarity of $\{\xi_t\}$ and Lemma 1

$$E|R_{m,k,2}| \le 2E\Big|\int_{t_{k-1}}^{t_k}(\nu+\sigma\xi_s)ds\Big|^2 = 2E\Big|\int_0^{t_1}(\nu+\sigma\xi_s)ds\Big|^2 \le c\frac{1}{m^2}, \tag{27}$$

and

$$E|R_{m,k,2}|^{2+\delta} \le cE\Big|\int_{t_{k-1}}^{t_k}(\nu+\sigma\xi_s)ds\Big|^{4+2\delta} = cE\Big|\int_0^{t_1}(\nu+\sigma\xi_s)ds\Big|^{4+2\delta} \le c\frac{1}{m^{4+2\delta}}. \tag{28}$$

Thus, noting that $E\xi_s = 0$, and using (27) we have

$$\sum_{k=1}^{[mT]} EU_{m,k} = \sum_{k=1}^{[mT]} E\log\Big(1+\int_{t_{k-1}}^{t_k}(\nu+\sigma\xi_s)ds\Big)$$
$$= \sum_{k=1}^{[mT]} E\Big(\int_{t_{k-1}}^{t_k}(\nu+\sigma\xi_s)ds + R_{m,k,2}\Big)$$

$$
\begin{aligned}
&= \sum_{k=1}^{[mT]} \Big\{ E \int_{t_{k-1}}^{t_k} (\nu + \sigma \xi_s) ds + O(m^{-2}) \Big\} \\
&= \nu \sum_{k=1}^{[mT]} \int_{t_{k-1}}^{t_k} ds + O(m^{-1}T) = \nu T + O(m^{-1}T).
\end{aligned}
$$

Now, put

$$V_{m,j,k} = \mathrm{cov}(U_{m,j}, U_{m,k}) \quad (j,k \geq 1)$$

and evaluate $V_{m,j,k}$ in the case $j < k-2$. We write $V_{m,j,k}$ as

$$
\begin{aligned}
V_{m,j,k} &= \mathrm{cov}\bigg(\log\bigg(1 + \int_{t_{j-1}}^{t_j} (\nu + \sigma\xi_s) ds\bigg), \log\bigg(1 + \int_{t_{k-1}}^{t_k} (\nu + \sigma\xi_{s'}) ds'\bigg)\bigg) \\
&= \mathrm{cov}\bigg(\int_{t_{j-1}}^{t_j} (\nu + \sigma\xi_s) ds + R_{m,j,2}, \int_{t_{k-1}}^{t_k} (\nu + \sigma\xi_{s'}) ds' + R_{m,k,2}\bigg) \\
&= \mathrm{cov}\bigg(\int_{t_{j-1}}^{t_j} (\nu + \sigma\xi_s) ds, \int_{t_{k-1}}^{t_k} (\nu + \sigma\xi_{s'}) ds'\bigg) \\
&\quad + \mathrm{cov}\bigg(\int_{t_{j-1}}^{t_j} (\nu + \sigma\xi_s) ds, R_{m,k,2}\bigg) + \mathrm{cov}\bigg(R_{m,j,2}, \int_{t_{k-1}}^{t_k} (\nu + \sigma\xi_{s'}) ds'\bigg) \\
&\quad + \mathrm{cov}(R_{m,j,2}, R_{m,k,2}) \\
&= D_{1,j,k} + D_{2,j,k} + D_{3,j,k} + D_{4,j,k} \quad \text{(say)}.
\end{aligned}
$$

Since for any j

$$E \int_{t_{j-1}}^{t_j} (\nu + \sigma\xi_s) ds = \int_{t_{j-1}}^{t_j} \nu ds,$$

we have

$$
\begin{aligned}
D_{1,j,k} &= \mathrm{cov}\bigg(\int_{t_{j-1}}^{t_j} (\nu + \sigma\xi_s) ds, \int_{t_{k-1}}^{t_k} (\nu + \sigma\xi_{s'}) ds'\bigg) \\
&= \sigma^2 \int_{t_{j-1}}^{t_j} \int_{t_{k-1}}^{t_k} \mathrm{cov}(\xi_s, \xi_{s'}) ds ds'.
\end{aligned}
$$

We note here that for $j < k-1$

$$\int_{t_{j-1}}^{t_j} (\nu + \sigma\xi_s) ds \in \mathcal{M}_{-\infty}^{t_j} \quad \text{and} \quad R_{m,k,2} \in \mathcal{M}_{t_{k-1}}^{\infty}.$$

Hence, by Lemma A, Lemma 1 and the fact that $\alpha(t)$ is monotone decreasing, we have

$$|D_{2,j,k}| \leq c_0 \bigg\| \int_{t_{j-1}}^{t_j} (\nu + \sigma\xi_s) ds \bigg\|_{2+\delta} \|R_{m,k,2}\|_{2+\delta} \alpha^{\frac{\delta}{2+\delta}}(t_{k-1} - t_j)$$

$$\leq c_0\Big(\frac{1}{m^{2+\delta}}\Big)^{2+\delta}\Big(\frac{1}{m^{4+2\delta}}\Big)^{\frac{1}{2+\delta}}\alpha^{\frac{\delta}{2+\delta}}(t_{k-1}-t_j)$$
$$\leq c_0\frac{1}{m}\frac{1}{m^2}m\int_{t_{k-2}}^{t_{k-1}}\alpha^{\frac{\delta}{2+\delta}}(t-t_j)dt \leq c_0\frac{1}{m^2}\int_{t_{k-2}}^{t_{k-1}}\alpha^{\frac{\delta}{2+\delta}}(t-t_j)dt.$$

Similarly, we have

$$|D_{3,j,k}| \leq c_0\frac{1}{m^2}\int_{t_{k-2}}^{t_{k-1}}\alpha^{\frac{\delta}{2+\delta}}(t-t_j)dt$$

and

$$|D_{4,j,k}| \leq c_0\frac{1}{m^2}\int_{t_{k-2}}^{t_{k-1}}\alpha^{\frac{\delta}{2+\delta}}(t-t_j)dt$$

Further, for $k=j,j+1,j+2$

$$\begin{aligned}V_{m,j,k} &\leq \|U_{m,j}\|_2\|U_{m,k}\|_2 = \|U_{m,1}\|_2^2\\ &\leq E\Big|\log\Big(1+\int_{t_{j-1}}^{t_j}|\nu+\sigma\xi_s|ds\Big)\Big|^2\\ &\leq E\Big(\int_0^{t_1}|\nu+\sigma\xi_s|ds\Big)^2 \leq c_0 m^{-2}.\end{aligned}$$

Combining these inequalities and using (21) we have

$$\begin{aligned}&\frac{1}{[mT]}\sum_{j=1}^{[mT]}\sum_{k=1}^{[mT]}V_{m,j,k}\\ &= \frac{1}{mT}\sum_{j=1}^{[mT]}(V_{m,j,j}+2V_{m,j,j+1}+2V_{m,j,j+2}) + \frac{1}{[mT]}\sum_{j=1}^{[mT]}\sum_{k=1}^{[mT]}D_{1,j,k}\\ &\quad +2\frac{1}{[mT]}\sum_{1\leq j<k-2\leq[mT]-2}(D_{2,j,k}+D_{3,j,k}+D_{4,j,k})\\ &= O(m^{-2}) + \frac{\sigma^2}{[mT]}\int_0^{[mT]/m}\int_0^{[mT]/m}\mathrm{cov}(\xi_s,\xi_{s'})dsds'\\ &\quad +O(m^{-2})\frac{1}{[mT]}\sum_{j=1}^{[mT]}\sum_{k=1}^{[mT]}\int_{t_{k-2}}^{t_{k-1}}\alpha^{\frac{\delta}{2+\delta}}(t-t_j)dt\\ &= O(m^{-2}) + \frac{\sigma^2T}{[mT]}\frac{1}{T}\int_0^{[mT]/m}\int_0^{[mT]/m}\mathrm{cov}(\xi_s,\xi_{s'})dsds'\\ &\quad +O(m^{-2})\int_0^T\alpha^{\frac{\delta}{2+\delta}}(t)dt\end{aligned}$$

$$= \frac{\sigma^2}{m}\frac{1}{T}\int_0^T\int_0^T \mathrm{cov}(\xi_s, \xi_{s'})dsds' + O(m^{-2}).$$

Hence, via the definition of ρ^2, we can define

$$\hat{\rho}_m^2 = \lim_{T\to\infty}\frac{1}{[mT]}\sum_{j=1}^{[mT]}\sum_{k=1}^{[mT]} V_{m,j,k} = \frac{\sigma^2}{m}(\rho^2 + O(m^{-2}))$$

for all m sufficiently large.

Since $\{U_{m,k} - EU_{m,k}\}$ is a stationary strong mixing sequence of zero mean random sequence, by Theorem A we have

$$\begin{aligned}
&\sum_{k=1}^{[mT]}\{U_{m,k} - EU_{m,k}\}\\
&= \hat{\rho}_m(W(mT) + o\big((mT)^{\frac{1}{2}-\lambda}\big)\\
&= \sqrt{\frac{\sigma^2}{m}(\rho^2 + O(m^{-2}))}\big(W(mT) + o\big((mT)^{\frac{1}{2}-\lambda}\big)\big)\\
&= \sqrt{\sigma^2\rho^2 + O(m^{-2})}\big(W(T) + o\big(m^{-\lambda}T^{\frac{1}{2}-\lambda}\big)\big) \quad a.s.
\end{aligned}$$

for all m sufficiently large where λ is a number such that $0 < \lambda < \frac{1}{2}$.

Combining (25) with the above relations, we have

$$\begin{aligned}
\log\frac{X(T)}{X(0)} &= T\nu + \sqrt{\sigma^2(\rho^2 + O(m^{-2}))}\big(W(T) + o\big(m^{-\lambda}T^{\frac{1}{2}-\lambda}\big)\big)\\
&\quad + O(m^{-1}T) + O(m^{-2})
\end{aligned}$$

holds almost surely for all m sufficiently large.

Finally, letting $m \to \infty$, we have (22) and the proof is completed. □

Next, we consider the process $\{X(t_k)\}$ satisfying

$$\Delta X(t_k) = \big((t_k - t_{k-1})\nu + \sqrt{t_k - t_{k-1}}\sigma\zeta_k\big)X(t_{k-1}) \quad (1 \le k \le n), \tag{29}$$

for all n where $\{\zeta_i\}$ is a strictly stationary strong mixing sequence with $E\zeta_1 = 0$ and $E\zeta_1^2 = 1$, and $\sigma > 0$ is some absolute constant.

Theorem 4. *Suppose the process $\{\zeta_i\}$ is strictly stationary with $E\zeta_1 = 0$ and $E\zeta_1^2 = 1$, and satisfies the strong mixing condition with mixing coefficient $\alpha(n)$. Furthermore, suppose that for some $\delta > 0$*

$$E|\zeta_1|^9 < \infty \text{ and } \sum_{n=1}^{\infty}\alpha^{\frac{2\delta}{2+\delta}}(n) < \infty. \tag{30}$$

Then, for the process satisfying (29)

$$X(T) = X(0)\exp\Big\{T\Big(\nu - \frac{\sigma^2}{2}\Big) + \sigma\rho W(T)\Big\} \quad a.s. \tag{31}$$

where

$$\rho^2 = \lim_{n\to\infty}\frac{1}{n}\Big\{n\mathrm{Var}\zeta_1 + 2\sum_{1\le i<j\le n}\mathrm{cov}(\zeta_i,\zeta_j)\Big\} > 0. \tag{32}$$

Proof. Firstly, let $k \ge 1$ be arbitrary. We note that

$$\begin{aligned} E\Big|\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k\Big|^9 &\le E\Big|\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_1\Big|^9 \\ &\le \Big(\frac{T}{n}\Big)^{\frac{9}{2}} E\Big|\sqrt{\frac{T}{n}}\nu + \sigma\zeta_1\Big|^9 \le c_0\Big(\frac{T}{n}\Big)^{\frac{9}{2}}. \end{aligned}$$

So, we have

$$\begin{aligned} &P\Big(\Big|\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k\Big|^3 \ge n^{-\frac{17}{16}}\Big) \\ &\le n^{\frac{51}{16}} E\Big|\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k\Big|^9 \le c_0 n^{\frac{51}{16}}\Big(\frac{T}{n}\Big)^{\frac{9}{2}} \le c_0 T^{\frac{9}{2}} n^{-\frac{21}{16}}, \end{aligned}$$

which, via the Borel-Cantelli lemma, implies that for all n sufficiently large

$$\Big|\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k\Big|^3 \le n^{-\frac{17}{16}} \quad a.s. \tag{33}$$

Thus, we can define

$$\log(1 + (t_k - t_{k-1})\nu + \sqrt{t_k - t_{k-1}}\zeta_k) = \log\Big(1 + \frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k\Big),$$

By the Taylor theorem we can write

$$\begin{aligned} &\log\Big(1 + \frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k\Big) \\ &= \Big(\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k\Big) - \frac{1}{2}\Big(\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k\Big)^2 + R_{n,k,3} \end{aligned}$$

where $R_{n,k,3}$ is the residual, that is,

$$R_{n,k,3} = \frac{1}{3}\Big(1 + \theta(\omega)\Big(\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k\Big)\Big)^{-3}\Big(\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k\Big)^3$$

where $\theta_k(\omega)$ is a random variable such that $|\theta_k(\omega)| \le 1$.

Now, we consider

$$\begin{aligned}
\log \frac{X(T)}{X(0)} &= \sum_{k=1}^{n} \log(1 + (t_k - t_{k-1})\nu + \sqrt{t_k - t_{k-1}}\zeta_k) \\
&= \sum_{k=1}^{n} \left\{ \left(\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k \right) - \frac{1}{2}\left(\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k \right)^2 + R_{n,k,3} \right\} \\
&= T\nu + \sum_{k=1}^{n} \sqrt{\frac{T}{n}}\sigma\zeta_k + \frac{1}{2}\sum_{k=1}^{n}\left(\frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_k \right)^2 + \sum_{k=1}^{n} R_{n,k,3} \\
&= T\nu + I_{n,1} + I_{n,2} + I_{n,3}, \quad \text{(say)}.
\end{aligned}$$

Since $\{\zeta_k\}$ is stationary, by (33) we have

$$|R_{n,k,3}| \le c_1 \left| \frac{T}{n}\nu + \sqrt{\frac{T}{n}}\sigma\zeta_1 \right|^3 \le c_1 n^{-\frac{17}{16}} \quad (k \ge 1) \quad a.s.$$

for all n sufficiently large. Hence, as $n \to \infty$ we have

$$|I_{n,3}| = O\big(n^{-\frac{1}{16}}\big) \quad a.s. \tag{34}$$

Next, noting that $E\zeta_1 = 0$ and $E\zeta_1^2 = 1$ and using the strong law of large numbers

$$\begin{aligned}
I_{n,2} &= -\frac{1}{2}\left\{ \sum_{k=1}^{n}\left(\frac{T}{n}\nu\right)^2 + 2\nu\sigma\sum_{k=1}^{n}\left(\frac{T}{n}\right)^{\frac{3}{2}} + \sum_{k=1}^{n}\frac{T}{n}\sigma^2\zeta_k^2 \right\} \\
&= O\big(T^2 n^{-1}\big) + O\big(T^{\frac{3}{2}} n^{-\frac{1}{2}}\big) - \frac{1}{2}T\sigma^2 \quad a.s.
\end{aligned} \tag{35}$$

Finally, by Theorem A

$$\begin{aligned}
I_{1,n} &= \sqrt{\frac{T}{n}}\sigma\sum_{k=1}^{n}\zeta_k = \sqrt{\frac{T}{n}}\sigma\{\rho W(n) + O(n^{\frac{1}{2}-\lambda})\} \\
&= \sigma\rho W(T) + O\big(T^{\frac{1}{2}} n^{-\lambda}\big) \quad a.s.
\end{aligned} \tag{36}$$

for all n sufficiently large.

Using (34)-(36) we have

$$\begin{aligned}
\log \frac{X(T)}{X(0)} &= T\nu + \{\sigma\rho W(T) + O\big(T^{\frac{1}{2}} n^{-\lambda}\big)\} \\
&\quad -\frac{1}{2}\{T\sigma^2 + O\big(T^2 n^{-\frac{1}{2}}\big)\} + O\big(n^{-\frac{1}{16}}\big) \quad a.s.
\end{aligned}$$

for all n suffiently large.

Hence, letting $n \to \infty$ we have (31) and the proof is completed. □

Corollary. *$\{\zeta_i\}$ is an i.i.d. sequence of random variables with $E\zeta_1 = 0$ and $E\zeta_1^2 = 1$. Then*

$$X(T) = X(0)\exp\left\{T\left(\nu - \frac{\sigma^2}{2}\right) + \sigma W(T)\right\} \quad a.s. \tag{37}$$

Specifically, if $\{\zeta_i\}$ is an i.i.d. sequence of $N(0,1)$-random variables, then the same coclusion holds.

Proof. Since $E\zeta_i = 0$, $\mathrm{Var}\zeta_i = 1$ and $\mathrm{cov}(\zeta_i, \zeta_j) = 0\,(i \neq j)$, we have $\rho^2 = 1$. Hence, the desired conclusion follows from Theorem 4. □

Remark. The condition (30) is not best possible ones. We can relax those conditions by the tedious calculations.

4. Applications

Now, we consider the option pricing. Let S be the price of a stock. We assume that S is continuous and is generated by the formula

$$S(t + dt) = S(t)\{1 + \nu dt + \sigma\zeta_t\sqrt{dt}\}, \tag{38}$$

with the initial condition $S(0) = S_0$, where $\sigma > 0$ is some constant and $\{\zeta_t\}$ is a stricly stationary strong mixing stochastic process with $E\zeta_1 = 0$ and $E\zeta_1^2 = 1$. In the sequel, we always assume that $\{\zeta_i\}$ satisfies the condition (30) and

$$\rho^2 = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{cov}(\zeta_i, \zeta_j) = 1.$$

Let $f(S, t)$ be the price of the claim at time t when the stock price is S.

To obtain the asymptotic Black-Scholes formula, we need the following lemma which is known (see, for example, N.H. Chan and H.Y.Wong (2006)).

Lemma B. *Let S be a lognormally distributed random variables such that $\log S$ is an $N(\nu, \rho^2)$-variable and let K be a given constant. Then*

$$E\{\max(S - K, 0)\} = E(S)\Phi(d_1) - K\Phi(d_2), \tag{39}$$

where $\Phi(\cdot)$ denotes the distribution function of a standard normal random variable and

$$d_1 = \frac{1}{\rho}(-\log K + \nu + \rho^2) = \frac{1}{\rho}\log\left(E\left(\frac{S}{K}\right) + \frac{\rho^2}{2}\right),$$
$$d_2 = \frac{-\log K + \nu}{\rho} = \frac{1}{\rho}\log\left(E\left(\frac{S}{K}\right) - \frac{\rho^2}{2}\right).$$

Remark. Since

$$E(S) = \exp\left(\nu + \frac{\rho^2}{2}\right),$$

we can rewrite (39) as

$$E\{\max(S - K, 0)\} = \exp\left(\nu + \frac{\rho^2}{2}\right)\Phi(d_1) - K\Phi(d_2).$$

Using the usual methods, from Theorem 4 and Lemma B we have the following theorems which are generalizations of the Black-Scholes option pricing formulas.

Theorem 5. *Consider a Europian option with payoff $F(S)$ and expiration time T. Suppose the continuous compounding interest rate r. Then, the current Europian option price is determined by*

$$f(S, 0) = e^{-rT}\hat{E}\{F(S(T)\}, \tag{40}$$

where $\hat{E}$ denotes the expectation under the risk-neutral probability that is derived from the risk-neutral process defined by (38) with $\nu = r$.

Theorem 6. *Consider a Europian call option with strike price K and expiration time T. If the underlying stock pays no dividends during the time $[0, T]$ and if there is continuously compounded risk-free rate r, then the price of this contract at time 0, $f(S, 0) = C(S, 0)$, is given by*

$$C(S, 0) = S\Phi(d_1) - Ke^{-rT}\Phi(d_2). \tag{41}$$

The proofs are omitted.

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