Yokohama Mathematical
Journal Vol. 57, 2011

DIAGONAL TRANSFORMATIONS IN HEXANGULATIONS ON THE SPHERE

By

Naoki Matsumoto

(Received July 21, 2010)

Abstract. In this article, we shall prove that any two hexangulations on the sphere with the same number of vertices can be transformed into each other by three kinds of transformations specifically defined.

1. Introduction

An *n-angulation* G is a 2-connected simple graph on a closed surface such that each face of G is bounded by a cycle of length n, where $n \geq 3$ is an integer. In an n-angulation, let $P = x_1y_1y_2\cdots y_{l-2}y_{l-1}x_k$ be a path of length l ($1 \leq l \leq \lfloor \frac{n}{2} \rfloor$, $l + k = n + 1$), which is shared by two faces F_1 and F_2, where the boundaries of F_1 and F_2, denoted ∂F_1 and ∂F_2, are supposed to be $\partial F_1 = x_1x_2x_3\cdots x_{k-2}x_{k-1}x_ky_{l-1}y_{l-2}\cdots y_2y_1$ and $\partial F_2 = x_1x_{2k-2}x_{2k-3}\cdots x_{k+2}x_{k+1}x_ky_{l-1}y_{l-2}\cdots y_2y_1$, respectively. See Figure 1, replacing a path P with a path $P' = x_2y_1y_2\cdots y_{l-2}y_{l-1}x_{k+1}$ is called a *diagonal transformation.* So, for n-angulations, there are $\lfloor \frac{n}{2} \rfloor$ kinds of diagonal transformations, depending on the length l of the path P. When this transformation breaks the simpleness or 2-connectedness of graphs, we don't apply it. Two n-angulations are said to be *equivalent* if they can be transformed into each other by diagonal transformations, up to homeomorphism.

There are many results on diagonal transformations in graphs on closed surfaces. For triangulations on closed surfaces, the following theorem was proved. By the definition of diagonal transformations, flipping an edge is a unique diagonal transformation for triangulations, and it is called a *diagonal flip.* Related topics are in [1, 3].

Theorem 1. (S. Negami [7]) *For any closed surface F^2, there exists a positive integer $N(F^2)$ such that any two triangulations G_1 and G_2 with $|V(G_1)| = |V(G_2)| \geq N(F^2)$ are equivalent to each other, up to homeomorphism.*

2010 Mathematics Subject Classification: 05C10
Key words and phrases: diagonal transformation, hexangulation, sphere

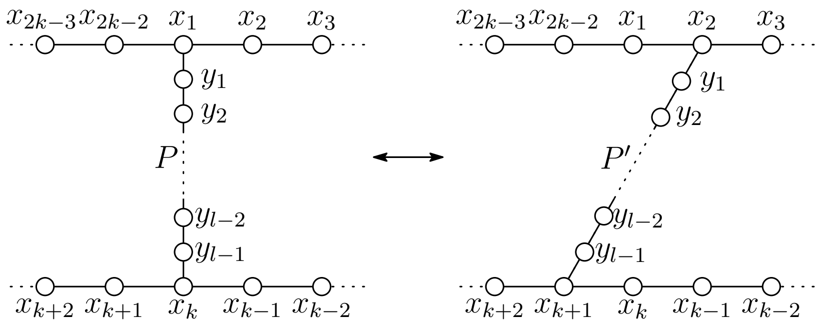

Figure 1 Diagonal transformations in n-angulations

For quadrangulations on the sphere, the following theorem was proved. Note that any quadrangulation on the sphere is bipartite. By the definition of diagonal transformations, sliding an edge and rotating a path of length 2 are two kinds of diagonal transformations in quadrangulations, and they are called a *diagonal slide* and a *diagonal rotation*, respectively. Related topics are in [4, 6], in which diagonal transformations in non-bipartite quadrangulations on non-spherical surfaces are also considered. Though those transformations clearly preserve the bipartiteness of graphs, any two non-bipartite quadrangulations are not necessarily equivalent to each other even if they have the same and sufficiently large number of vertices. The equivalence can be descibed by a notion called a "cycle parity", where the detailed argument can be found in [6].

Theorem 2. (A. Nakamoto [5]) *For any closed surface F^2, there exists a positive integer $M(F^2)$ such that any two bipartite quadrangulations G_1 and G_2 with $|V(G_1)| = |V(G_2)| \geq M(F^2)$ are equivalent to each other, up to homeomorphism.*

For *quintangulations* which are 5-angulations in our terminology, the following theorem was proved. Diagonal transformations in quintangulations are the same as those for quadrangulations.

Theorem 3. (J. Kanno et al. [2]) *Any two quintangulations on the sphere with the same number of vertices are equivalent to each other, up to homeomorphism.*

In this paper, we would like to consider 6-angulations which are called *hexangulations*. We always consider a fixed vertex 2-coloring which assigns black and white to the vertices of hexangulations since hexangulations on the sphere are always bipartite.

By the definition of diagonal transformations, diagonal transformations in hexangulations are Transformations A, B and C which are shown in Figures 2, 3 and 4, respectively. The shaded regions mean a part of a hexangulation other than two neighboring faces.

Our main theorem can be described as follow.

THEOREM 4. *Any two hexangulations on the sphere with the same number of vertices are equivalent to each other, up to homeomorphism.*

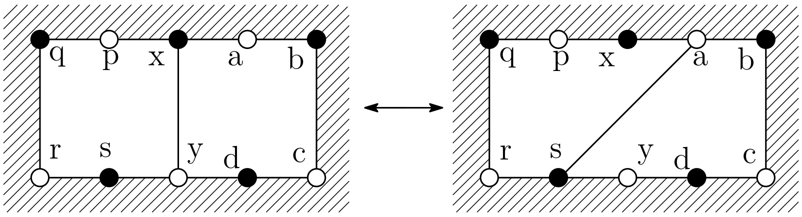

Figure 2 Transformation A

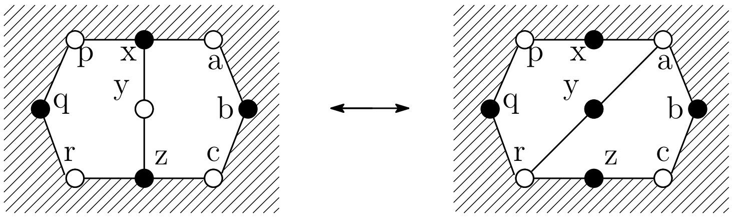

Figure 3 Transformation B

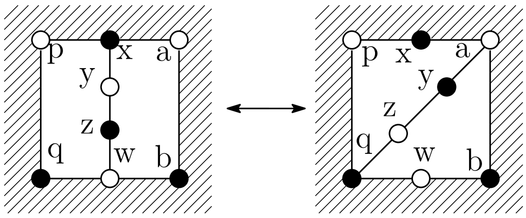

Figure 4 Transformation C

2. Necessity of the transformations

In this section, we shall show that each of Transformations A, B and C is necessary in Theorem 4. That is, if we omit any one of the three transformations, then we can find a hexangulation which cannot be transformed into any other hexangulation only by the remaining two transformations.

Proposition 5. *For any two of Transformations A, B and C, there are hexangulations which cannot be transformed by them.*

Proof. Firstly we construct a hexangulation which needs Transformation C. See Figure 5, which shows a *standard form* of hexangulations on the sphere. Clearly, this admits only Transformation C, since each of Transformations A and B yields a vertex of degree one.

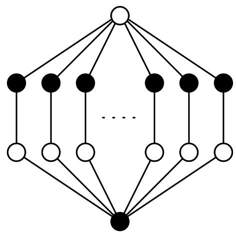

Figure 5 The standard form

Secondly we construct a hexangulation which needs Transformation A. Figure 6 shows a hexangulation obtained from a spherical quadrangulation by adding a path of length 4 whose three middle vertices are of degree 2 into each face as a diagonal. It is easy to see that neither of Transformations B and C can be applied to it.

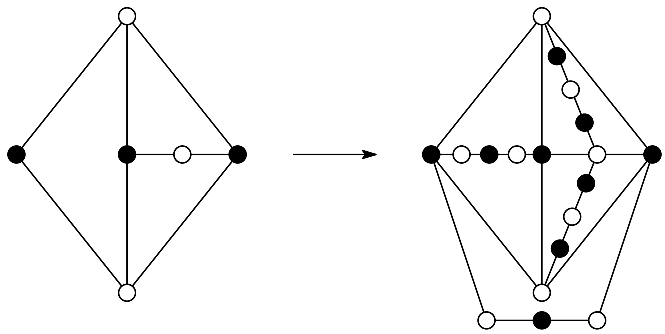

Figure 6 A hexangulation obtained from a spherical quadrangulation

Finally we construct a hexangulation which needs Transformation B. Figure 7 shows a hexangulation obtained from a spherical triangulation by subdividing each edge with exactly one vertex. We can see that only Transformation B can be applied to it.

Therefore we have been able to show that we cannot omit any one of the three diagonal transformations from Theorem 4. □

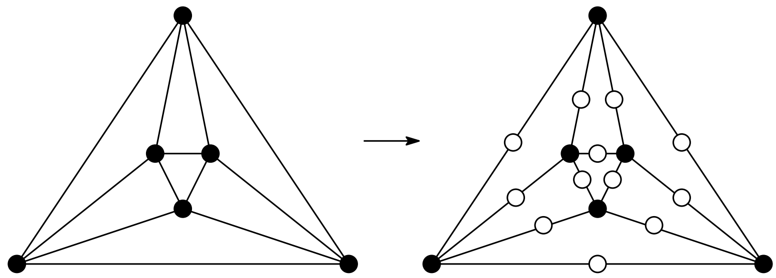

Figure 7 A hexangulation obtained from a spherical triangulation

3. Lemmas

Before we prove our main theorem, we show the following lemmas. Let xy be an edge in a hexangulation, and we suppose that xy can be flipped by Transformation A to join two vertices a and b. In this case, we denote $xy \to ab$. For Transformation B (resp., C) applied to a path xvy (resp., $xvuy$), we similarly denote $xvy \to avb$ (resp., $xvuy \to avub$).

Lemma 6. *Let G be a hexangulation on the sphere and let $x \in V(G)$ with $\deg(x) \geq 3$.*

(1) *Let $e = xy$ be an edge with $\deg(y) \geq 3$. Then e can be flipped by Transformation A to reduce the degree of x.*
(2) *Let $P = xyz$ be a path of length 2 with $\deg(y) = 2$ and $\deg(z) \geq 3$. Then P can be flipped by Transformation B to reduce the degree of x.*
(3) *Let $P = xyzw$ be a path of length 3 with $\deg(y) = \deg(z) = 2$ and $\deg(w) \geq 3$. Then P can be flipped by Transformation C to reduce the degree of x.*

Proof. (1) We apply Transformation A. If xy is shared by two faces $xabcdy$ and $xpqrsy$, there are two choices to move xy, that is, $xy \to as$ or $xy \to pd$ (see Figure 2). By the planality, one of those two transformations is possible without loss the simpleness and 2-connectedness of graphs. That is, if $xy \to as$ is impossible, then G has an edge as. In this case, the 4-cycle $axys$ separates p and d in the interior and exterior. Hence we have $xy \to pd$.

(2) We apply Transformation B. If xyz is shared by two faces $xabczy$ and $xpqrzy$, there are two choices to move xyz, that is, $xyz \to ayr$ or $xyz \to pyc$ (see Figure 3). If both are impossible, we have both $a = r$ and $c = p$. Similarly to (1), it is impossible since $a \neq c$ by the 2-connectedness of graphs.

(3) Transformation C is always possible without breaking the simpleness and 2-connectedness of graphs (see Figure 4). □

We define Transformation D as shown in Figure 8, and let us consider whether any two hexangulations with the same number of vertices can be transformed into each other by Transformations A, B, C and D. By the following lemma, this proves Theorem 4.

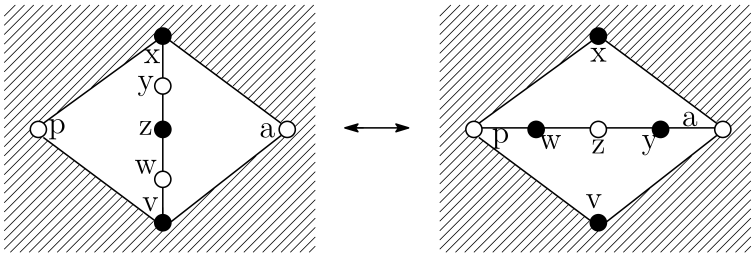

Figure 8 Transformation D

LEMMA 7. *Let G be a hexangulation on the sphere and let $x \in V(G)$ with $\deg(x) \geq 3$. Let $P = xyzwv$ be a path of length 4 with $\deg(y) = \deg(z) = \deg(w) = 2$ and $\deg(v) \geq 3$. Then P can be flipped by Transformation D to reduce the degree of x. Moreover, Transformation D can be derived from Transformations A, B and C.*

Proof. It is easy to see that Transformation D is always possible without breaking the simpleness and 2-connectedness of graphs.

We consider whether Transformation D can be obtained by a sequence of Transformations A, B and C. Suppose that the union of two faces sharing a path $xyzwv$ of length four is bounded by a 4-cycle $xavp$ as shown in Figure 8. We consider a face neighboring to the quadrilateral region Γ bounded by the 4-cycle $xavp$. By the planarity, we can always find a face f such that the common edges of f and Γ induce a connected graph. Moreover, we can see that the number of such common edges is at most two by the simpleness and 2-connectedness of graphs, and hence we have the following.

CASE 1. There is a face f sharing exactly one edge with Γ.

Without loss of generality, we may suppose that f contains the edge pv. Transformation D can be derived from Transformations A, B and C, as shown in Figure 9. Since the intersection of f and Γ is only pv, all transformations in Figure 9 preserve the simpleness and 2-connectedness of graphs.

CASE 2. There is a face f sharing exactly two edges with Γ.

Without loss of generality, we may suppose that f contains the path pva. Transformation D can be derived from Transformations A, B and C, as shown

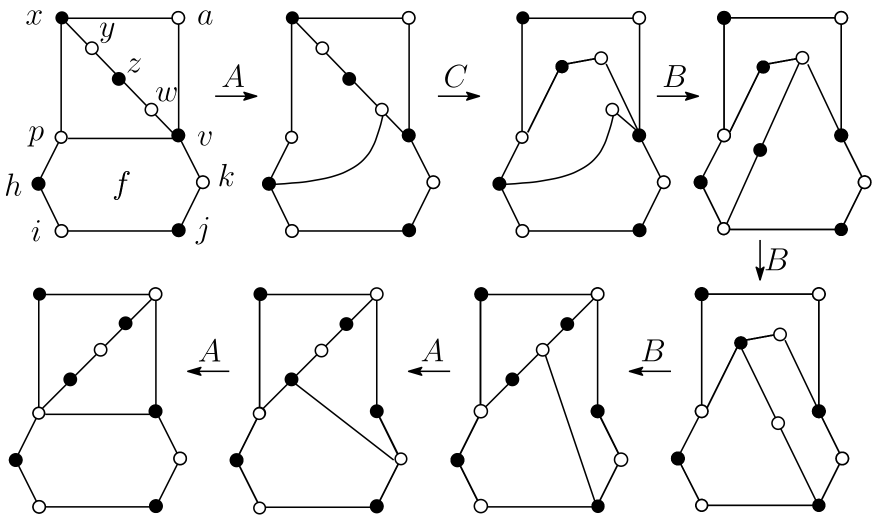

Figure 9 Case 1

in Figure 10. Since f is bounded by a cycle, all transformations preserve the simpleness and 2-connectedness of graphs.

Therefore, Transformation D can be derived from a sequence of Transformations A, B and C in all cases. □

4. Proof of the theorem

In this section, we shall prove the following theorem. By Lemma 7, Theorem 4 is equivalent to Theorem 8.

THEOREM 8. *Any two hexangulations on the sphere with the same number of vertices can be transformed into each other by Transformations A, B, C and D, up to homeomorphism.*

Proof. Let G be a hexangulation on the sphere with an outer cycle $NxySzw$. By induction on $|V(G)|$, we shall prove that G can be transformed into the standard form (shown in Figure 5), by Transformations A, B, C and D, fixing the outer 6-cycle. If G is isomorphic to a 6-cycle, then G can be regarded as the standard form, and hence we may suppose that $|V(G)| > 6$. For a face or a 2-cell region f of G, let ∂f denote the boundary cycle of f.

STEP 1. We make $\deg(x) = \deg(y) = 2$.

First, applying Transformations A, B, C and D, we can make $\deg(x) = 2$ by

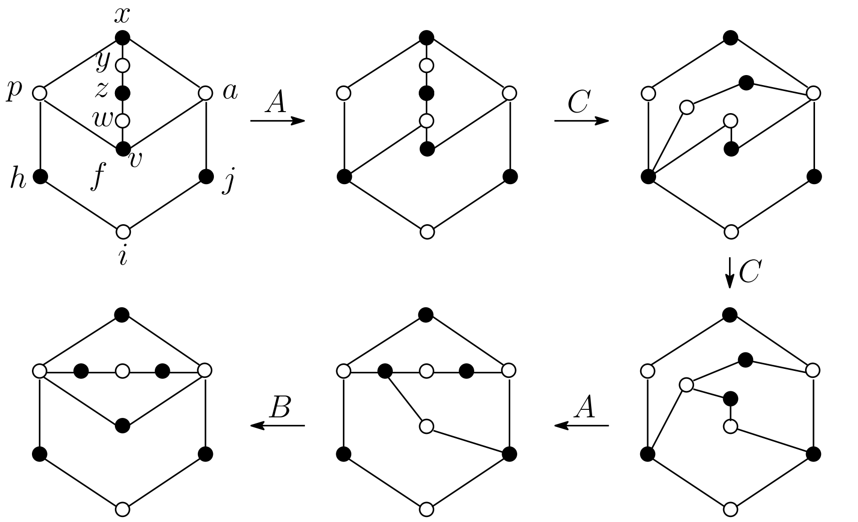

Figure 10 Case 2

Lemmas 6 and 7. We may suppose that $\deg(y) \neq 2$. Let F_1 be the finite face containing x, where ∂F_1 is supposed to be $yuhkNx$. Since $\deg(y) \neq 2$, we have $u \neq S$. So we let F_2 be a finite face sharing uy with F_1, where $\partial F_2 = yuabcd$ (see the left in Figure 11).

Now, if $d \neq S$, we can apply Transformations A, B, C and D to a path or an edge which contains yd without increasing $\deg(x)$ by Lemmas 6 and 7. By repeating this operation, we have $d = S$, that is, $\deg(y) = 3$ (see the right in Figure 11).

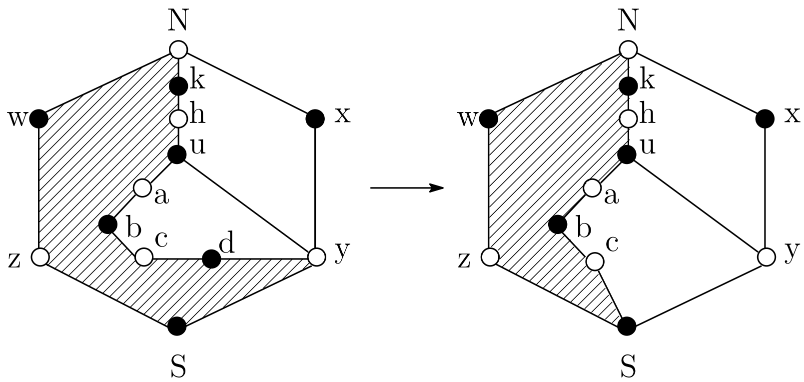

Figure 11 Make $\deg(y) = 3$

Let P be the path shared by ∂F_1 and ∂F_2, whose middle vertices are of degree 2 and whose end vertices are y and a vertex of degree 3. We consider the following four cases, according to the length of P.

Case 1. $P = yu$.

In the right hand of Figure 11, we consider whether Transformation A can be applied to yu to join h and S. If this is applicable, then we are done. Otherwise, G has an edge hS. In this case, yu can be switched to an edge ax by Transformation A, since a 4-cycle $Shuy$ separates a and x in the interior and exterior. Following this, we can switch ax to an edge bN by Transformation A since a 6-cycle $Shuaxy$ separates b and N in its interior and exterior. Thus, we can make $\deg(x) = \deg(y) = 2$.

Case 2. $P = yuv$ with $\deg(u) = 2$.

See Figure 12, which is obtained from the configuration in the right hand of Figure 11 by identifying a with h. Consider Transformation B to flip yuv to join k and S. If this is applicable, then we are done. Otherwise, we have $k = S$. In this case, we have $b \neq x$ and $c \neq N$ because a 4-cycle $Svuy$ separates $\{b, c\}$ and $\{x, N\}$ in the interior and exterior. Hence yuv can be switched to a path Nuc by applying Transformation B twice. Thus, we can make $\deg(x) = \deg(y) = 2$.

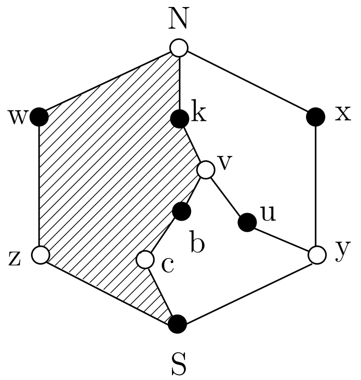

Figure 12 $P = yuv$

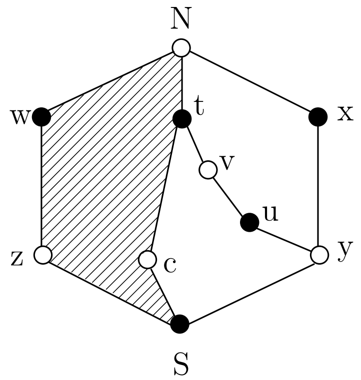

Figure 13 $P = yuvt$

Case 3. $P = yuvt$ with $\deg(u) = \deg(v) = 2$.

See Figure 13, which is obtained from the configuration in Figure 12 by identifying b with k. Consider Transformation C to flip $yuvt$ to join S and N. By Lemma 6, we can obtain $\deg(x) = \deg(y) = 2$.

Case 4. $P = yuvtN$ with $\deg(u) = \deg(v) = \deg(t) = 2$.

See Figure 14, which is obtained from the configuration in Figure 13 by identifying c with N. Consider Transformation A to flip an edge NS to make $\deg(t) = 3$. Then we apply Transformation C to $yuvt$ to join N and S as in Case 3.

Therefore we can make $\deg(x) = \deg(y) = 2$ in all cases.

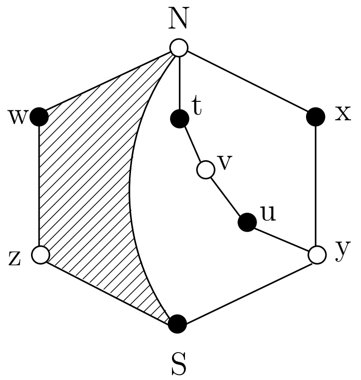

Figure 14 $P = yuvtN$

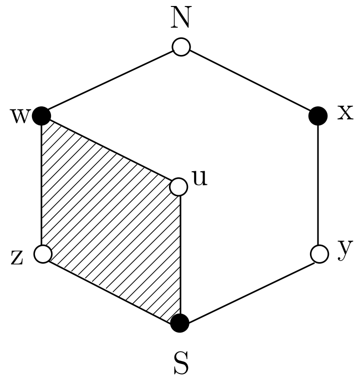

Figure 15 $\deg(N) = 2$

Step 2. We make $\deg(N) \geq 3$ and $\deg(S) \geq 3$, keeping $\deg(x) = \deg(y) = 2$.

We have $\deg(x) = \deg(y) = 2$ since we did in Step 1. If we have $\deg(N) \geq 3$ and $\deg(S) \geq 3$, then we are done. If $\deg(N) = \deg(S) = 2$, then G must be a 6-cycle, contrary to the assumption on $|V(G)| > 6$. Hence, by symmetry, we may suppose $\deg(N) = 2$ and $\deg(S) \geq 3$ (see Figure 15). Here we may suppose $\deg(w) \geq 4$ or $\deg(S) \geq 4$ or $\deg(u) \geq 3$. (For otherwise, i.e., if $\deg(w) = \deg(S) = 3$ and $\deg(u) = 2$, then G would have a face whose boundary is not a cycle, a contradiction.) Moreover, if $\deg(w) \geq 4$, then we can make $\deg(w) = 3$ as in Step 1 by Lemmas 6 and 7. Thus we may suppose $\deg(u) \geq 3$ or $\deg(S) \geq 4$.

First we assume $\deg(u) \geq 3$. We consider Transformation A to flip wu since $\deg(w) = 3$ and $\deg(u) \geq 3$. Let $uwzabc$ be a face sharing the edge uw with the face $NxySuw$. Then uw can be switched to make an edge Nc by Transformation A, since a 4-cycle $zwuS$ separates c and N in the interior and exterior. Hence we can make $\deg(N) \geq 3$ and $\deg(S) \geq 3$.

Second we may suppose $\deg(u) = 2$ and $\deg(S) \geq 4$. Let $Suwzab$ be a face sharing the path Suw with the face $NxySuw$. Since $N \neq b$, then we have $wuS \to Nub$ by Transformation B.

Therefore, we can make $\deg(N) \geq 3$ and $\deg(S) \geq 3$.

Step 3. Let G' be the hexangulation obtained from G by removing the path $NxyS$, and apply the procedures in Steps 1 and 2 to G'.

By the inductive hypothesis, we can transform G' into a standard form, fixing the boundary cycle of G'. Hence G can be transformed into the standard form by Transformations A, B, C and D. $\square$

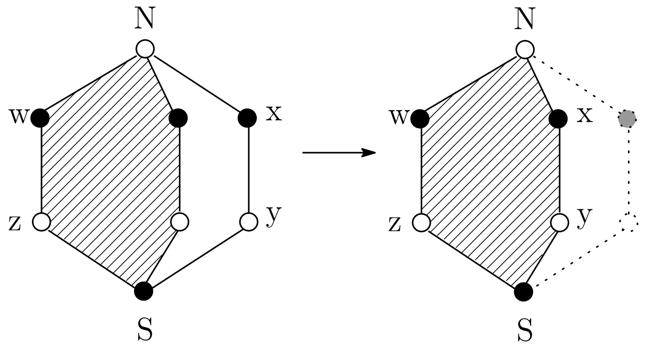

Figure 16 Operation in Step 3 for getting G' from G

5. Concluding Remarks

In this paper, we have proved that any two hexangulations G and G' on the sphere with the same number of vertices can be transformed into each other by three kinds of transformations, Transformations A, B and C. Moreover, without any one of the three transformations, we cannot always transform G and G' as described in Proposition 5.

Observe that every hexangulation on the sphere is bipartite, and that all of Transformations A, B and C preserve the bipartiteness of hexangulations. Moreover, let $(V_B(G), V_W(G))$ be the bipartition of a given hexangulation G on the sphere. Then Transformations A and C preserve a *bipartition size* $(|V_B(G)|, |V_W(G)|)$, but Transformation B changes the bipartition size, since Transformation B changes the color of a vertex of degree 2 included on the path of length two which is switched by this transformation.

As a natural question, can any two hexangulations with the same bipartition size be transformed into each other only by Transformations A and C? However, as we saw in Proposition 5, there exists a hexangulation on the sphere which cannot be transformed into any other hexangulation only by Transformations A and C. Hence a further problem would be to define a set of transformations preserving a bipartition size which guarantees that any two hexangulations with the same bipartition size can be transformed into each other by them.

How about n-angulations on the sphere with $n \geq 7$? For a fixed n, it is not surprising to be able to prove that any two spherical n-angulations with the same number of vertices can be transformed into each other by $\lfloor \frac{n}{2} \rfloor$ kinds of transformations, but a proof should be a routine with a case-by-case argument. We hope to find a good breakthrough to deal with n-angulations for all $n \geq 3$. Simultaneously, which implies all the earlier results for transformations in n-angulations.

How about n-angulations on non-spherical surfaces with $n \geq 5$. Such surfaces do not admit "Jordan Curve Theorem", and hence the problems must be difficult. The breakthrough for triangulations and quadrangulations to deal with those on non-spherical surfaces are to find a relation with diagonal transformations and some reductions of the graphs on surfaces. Such arguments seem to be needed also for hexangulations.

For solutions for those problems, we would like to expect further researches on diagonal transformations in hexangulations on surfaces.

References

[1] R. Brunet, A. Nakamoto, S. Negami, Diagonal flips of triangulations on closed surfaces preserving specified properties, *J. Combin. Theory Ser. B.* **68** (1996), 295–309.

[2] J. Kanno, A. Nakamoto, J. Su, K. Yamamoto, Diagonal transformations of quintangulation on the sphere, *preprint*.

[3] H. Komuro, A. Nakamoto, S. Negami, Diagonal flips in triangulations on closed surfaces with minimum degree at least 4, *J. Combin. Theory Ser. B* **76** (1999), 68–92.

[4] A. Nakamoto, Diagonal transformations and cycle parities of quadrangulations on surfaces, *J. Combin. Theory Ser. B* **67** (1996), 202–211.

[5] A. Nakamoto, Diagonal transformations in quadrangulations on surfaces, *J. Graph Theory* **21** (1996), 289–299.

[6] A. Nakamoto, Quadrangulations on closed surfaces, *Interdiscip. Inform. Sci.* **7** (2001), 77–98.

[7] S. Negami, Diagonal flips of triangulations on surfaces, a survey, *Yokohama Math. J.* **47** (1999), 1–40.

Graduate School of Environment and Information Sciences,
Yokohama National University,
79-7 Tokiwadai, Hodogaya-Ku, Yokohama 240-8501, Japan.
E-mail: matsumoto-naoki-gn@ynu.ac.jp