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THE DISTINGUISHING CHROMATIC NUMBERS OF TRIANGULATIONS ON THE SPHERE

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Abstract. The distinguishing chromatic number of a graph G is defined as the minimum number d such that G admits a vertex coloring with d colors which no automorphism of G other than the identity map preserves, and is denoted by $\chi_D(G)$. We shall show that $\chi_D(G) \leq \chi(G)+1$ for any triangulation G on the sphere unless it is isomorphic to either $\overline{K}_2+C_{2r}$ for $r \geq 2$ or the face subdivision of the 3-cube Q_3.

Introduction

Let G be a simple graph. An assignment $c : V(G) \to \{1, 2, \ldots, d\}$ is called a (proper) *coloring* of G if $c(u) \neq c(v)$ for any two adjacent vertices u and v in G. Furthermore, if no automorphism of G other than the identity map of G preserves the colors given by a coloring c, we call c a (d-)*distinguishing coloring* of G. If there exists a d-distinguishing coloring of G, then G is said to be d-*distinguishing colorable*. We define the *distinguishing chromatic number* of G as the minimum number d such that G is d-distinguishing colorable, and denote it by $\chi_D(G)$. The *chromatic number* $\chi(G)$ of G is defined as the minimum number k such that G is k-colorable, as usual.

The distinguishing chromatic number of a graph has been introduced in [4]. We may define a similar notion called the *distinguishing number*, replacing a vertex coloring in the previous with a color assignment, which is not assumed to be a proper coloring. This has a long history and there have been established many results; [1, 3, 10] for example.

A graph G is said to be *faithfully embedded* on a closed surface if G is embedded on the surface so that if any automorphism of G extends an auto-homeomorphism of the surface. This notion has been introduced in [7]. A 3-connected graph G embedded on a closed surface is said to be *polyhedral* if any two distinct faces meet each other in at most one vertex or one edge. Negami

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and Sakurai [8] have shown that $\chi_D(G)$ is enough close to $\chi(G)$ for a polyhedral graph G faithfully embedded on a closed surface:

Theorem 1. (Negami and Sakurai [8]) *Let G be a polyhedral graph on a closed surface. If G is faithfully embedded on the surface, then $\chi_D(G) \leq \max\{6, \chi(G)+2\}$.*

It is well-known that every 3-connected planar graph has a unique dual, proved by Whitney [11], and this implies that it can be faithfully embedded on the sphere. Since it is 4-colorable by Four Color Theorem [2, 9], Theorem 1 implies that every 3-connected planar graph is 6-distinguishing colorable. However, this fact has been improved with two exceptions $K_{2,2,2}$ and $\overline{K}_2 + C_6$, as follows:

Theorem 2. (Fijavž, Negami and Sano [5]) *Every 3-connected planar graph G is 5-distinguishing colorable unless G is isomorphic to $K_{2,2,2}$ or $\overline{K}_2 + C_6$.*

A *triangulation* on a closed surface is a polyhedral graph embedded on the surface that every face is bounded by a cycle of length 3. For example, the *double wheel* $\overline{K}_2 + C_n$ can be embedded on the sphere as a triangulation so that the cycle C_n of length n lies along the equator and the two vertices corresponding to "$\overline{K}_2$" are placed at the north and south poles and are joined to all vertices along the equator.

In this paper, we shall establish a more accurate theorem on the relationship between the distinguishing chromatic number and the chromatic number of triangulations on the sphere:

Theorem 3. *Let G be a triangulation on the sphere. Then $\chi_D(G) \leq \chi(G)+1$ unless G is isomorphic to either $\overline{K}_2 + C_{2r}$ for $r \geq 2$ or the face subdivision of Q_3.*

Let $S(Q_3)$ denote the face subdivision of the 3-cube Q_3, that is, one obtained from the cube by adding one vertex to each of its faces so that it is adjacent to the four vertices lying on the face boundary. Note that $K_{2,2,2}$ is isomorphic to $\overline{K}_2 + C_4$. It is not so difficult to see the following for the exceptions in the theorem:

- $\chi_D(G) = \chi(G) + 2 = 5$ if G is isomorphic to either $\overline{K}_2 + C_{2r}$ for $r \geq 4$, or $S(Q_3)$.
- $\chi_D(G) = \chi(G) + 3 = 6$ if G is isomorphic to $K_{2,2,2}$ or $\overline{K}_2 + C_6$.

In the next section, we shall prepare some technical lemmas on the distinguishing chromatic number of triangulations faithfully embedded on closed surface. Section 2 is devoted to a proof of Theorem 3. Finally, we shall discuss the distinguishing chromatic number of the exceptions in the theorem in Section 3.

1. Lemmas

In this section, we shall present some lemmas on triangulations on general closed surfaces.

Let G be a triangulation on a closed surface F^2. Then each vertex v of G is surrounded by a cycle consisting of all of its neighbors. This cycle is called the *link* of v in G and is denoted by $\mathrm{lk}(v)$. A vertex u (or a face uvw) of G is said to be *fixed* by an automorphism $\sigma \in \mathrm{Aut}(G)$ if $u = \sigma(u)$ (or if $u = \sigma(u)$, $v = \sigma(v)$ and $w = \sigma(w)$).

We shall use the following lemma implicitly to discuss the distinguishability of triangulations faithfully embedded on closed surfaces:

Lemma 4. *Let G be a triangulation faithfully embedded on a closed surface and $\sigma \in \mathrm{Aut}(G)$ an automorphism of G.*

(i) *If σ fixes a face of G, then σ fixes all vertices of G.*
(ii) *If σ fixes a vertex of G and all vertices lying along its link, then σ fixes all vertices of G.*

Proof. (i) Suppose that $\sigma \in \mathrm{Aut}(G)$ fixes a face A of G. Then any face incident to A is also fixed by σ since G is faithfully embedded. Repeating this, we conclude that all faces are fixed by σ and hence all vertices of G are fixed by σ.

(ii) If σ fixes a vertex v of G and all vertices lying along its link, then σ fixes each of faces incident to v and hence σ fixes all vertices of G by (i). □

As the following lemma suggests, degree conditions for vertices often work well to analyze the distinguishability of graphs.

Lemma 5. *Let G be a triangulation faithfully embedded on a closed surface F^2. If G has a vertex of odd prime degree, then $\chi_D(G) \leq \chi(G) + 1$.*

Proof. Let $c : V(G) \to \{1, 2, \ldots, \chi(G)\}$ be any vertex coloring of G with precisely $\chi(G)$ colors and u a vertex of odd prime degree, say $\deg_G(u) = p$. Define another vertex coloring $c' : V(G) \to \{1, 2, \ldots \chi(G), \chi(G) + 1\}$ of G by $c'(u) = \chi(G) + 1$ and $c'(x) = c(x)$ for each vertex x other than u.

Let $C = x_0 x_1 \cdots x_{p-1}$ be the link of u and σ any automorphism of G preserving the colors given by c'. Since u is a unique vertex of G colored by "$\chi(G)+1$", σ fixes u and hence it acts on the link C of u as either a rotation or a reflection if it is not the identity map over C.

First, suppose that σ is a rotation over the cycle C of length p. Since p is odd prime, σ must be of order p and $\{\sigma^i(x_0) : i = 0, \ldots, p-1\}$ consists of p distinct vertices. Then there exists $i \in \{1, \ldots, p-1\}$ such that $\sigma^i(x_0)$ is adjacent to x_0 and hence they would have the same color since σ preserves the colors. However, this contradicts that c' is a vertex coloring.

Next, suppose that σ is a reflection over C. Since p is odd, say $2k+1$, we may assume that $\sigma(x_i) = x_{-i}$ after re-labeling, where the indices are taken modulo p. In particular, σ fixes x_0 and exchanges x_k and x_{k+1}. The latter implies that the adjacent pair x_k and x_{k+1} would have the same color since σ preserves the colors. However, this contradicts that c' is a vertex coloring, again.

Therefore, σ must be the identity map over C and hence that over G by Lemma 4. This implies that c' is a distinguishing coloring of G and we have $\chi_D(G) \leq \chi(G) + 1$. $\square$

We shall focus on the structures around vertices of degree 4 in our proof of Theorem 3 given below. The next lemma is very useful to do it but can be proved easily. We can find it in [6] for example.

Lemma 6. *Let G be a triangulation with minimum degree at least 4 on a closed surface and H any component of the subgraph induced by the vertices of degree 4 in G. Then, one of the following four holds:*

(i) *H is either an isolated point or a path.*
(ii) *H is a cycle of length 3 bounding a face.*
(iii) *H is a cycle of length at least 5 and G is isomorphic to a double wheel with rim H on the sphere.*
(iv) *$H = G$ and it is isomorphic to $K_{2,2,2}$ on the sphere.* $\square$

2. Proof

We shall prove Theorem 3 throughout this section. Since any triangulation on the sphere is 3-connected, it is faithfully embedded on the sphere, as is mentioned in introduction. Thus, we may use lemmas in the previous section freely for triangulations on the sphere even if we do not assume the faithfulness of their embedding explicitly.

The following two lemmas will be used to skip many cases in our arguments given later:

Lemma 7. *If a triangulation G on the sphere is not 4-connected, then $\chi_D(G) \leq$*

$\chi(G) + 1$.

Proof. Let G be a triangulation on the sphere. Suppose that G is not 4-connected and is not isomorphic to K_4 since $\chi_D(K_4) = \chi(K_4) = 4$ clearly. Then there is a cycle uvw of length 3 not bounding any face and $\{u, v, w\}$ forms a 3-cut of G. Choose it to minimize the number of faces contained inside one of the regions bounded by uvw, say A, and let x be a vertex of G inside the region A.

Put $k = \chi(G)$. First make a vertex coloring of G with colors $1, \ldots, k$ and re-color only x with color $k + 1$. It is clear that any automorphism σ of G preserving the colors fixes x and that $\sigma(uvw)$ coincides with uvw by the choice of uvw. Furthermore, σ fixes each of u, v and w since they have different colors. This implies that σ fixes each face inside A and hence must be the identity map of G by Lemma 4. Therefore, the coloring of G with color $1, \ldots, k + 1$ is a distinguishing coloring and we have $\chi_D(G) \leq \chi(G) + 1$. □

Lemma 8. *If a triangulation G on the sphere has a vertex of degree at least 7 and is not isomorphic to any double wheel $\overline{K}_2 + C_r$, then $\chi_D(G) \leq \chi(G) + 1$.*

Proof. Let G be a triangulation on the sphere and v a vertex of degree $d \geq 7$ in G. Let $\mathrm{lk}(v) = u_0 \cdots u_{d-1}$ be the link of v with indices modulo d. We may assume that G is 4-connected by Lemma 7 and hence u_i and u_j are not adjacent if $|i - j| \geq 2$; otherwise, $\{v, u_i, u_j\}$ would form a 3-cut of G.

Let w_i be a common neighbor of u_i and u_{i+3} other than v if any. Such a vertex w_i does not lie on the cycle $\mathrm{lk}(v)$ and does outside it by the assumption of G being 4-connected. If w_i's exist for all i's, then G must be isomorphic to $\overline{K}_2 + C_d$ so that $\{v, w_i\}$ corresponds to "$\overline{K}_2$" and $\mathrm{lk}(v)$ to C_d, by the planarity. However, this case is excluded in the lemma. Thus, u_i and u_{i+3} do not have such a common neighbor w_i for some i, say $i = 0$ without loss of generality.

Put $k = \chi(G)$. First make a vertex coloring of G with colors $1, \ldots, k$ and re-color u_0 and u_3 with color $k+1$. Let σ be any automorphism of G preserving the colors. The σ fixes $\{u_0, u_3\}$ as a set since no other vertices have color $k + 1$. By our assumption on u_0 and u_3, we have $\sigma(v) = v$; otherwise $\sigma(v)$ would be w_0. This implies that σ maps the link of v to itself. Since $d \geq 7$, σ does not exchange the two segments that $\{u_0, u_3\}$ cuts $\mathrm{lk}(v)$ into. Since u_1 and u_2 have different colors, σ fixes the segments $u_0u_1u_2u_3$ pointwise. Thus, σ fixes v and its link and hence it is the identity map of G by Lemma 4. Therefore, the coloring of G with $k+1$ colors is a distinguishing coloring and hence we have $\chi_D(G) \leq \chi(G)+1$. □

Proof of Theorem 3. Let G be a triangulation on the sphere. Then G is 4-colorable by Four Color Theorem. We denote the set of vertices of degree i by

V_i. By Euler's formula, we have the following equality:

$$3|V_3| + 2|V_4| + |V_5| = 12 + \sum_{i \geq 7} (i-6)|V_i| \tag{1}$$

We may assume that $|V_i| = 0$ for $i \geq 7$ by Lemma 8, and $|V_3| = |V_5| = 0$ in addition by Lemma 5. Thus, G consists of vertices of degree 4 and 6 and there are precisely six vertices of degree 4. We may assume that G is 4-connected, by Lemma 7.

It is well-known that a triangulation on the sphere is 3-colorable if and only if its vertices have all even degree. Thus, G has a 3-coloring $c : V(G) \to \{1, 2, 3\}$. We shall modify it to be a distinguishing coloring, assuming that the vertices are colored by c in advance.

Focus on the structures around vertices of degree 4, which have been classified as in Lemma 6. Let H be any component of the subgraph induced by V_4. If (iii) or (iv) in the lemma happens, G itself is isomorphic to either $\overline{K}_2 + C_r$ or $K_{2,2,2}$, which are excluded as exceptions in the theorem. Thus, it suffices to discuss the cases of (i) and (ii).

First suppose that H is a cycle $x_0x_1x_2$ of length 3 bounding a face of G, corresponding to (ii). Then there is a cycle $y_0y_1y_2$ of length 3 bounding a triangular region which contains only three vertices x_0, x_1 and x_2 and y_i is adjacent to x_j and x_k for $\{i, j, k\} = \{1, 2, 3\}$. Define $c' : V(G) \to \{1, 2, 3, 4\}$ of G by re-coloring x_0 with "4", and let σ be any automorphism of G preserving the colors given by c'.

Since x_0 is a unique vertex colored with "4", σ fixes x_0 and hence maps the link of x_0 onto itself. The link of x_0 is a cycle $x_1x_2y_1y_0$ and hence $c'(x_1) = c'(y_1)$ and $c'(x_2) = c'(y_0)$. By the assumption on H, we have $4 = \deg_G(x_1) \neq \deg_G(y_1)$ and $4 = \deg_G(x_2) \neq \deg_G(y_0)$. Under this situation, σ fixes the four vertices lying on $\mathrm{lk}(x_0)$ and hence it must be the identity map over G by Lemma 4. Therefore, c' is a distinguishing coloring of G and we have $\chi_D(G) \leq \chi(G) + 1$.

Now suppose that H is a path $u_0u_1 \cdots u_{r-1}$, corresponding to (i) in Lemma 6; H may be a single vertex. Then it is clear that there are a cycle $a_0b_0a_1b_1$ of length 4 bounding a quadrilateral region which contains only the vertices lying along H and such that a_0 and a_1 are adjacent to u_0 and u_{r-1} respectively, and b_0 and b_1 are adjacent to all vertices on H. Under our assumption on G, each of a_0, a_1, b_0 and b_1 has degree 6.

First, assume that there H is a path consisting of at least two vertices, that is, $r \geq 2$. However, we have $r = 2$ or 3 since $\deg_G(b_0) = 6$. For, if $r = 4$, then G would be isomorphic to $\overline{K}_2 + C_6$. Without loss of generality, we may assume that $c(b_0) = c(b_1) = 3$, $c(a_0) = 1$, $c(u_0) = 2$ and that the path $a_0u_0u_1 \cdots u_{r-1}a_1$ is colored with 1 and 2 alternately. Let w be the common neighbor of b_0 and a_1

other than u_{r-1} such that b_0a_1w bounds a face. Define a 4-coloring $c' : V(G) \to \{1, 2, 3, 4\}$ by re-coloring u_0 and w by "4". Let σ be any automorphism of G preserving the colors given by c'.

If $r = 2$, then $c(a_1) = c'(a_1) = 2$ and w was colored with "1" in c since a_1 and w lie along the link of b_0. Then σ fixes $\{u_0, w\}$ as a set since they have the same color "4". However, u_0 and w had different colors in c and their links are colored in different ways. This implies that σ does not change u_0 and w and hence it fixes each of u_0 and w. Furthermore, it is clear that σ fixes each vertex lying on the path $a_0u_0u_1a_1$. If σ exchanges b_0 and b_1, then it acts on the link of a_1 as a reflection fixing u_1 and w. However, we would have $\deg_G(a_1) = 4$, contrary to the assumption on H. Thus, σ fixes each of b_0 and b_1 and becomes the identity map over G since it fixes the faces $b_0u_0u_1$ and $b_1u_0u_1$.

If $r = 3$, then $a_0u_0u_1u_2a_1w$ forms the link of b_0. If σ exchanged u_0 and w, then w would have degree 4 and not be isolated in the subgraph induced by V_4 as well as u_0. Three neighbors a_0, b_0 and a_1 of w have degree 6 and they lie along the link of w in this order. This implies that σ would exchange a_0 and b_0, but it is impossible since a_0 and b_0 have different colors. Thus, σ does not exchange u_0 and w. By the same argument as in the previous, σ becomes the identity map over G in this case, too. Therefore, c' is a distinguishing coloring of G in either case and we have $\chi_D(G) \leq \chi(G) + 1$.

The remaining case is when each vertex of degree 4 is isolated in the subgraph induced by V_4. Let u_0 be any vertex of degree 4 with the link $a_0a_1a_2a_3$ in this case. Then we have $\deg_G(a_i) = 6$ and there are four faces incident to $\mathrm{lk}(u_0)$, say $a_0a_1u_1$, $a_1a_2u_2$, $a_2a_3u_3$ and $a_3a_0u_4$.

Let $c' : V(G) \to \{1, 2, 3, 4\}$ be the 4-coloring of G obtained from c by re-coloring u_0 and u_1 with "4", and σ any automorphism of G preserving the colors given by c'. Suppose that $\deg_G(u_1) \neq 4$. Then σ does not exchange u_0 and u_1 and hence it fixes each of them. If σ is not the identity map over G, then it does not fix $u_0a_0a_1$ and we have $u_1 = \sigma(u_1) = u_3$ since any vertex cannot appear twice in the link of a vertex. However, $\{u_1, a_1, a_2\}$ would form a 3-cut of G, which is contrary to G being 4-connected. Thus, σ must be the identity map and c' is a distinguishing coloring of G. Therefore, we conclude that $\chi_D(G) \leq \chi(G) + 1$ unless $\deg_G(u_1) = 4$.

Similarly, if one of u_1, u_2, u_3 and u_4 does not have degree 4, then we obtain the desired inequality. Otherwise, all of u_1, u_2, u_2 and u_4 have degree 4 and this happens around any chosen vertex u_0 of degree 4 when $\chi(G) > \chi(G) + 1$. Under this situation, it is easy to see that G is isomorphic to the face subdivision of Q_3, which is excluded as an exception in the theorem. Now we have concluded that $\chi_D(G) \leq \chi(G) + 1$ when G is none of the exceptions. $\square$

3. Exceptions

Finally, we should determine the distinguishing chromatic numbers of the exceptions in Theorem 3. We can find the formula for $\chi_D(\overline{K}_2+C_n)$ in [8], which guarantees the facts on $\overline{K}_2+C_{2r}$ with $r \geq 2$ shown in introduction. Here, we shall discuss the distinguishing chromatic number $\chi_D(S(Q_3))$ of the face subdivision of the 3-cube.

The 3-cube Q_3 consists of eight vertices v_{ijk} labeled with $i, j, k \in \{0, 1\}$ so that two vertices are adjacent whenever their labels differ by exactly one place, as depicted in Figure 1. This is uniquely embedded on the sphere with six faces F_1 to F_6 indicated as follows:

$$\begin{aligned} &F_1 : 000\ 100\ 110\ 010, \quad F_2 : 000\ 100\ 101\ 001, \quad F_3 : 000\ 010\ 011\ 001 \\ &F_6 : 001\ 101\ 111\ 011, \quad F_5 : 010\ 110\ 111\ 011, \quad F_4 : 100\ 110\ 111\ 101 \end{aligned}$$

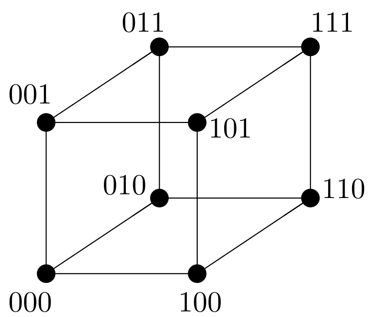

Figure 1 3-Bit labeling of the 3-cube

To construct the face subdivision of Q_3, we add a vertex u_s to the center of each face F_s and join u_s to the four vertices lying along its boundary cycle for $s = 1, \ldots, 6$. Note that F_s and F_{7-s} are placed in parallel as a dice.

THEOREM 9. *$\chi_D(Q_3) = 4$ and $\chi_D(S(Q_3)) = 5$ for the 3-cube Q_3 and its face subdivision $S(Q_3)$.*

Proof. We shall discuss the formulas for Q_3 and for $S(Q_3)$ in the theorem together. Let $c : V(Q_3) \to \{1, 2, \cdots, k\}$ be any k-coloring of Q_3 and $\tilde{c}$ any extension of c to $S(Q_3)$, which uses k or more colors. However, we assume that $\tilde{c}$ uses at most four colors, say "1" to "4", to show that $\chi_D(S(Q_3)) > 4$.

First, suppose that c is a 2-coloring with two colors "1" and "2". It is clear that such a 2-coloring of Q_3 is unique, up to relabeling of colors, and that there are many automorphisms of Q_3 preserving its colors. This implies that $\chi_D(Q_3) > 2$. Since each face of Q_3 has both two colors "1" and "2" on its boundary cycle, $\tilde{c}$ must assign colors "3" or "4" to each u_s.

Suppose that $\tilde{c}$ is a distinguishing coloring of $S(Q_3)$. Assume that $c(u_1) = 3$ and $c(u_6) = 4$ and consider the half rotation ρ of the cube with the axis passing through u_1 and u_6. Then ρ preserves the colors over Q_3, but does not the colors of u_s's since $\tilde{c}$ is a distinguishing coloring. This implies that either $\tilde{c}(u_2) \neq \tilde{c}(u_5)$ or $\tilde{c}(u_3) \neq \tilde{c}(u_4)$, say the former.

In this case, we may assume that $c(u_2) = 3$ and $c(u_5) = 4$, up to symmetry. However, the reflexion fixing two edges $v_{000}v_{100}$ and $v_{011}v_{111}$ preserves all colors of $\tilde{c}$, contrary to $\tilde{c}$ being a distinguishing coloring. Thus, we have $\tilde{c}(u_1) = \tilde{c}(u_6)$ and hence $\tilde{c}(u_s) = \tilde{c}(u_{7-s})$ for all pairs $(s, 7-s)$. Then, we may assume that only u_1 and u_6 get "3" and the others get "4" in $\tilde{c}$, up to symmetry, but the same rotation ρ as above would preserve the colors, contrary to $\tilde{c}$ being a distinguishing coloring. Therefore, we can conclude that $\tilde{c}$ is not a distinguishing coloring if c is a 2-coloring.

Now suppose that c uses 3 or 4 colors. Then there is a face of Q_3, say F_1, whose boundary cycle have three colors. We may assume that $c(v_{000}) = 1$, $c(v_{100}) = c(v_{010}) = 2$ and $c(v_{110}) = 3$ without loss of generality; if these four vertices got four distinct colors, then we could not assign any color to u_i. Then we must have $\tilde{c}(u_1) = 4$. Consider the reflexion σ of the cube fixing the diagonal $v_{000}v_{110}$ of the face F_1, which fixes v_{001} and v_{111}. Since this reflexion σ should not preserve the colors, we have either (i) $c(v_{101}) \neq c(v_{011})$ or (ii) $\tilde{c}(u_2) \neq \tilde{c}(u_3)$, up to symmetry; the latter works only for $S(Q_3)$.

In Case (i), there are two possibilities essentially; either $c(v_{011}) = 1$ or $c(v_{011}) = 4$ with $c(v_{101}) = 3$ in each case. In the first case, we can decide the colors of all vertices of $S(Q_3)$ but u_3 and u_4 in order;

$$\tilde{c}(u_5) = 4,\ \tilde{c}(u_2) = 4,\ c(v_{001}) = 2,\ \tilde{c}(u_6) = 4,\ c(v_{111}) = 2$$

However, the rotation fixing u_3 and u_4 would preserve this coloring, not depending on the colors of u_3 and u_4. Therefore, if c is a 3-coloring, then it cannot be a distinguishing coloring of Q_3 and $\tilde{c}$ cannot be that of $S(Q_3)$. It follows from this that $\chi_D(Q_3) > 3$ in particular.

On the other hand, if $c(v_{001}) = 4$ and $c(v_{101}) = 3$, then we can decide the colors of all vertices of $S(Q_3)$ but u_4:

$$c(v_{001}) = 2,\ \tilde{c}(u_2) = 4,\ \tilde{c}(u_3) = 3,\ \tilde{c}(u_5) = 1, c(v_{111}) = 2,\ \tilde{c}(u_6) = 1$$

In this case, the reflexion fixing two edges $v_{000}v_{100}$ and $v_{011}v_{111}$ would preserves this coloring, not depending on $\tilde{c}(u_4)$. Thus, any 4-coloring $\tilde{c}$ obtained as an extension of c of this type cannot be a distinguishing coloring of $S(Q_3)$.

Consider Case (ii), excluding Case (i). Then v_{101} and v_{011} must get the same color in c, "1", "3" or "4". However, if $c(v_{101}) = c(v_{101}) = 3$ or 4, then we would have $\tilde{c}(u_2) = \tilde{c}(u_3)$, contrary to the assumption of Case (ii). Thus, we may assume that $c(v_{101}) = c(v_{101}) = 1$, $\tilde{c}(u_2) = 3$ and $\tilde{c}(u_3) = 4$, up to symmetry. This implies that $\tilde{c}(u_4) = \tilde{c}(u_5) = 4$ and $c(v_{001}) = c(v_{111}) = 2$.

There are two possibilities on $\tilde{c}(u_6)$, which should be either "3" or "4". If $\tilde{c}(u_6) = 3$, then the reflexion fixing edges $v_{001}v_{101}$ and $v_{010}v_{110}$ would preserve the colors. If $\tilde{c}(u_6) = 4$, then the reflexion fixing edges $v_{001}v_{011}$ and $v_{100}v_{110}$ would preserve the colors. Thus, each case contradicts the assumption of $\tilde{c}$ being a distinguishing coloring.

Now we have concluded that any 4-coloring of $S(Q_3)$ cannot be a distinguishing coloring and hence $\chi_D(S(Q_3)) > 4$. We have already known that $\chi_D(Q_3) > 3$. Consider the following colorings c and $\tilde{c}$:

$$\begin{aligned}
&c(v_{000}) = c(v_{110}) = 1,\ c(v_{100}) = c(v_{001}) = 2,\\
&c(v_{010}) = c(v_{111}) = 3,\ c(v_{011}) = c(v_{101}) = 4\\
&\tilde{c}(u_s) = 5 \quad (s = 1, \dots, 6)
\end{aligned}$$

There are four faces of Q_3 each of which contains a diagonal joining two vertices colored with the same color. Since the colors are all distinct, any color-preserving automorphism σ of Q_3 must fix each of these faces and it must be the identity map of Q_3. This implies that c is a distinguishing coloring of Q_3 and hence $\chi_D(Q_3) = 4$.

On the other hand, any automorphism σ of $S(Q_3)$ maps Q_3 onto Q_3 and its restriction to Q_3 can be regarded as an automorphism of Q_3. Thus, if σ preserves the colors given by $\tilde{c}$, then σ fixes each vertex of Q_3 and also fixes each of u_s's to be the identity map of $S(Q_3)$. Therefore, $\tilde{c}$ is a distinguishing coloring of $S(Q_3)$ and hence $\chi_D(S(Q_3)) = 5$. $\square$

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