Yokohama Mathematical
Journal Vol. 57, 2011

FRACTAL SHEETS

By

Akio Kato

(Received November 1, 2010)

Abstract. We weave well-known fractals into non-porous "Fractal Sheets."

1. Introduction

We present a new type of fractal sets without any holes of positive sizes, which we call "Fractal Sheets." Up to now many kinds of fractals have been constructed, but most of them were very porous. Our idea is to keep pasting together known fractals infinitely many times until their holes disappear. Given a fractal F, we can use it as an ingredient to weave a non-porous *sheet* $\Omega(F)$ *modeled on* F. First, in Sections 2,3 and 4, we construct such a sheet $\Omega(S)$ modeled on the Sierpinski gasket S and study its structure from the geometric point of view. Next, in Section 5, we choose the Sierpinski carpet K. The resultant sheet $\Omega(K)$ happens to be identical with "the universal 1-dimensional pseudo-boundary" of the Euclidean plane introduced by Geoghagen and Summerhill [2]. We hope our geometric approach makes this space accessible to those who concern Fractals. In Section 6 we present a geometric embedding of $\Omega(S)$ into $\Omega(K)$. Further applications utilizing the circle packing will be discussed in the final Section 7.

2. Sierpiński Triangle "Sheet"

We want to make a non-porous fractal sheet modeled on the Sierpinski gasket. Let Δ be the equilateral solid triangle in the complex plane $\mathbb{C}$ of vertices $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = \rho = \exp(\mathrm{i}\frac{\pi}{3}) = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$ ("ρ" suggesting the "rhombic" coordinate system). Then the well-known Sierpinski gasket S is the invariant set of the iterated function system (f_1, f_2, f_3) of the similarities with ratio $1/2$ and centers at the three vertices. Precisely, $S = f_1(S) \cup f_2(S) \cup f_3(S)$, and $f_i(z) = \alpha_i + \frac{1}{2}(z - \alpha_i)$ $(i = 1, 2, 3)$. Besides these three functions, we introduce one more function

---

2010 Mathematics Subject Classification: 52C45
Key words and phrases: Fractal

$f_0 : \mathbb{C} \to \mathbb{C}$ such that

$$f_0(z) = \xi + \frac{1}{2}\,\rho\,(z - \xi) = \frac{1}{2}(\rho z + 1)$$

where $\xi = (1+\rho)/3 = \frac{1}{2} + \frac{1}{2\sqrt{3}}\mathrm{i}$ is the center of gravity of the triangle Δ. This map plays an essential role in our construction. Geometrically, f_0 is the rotation $\pi/3$ around the fixed point ξ followed by the contraction $1/2$. Recalling the construction "by tremas" of the Sierpinski gasket, we start with the triangle Δ. Devide it into four triangles with size (= edge length) $1/2$ using midpoints of the sides of Δ, that is, $\Delta = \Delta(0) \cup \Delta(1) \cup \Delta(2) \cup \Delta(3)$ where $\Delta(i) = f_i(\Delta)$ $(i = 0, 1, 2, 3)$. Observe that $\Delta(0)$ is the "upside down" middle triangle with size $1/2$. Let $4^{<\omega} = \{0, 1, 2, 3\}^{<\omega} = \bigcup_{p\in\omega} 4^p$, where $p = \{0, 1, \cdots, p-1\}$ and $\omega = \{0, 1, 2, \cdots\}$, denotes the set of all finite strings consisting of $0, 1, 2, 3$. For each string $\sigma = (\sigma(0), \sigma(1), \cdots, \sigma(p-1)) \in 4^p \subset 4^{<\omega}$ put $f_\sigma = f_{\sigma(0)} \circ f_{\sigma(1)} \cdots \circ f_{\sigma(p-1)}$ and $\Delta(\sigma) = f_\sigma(\Delta)$. Now make the Sierpinski gasket S in the usual way: delete all the interiors of the triangles $\Delta(\sigma)$ $(\sigma \in 4^{<\omega})$ such that only the final coordinate of σ has the value 0. In this S look at the tremas of size $1/2$; this gasket has only one such, which is $\Delta(0)$. Here, attach a copy of the Sierpinski gasket S using the map f_0. Now, in the resultant space $S^{(1)} = S \cup f_0(S)$, look at the tremas of size $1/2^2$. We have four such tremas $\Delta(i, 0) = f_i(f_0(\Delta))\,(i < 4)$. See Figure 1. Attach to these 4 tremas the copies of the Sierpinski gasket $f_{i0}(S) = f_i(f_0(S))\,(i < 4)$ of the same size. Then we get $S^{(2)} = S^{(1)} \cup \bigcup_{i<4} f_{i0}(S)$, which now has 4^2 tremas $\Delta(i, j, 0)\,(i, j < 4)$ of size $1/2^3$. See Figure 2. Again fill up these 4^2 holes: $S^{(3)} = S^{(2)} \cup \bigcup_{i,j<4} f_{ij0}(S)$. Repeating these procedures ad infinitum, we will finally get $\bigcup_{p<\omega} S^{(p)}$, where all tremas of positive sizes are filled up by the copies of Sierpinski gasket $S = S^{(0)}$. This is what we wanted. Let us call this space the *Sierpiński triangle sheet* and denote $\omega(S)$, or simply, $\mathbb{T}$;

$$\mathbb{T} = \omega(S) = \bigcup_{p<\omega} S^{(p)} \subset \Delta.$$

So, as a result we just pasted together countably many Sierpinski gaskets along their edges. Though here we needed an infinite iterative procedure, we have an alternative, all-at-once construction of $\mathbb{T}$ as follows.

Let 4^ω be the compact space of all functions from $\omega = \{0, 1, 2, \cdots\}$ to $4 = \{0, 1, 2, 3\}$ with the product topology generated by the open bases of the form $\sigma \times\ 4^{\omega\setminus p}$ where $\sigma \in 4^p$ and $p \in \omega$. Let Φ be the continuous map from 4^ω onto Δ which assigns to each $s \in 4^\omega$ the single point $\Delta(s) = \bigcap_{p\in\omega} \Delta(s \restriction p)$ in Δ. This is at most six-to-one map. Consider the subset Σ of 4^ω consisting of all the functions s such that $s(i) > 0$ for *almost* all $i \in \omega$, that is,

$$\Sigma = \bigcup_{p\in\omega} (4^p \times \{1, 2, 3\}^{\omega\setminus p})\ \subset 4^\omega.$$

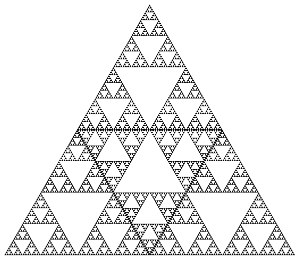

Figure 1 The first stage $S^{(1)}$ for $\omega(S)$.

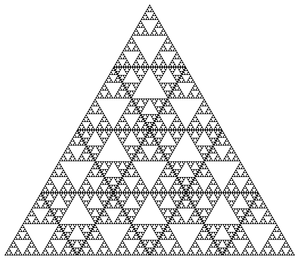

Figure 2 The second stage $S^{(2)}$ for $\omega(S)$.

Then the image of this set $\Phi(\Sigma)$ is nothing but our Sierpinski triangle sheet $\mathbb{T}$. The subspace $S^{(p)}$ used in the above inductive construction is identical with $\Phi(4^p \times \{1,2,3\}^{\omega\setminus p})$. For a finite string $\sigma \in 4^p$ denote by $S(\sigma)$ the image $\Phi(\sigma \times \{1,2,3\}^{\omega\setminus p})$, and note then that this is a copy of the Sierpinski gasket fitted in with the triangle $\Delta(\sigma)$. So, our Sierpinski triangle sheet $\mathbb{T}$ is the union of countably many copies of Sierpinski gaskets $S(\sigma)\,(\sigma \in 4^{<\omega})$.

Since the operation of "taking countable union" preserves most properties of fractals, those most properties of $\mathbb{T}$ will be inherited from the Sierpinski gasket. For example, the Haudorff dimension of the Sierpinski triangle sheet is equal to that of the Sierpinski gasket $\log 3/\log 2$. But, of course, some aspects are slightly different from the Sierpinski gasket. First of all, the sheet is not compact missing many points of Δ; in particular, the center of every triangle $\Delta(\sigma)$ never belongs to the sheet. Remark also that our triangle sheet is invariant with respect to the function system (f_0, f_1, f_2, f_3), that is,

$$\mathbb{T} = f_0(\mathbb{T}) \cup f_1(\mathbb{T}) \cup f_2(\mathbb{T}) \cup f_3(\mathbb{T});$$

but it is not the unique such set, because obviously the compact solid triangle Δ itself is also invariant! Start with the Sierpinski gasket S and repeat applying the four functions f_0, f_1, f_2, f_3; then you will finally reach our sheet. Hence, the triangle sheet $\mathbb{T}$ is characterized as the minimal set in the plane which contains the Sierpinski gasket S and is closed with respect to the function system (f_0, f_1, f_2, f_3).

Remark 1. Our construction is closely related with that of Lipscomb; in Chapter 13 of [3] he defines a "2-simplex iterated function system" and the address map, which are essentially the same with our (f_0, f_1, f_2, f_3) and Φ. Lipscomb aims to encode the 2-simplex Δ, and phrases the role of the extra digit "0" (which is the digit "3" in [3]) as encoding an *iterated pasting*.

Remark 2. We note that the subspaces $S^{(p)}\,(p > 0)$ appeared on the way of construction are topologically quite different from the gasket $S = S^{(0)}$ itself. Indeed, let $S_* = S_1 \cup S_2$ be a union of two copies S_1, S_2 of S such that their intersection $S_1 \cap S_2$ forms an edge e, and suppose there was a homeomorphic embedding g of S_* into S. As is well known, the Sierpinski gasket is quite rigid:

Fact 1. (Dębski-Mioduszewski [1]) *Every homeomorphic embedding of S into S is a similarity onto some standard subtriangle of S.*

Hence, $g(S_1)$ and $g(S_2)$ are such standard subtriangles, and, by the structure of Sierpinski gasket their intersection must be only one vertex v. Let $g(v_*) = v$. Then, $g(S_*)\backslash\{v\}$ is disconnected, while $S_*\backslash\{v_*\}$ is still connected by $e\backslash\{v_*\}$.This contradiction proves that S_* is not embeddable into S. Consequently, none of $S^{(p)}\,(p > 0)$ are embeddable into S. The situation will turn out to be quite different if we use the Sierpinski "carpet" instead of the gasket (see Section 5).

3. Spread Sheet

Now we want to spread the triangle sheet $\mathbb{T} = \omega(S)$ over the entire plane.

Let $\mathbb{Q}_2$ denote the subring of the rationals $\mathbb{Q}$ consisting of all "binary rationals" $m\,2^{-p}\,(m \in \mathbb{Z},\, p \in \omega)$. Using this $\mathbb{Q}_2$ and the vertices of the triangle Δ, we generate in the plane $\mathbb{C}$ a "rhombic or triangular" lattice $\mathbb{V}$ ("$\mathbb{V}$" for "vertices") such that

$$\mathbb{V} = \mathbb{Q}_2[\rho] = \mathbb{Q}_2 + \mathbb{Q}_2\,\rho = \{\,(l + m\rho)\,2^{-p} \mid l, m \in \mathbb{Z},\, p \in \omega\}.$$

For each $p \in \omega$ put $\mathbb{V}(p) = 2^{-p}\,\mathbb{Z}[\rho]$; then $\mathbb{V} = \bigcup_{p\in\omega} \mathbb{V}(p)$. Note that $\rho^2 = \rho - 1$ and $\mathbb{V}(0) = \mathbb{Z}[\rho] = \mathbb{Z}[\rho^2] = \mathbb{Z} + \mathbb{Z}\,\rho$ is the so-called "ring of Eisenstein integers." The ring structure of $\mathbb{Z}[\rho]$ will be inherited to the union $\mathbb{V}$ of the multiples $\mathbb{V}(p) = 2^{-p} \cdot \mathbb{V}(0)$. So, the lattice $\mathbb{V}$ is a subring of $\mathbb{C}$. Very well suited to our construction is that $\mathbb{V}$ is closed with respect to the operation of "taking midpoints."

Remark 3. Note that $1/3$ is not in $\mathbb{Q}_2$, hence not in $\mathbb{V}$, and that neither the center $\xi = (1+\rho)/3$ of Δ nor the imaginary unit $\mathrm{i} = (-1+2\rho)/\sqrt{3}$ belongs to $\mathbb{V}$.

Our $\mathbb{V}$ is the least "subring" of $\mathbb{C}$ containing $1/2$ and ρ, while the least "subfield" of $\mathbb{C}$ containing ρ is $\mathbb{Q}[\rho] = \mathbb{Q} + \mathbb{Q}\,\rho$. See that $\mathrm{i} \in \mathbb{C} \setminus \mathbb{Q}[\rho]$ and $\xi \in \mathbb{Q}[\rho] \setminus \mathbb{V}$.

Let us utilize this dense triangular lattice $\mathbb{V} = \bigcup_{p\in\omega} \mathbb{V}(p)$. Translate the triangle sheet $\mathbb{T}$ by $\mathbb{V}$ over the entire plane $\mathbb{C}$. We call the resultant space the *Sierpiński spread sheet* or simply *Sierpiński sheet* and denote it as $\Omega(S)$. Since the sheet $\mathbb{T}$ contains the rhombus subsheet $f_0(\mathbb{T}) \cup f_1(\mathbb{T})$ of size $1/2$, the whole sheet $\Omega(S)$ will be covered with the copies of $\mathbb{T}$ only by the translations through the discrete lattice $\mathbb{V}(1)$ of mesh size $1/2$. The following equivalent descriptions show that $\Omega(S)$ can be constructed in various ways. Here, $C_6 = \{\rho^k \,|k < 6\}$ is the cyclic group of order 6 in the plane, or the set of vertices of the hexagon.

$$\Omega(S) \;=\; \mathbb{T} + \mathbb{V} \;=\; \mathbb{T} + \mathbb{V}(1) = \;(\pm\mathbb{T}) + \mathbb{V}(0) \;=\; (\mathbb{T}\cdot C_6) + \mathbb{V}(0)$$

$$= \bigcup_{n\in\omega} 2^n \cdot (\mathbb{T}\cdot C_6) = \;(\pm S) + \mathbb{V} \;=\; (S\cdot C_6) + \mathbb{V}$$

For example, $(S \cdot C_6) + \mathbb{V}$ means the set of all points of the form $z \cdot c + v$ in $\mathbb{C}$ such that $z \in S$, $c \in C_6$ and $v \in \mathbb{V}$. In other words, first, rotate S to make the hexagonal gasket $S \cdot C_6$, next translate it by $\mathbb{V}$; then we get our spread sheet.

Let us present an alternative simple construction of our spread sheet. For each $p \in \omega$ consider the infinite 2-dimensional cell complex M_p in the plane naturally induced by the triangular lattice $\mathbb{V}(p)$ whose 2-dimensional cells are the closed triangles of size 2^{-p}. Replace each of these cells by a copy of the Sierpinski gasket S, and denote the resultant space as $M_p[S]$. Precisely,

$$M_p[S] = (\pm\, 2^{-p}) \cdot S + \mathbb{V}(p).$$

Then our sheet $\Omega(S)$ is identical with the union $\bigcup_{p\in\omega} M_p[S]$. Note here that the sequence $M_0[S] \subset M_1[S] \subset M_2[S] \subset \cdots$ is increasing and obtained from successive contractions by $1/2$ ("contractions" making sets bigger!). Moreover, each of these contractions $M_p[S] \to M_{p+1}[S]$ is an onto homeomorphism. Hence the sheet $\Omega(S)$ is the increasing union of the homeomorphs $M_p[S] = 2^{-p} \cdot M_0[S]\,(p \in \omega)$. Figure 3 illustrates

$$M_1[S] = 2^{-1} \cdot M_0[S] = 2^{-1}(\pm\, S + \mathbb{Z}[\rho]).$$

Through every construction of the above we can see that $\Omega(S)$ is, as in the case of $\omega(S)$, a union of countably many copies of Sierpinski gaskets and so, its Hausdorff dimension is equal with that of the Sierpinski gasket.

Remark 4. The sheet $\Omega(S)$ can be characterized as the minimal set in the plane $\mathbb{C}$ which contains S and is closed with respect to the following three kinds of operations: the rotation ρ, the translations by the discrete lattice $\mathbb{Z}[\rho]$ of Eisenstein

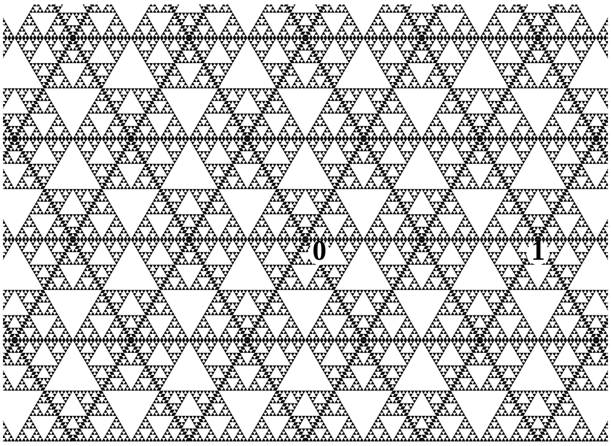

Figure 3 An intermediate stage for $\Omega(S)$.

integers and the contraction 1/2. In other words, starting with S and applying these operations, one can generate the sheet $\Omega(S)$.

4. Geometric structures of the Sheet

We first examine "homogeneity" aspects of the sheet $\Omega(S)$.

Let $\mathbb{E}(p)$ ("$\mathbb{E}$" for "edges") be the set of points in its 1-dimensional skeleton of the complex M_p, and put $\mathbb{E} = \bigcup_{p\in\omega}\mathbb{E}(p)$. That is, $\mathbb{E}$ is the union of all the "edges" of triangles in the sheet $\Omega(S)$, along which our "weaving" job was done.This skeleton $\mathbb{E}$ induces the metric d in the entire plane $\mathbb{C}$ such that $d(x, y)$ is the infimum of lengths of all continuous paths $C(x, y)$ connecting the points x and y with the property that $C(x, y)\backslash\{x, y\}$ lies in the set $\mathbb{E}$. Obviously this metric is compatible with the natural Euclidean metric, and we call it the metric *induced* by $\mathbb{E}$. Now let $U(x\,;\epsilon)$ denote the open neighborhood at a point $x \in \mathbb{C}$ of radius ϵ with respect to this metric d, that is, $U(x\,;\epsilon) = \{y \in \mathbb{C} \mid d(x, y) < \epsilon\}$. Then the open "ball" $U(0\,;1)$ of radius 1 at the origin 0 is just the inside of the hexagon $\Delta \cdot C_6$. Observe that

$$(U(0\,;2^{-p}) \cap \Omega(S)) + v = U(v\,;2^{-p}) \cap \Omega(S)$$

for each "vertex" $v \in \mathbb{V}$; hence in $\Omega(S)$ each "vertex" v has a neighborhood system of hexagons $U(v\,;2^{-p}) \cap \Omega(S)$ $(p \in \omega)$ similar to that of at the origin $U(0\,;2^{-p}) \cap \Omega(S)$ $(p \in \omega)$. So,

Property 1. $\Omega(S)$ *is topologically homogeneous with respect to the translations by its dense subset* $\mathbb{V}$.

This is a bit notable property of our sheet $\Omega(S)$ compared with that of the Sierpinski gasket S, where every vertex except the three corners separate S locally. No point can separate our sheet locally! Though we don't know presently if our sheet $\Omega(S)$ is topologically homogeneous (we believe there would be no homeomorphism of $\Omega(S)$ which maps the origin to an inner point of S), we can show in the next Property 2 that the sheet essentially admits only a few homeomorphisms. Consider a similarity h of the plane $\mathbb{C}$ which takes the form such that

$$(*) \quad h(z) = \alpha\, z + \beta \ \ or \ \ \alpha\, \bar{z} + \beta$$

for some α, β in $\mathbb{V}$, and furthermore, the coefficient α takes the form of $2^m\gamma$ where $m \in \mathbb{Z}$ and $\gamma \in C_6$. Note that, since the group of units in the ring $\mathbb{Z}[\rho]$ is C_6, the group of units in the ring $\mathbb{V} = \mathbb{Q}_2[\rho]$ coincides with all of its multiples $2^m \cdot C_6$ $(m \in \mathbb{Z})$. Hence taking account of Property 1, it follows that every such similarity $(*)$ provides an autohomeomorphism of $\Omega(S)$. The converse is true essentially:

Property 2. *Let h be a homeomorphic embedding of $\Omega(S)$ into $\Omega(S)$ such that each standard subgasket of sufficiently small size is mapped into some standard subgasket. Then h is a similarity as the above $(*)$.*

Here, a "standard subgasket" means a copy of the Sierpinski gasket fitted in with a triangular cell of the complex M_p for some $p \in \omega$; precisely, such gaskets are $\pm S(\sigma)\,(\sigma \in 4^{<\omega})$ and their translations by the discrete lattice $\mathbb{V}(0) = \mathbb{Z}[\rho]$. Note, first of all, that our sheet is connected in the following strong sense: Every two points x, y of $\Omega(S)$ can be joined by a sequence of standard subgaskets, with any small size, $S_1, S_2, \cdots, S_k$ such that $x \in S_1, y \in S_k$ and $S_i \cap S_{i+1}$ is an edge for each $i = 1, \cdots, k-1$. We may simply express that the Sierpinski sheet is "connected by edges," while the Sierpinski gasket is connected only by vertices (see Remark 2).

Proof of Property 2. Suppose h is an embedding as above, and let $\epsilon > 0$ be such that every standard subgasket of size $< \epsilon$ is mapped into some standard subgasket. Take any two points x, y in $\Omega(S)$. Since $\Omega(S)$ is connected by edges, we can find standard subgaskets $S_i\,(i < k)$ of size $< \epsilon$ which join $x \in S_1$ and $y \in S_k$ such that $S_i \cap S_{i+1}$ is an edge for each $i < k$. By Fact 1 the restriction of h to S_i is a similarity of the form $h(z) = \alpha_i\, z + \beta_i$ *or* $\alpha_i\, \bar{z} + \beta_i$ for some $\alpha_i, \beta_i \in \mathbb{V}$. Since h is a homeomorphism, taking account of the orientations of two triangles with a common edge, it does not happen for example that h is $\alpha_i\, z + \beta_i$ on the one side S_i, while $\alpha_{i+1}\, \bar{z} + \beta_{i+1}$ on the other side S_{i+1}. Moreover, $h \restriction S_i$ coincides with $h \restriction S_{i+1}$ on the edge $S_i \cap S_{i+1}$, and so, we have that $\alpha_1 = \cdots = \alpha_{k-1}$ and

$\beta_1 = \cdots = \beta_{k-1}$. Thus we can conclude that h is of the form $\alpha\, z + \beta$ or $\alpha\, \bar{z} + \beta$ for some $\alpha, \beta \in \mathbb{V}$. Since h is a similarity from S_1 onto $h(S_1)$, the coefficient α can be written as $2^{p-q}\gamma$ ($\gamma \in C_6$) if the sizes of S_1 and $h(S_1)$ are 2^{-p} and 2^{-q} respectively. □

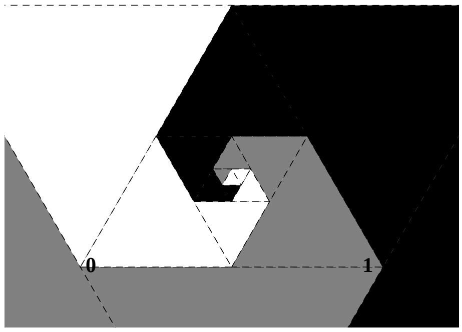

Figure 4 The Baravelle spirals by f_0.

Typical examples of autohomeomorphisms of $\Omega(S)$ described in the above Property 2 are the maps f_0, f_1, f_2, f_3 which we first introduced in Section 2. Note that these contractive maps never become autohomeomorphisms on any bounded subsets like $\omega(S)$ or S. The map $f_0(z) = \frac{1}{2}(\rho z+1)$ especially provides an interesting autohomeomorphism of $\Omega(S)$ whose orbits approach to the attracting point ξ lying outside of $\Omega(S)$. Figure 4 shows a decomposition of $\Omega(S)$ into three sets invariant by f_0, which are well known as the "Baravelle spirals." Should be noted, on the other hand, is that neither the very simple map $h(z) = 3\, z$ nor $z/3$ induces an autohomeomorphism of $\Omega(S)$.

Now, to look for some "dual" structure of our sheet, we use the multiplication by $\xi = \exp(\mathrm{i}\frac{\pi}{6})/\sqrt{3}$, which is the rotation $\pi/6$ followed by contraction $1/\sqrt{3}$. For each $p \in \omega$, multiply ξ to the entire structure of the complex M_p, and put $\mathbb{V}^*(p) = \xi \cdot \mathbb{V}(p)$, $\mathbb{V}^* = \xi \cdot \mathbb{V}$, $\mathbb{E}^*(p) = \xi \cdot \mathbb{E}(p)$ and $\mathbb{E}^* = \xi \cdot \mathbb{E}$. Observe the followings:

(1) $\mathbb{Z}[\xi] = \mathbb{Z} + \mathbb{Z}\,\xi = \xi \cdot \mathbb{Z}[\rho]$ (notice $1 = \xi \cdot (2-\rho) \in \xi \cdot \mathbb{Z}[\rho]$), and

$$\mathbb{Z}[\rho] \subset \mathbb{Z}[\xi] \subset \mathbb{Z}[\xi] \cdot \mathbb{Z}[\xi] = \frac{1}{3}\,\mathbb{Z}[\rho].$$

Be careful especially that $\mathbb{Z}[\xi]$ is not a ring because $\xi^2 = \rho/3 \in \frac{1}{3}\,\mathbb{Z}[\rho]\backslash\mathbb{Z}[\xi]$.

(2) $\mathbb{V}^*(p) = 2^{-p} \cdot \mathbb{Z}[\xi]$ and

$$\mathbb{V}^* = \bigcup_{p\in\omega} \mathbb{V}^*(p) = \mathbb{Q}_2[\xi] = \mathbb{Q}_2 + \mathbb{Q}_2\,\xi$$

consists of points with the form $(l + m\xi)\, 2^{-p}$ where $l, m \in \mathbb{Z}$, $p \in \omega$.
(3) Every point $z \in \mathbb{V}^*\backslash\mathbb{V}$ is the center of some standard triangle, that is, $z = v \pm 2^{-p} \cdot \xi$ for some $v \in \mathbb{V}$ and some $p \in \omega$. Therefore, new points of $\mathbb{V}^*$ added to $\mathbb{V}$ are always outside of the sheet $\Omega(S)$.
(4) $\mathbb{E}^*$ contains only countably many points of the sheet $\Omega(S)$; indeed, $\mathbb{E}^* \cap \Omega(S) \subset \mathbb{V}$. This is because the Sierpinski gasket has only countably many points on its bisectors, and all of these points are vertices.

Now consider the 2-dimensional cell complex M_p^*, "dual" to M_p, induced by the triangular lattice $\mathbb{V}^*(p)$. The triangular cells of M_p and M_p^* are of sizes 2^{-p} and $2^{-p}/\sqrt{3}$ respectively. The so-called "Voronoi" cells of centers $v \in \mathbb{V}(p)$ can be identified with the hexagons $v + \xi \cdot 2^{-p}(\Delta \cdot C_6)$, the inside of which is the open "ball" at v of radius $2^{-p}/\sqrt{3}$ with respect to the metric induced by the grid $\mathbb{E}^*$. The above observation (4) tells that geometric boundaries of the triangular cells in M_p^* as well as those of the "Voronoi" hexagons contain only countably many points of the sheet $\Omega(S)$. So, they are very loosely connected each other, as well indicating that $\Omega(S)$ has the topological "inductive" dimension 1. We note especially that $\Omega(S)\backslash\mathbb{V}$ is 0-dimensional, and hence, totally disconnected. This fact implies, for example, that any continuous path in $\Omega(S)$ inevitably hits the vertices, that is, the points of the countable set $\mathbb{V}$.

5. The "Sheet" modeled on the Sierpiński Carpet

Hitherto, using the Sierpinski gasket we made the Sierpinski sheet. If we select other fractals as ingredients, we can make other kinds of "sheets." Now let us choose the Sierpinski carpet K. As was mentioned in the Introduction, the resultant space will be identical with the space known (in the field of infinite-dimensional topology) as "the universal 1-dimensional pseudo-boundary" of the Euclidean plane [2]. Recall that the Sierpinski carpet K itself is universal in the sense that every nowhere dense compact subsets in the plane is topologically embeddable into K.

Let us start with the unit square $I^2 \subset \mathbb{C}$. Index the nine points $a + b\,\mathrm{i}\,(a, b = 0, \frac{1}{2}, 1)$ by $\alpha_k\,(k < 9)$ so that α_0 is the middle point $\frac{1}{2} + \frac{1}{2}\,\mathrm{i}$. Let g_k be the contraction by $1/3$ with centers α_k, that is, $g_k(z) = \alpha_k + \frac{1}{3}(z - \alpha_k)$. The unit square is decomposed into nine small squares $g_k(I^2)\,(k < 9)$ of size $1/3$, and $g_0(I^2)$ is the middle one. The Sierpinski carpet K is the invariant set of the iterated function system of eight functions $(g_k : 0 < k < 9)$. Now let the function g_0 join in. Using the same techniques as in Sections 2 and 3, we can weave (along edges) countably many copies of K into the *square sheet* $\omega(K)$ and the *spread sheet* $\Omega(K)$. Figure 6 shows the first stage towards the sheet $\omega(K)$. More precisely: Let

$9^{<\omega} = \{0, 1, 2, \cdots, 8\}^{<\omega}$ be the set of all finite strings consisting of $0, 1, 2, \cdots, 8$, and for each string $\sigma = (\sigma(0), \sigma(1), \cdots, \sigma(p-1))$ put $g_\sigma = g_{\sigma(0)} \circ g_{\sigma(1)} \cdots \circ g_{\sigma(p-1)}$ and $I^2(\sigma) = g_\sigma(I^2)$. Let Ψ be the continuous (at most four-to-one) map from 9^ω onto I^2 which assigns to each $s \in 9^\omega$ the single point $I^2(s) = \bigcap_{p\in\omega} I^2(s \restriction p)$. Consider the subset Σ of 9^ω consisting of all the functions s such that $s(i) > 0$ for *almost* all $i \in \omega$. Then the image $\Psi(\Sigma)$ is our square sheet $\omega(K)$, while the image $\Psi(\{1, 2, \cdots, 8\}^\omega)$ is the Sierpinski carpet K itself. If we put $K(\sigma) = g_\sigma(K)$, we know that $\omega(K)$ is the union of countably many copies of the Sierpinski carpet $K(\sigma)\,(\sigma \in 9^{<\omega})$. Put $K^{(p)} = \Psi(9^p \times \{1, 2, \cdots, 9\}^{\omega\setminus p})$ which corresponds to the p-th step $S^{(p)}$ in the iterative construction of $\omega(S)$ in Section 2. Though we observed there that $S^{(p)}\,(p > 0)$ was not homeomorphic with the gasket S itself (see Remark 2), in the present case, each $K^{(p)}\,(p \in \omega)$ is homeomorphic with the Sierpinski carpet K itself, due to the following result of Whyburn. Recall that, by definition, a *Sierpinski curve* is a compact, connected, locally connected, nowhere-dense subset of the plane that has the property that any two boundaries of complementary domains are pairwise-disjoint simple closed curves. The Sierpinski carpet is the most well known example of a Sierpinski curve.

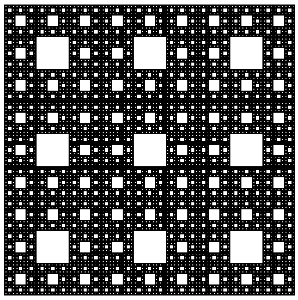

Figure 5 The first stage $K^{(1)}$ for $\omega(K)$.

Fact 2. (Whyburn [7]) *Any two Sierpinski curves are homeomorphic.*

So, we can say that $\omega(K) = \bigcup_{p\in\omega} K^{(p)}$ is the increasing union of countably many homeomorphic copies of the Sierpinski carpet.

Let $\mathbb{Q}_3$ denote the subring of $\mathbb{Q}$ consisting of all "ternary rationals" $m\,3^{-p}$ $(m \in \mathbb{Z},\, p \in \omega)$. Using this $\mathbb{Q}_3$ and the four vertices of I^2, we generate in the plane a lattice $\mathbb{V}^{(3)}$ such that $\mathbb{V}^{(3)} = \mathbb{Q}_3[\mathrm{i}] = \mathbb{Q}_3 + \mathbb{Q}_3\,\mathrm{i} = \{\, (l + m\,\mathrm{i})\,3^{-p} \mid l, m \in \mathbb{Z},\, p \in \omega\}$. For each $p \in \omega$ put $\mathbb{V}^{(3)}(p) = 3^{-p} \cdot \mathbb{Z}[\mathrm{i}]$; then $\mathbb{V}^{(3)} = \bigcup_{p\in\omega} \mathbb{V}^{(3)}(p)$, and $\mathbb{V}^{(3)}(0) = \mathbb{Z}[\mathrm{i}]$ is the ring of Gaussian integers. Note that, since $\mathbb{Q}_2 \cap \mathbb{Q}_3 = \mathbb{Z}$, the number $1/2$ is not in $\mathbb{Q}_3$, and the center $\frac{1}{2} + \frac{1}{2}\,\mathrm{i}$ does not belong to $\mathbb{V}^{(3)}$. Translate the carpet K by this countable dense lattice $\mathbb{V}^{(3)}$, or translate the square sheet

$\omega(K)$ only by the discrete square lattice $\mathbb{V}^{(3)}(0) = \mathbb{Z}[\mathrm{i}]$; then we get the *spread sheet* $\Omega(K)$ modeled on the carpet K:

$$\Omega(K) = \omega(K) + \mathbb{Z}[\mathrm{i}] = K + \mathbb{V}^{(3)} = \bigcup_{n\in\omega} 3^n \cdot (\omega(K) \cdot C_4),$$

where $C_4 = \{\pm 1, \pm \mathrm{i}\}$. So, both sheets $\Omega(K)$ and $\omega(K)$ are obtained by weaving countably many copies of Sierpinski carpets along their edges, and their Hausdorff dimension is equal to that of the Sierpinski carpet $\log 8/\log 3 = 1.8927\cdots$. Let $K_*^{(p)}$ be the finite union of the translates of $K^{(p)}$ by $l + m\,\mathrm{i} \in \mathbb{Z}[\mathrm{i}]$ such that $|l|$, $|m| \leqslant p$. Then, by Fact 2, this is a homeomorph of K; hence, as in the case of $\omega(K)$, the spread sheet $\Omega(K) = \bigcup_{p\in\omega} K_*^{(p)}$ also takes the form of the increasing union of countably many homeomorphic copies of the Sierpinski carpet.

In the theory of infinite-dimensional topology this sheet $\Omega(K)$ is usually described as follows. For each $p \in \omega$ let L_p be the infinite 2-dimensional cell complex in the plane induced by the square lattice $\mathbb{V}^{(3)}(p)$, whose 2-dimensional cells are the closed squares of size 3^{-p}. Let $L_p^{(1)}$ denote the 1-skeleton (that is, all the edges and vertices) of L_p. Consider the "star" of $L_p^{(1)}$ in the subdivision L_{p+1} of L_p:

$N_p = star(L_p^{(1)}, L_{p+1})$ which is the union of all the 2-dimensional cells in L_{p+1} meeting $L_p^{(1)}$. This star forms a regular (or collar) neighborhood of the skeleton $L_p^{(1)}$. Now take the *limit inferior* of the collection $\{N_p \,|\, p \in \omega\}$:

$$B_1^2 = \bigcup_{p\in\omega} \Big(\bigcap_{i\geqslant p} N_i \Big)$$

which consists of points belonging to *almost* all of N_p's. This B_1^2 is called "the universal 1-dimensional pseudo-boundary" of the Euclidean plane (see [2]). Similarly to the case of the complex M_p in Section 3, replace each of the 2-dimensional cells in L_p by a copy of the Sierpinski carpet K, and denote the resultant space as $L_p[K] = 3^{-p} \cdot K + \mathbb{V}^{(3)}(p)$. Then it can be seen that the above set $\bigcap_{i\geqslant p} N_i$ is the same as this $L_p[K]$. Hence, the universal pseudo-boundary B_1^2 is identical with our sheet $\Omega(K) = \bigcup_{p\in\omega} L_p[K]$, which is, as in the case of $\Omega(S)$, the union of the increasing sequence $L_0[K] \subset L_1[K] \subset L_2[K] \subset \cdots$ obtained from successive contractions by $1/3$.

Though it seems well known, since from its original construction by Geoghagen and Summerhill, that the universal pseudo-boundary is topologically homogeneous, we don't know any literature which stated this result explicitly. So, for the reader's convenience, we present the proof of the following property.

Property 3. *The sheet $\Omega(K)$ is topologically homogeneous. More precisely, for any two points x, y of $\omega(K)$ inside the unit square I^2 there exists an auto-*

homeomorphism h of $\omega(K)$ such that $h(x) = y$ and h fixes every point on the boundary of the unit square I^2.

From our construction it is easy to see that $\Omega(K)$ is homogeneous with respect to the dense lattice $\mathbb{V}^{(3)}$. But, as is well known, the Sierpinski carpet K itself is not homogeneous (see the following Fact 3). So the above property is not obvious. Recall that a Sierpinski curve is a homeomorph of the Sierpinski carpet K in the plane, and its points on the boundaries of complementary domains are called *rational*, while other points are *irrational* or *inner*. The rational part of a Sierpinski curve consists of countably many disjoint circles, which we call *boundary circles*.

FACT 3. (Whyburn [7]) *K is homogeneous between points x and y if and only if both belong either to the rational part of K or to the irrational part of K.*

FACT 4. (Whyburn [7]) *Let X_1 and X_2 be any Sierpinski curves in the plane. Then, every homeomorphism from any boundary circle of X_1 to any boundary circle of X_2 can be extended to a homeomorphism from X_1 onto X_2.*

FACT 5. (Visser [5]) *Let X be any Sierpinski curve in the plane and let S_0 be any boundary circle of X. Then for any inner points x, y of X there exists an autohomeomorphism h of X such that $h(x) = y$ and h fixes every point of S_0.*

Now we can prove Property 3. Due to the homogeneity with respect to the dense lattice $\mathbb{V}^{(3)}$, we need only show its latter assertion. So, let x, y be any two points of $\omega(K)$ inside the unit square. Recall that $\omega(K) = \bigcup_{p\in\omega} K^{(p)}$ is the increasing union of copies of the Sierpinski carpet, and observe that the boundary circles of $K^{(p)}$, which are boundaries of some squares of size 3^{-q} $(q > p)$, become an inner (or irrational) part of $K^{(q)}$. Hence, taking sufficiently large $q \in \omega$, we can assume that both of x, y are the inner points of $K^{(q)}$. So, by Fact 5 there exists an autohomeomorphism h_q of $K^{(q)}$ such that $h_q(x) = y$ and h_q fixes every point on the boundary of the unit square. Now think of the restrictions of h_q to the boundary circles of $K^{(q)}$ with size 3^{-q-1}, and extend them using Fact 4 to get an autohomeomorphism h_{q+1} of $K^{(q+1)}$. Next consider the restrictions of h_{q+1} to the boundary circles of $K^{(q+2)}$ with size 3^{-q-2}, and extend them to an autohomeomorphism h_{q+2} of $K^{(q+2)}$. Repeating these procedures infinitely many times, we will finally reach the desired homeomorphism of $\omega(K)$. $\square$

So, intuitively speaking, Property 3 holds because we pasted together the rational parts (the geometric boundaries) of Sierpinski carpets to end up with no distinction between the rational points and the irrational points.

6. Comparison of the two sheets

Let us now compare what we have constructed, $\Omega(S)$ and $\Omega(K)$. We can show that they are quite distinct topologically.

Property 4. *The carpet K is not embeddable into the Sierpinski sheet $\Omega(S)$; consequently, $\Omega(K)$ is not embeddable into $\Omega(S)$. But $\Omega(S)$ is embeddable into $\Omega(K)$.*

It is easy to see that K is not embedable into $\Omega(S)$, because the countable set of vertices $\mathbb{V}$ totally disconnects $\Omega(S)$ (see the end of Section 4), while the carpet K remains connected even after deleting any of its countable subset. Now let us see that $\Omega(S)$ is embeddable into $\Omega(K)$. This follows from the fact ([4, 6]) that $\Omega(K) = B_1^2$ is known to be very universal in the sense that every subspace in the plane which is a union of countably many nowhere dense compact subsets is topologically embeddable into B_1^2, because $\Omega(S)$ is such a union. Here, we want to give another proof by presenting a geometric concrete embedding. In the triangular cell complex M_i defined in Section 3 take the 1-skeleton $M_i^{(1)}$ and consider its *star* of $M_i^{(1)}$ in M_{i+2} (not in M_{i+1}). Put

$$\widehat{\Omega}(S) = \bigcup_{p\in\omega}\Big(\bigcap_{i\geqslant p} star(M_i^{(1)}, M_{i+2})\Big).$$

Then, since the complement of $star(M_i^{(1)}, M_{i+2})$ consists of disjoint triangles, we can apply Facts 2 and 4 to see that this $\widehat{\Omega}(S)$ is homeomorphic to $\Omega(K) = \bigcup_{p\in\omega} L_p[K]$. Now, instead of "star," consider "corona" of $M_i^{(1)}$ in M_{i+1}:

$$corona(M_i^{(1)},\, M_{i+1})$$

which we define to be the union of all the 2-dimensional cells in M_{i+1} some of whose *edges* are in the realization of $M_i^{(1)}$. So, the "corona" discards the 2-dimensional cells which meets only with some points in $M_i^{(1)}$, while the "star" takes them. Then, $\Omega(S)$ is expressed as

$$\Omega(S) = \bigcup_{p\in\omega}\Big(\bigcap_{i\geqslant p} corona(M_i^{(1)}, M_{i+1})\Big),$$

where $\bigcap_{i\geqslant p} corona(M_i^{(1)}, M_{i+1})$ is identical with $M_p[S]$. Since $corona(M_i^{(1)}, M_{i+1})$ is included in $star(M_i^{(1)}, M_{i+2})$), we get the desired embedding

$$\Omega(S) \subset \widehat{\Omega}(S) \approx \Omega(K).$$

Remark 5. The inclusion map

$$\bigcap_{i\geqslant 0} corona(M_i^{(1)}, M_{i+1}) \subset \bigcap_{i\geqslant 0} star(M_i^{(1)}, M_{i+2})$$

restricted to the triangle Δ, provides a natural embedding of the Sierpinski gasket S into a homeomorph of the carpet K.

7. Appendix

Our techniques will be easily extended to higher dimensional cases or to various hyperbolic tessellations. As one of generalizations of our way of construction, we want to mention examples produced by "circle packing." Let us consider a circle packing $\mathcal{C} = \{C_i \,|i \in \omega\}$ in a disk D_0 of the plane where C_i is a boundary circle of a disk D_i such that:

(1) $\bigcup_{i>0} D_i$ is a dense subset of D_0;

(2) open disks $\overset{\circ}{D}_i = D_i \backslash C_i$ $(i > 0)$ are disjoint.This packing naturally determines a compact nowhere dense subset $A = D_0 \setminus \bigcup_{i>0} \overset{\circ}{D}_i$, which we call the "CPG" (Circle Packing Gasket) for short. In case that the first three circles C_0, C_1, C_2 touch each other, and each circle C_i $(i \geqslant 3)$ kisses exactly three other circles among C_j $(0 \leqslant j < i)$, the corresponding CPG is the well known Apollonian gasket, which is made from two homeomorphic copies of Sierpinski gaskets identifying the corresponding three vertices. In case all the boundary circles are disjoint, the CPG is one of the Sierpinski curves described before Fact 2 in Section 5, hence is homeomorphic with the Sierpinski carpet.

Now choose a CPG A_0. Attach to every hole of A_0 of the maximal size some CPG along the boundary circle (there may be various ways of attaching). In the resultant space A_1 do the same job to its holes to get A_2. Keeping these procedures infinitely, taking care so that any holes of finite sizes disappear, we will end up getting a non-porous sheet A_∞, which is what we want. If we start with the hexagonal circle packing of the entire plane, we will get a spread-sheet version of A_∞. This way of construction through circle packing will provide not only the homeomorphic copies of hitherto examples like $\omega(S)$, $\Omega(S)$, $\omega(K)$, $\Omega(K)$ but also a variety of new interesting fractal sheets. Of course, for practical use, fractal sets which appear on some finite intermediate stages towards the sheet A_∞ will be sufficient. Figure 6 illustrates one of such examples. We imagine that this figure looks like the "coacervates," the colloidal droplets surrounded by tight skins of water molecules, may be found in the ocean of the infant Earth, as asserted by the Russian biochemist Oparin.

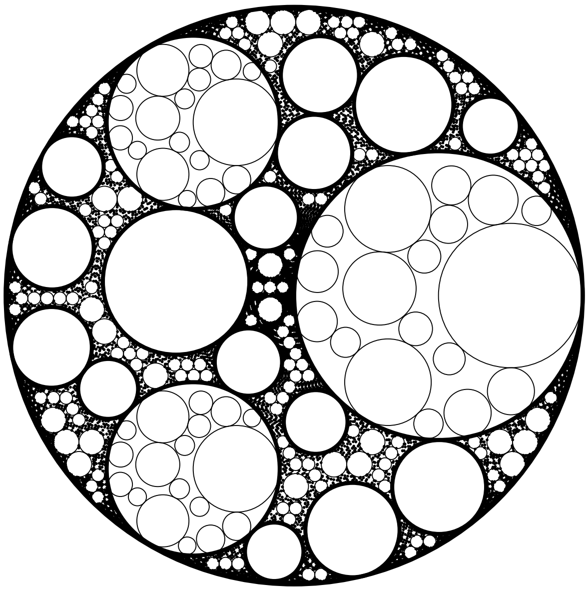

Figure 6 "Coacervates."

References

[1] W. Dębski, J. Mioduszewski, Simple plane images of the Sierpinski trianglar curve are nowhere dense, *Clloq. Math.* **59** (1990), 125–140.

[2] R. Geoghagen, R. Summerhill, Pseudo-boundaries and pseudo-interiors in Euclidean spaces and topological manifolds, *Trans. AMS* **194** (1974), 141–165.

[3] S.L. Lipscomb, *Fractals and Universal spaces in Dimension Theory*, Springer 2009

[4] H. Toruńczyk, *Skeletons and absorbing sets in complete metric spaces*, Doctoral thesis, Institute of Mathematics of the Polish Academy of Sciences, unpublished.

[5] D. Visser, *Representations of Erdös spaces by homeomorphism groups and by lower semi-continuous functions on product spaces*, Doctoral thesis, Vrije Universiteit, Amsterdam (2009).

[6] J.E. West, The ambient homeomorphy of an incomplete subspace of an infinite-dimensional Hilbert space, *Pacific J. Math.* **34** (1970), 247–267.

[7] G.T. Whyburn, Topological characterization of the Sierpinski curve, *Fund. Math.* **45** (1958), 320–324.

Department of Mathematics,
National Defense Academy,
Yokosuka 239-8686,
Japan
E-mail: akiokato@nda.ac.jp