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THE DESINGULARIZATIONS OF CLOSURES OF MORIN SINGULAR SETS

By

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Abstract. For a miniversal unfolding of a holomorphic function germ, we consider the closures of Morin singular sets. We construct their desingularizations. Our basic tool are Bell polynomials.

1. Introduction

Let N^n, M^m be complex analytic manifolds, and let $F : N^n \to M^m$ be a generic holomorphic map. For each positive integer k we have the Thom-Boardman singular set $\Sigma^{1^k}F$ ($n \leq m$) or $\Sigma^{n-m+1,1^{k-1}}F$ ($n \geq m$) in N, where $1^k = 1, \ldots, 1$ with k ones. We call this set the *Morin singular set* and denote it by Σ_k (cf. [6]).

It is well known that these sets Σ_k are nonsingular, while their closures $\overline{\Sigma_k}$ are not. However it is possible to construct desingularizations of $\overline{\Sigma_k}$. The usual way of constructing desingularizations of $\overline{\Sigma_k}$ is to represent a desingularization as the zeros of a generic section of a vector bundle (see [9], [4], [11], [5]). We call such a vector bundle *specifying* $\overline{\Sigma_k}$.

In the case $n \leq m$, Gaffney ([4]) constructed desingularizations of $\overline{\Sigma_k}$ ($k = 1, 2, 3, 4$). The vector bundles are defined locally, and it is not clear why they are defined independently of coordinates. In [11] Turnbull introduced the natural higher tangent bundles, and constructed desingularizations of not only $\overline{\Sigma_k}$ ($k = 1, 2, 3, 4$) but also $\overline{\Sigma_5}$ (the last one is restricted to $\Sigma^1F \cup \Sigma^2F$).

On the other hand, in the case $n \geq m$, it is more complicated to construct desingularizations of $\overline{\Sigma_k}$. In [5], for F given locally by a generic unfolding of a function germ, Kazarian constructed desingularizations of $\overline{\Sigma_k}$ ($k = 1, 2, 3$) and defined a vector bundle specifying $\overline{\Sigma_4}$.

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The purpose of this paper is, in the local situation where F is a miniversal unfolding of a function germ, to construct desingularizations of not only $\overline{\Sigma_k}$ ($k = 1, 2, 3, 4$) but also $\overline{\Sigma_5}, \overline{\Sigma_6}$. We do these in terms of Turnbull's higher tangent bundles.

Our basic tool are Bell polynomials. These polynomials appear in the equations defining Σ_k (see §3), which we use to guide our definitions of vector bundles specifying $\overline{\Sigma_k}$. Choosing local coordinates, we can consider a Bell polynomial as an element of a basis of local sections of a vector bundle (see §7.2). Moreover operations of Bell polynomials correspond to operations of vector bundles. That is, by using Bell polynomials we can organize the definitions of vector bundles specifying $\overline{\Sigma_k}$.

By the symmetry of the equations defining Σ_k, we need blowups only for $\overline{\Sigma_{2i}}$. Unlike the cases $\overline{\Sigma_2}, \overline{\Sigma_4}$, the case $\overline{\Sigma_6}$ needs an extra blowup which is along the locus corresponding to the closure $\overline{E_7}$ (E_7 is Arnold's notation [1, §15.1], and we denote by E_7 the set of E_7 points). Therefore the case $\overline{\Sigma_6}$ is of particular interest.

There is a famous application of desingularizations of $\overline{\Sigma_k}$. Gaffney, Turnbull, and Kazarian used their desingularizations to calculate Thom polynomials. In order to calculate Thom polynomials further in the case $n \geq m$, one may use our desingularizations.

In §2 we recall Bell polynomials and take up their two properties. In §3 we determine the equations defining Σ_k. In §4 we summarize Turnbull's higher tangent bundles. In §5 we define our vector bundles specifying $\overline{\Sigma_k}$ and state our theorems. In §6 we prove the well-definedness of the vector bundles and the transversality of the sections. The section 7 is the core of this paper, in which we establish the relationship between the equations defining Σ_k and the vector bundles specifying $\overline{\Sigma_k}$. In §8 Appendix, as a byproduct of our desingularizations of $\overline{\Sigma_k}$, we give desingularizations of the closures $\overline{D_5}, \overline{E_6}, \overline{D_6}, \overline{E_7}, \overline{E_8}, \overline{X_9}$.

2. Bell polynomials

We recall Bell polynomials and take up their two properties. Bell polynomials appear in Faà de Bruno's formula.

2.1 Faà de Bruno's formula

For C^∞ functions $f, g : \mathbb{R} \to \mathbb{R}$, write

$$(fg)_i = \frac{d^i f(g(x))}{dx^i}, \ f_i = \left. \frac{d^i f(y)}{dy^i} \right|_{y=g(x)}, \ g_i = \frac{d^i g(x)}{dx^i}.$$

By differentiating $f(g(x))$ successively, we have

$$
\begin{aligned}
(fg)_1 &= [f(g(x))]' &&= f_1 g_1, \\
(fg)_2 &= [f'(g(x))g'(x)]' &&= f_2 g_1^2 + f_1 g_2, \\
(fg)_3 &= [f''(g(x))(g'(x))^2 + f'(g(x))g''(x)]' &&= f_3 g_1^3 + 3 f_2 g_1 g_2 + f_1 g_3, \\
&\text{etc.}
\end{aligned}
$$

These are generalized to *Faà de Bruno's formula* ([3])

$$
(fg)_k = \sum \frac{k! f_s}{j_1! \cdots j_k!} \left(\frac{g_1}{1!}\right)^{j_1} \cdots \left(\frac{g_k}{k!}\right)^{j_k},
$$

where the sum ranges over $s = 1, 2, \ldots, k$ and $j_1, j_2, \ldots, j_k \in \mathbb{N} \cup \{0\}$ such that $j_1 + j_2 + \cdots + j_k = s$ and $j_1 + 2j_2 + \cdots + kj_k = k$.

It is known that this formula also holds in the case that f, g are maps (see [10]).

2.2 Bell polynomials

Write v for the set of k variables $(v_1, \ldots, v_k)$. The *Bell polynomial* $Y_k(v)$ ([2, §7]) is defined by

$$
Y_k(v) = \sum \frac{k!}{j_1! \cdots j_k!} \left(\frac{v_1}{1!}\right)^{j_1} \cdots \left(\frac{v_k}{k!}\right)^{j_k},
$$

where the sum ranges over $s = 1, 2, \ldots, k$ and $j_1, j_2, \ldots, j_k \in \mathbb{N} \cup \{0\}$ such that $j_1 + j_2 + \cdots + j_k = s$ and $j_1 + 2j_2 + \cdots + kj_k = k$. Set $Y_0(v) = 1$.

For example, $Y_1(v) = v_1, Y_2(v) = v_2 + v_1^2, Y_3(v) = v_3 + 3v_1 v_2 + v_1^3$, etc. Each v_i corresponds to g_i in the Faà de Bruno's formula.

For fixed s, set

$$
Y_{k,s}(v) = \sum \frac{k!}{j_1! \cdots j_k!} \left(\frac{v_1}{1!}\right)^{j_1} \cdots \left(\frac{v_k}{k!}\right)^{j_k},
$$

where the sum ranges over $j_1, j_2, \ldots, j_k \in \mathbb{N} \cup \{0\}$ such that $j_1 + j_2 + \cdots + j_k = s$ and $j_1 + 2j_2 + \cdots + kj_k = k$. Then $Y_k(v) = \sum_{s=1}^k Y_{k,s}(v)$, and the Faà de Bruno's formula can be written in the form

$$
(fg)_k = \sum_{s=1}^{k} f_s Y_{k,s}(g_1, \ldots, g_k). \tag{2. 1}
$$

For the sets of k variables $v = (v_1, \ldots, v_k)$ and $w = (w_1, \ldots, w_k)$, write $v + w = (v_1 + w_1, \ldots, v_k + w_k)$. The following two properties will be useful in §7.

Lemma 2.1. ([8, p.36,p.45])

(i) $Y_k(v) = \sum_{i=0}^{k-1} \binom{k-1}{i} v_{k-i} Y_i(v)$.

(ii) $Y_k(v+w) = \sum_{i=0}^{k} \binom{k}{i} Y_i(v) Y_{k-i}(w)$.

Proof. Note that for $f(y) = e^y$, we have $f_i = e^{g(x)}$. Substituting $f(y) = e^y$ in (2. 1), we get that

$$\begin{aligned}(e^{g(x)})_k &= \sum_{s=1}^{k} e^{g(x)} Y_{k,s}(g_1, \ldots, g_k)\\ &= e^{g(x)} Y_k(g_1, \ldots, g_k).\end{aligned}$$

Hence

$$Y_k(g_1, \ldots, g_k) = e^{-g(x)} \frac{d^k e^{g(x)}}{dx^k}.$$

(i) Using Leibniz's formula for differentiation of a product, we have

$$\begin{aligned}Y_k(g_1, \ldots, g_k) &= e^{-g(x)} \frac{d^{k-1}}{dx^{k-1}} (g_1 \cdot e^{g(x)})\\ &= \sum_{i=0}^{k-1} \binom{k-1}{i} \left(\frac{d^{(k-1)-i}}{dx^{(k-1)-i}} g_1\right) \cdot \left(e^{-g(x)} \frac{d^i}{dx^i} e^{g(x)}\right)\\ &= \sum_{i=0}^{k-1} \binom{k-1}{i} g_{k-i} Y_i(g_1, \ldots, g_k).\end{aligned}$$

Since $g(x)$ is arbitrary, (i) is proved.

(ii) Let $h(x)$ be a C^∞ function, and write $h_i = \frac{d^i}{dx^i} h(x)$. Using Leibniz's formula for differentiation of a product, we have

$$\begin{aligned}Y_k(g_1 + h_1, \ldots, g_k + h_k) &= e^{-(g(x)+h(x))} \frac{d^k}{dx^k} (e^{g(x)} \cdot e^{h(x)})\\ &= \sum_{i=0}^{k} \binom{k}{i} \left(e^{-g(x)} \frac{d^i}{dx^i} e^{g(x)}\right) \cdot \left(e^{-h(x)} \frac{d^{k-i}}{dx^{k-i}} e^{h(x)}\right)\\ &= \sum_{i=0}^{k} \binom{k}{i} Y_i(g_1, \ldots, g_k) Y_{k-i}(h_1, \ldots, h_k).\end{aligned}$$

Since $g(x), h(x)$ are arbitrary, (ii) is proved. $\square$

For example, in the case $k = 3$ we consider Lemma 2.1. Assertion (i) says that $Y_3(v) = v_3Y_0(v) + 2v_2Y_1(v) + v_1Y_2(v)$. This can be encoded by Table 1.

Table 1 The encoding of $Y_3(v)$

1	v_3
v_1	$2v_2$
$v_2 + v_1^2$	v_1

On the other hand, assertion (ii) says that $Y_3(v+w) = Y_0(v)Y_3(w)+3Y_1(v)Y_2(w) + 3Y_2(v)Y_1(w) + Y_3(v)Y_0(w)$. More precisely, $(v_3 + w_3) + 3(v_1 + w_1)(v_2 + w_2) + (v_1+w_1)^3 = 1(w_3+3w_1w_2+w_1^3)+3v_1(w_2+w_1^2)+3(v_2+v_1^2)w_1+(v_3+3v_1v_2+v_1^3)1$.

3. The equations defining Morin singular sets

In this section, for an unfolding of a function germ we give the equations defining Morin singular sets.

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a complex holomorphic function germ with an isolated singular point at $0 \in \mathbb{C}^n$, and let $F : (\mathbb{C}^n \times \mathbb{C}^p, (0, 0)) \to (\mathbb{C} \times \mathbb{C}^p, (0, 0))$ be a holomorphic unfolding of f_0 given by

$$F(x, t) = (f(x, t), t) \text{ with } f(x, 0) = f_0(x),$$

where $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, $t = (t_1, \dots, t_p) \in \mathbb{C}^p$. Let r be a sufficiently large integer, and let $J^r(\mathbb{C}^{n+p}, \mathbb{C}^{1+p})$ be the jet space. In this section, we assume that the jet extension $j^rF : (\mathbb{C}^n \times \mathbb{C}^p, (0, 0)) \to J^r(\mathbb{C}^{n+p}, \mathbb{C}^{1+p})$ is transverse to all the Morin singularities Σ^n and $\Sigma^{n,1^k}$ $(k \geq 1)$ in $J^r(\mathbb{C}^{n+p}, \mathbb{C}^{1+p})$ (cf. [6]).

The Morin singular sets Σ^nF and $\Sigma^{n,1^k}F$ $(k \geq 1)$ are given by the inverse image of the Morin singularities under j^rF. We write

$$\Sigma_1 = \Sigma^nF,\ \Sigma_{k+1} = \Sigma^{n,1^k}F\ (k \geq 1).$$

By the assumption of transversality, these sets are nonsingular.

For each point $a \in \mathbb{C}^n \times \mathbb{C}^p$ near $(0, 0)$, we define the affine map germ $\mathfrak{a}$: $(\mathbb{C}^n, 0) \to (\mathbb{C}^n \times \mathbb{C}^p, a)$ by $x \mapsto a + (x, 0)$. By modifying Gaffney's proposition ([4, Proposition 1.1]), we get the following result. This is based on the "probe" due to Porteous ([7]).

Proposition 3.1.

(i) *A point $a \in \mathbb{C}^n \times \mathbb{C}^p$ near $(0, 0)$ is in Σ_1 if and only if $d(f \circ \mathfrak{a})(x)|_{x=0} = 0$, where $d(f \circ \mathfrak{a})(x)$ is the derivative of $f \circ \mathfrak{a}$ at $x \in \mathbb{C}^n$.*

(ii) *A point* $a \in \Sigma_1$ *near* $(0,0)$ *is in* Σ_2 *if and only if* (a) *there exists an analytic curve* $c : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ *with* $\frac{dc}{du}(0) \neq 0$ *such that* $\frac{d}{du}[d(f \circ \mathfrak{a})(c(u))]\big|_{u=0} = 0$, *and* (b) *all such curves have the unique tangent direction at* $u = 0$.

(iii) *Let* $k \geq 2$. *A point* $a \in \Sigma_2$ *near* $(0,0)$ *is in* Σ_{k+1} *if and only if there exists an analytic curve* $c : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ *with* $\frac{dc}{du}(0) \neq 0$ *such that* $\frac{d^i}{du^i}[d(f \circ \mathfrak{a})(c(u))]\Big|_{u=0} = 0$ *for every* i *with* $1 \leq i \leq k$.

Proof. (i) By definition of $\mathfrak{a}$, we have $d(f \circ \mathfrak{a})(x) = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})(\mathfrak{a}(x))$. So $d(f \circ \mathfrak{a})(x)|_{x=0} = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})(a)$. Since $\Sigma^n F = \{(x,t) \in \mathbb{C}^n \times \mathbb{C}^p \mid \frac{\partial f}{\partial x_1}(x,t) = \cdots = \frac{\partial f}{\partial x_n}(x,t) = 0\}$, assertion (i) is proved.

(iii) Before proving (ii), we show (iii).

Suppose that $a \in \Sigma^{n,1}F (= \Sigma_2)$. Then there exists l (≥ 1) such that $a \in \Sigma^{n,1^l,0}F$. We can choose coordinates $x = (x_1, \dots, x_n)$ of $\mathbb{C}^n$ centered at 0 such that $f \circ \mathfrak{a}(x) = x_1^{l+2} + x_2^2 + \cdots + x_n^2$. Then

$$d(f \circ \mathfrak{a})(x) = ((l+2)x_1^{l+1}, 2x_2, \dots, 2x_n). \tag{3.1}$$

If we choose the curve c such that

$$c(u) = (u, 0, \dots, 0),$$

then $d(f \circ \mathfrak{a})(c(u)) = ((l+2)u^{l+1}, 0, \dots, 0)$.

Now if $a \in \Sigma^{n,1^k}F$ then $k \leq l$. Hence $\frac{d^i}{du^i}[d(f \circ \mathfrak{a})(c(u))]\Big|_{u=0} = 0$ for every i with $1 \leq i \leq k$.

Conversely if $a \notin \Sigma^{n,1^k}F$ then $a \in \Sigma^{n,1^l,0}F$ for some l with $l < k$. By (3. 1), there are no curves that satisfy the condition.

(ii) The "only if" part is similar to the proof of (iii). The "if" part follows from the uniqueness of the tangent direction. □

Before giving an example, we recall higher derivatives: Let S, T be finite dimensional vector spaces over $\mathbb{C}$, U an open subset of S, and $\varphi : U \to T$ a holomorphic map. We denote by $\varphi_1 : U \to L(S,T)$ the first derivative of φ. The second derivative of φ is $\varphi_2 : U \to L(S, L(S,T)) = L(S \otimes S, T)$. It is well known that φ_2 is symmetric. So $\varphi_2 : U \to L(S \circ S, T)$, where $\circ$ denotes the symmetric product. In general, the i th derivative is $\varphi_i : U \to L(\circ^i S, T)$.

Example 3.1. The first three derivatives $\frac{d^i}{du^i}(d(f\circ\mathfrak{a})(c(u)))$ are:

$$\begin{aligned}
\frac{d}{du}(d(f\circ\mathfrak{a})(c(u))) &= ((f\circ\mathfrak{a})_1\circ c)_1\\
&= ((f_1\circ\mathfrak{a}\circ c)(\mathfrak{a}_1\circ c))_1\\
&= f_2(\mathfrak{a}_1c_1)\mathfrak{a}_1 + f_1(\mathfrak{a}_2c_1)\\
&= f_2(\mathfrak{a}_1c_1)\mathfrak{a}_1,\\
\frac{d^2}{du^2}(d(f\circ\mathfrak{a})(c(u))) &= (f_2(\mathfrak{a}_1c_1)\mathfrak{a}_1)_1\\
&= f_3(\mathfrak{a}_1c_1)^2\mathfrak{a}_1 + f_2(\mathfrak{a}_1c_2)\mathfrak{a}_1,\\
\frac{d^3}{du^3}(d(f\circ\mathfrak{a})(c(u))) &= (f_3(\mathfrak{a}_1c_1)^2\mathfrak{a}_1 + f_2(\mathfrak{a}_1c_2)\mathfrak{a}_1)_1\\
&= f_4(\mathfrak{a}_1c_1)^3\mathfrak{a}_1 + 3f_3(\mathfrak{a}_1c_1)(\mathfrak{a}_1c_2)\mathfrak{a}_1 + f_2(\mathfrak{a}_1c_3)\mathfrak{a}_1.
\end{aligned}$$

Here the superscript of a vector refers to the number of copies of the vector and, for example, $f_3(\mathfrak{a}_1c_1)^2\mathfrak{a}_1$ is shorthand for

$$(f_3\circ\mathfrak{a}\circ c)[(\mathfrak{a}_1\circ c)c_1, (\mathfrak{a}_1\circ c)c_1, \mathfrak{a}_1\circ c](u).$$

Note that $(f_i)_1 = f_{i+1}(\mathfrak{a}_1c_1)$ since $(f_i\circ\mathfrak{a}\circ c)_1 = (f_{i+1}\circ\mathfrak{a}\circ c)(\mathfrak{a}_1\circ c)c_1$, and that $\mathfrak{a}_2 = \mathfrak{a}_3 = \cdots = 0$ since $\mathfrak{a}$ is affine.

Write $f_t(x) = f(x,t)$, and consider f_t as a function with respect to $x\in\mathbb{C}^n$. In Example 3.1, each $f_i\mathfrak{a}_1^i$, for instance $f_4\mathfrak{a}_1^4$ appearing in the term $f_4(\mathfrak{a}_1c_1)^3\mathfrak{a}_1$, is exactly the i th derivative of f_t. Hence we can write

$$f_i\mathfrak{a}_1^i = (f_t)_i.$$

Replacing c_i by vectors v_i and the absence of c_i by a vector V, we get the equations defining Σ_k.

The equations defining Σ_k. *Near $(0,0)$, the points of Σ_k are characterized as follows.*

- *A point of Σ_1 is in Σ_2 if and only if at the point, $0\neq\exists v_1\in\mathbb{C}^n$, unique up to scalar multiplication, such that*

$$(f_t)_2(v_1V) = 0 \ (\forall V\in\mathbb{C}^n). \tag{3.2}$$

- *A point of Σ_2 is in Σ_3 if and only if at the point, also $\exists v_2\in\mathbb{C}^n$ such that*

$$(f_t)_2(v_2V) + (f_t)_3(v_1^2V) = 0 \ (\forall V\in\mathbb{C}^n). \tag{3.3}$$

- *A point of Σ_3 is in Σ_4 if and only if at the point, also $\exists v_3\in\mathbb{C}^n$ such that*

$$(f_t)_2(v_3V) + (f_t)_3(3v_1v_2V) + (f_t)_4(v_1^3V) = 0 \ (\forall V\in\mathbb{C}^n). \tag{3.4}$$

- *A point of* Σ_4 *is in* Σ_5 *if and only if at the point, also* $\exists v_4 \in \mathbb{C}^n$ *such that*

$$(3.5)\quad \begin{aligned}&(f_t)_2(v_4V)+(f_t)_3(4v_1v_3V)+(f_t)_3(3v_2^2V)\\&+(f_t)_4(6v_1^2v_2V)+(f_t)_5(v_1^4V)=0\ (\forall V\in\mathbb{C}^n).\end{aligned}$$

- *A point of* Σ_5 *is in* Σ_6 *if and only if at the point, also* $\exists v_5 \in \mathbb{C}^n$ *such that*

$$(3.6)\quad \begin{aligned}&(f_t)_2(v_5V)+(f_t)_3(5v_1v_4V)+(f_t)_3(10v_2v_3V)+(f_t)_4(10v_1^2v_3V)\\&+(f_t)_4(15v_1v_2^2V)+(f_t)_5(10v_1^3v_2V)+(f_t)_6(v_1^5V)=0\ (\forall V\in\mathbb{C}^n).\end{aligned}$$

4. Turnbull's higher tangent bundles

In this section we summarize Turnbull's higher tangent bundles ([11, §1]). Whereas Turnbull has worked over $\mathbb{R}$, for our purposes we work over $\mathbb{C}$.

First, we recall symmetric products; $\circ^s\mathbb{C}^n$ is a subspace of $\otimes^s\mathbb{C}^n$, and consists of the elements which are invariant under permutations of the factors. If $v_1,\dots,v_s \in \mathbb{C}^n$ then we write

$$v_1\cdots v_s=\frac{1}{s!}\sum_{\sigma\in\mathfrak{S}_s} v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(s)}\in\circ^s\mathbb{C}^n,$$

where $\mathfrak{S}_s$ is the symmetric group of degree s. If $v_1,\dots,v_n\in\mathbb{C}^n$ is a basis of $\mathbb{C}^n$ then $\{v_{i_1}\cdots v_{i_s}|1\le i_1\le\cdots\le i_s\le n\}$ is a basis of $\circ^s\mathbb{C}^n$.

4.1 The higher tangent bundles

Let r be a positive integer. The vector space $S_r\mathbb{C}^n$ is defined by

$$S_r\mathbb{C}^n=\mathbb{C}^n\oplus\mathbb{C}^n\circ\mathbb{C}^n\oplus\circ^3\mathbb{C}^n\oplus\cdots\oplus\circ^r\mathbb{C}^n.$$

Here $\circ$ takes precedence over $\oplus$.

Let $U\subset\mathbb{C}^n$ and $W\subset\mathbb{C}^m$ be open subsets, and let $f: U\to W$ be a holomorphic map. The linear map $S_rf: S_r\mathbb{C}^n\to S_r\mathbb{C}^m$ at $x\in U$ is defined by

$$S_rf=(\sum f_i,\sum C_{i_1i_2}f_{i_1}f_{i_2},\sum C_{i_1i_2i_3}f_{i_1}f_{i_2}f_{i_3},\dots,f_1^r).$$

Here f_j is the j th derivative of f at x. The coefficient $C_{i_1\cdots i_s}$ is the number of distinct partitions of a set of order $i_1+\cdots+i_s$ into subsets of order $i_1,\dots,i_s$. The first sum is over $\{i\mid 1\le i\le r\}$, the second is over $\{i_1,i_2\mid i_1+i_2\le r, 1\le i_1\le i_2\}$, the third is over $\{i_1,i_2,i_3\mid i_1+i_2+i_3\le r,1\le i_1\le i_2\le i_3\}$ and so on. A monomial $f_{i_1}f_{i_2}\cdots f_{i_s}$ acts on $\circ^{i_1+\cdots+i_s}\mathbb{C}^n$ to give an element of $\circ^s\mathbb{C}^m$.

For example, if $r = 3$ then $S_3 f = (f_1 + f_2 + f_3,\ f_1^2 + 3f_1 f_2,\ f_1^3)$. Note that $(3f_1f_2)(v_1v_2v_3) = f_1(v_1)f_2(v_2v_3)+f_1(v_2)f_2(v_1v_3)+f_1(v_3)f_2(v_1v_2)$ $(v_1v_2v_3 \in \circ^3\mathbb{C}^n)$.

Let $k = i_1+i_2+\cdots+i_s$, and let j_l be the number of times that a particular integer l occurs in $i_1, \ldots, i_s$. Then the coefficient $C_{i_1\cdots i_s}$ equals $\frac{k!}{j_1!\cdots j_k!}\left(\frac{1}{1!}\right)^{j_1}\cdots\left(\frac{1}{k!}\right)^{j_k}$, which appears in the Faà de Bruno's formula in §2.1. By the map version of the formula, we can show that $S_r f$ satisfies the equality

$$(\phi_1 + \cdots + \phi_r) \circ S_r f = (\phi \circ f)_1 + \cdots + (\phi \circ f)_r$$

for any holomorphic function $\phi : W \to \mathbb{C}$ ([11, Proposition 3]). By using this equality, Turnbull proved the chain rule

$$S_r(g \circ f) = S_r g \circ S_r f$$

for holomorphic maps $f : U \to W$ and $g : W \to \mathbb{C}^l$, and obtained the following result.

Theorem 4.1. (Turnbull [11, §1.5]) *Let N^n, M^m, and L^l be complex analytic manifolds, and let $f : N \to M$ and $g : M \to L$ be holomorphic maps.*

- (i) *There exists an vector bundle T_rN over N, with fibre $S_r\mathbb{C}^n$ and with transition map $S_r(\varphi \circ (\varphi'|_{U\cap U'})^{-1})$ for any pair of charts $(U, \varphi), (U', \varphi')$ of N.*
- (ii) *There exists a bundle map $T_r f : T_rN \to T_rM$ which is given locally by $S_r f$.*
- (iii) *The chain rule $T_r(g \circ f) = T_r g \circ T_r f$ holds.*
- (iv) *There exist natural inclusions $T_1N \hookrightarrow T_2N \hookrightarrow \cdots$ such that $T_r f|_{T_{r-1}N} = T_{r-1}f$.*

4.2 The map $\sigma : T_rN \to T_rN \bigcirc T_rN$

We will in what follows distinguish the two types of symmetric products; $\circ$ inside $S_r\mathbb{C}^n (= \mathbb{C}^n \oplus \mathbb{C}^n\circ\mathbb{C}^n \oplus \cdots \oplus \circ^r\mathbb{C}^n)$ and $\bigcirc$ outside $S_r\mathbb{C}^n$ (for example, $\bigcirc$ of $S_r\mathbb{C}^n \bigcirc S_r\mathbb{C}^n$).

It is not possible to define a multiplication $T_rN \bigcirc T_sN \to T_{r+s}N$ independently of coordinates. But Turnbull defined instead a map $\sigma : T_rN \to T_rN \bigcirc T_rN$. For subbundles ξ, η of T_rN, we will use $\sigma^{-1}(\xi \bigcirc \eta)$ like a multiplication of ξ and η.

The space $S_r\mathbb{C}^n$ is spanned by elements of the form $v_1 \cdots v_s$ $(v_1, \ldots, v_s \in \mathbb{C}^n,\ s \le r)$. So we can define the linear map $\sigma : S_r\mathbb{C}^n \to S_r\mathbb{C}^n \bigcirc S_r\mathbb{C}^n$ by

$$\sigma(v_1 \cdots v_s) = \begin{cases} 0 & (s = 1), \\ \sum (v_{i_1} \cdots v_{i_l}) \bigcirc (v_{i_{l+1}} \cdots v_{i_s}) & (s \ge 2). \end{cases}$$

Here the sum is over all partitions of $\{1, \ldots, s\}$ into disjoint, nonempty subsets $\{i_1, \ldots, i_l\}$, $\{i_{l+1}, \ldots, i_s\}$ $(1 \le l \le \frac{s}{2})$.

For example, $\sigma(v_1^2v_2^2) = 2v_1 \bigcirc v_1v_2^2 + 2v_2 \bigcirc v_1^2v_2 + v_1^2 \bigcirc v_2^2 + 2v_1v_2 \bigcirc v_1v_2$.

Let θ be an analytic change of coordinates. By direct computation, we can get that

$$\sigma \circ S_r\theta = (S_r\theta \bigcirc S_r\theta) \circ \sigma, \tag{4.1}$$

where $S_r\theta \bigcirc S_r\theta$ denotes the map $S_r\mathbb{C}^n \bigcirc S_r\mathbb{C}^n \to S_r\mathbb{C}^n \bigcirc S_r\mathbb{C}^n$ given by $(S_r\theta \bigcirc S_r\theta)(a \bigcirc b) = S_r\theta(a) \bigcirc S_r\theta(b)$. So σ does not depend on the particular choice of coordinates, and we obtain the map $\sigma : T_rN \to T_rN \bigcirc T_rN$ given locally by $\sigma : S_r\mathbb{C}^n \to S_r\mathbb{C}^n \bigcirc S_r\mathbb{C}^n$. By definition of σ, we see that $\mathrm{Ker}\sigma = T_1N$.

For later use we state the general result of (4. 1).

Lemma 4.1. (Turnbull [11, §1.6, §2.7]) *For a holomorphic map $f : N \to M$, we have $\sigma_M \circ T_rf = (T_rf \bigcirc T_rf) \circ \sigma_N$, where σ_N and σ_M act on T_rN and T_rM, respectively.*

4.3 The image of Bell polynomials under σ

In almost the same way as the map σ, we define the map $\tilde{\sigma} : S_r\mathbb{C}^n \to S_r\mathbb{C}^n \otimes S_r\mathbb{C}^n$ by

$$\tilde{\sigma}(v_1 \cdots v_s) = \begin{cases} 0 & (s = 1), \\ \sum (v_{i_1} \cdots v_{i_l}) \otimes (v_{i_{l+1}} \cdots v_{i_s}) & (s \geq 2). \end{cases}$$

Here the sum is over all ordered partitions (A_1, A_2) of $\{1, \ldots, s\}$ into disjoint, nonempty subsets $A_1 = \{i_1, \ldots, i_l\}$, $A_2 = \{i_{l+1}, \ldots, i_s\}$ $(1 \leq l \leq s-1)$. Then $\sigma = \frac{1}{2}\pi \circ \tilde{\sigma}$, where π denotes the canonical projection $S_r\mathbb{C}^n \otimes S_r\mathbb{C}^n \to S_r\mathbb{C}^n \bigcirc S_r\mathbb{C}^n$ given by $\pi(a \otimes b) = \frac{1}{2!}(a \otimes b + b \otimes a)$.

The space $S_r\mathbb{C}^n \otimes S_r\mathbb{C}^n$ is spanned by elements of the form $v_1 \cdots v_l \otimes v_{l+1} \cdots v_{l+m}$ $(v_1, \ldots, v_{l+m} \in \mathbb{C}^n;\ l, m \leq r)$. We associate $v_1 \cdots v_l \otimes v_{l+1} \cdots v_{l+m}$ with the monomial $v_1 \cdots v_l w_{l+1} \cdots w_{l+m}$. By extending this linearly, we can associate the image $\tilde{\sigma}(v_1 \cdots v_s)$ with the polynomial $(v_1 + w_1) \cdots (v_s + w_s) - (v_1 \cdots v_s) - (w_1 \cdots w_s)$.

We now fix vectors $v_1, \ldots, v_k$ $(k \leq r)$ of $\mathbb{C}^n$, and set $v = (v_1, \ldots, v_k)$. The Bell polynomial $Y_k(v)$ can be considered as an element of $S_r\mathbb{C}^n$.

The following lemma will be useful to define the vector bundles specifying $\overline{\Sigma_k}$ (see Note 7.1).

Lemma 4.2. *For the map σ, we have $\sigma(Y_k(v)) = \frac{1}{2}\sum_{i=1}^{k-1}\binom{k}{i}Y_i(v) \bigcirc Y_{k-i}(v)$.*

Proof. As above, we can associate the image $\tilde{\sigma}(Y_k(v))$ with the polynomial $Y_k(v+w) - Y_k(v) - Y_k(w)$. By Lemma 2.1 (ii), the polynomial $Y_k(v+w) - Y_k(v) - Y_k(w)$

equals $\sum_{i=1}^{k-1}\binom{k}{i}Y_i(v)Y_{k-i}(w)$, with which $\sum_{i=1}^{k-1}\binom{k}{i}Y_i(v)\otimes Y_{k-i}(v)$ is associated. Thus $\tilde{\sigma}(Y_k(v)) = \sum_{i=1}^{k-1}\binom{k}{i}Y_i(v)\otimes Y_{k-i}(v)$. Since $\sigma = \frac{1}{2}\pi\circ\tilde{\sigma}$, the lemma is proved. $\square$

For example, if $k=4$ then $\sigma(Y_4(v)) = \frac{1}{2}(4Y_1(v)\bigcirc Y_3(v) + 6Y_2(v)\bigcirc Y_2(v) + 4Y_3(v)\bigcirc Y_1(v)) = 4Y_1(v)\bigcirc Y_3(v) + 3Y_2(v)\bigcirc Y_2(v)$.

5. Construction of the desingularizations

In this section, we construct our desingularizations of $\overline{\Sigma_1},\dots,\overline{\Sigma_6}$. Some of the proofs will be given in the later sections.

By a *desingularization* of an analytic set A, we mean a holomorphic map $\pi:\tilde{A}\to A$ such that (i) π is proper and surjective, (ii) $\tilde{A}$ is a nonsingular analytic set, and (iii) there exists an open dense subset U of A satisfying the conditions that $\pi^{-1}(U)$ is open dense in $\tilde{A}$ and $\pi|_{\pi^{-1}(U)}:\pi^{-1}(U)\to U$ is biholomorphic.

Let $f_0:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a holomorphic function germ with an isolated singular point at $0\in\mathbb{C}^n$, and let $F:(\mathbb{C}^n\times\mathbb{C}^p,(0,0))\to(\mathbb{C}\times\mathbb{C}^p,(0,0))$ be the $\mathcal{R}^+$-miniversal unfolding (cf. [1, §19.4]) of f_0 defined by

$$F(x,t) = (f(x,t),t),\ f(x,t) = f_0(x) + \sum_{i=1}^{p} t_i g_i(x)$$

such that

(i) $g_i\in\mathbb{C}\{x\}$ and $g_i(0)=0$ for every i,

(ii) $1,g_1,\dots,g_p$ is a basis of $\mathbb{C}\{x\}/(\frac{\partial f_0}{\partial x_1},\dots,\frac{\partial f_0}{\partial x_n})$ as a $\mathbb{C}$-vector space,

where $x=(x_1,\dots,x_n)\in\mathbb{C}^n$, $t=(t_1,\dots,t_p)\in\mathbb{C}^p$, $\mathbb{C}\{x\}$ is the ring of convergent power series, and $(\frac{\partial f_0}{\partial x_1},\dots,\frac{\partial f_0}{\partial x_n})$ is the ideal of $\mathbb{C}\{x\}$ generated by $\frac{\partial f_0}{\partial x_1},\dots,\frac{\partial f_0}{\partial x_n}$. Then this F satisfies the assumption that the jet extension j^rF is transverse to all the Morin singularities Σ^n and $\Sigma^{n,1^k}(k\geq 1)$, and hence we can use the equations (3. 2)-(3. 6) defining Σ_k.

By considering the stable equivalence ([1, §11.1]) we can assume that $n\geq 3$, and for our purpose we assume that $(0,0)\in\overline{\Sigma_6}$.

Write $f_t(x)=f(x,t)$, and consider f_t as a function with respect to $x\in\mathbb{C}^n$. Let r be a sufficiently large integer. We have the map $T_r(f_t):T_r\mathbb{C}^n\to T_r\mathbb{C}$ over $\mathbb{C}^n$. By considering its pullback over $\mathbb{C}^n\times\mathbb{C}^p$ under the standard projection $\mathbb{C}^n\times\mathbb{C}^p\to\mathbb{C}^n$, we will in what follows consider all sets, bundles, and bundle maps above a sufficiently small, open neighbourhood of $(0,0)$ in $\mathbb{C}^n\times\mathbb{C}^p$.

5.1 The desingularization of $\overline{\Sigma_1}$

By Proposition 3.1 (i),

$$\Sigma_1 = \{(x,t) \in \mathbb{C}^n \times \mathbb{C}^p \mid T_1(f_t) = 0 \text{ at } (x,t)\}.$$

By the transversality of $j^r F$, the set Σ_1 is nonsingular. Since Σ_1 is closed, it follows that $\overline{\Sigma_1} = \Sigma_1$. So we write $\tilde{\Sigma}_1$ for Σ_1, and call it a desingularization of $\overline{\Sigma_1}$.

5.2 The desingularization of $\overline{\Sigma_2}$

By restricting bundles to $\tilde{\Sigma}_1$, we now work over $\tilde{\Sigma}_1$. Let us denote $T_1\mathbb{C}^n$ by ξ_1 and $T_1\mathbb{C}$ by ι_1. Then $T_1(f_t)(\xi_1) = \{0\}$.

We have the projective bundle of ξ_1 which we denote by $\mathbb{P}(\xi_1)$. Write π_2 for the projection of $\mathbb{P}(\xi_1)$ onto $\tilde{\Sigma}_1$. Over $\mathbb{P}(\xi_1)$ we have the tautological line bundle ξ_1^1 whose fibre is corresponding to the point of $\mathbb{P}(\xi_1)$.

Let η_2 be the inverse image of $\xi_1^1 \bigcirc \xi_1$ under the map $\sigma : T_r\mathbb{C}^n \to T_r\mathbb{C}^n \bigcirc T_r\mathbb{C}^n$ (cf. the equation (3. 2) defining Σ_2). Since $\xi_1^1 \bigcirc \xi_1$ is contained in the image of σ, it follows that η_2 is a $2n$ dimensional subbundle of $T_2\mathbb{C}^n$. Then $\xi_1 \subset \eta_2$. Since $\xi_1^1 \subset \mathrm{Ker}T_1(f_t)$, by Lemma 4.1, $T_2(f_t)$ maps η_2 into ι_1. So let $\bar{T}_2(f_t) : \eta_2/\xi_1 \to \iota_1$ be the map induced by $T_2(f_t)$. Then we have a section, say Φ_2, of $\mathrm{Hom}(\eta_2/\xi_1, \iota_1)$ over $\mathbb{P}(\xi_1)$, induced by $\bar{T}_2(f_t)$.

We are interested in the zeros of this section Φ_2. At such points $T_2(f_t)(\eta_2) = \{0\}$. Let $\tilde{\Sigma}_2$ be the zeros of the section.

$$\begin{array}{ccc} & & \mathrm{Hom}(\eta_2/\xi_1, \iota_1) \\ & & \downarrow \upharpoonleft \Phi_2 \\ \tilde{\Sigma}_2 & \hookrightarrow & \mathbb{P}(\xi_1) \\ & & \downarrow \pi_2 \\ & & \tilde{\Sigma}_1 \end{array}$$

Set $S = \cup_{i \geq 2} \Sigma^{n,i} F$. It is easy to see that S is an analytic subset of $\tilde{\Sigma}_1$ and satisfies $\Sigma_2 = \overline{\Sigma_2} \setminus S$. Also set $\tilde{S}_2 = (\pi_2|_{\tilde{\Sigma}_2})^{-1}(S)$.

THEOREM 5.1. *The restriction of π_2 to $\tilde{\Sigma}_2$ is a desingularization of $\overline{\Sigma_2}$.*

Proof. We will prove in §6.1 that Φ_2 is transverse to the zero section of $\mathrm{Hom}(\eta_2/\xi_1, \iota_1)$. Hence $\tilde{\Sigma}_2$ is nonsingular.

By definition of η_2, π_2 maps $\tilde{\Sigma}_2 \setminus \tilde{S}_2$ into Σ_2. Above each point of $\Sigma_2 (\subset \tilde{\Sigma}_1)$, there exists a unique line ξ_1^1 such that Φ_2 vanishes. Hence $\pi_2|_{\tilde{\Sigma}_2 \setminus \tilde{S}_2} : \tilde{\Sigma}_2 \setminus \tilde{S}_2 \to \Sigma_2$

is surjective and has the inverse map. It follows that $\pi_2|_{\tilde{\Sigma}_2\setminus\tilde{S}_2} : \tilde{\Sigma}_2 \setminus \tilde{S}_2 \to \Sigma_2$ is biholomorphic.

By choosing local coordinates of $\tilde{\Sigma}_2$, we see that $\tilde{S}_2$ is a nowhere dense analytic subset of $\tilde{\Sigma}_2$, and hence $\tilde{\Sigma}_2 \setminus \tilde{S}_2$ is open dense in $\tilde{\Sigma}_2$.

Since π_2 is proper and Σ_2 is open dense in $\overline{\Sigma_2}$, it follows that π_2 maps $\tilde{\Sigma}_2$ onto $\overline{\Sigma_2}$. The theorem is proved. □

5.3 The desingularization of $\overline{\Sigma_3}$

The next stage does not need a blowup such as π_2. By raising bundles to $\tilde{\Sigma}_2$, we now work over $\tilde{\Sigma}_2$. Then $T_2(f_t)(\eta_2) = \{0\}$.

Let ξ_2 be the inverse image of $\xi_1^1 \bigcirc \xi_1^1$ under σ. Since $\xi_1^1 \bigcirc \xi_1^1$ is contained in the image of σ, it follows that ξ_2 is an $(n+1)$ dimensional subbundle of $T_2\mathbb{C}^n$. Let η_3 be the inverse image of $\xi_1^1 \bigcirc \xi_2$ under σ (cf. §7.1 together with the equation (3. 3) defining Σ_3). Similarly η_3 is a $(2n+1)$ dimensional subbundle of $T_3\mathbb{C}^n$. Then $\xi_1 \subset \xi_2 \subset \eta_2 \subset \eta_3$.

Since $\xi_1^1 \subset \mathrm{Ker}T_1(f_t)$, by Lemma 4.1, $T_3(f_t)$ maps η_3 into ι_1. So let $\bar{T}_3(f_t) : \eta_3/\eta_2 \to \iota_1$ be the map induced by $T_3(f_t)$. Then we have a section, say Φ_3, of $\mathrm{Hom}(\eta_3/\eta_2, \iota_1)$ over $\tilde{\Sigma}_2$, induced by $\bar{T}_3(f_t)$. Let $\tilde{\Sigma}_3$ be the zeros of the section Φ_3.

$$\begin{array}{ccc} & & \mathrm{Hom}(\eta_3/\eta_2, \iota_1) \\ & & \downarrow\upharpoonright \Phi_3 \\ \tilde{\Sigma}_3 & \hookrightarrow & \tilde{\Sigma}_2 \end{array}$$

We have $\Sigma_3 = \overline{\Sigma_3} \setminus S$, and set $\tilde{S}_3 = (\pi_2|_{\tilde{\Sigma}_3})^{-1}(S)$.

THEOREM 5.2. *The restriction of π_2 to $\tilde{\Sigma}_3$ is a desingularization of $\overline{\Sigma_3}$.*

Proof. The same argument as in §6.1 shows that Φ_3 is transverse to the zero section of $\mathrm{Hom}(\eta_3/\eta_2, \iota_1)$. On the other hand, in §7.1 we will prove the relation $\pi_2(\tilde{\Sigma}_3 \setminus \tilde{S}_3) = \Sigma_3$, the existence of the inverse map of $\pi_2|_{\tilde{\Sigma}_3\setminus\tilde{S}_3} : \tilde{\Sigma}_3 \setminus \tilde{S}_3 \to \Sigma_3$, and the density of $\tilde{\Sigma}_3 \setminus \tilde{S}_3$ in $\tilde{\Sigma}_3$. As in the proof of Theorem 5.1, these imply the theorem. □

5.4 The desingularization of $\overline{\Sigma_4}$

This stage is similar to that of $\overline{\Sigma_2}$. We now work over $\tilde{\Sigma}_3$. Then $T_3(f_t)(\eta_3) = \{0\}$.

Let $\bar{\xi}_2 = \xi_2/\xi_1^1$. We have the projective bundle $\mathbb{P}(\bar{\xi}_2)$ and the tautological line bundle $\bar{\xi}_2^1$ over it. Write π_4 for the projection of $\mathbb{P}(\bar{\xi}_2)$ onto $\tilde{\Sigma}_3$. Let ξ_2^2 be the inverse image of $\bar{\xi}_2^1$ under the quotient map $\xi_2 \to \bar{\xi}_2$. This is a 2 dimensional subbundle of ξ_2. Let ζ_4 be the inverse image of $\xi_1^1 \bigcirc \eta_3 + \xi_2^2 \bigcirc \xi_2$ under σ (cf. §7.2 together with the equations (3. 3), (3. 4)). Then ζ_4 is a $(3n+1)$ dimensional subbundle of $T_4\mathbb{C}^n$ (see §6.2), and satisfies $\eta_3 \subset \zeta_4$.

Since $\xi_1^1 \subset \mathrm{Ker}T_1(f_t)$ and $\xi_2^2 \subset \mathrm{Ker}T_2(f_t)$, by Lemma 4.1, $T_4(f_t)$ maps ζ_4 into ι_1. So let $\bar{T}_4(f_t) : \zeta_4/\eta_3 \to \iota_1$ be the map induced by $T_4(f_t)$. Then we have a section, say Φ_4, of $\mathrm{Hom}(\zeta_4/\eta_3, \iota_1)$ over $\mathbb{P}(\bar{\xi}_2)$, induced by $\bar{T}_4(f_t)$. Let $\tilde{\Sigma}_4$ be the zeros of the section Φ_4.

$$\begin{array}{ccc} & & \mathrm{Hom}(\zeta_4/\eta_3, \iota_1) \\ & & \downarrow \upharpoonright \Phi_4 \\ \tilde{\Sigma}_4 & \hookrightarrow & \mathbb{P}(\bar{\xi}_2) \\ & & \downarrow \pi_4 \\ & & \tilde{\Sigma}_3 \end{array}$$

We have $\Sigma_4 = \overline{\Sigma_4} \setminus S$, and set $\tilde{S}_4 = (\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4})^{-1}(S)$.

THEOREM 5.3. *The restriction of $\pi_2 \circ \pi_4$ to $\tilde{\Sigma}_4$ is a desingularization of $\overline{\Sigma_4}$.*

The theorem follows from the same argument as in the proof of Theorem 5.1. See §7.2 for the relation $\pi_2 \circ \pi_4(\tilde{\Sigma}_4 \setminus \tilde{S}_4) = \Sigma_4$, the existence of the inverse map of $\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4 \setminus \tilde{S}_4} : \tilde{\Sigma}_4 \setminus \tilde{S}_4 \to \Sigma_4$, and the density of $\tilde{\Sigma}_4 \setminus \tilde{S}_4$ in $\tilde{\Sigma}_4$.

5.5 The desingularization of $\overline{\Sigma_5}$

This stage is similar to that of $\overline{\Sigma_3}$, and again does not need a blowup. We now work over $\tilde{\Sigma}_4$. Then $T_4(f_t)(\zeta_4) = \{0\}$.

Let ξ_3 be the inverse image of $\xi_1^1 \bigcirc \xi_2^2$ under σ. Since $\xi_1^1 \bigcirc \xi_2^2$ is contained in the image of σ, it follows that ξ_3 is an $(n+2)$ dimensional subbundle of $T_3\mathbb{C}^n$. Then $\xi_2 \subset \xi_3 \subset \eta_3$. Let η_4 be the inverse image of $\xi_1^1 \bigcirc \xi_3 + \xi_2^2 \bigcirc \xi_2^2$ under σ. Then η_4 is a $(2n+2)$ dimensional subbundle of $T_4\mathbb{C}^n$ (see §6.3), and satisfies $\eta_3 \subset \eta_4 \subset \zeta_4$. Finally, let ζ_5 be the inverse image of $\xi_1^1 \bigcirc \eta_4 + \xi_2^2 \bigcirc \xi_3$ under σ (cf. §7.3). Then ζ_5 is a $(3n+2)$ dimensional subbundle of $T_5\mathbb{C}^n$ (see §6.3), and satisfies $\zeta_4 \subset \zeta_5$.

Since $\xi_1^1 \subset \mathrm{Ker}T_1(f_t)$ and $\xi_2^2 \subset \mathrm{Ker}T_2(f_t)$, by Lemma 4.1, $T_5(f_t)$ maps ζ_5 into ι_1. So let $\bar{T}_5(f_t) : \zeta_5/\zeta_4 \to \iota_1$ be the map induced by $T_5(f_t)$. We have a section, say Φ_5, of $\mathrm{Hom}(\zeta_5/\zeta_4, \iota_1)$ over $\tilde{\Sigma}_4$, induced by $\bar{T}_5(f_t)$. Let $\tilde{\Sigma}_5$ be the zeros of the section Φ_5.

$$\begin{array}{ccc} & & \mathrm{Hom}(\zeta_5/\zeta_4, \iota_1) \\ & & \downarrow \upharpoonright \Phi_5 \\ \tilde{\Sigma}_5 & \hookrightarrow & \tilde{\Sigma}_4 \end{array}$$

We have $\Sigma_5 = \overline{\Sigma_5} \setminus S$, and set $\tilde{S}_5 = (\pi_2 \circ \pi_4|_{\tilde{\Sigma}_5})^{-1}(S)$.

THEOREM 5.4. *The restriction of* $\pi_2 \circ \pi_4$ *to* $\tilde{\Sigma}_5$ *is a desingularization of* $\overline{\Sigma_5}$*.*

The proof is similar to that of Theorem 5.1. See §7.3 for the relation $\pi_2 \circ \pi_4(\tilde{\Sigma}_5 \setminus \tilde{S}_5) = \Sigma_5$, the existence of the inverse map of $\pi_2 \circ \pi_4|_{\tilde{\Sigma}_5 \setminus \tilde{S}_5} : \tilde{\Sigma}_5 \setminus \tilde{S}_5 \to \Sigma_5$, and the density of $\tilde{\Sigma}_5 \setminus \tilde{S}_5$ in $\tilde{\Sigma}_5$.

5.6 The desingularization of $\overline{\Sigma_6}$

We now work over $\tilde{\Sigma}_5$. Let $\bar{\xi}_3 = \xi_3/\xi_2^2$. We have the projective bundle $\mathbb{P}(\bar{\xi}_3)$, and the tautological line bundle $\bar{\xi}_3^1$ over it. Write π_5' for the projection of $\mathbb{P}(\bar{\xi}_3)$ onto $\tilde{\Sigma}_5$. Let ξ_3^3 be the inverse image of $\bar{\xi}_3^1$ under the quotient map $\xi_3 \to \xi_3/\xi_2^2$. This is a 3 dimensional subbundle of ξ_3.

By looking carefully at how the bundles was defined in the previous sections, we are tempted to define ω_6 to be $\sigma^{-1}(\xi_1^1 \bigcirc \zeta_5 + \xi_2^2 \bigcirc \eta_4 + \xi_3^3 \bigcirc \xi_3)$. However it is impossible, for this is $(4n+3)$ dimensional over some points of $\tilde{S}_5$; although it is $(4n+2)$ dimensional over $\tilde{\Sigma}_5 \setminus \tilde{S}_5$ (cf. Note 6.2 and Claim 7.2).

To overcome this difficulty, we divide the construction of the desingularization of $\overline{\Sigma_6}$ into three steps: first in §5.6.1 to blow up $\tilde{\Sigma}_5$ in order to get the line bundle $\bar{\xi}_3^1$ determined by the equations (3.4), (3.5); second in §5.6.2 to perform an extra blowup which avoids the difficulty; third in §5.6.3 to construct the desingularization of $\overline{\Sigma_6}$.

5.6.1 A preliminary blowup

Working over $\tilde{\Sigma}_5$, we have $T_5(f_t)(\zeta_5) = \{0\}$. Let ω_5 be the inverse image of $\xi_1^1 \bigcirc \zeta_4 + \xi_2^2 \bigcirc \eta_3 + \xi_3^3 \bigcirc \xi_2$ under σ (cf. §7.4). Then ω_5 is a $(4n+1)$ dimensional subbundle of $T_5\mathbb{C}^n$ (see §6.4), and satisfies $\zeta_5 \subset \omega_5$.

Since $\xi_1^1 \subset \mathrm{Ker}T_1(f_t)$, $\xi_2^2 \subset \mathrm{Ker}T_2(f_t)$, and $\xi_3^3 \subset \mathrm{Ker}T_3(f_t)$, by Lemma 4.1, $T_5(f_t)$ maps ω_5 into ι_1. So let $\bar{T}_5(f_t) : \omega_5/\zeta_5 \to \iota_1$ be the map induced by $T_5(f_t)$. Then we have a section, say Φ_5', of $\mathrm{Hom}(\omega_5/\zeta_5, \iota_1)$ over $\mathbb{P}(\bar{\xi}_3)$, induced by $\bar{T}_5(f_t)$. Let $\tilde{\Sigma}_5'$ be the zeros of the section Φ_5'.

$$\begin{array}{ccc} & & \mathrm{Hom}(\omega_5/\zeta_5, \iota_1) \\ & & \downarrow\upharpoonleft \Phi_5' \\ \tilde{\Sigma}_5' & \hookrightarrow & \mathbb{P}(\bar{\xi}_3) \\ & & \downarrow \pi_5' \\ & & \tilde{\Sigma}_5 \end{array}$$

Set $\tilde{S}_5' = (\pi_2 \circ \pi_4 \circ \pi_5'|_{\tilde{\Sigma}_5'})^{-1}(S)$.

Theorem 5.5. *The restriction of $\pi_2 \circ \pi_4 \circ \pi_5'$ to $\tilde{\Sigma}_5'$ is also a desingularization of $\overline{\Sigma_5}$.*

The proof is similar to that of Theorem 5.1. See §7.4 for the relation $\pi_2 \circ \pi_4 \circ \pi_5'(\tilde{\Sigma}_5' \setminus \tilde{S}_5') = \Sigma_5$, the existence of the inverse map of $\pi_2 \circ \pi_4 \circ \pi_5'|_{\tilde{\Sigma}_5' \setminus \tilde{S}_5'} : \tilde{\Sigma}_5' \setminus \tilde{S}_5' \to \Sigma_5$, and the density of $\tilde{\Sigma}_5' \setminus \tilde{S}_5'$ in $\tilde{\Sigma}_5'$.

5.6.2 An extra blowup

The blowup below is an analogue of Turnbull's extra blowup ([11, §2.7]), and is given by using the following map σ_2.

Considering $T_rN \bigcirc T_rN$ as a subbundle of $T_rN \otimes T_rN$, we define the map $\sigma_2 : T_rN \bigcirc T_rN \to (T_rN \bigcirc T_rN) \wedge T_rN$ by

$$\sigma_2 = \sigma \otimes id - id \otimes \sigma.$$

We summarize some properties of σ_2. From the definition, it follows that

$$\sigma_2(a \bigcirc b) = \sigma(a) \wedge b + \sigma(b) \wedge a \ (a, b \in T_rN). \tag{5.1}$$

Indeed, $\sigma_2(a \bigcirc b) = \frac{1}{2}\sigma_2(a \otimes b + b \otimes a) = \frac{1}{2}(\sigma(a) \otimes b - a \otimes \sigma(b) + \sigma(b) \otimes a - b \otimes \sigma(a)) = \sigma(a) \wedge b + \sigma(b) \wedge a$. Hence $\mathrm{Im}\sigma_2 \subset (T_rN \bigcirc T_rN) \wedge T_rN$. On the other hand, by an easy calculation, we get that

$$(a \bigcirc b) \wedge c + (b \bigcirc c) \wedge a + (c \bigcirc a) \wedge b = 0 \ (a, b, c \in T_rN). \tag{5.2}$$

Hence $(a \bigcirc a) \wedge a = 0$ and $(a \bigcirc b) \wedge a = -\frac{1}{2}(a \bigcirc a) \wedge b$. From these, for example, it follows that $\sigma_2(\sigma(a^4)) = \sigma_2(4a \bigcirc a^3 + 3a^2 \bigcirc a^2) = 12(a \bigcirc a^2) \wedge a + 6(a \bigcirc a) \wedge a^2 = 0$. More generally, Turnbull proved the following result.

Proposition 5.1. (Turnbull [11, §2.7]) $\mathrm{Ker}\sigma_2 \cap [T_rN \bigcirc T_rN]_r = \mathrm{Im}\sigma$, *where $[T_rN \bigcirc T_rN]_r$ denotes the subset of $T_rN \bigcirc T_rN$ of elements of order at most r.*

We now work over $\tilde{\Sigma}_5'$. Setting

$$\alpha_5 = \xi_1^1 \bigcirc \zeta_4 + \xi_2^2 \bigcirc \eta_3 + \xi_3^3 \bigcirc \xi_2 \text{ and}$$
$$\alpha_6 = \xi_1^1 \bigcirc \zeta_5 + \xi_2^2 \bigcirc \eta_4 + \xi_3^3 \bigcirc \xi_3,$$

the quotient space α_6/α_5 is three dimensional, and setting

$$\beta_5 = (\xi_1^1 \bigcirc \eta_3) \wedge \xi_1^1 + (\xi_1^1 \bigcirc \xi_2) \wedge \xi_2^2 + (\xi_2^2 \bigcirc \xi_2) \wedge \xi_1^1 \text{ and}$$
$$\beta_6 = (\xi_1^1 \bigcirc \eta_4) \wedge \xi_1^1 + (\xi_1^1 \bigcirc \xi_3) \wedge \xi_2^2 + (\xi_2^2 \bigcirc \xi_3) \wedge \xi_1^1,$$

the quotient space β_6/β_5 is three dimensional (cf. §6.5). From (5.1) and (5.2), it follows that

$$\sigma_2(\alpha_5) \subset \beta_5 \text{ and } \sigma_2(\alpha_6) \subset \beta_6.$$

For example, $\sigma_2(\alpha_6) \subset (\xi_1^1 \bigcirc \eta_4 + \xi_2^2 \bigcirc \xi_3) \wedge \xi_1^1 + (\xi_1^1 \bigcirc \xi_1^1) \wedge \eta_4 + (\xi_1^1 \bigcirc \xi_3 + \xi_2^2 \bigcirc \xi_2^2) \wedge \xi_2^2 + (\xi_1^1 \bigcirc \xi_2^2) \wedge \xi_3 + (\xi_1^1 \bigcirc \xi_2^2) \wedge \xi_3^3 = (\xi_1^1 \bigcirc \xi_1^1) \wedge \eta_4 + (\xi_1^1 \bigcirc \xi_2^2) \wedge \xi_3 + (\xi_2^2 \bigcirc \xi_3) \wedge \xi_1^1 + (\xi_1^1 \bigcirc \xi_3) \wedge \xi_2^2 = \beta_6$.

Since $\sigma_2(\alpha_5 + \xi_1^1 \bigcirc \zeta_5) \subset \beta_5 + (\xi_1^1 \bigcirc \eta_4) \wedge \xi_1^1 + (\xi_2^2 \bigcirc \xi_3) \wedge \xi_1^1$, by setting

$$\bar{\alpha}_6 = \alpha_6/(\alpha_5 + \xi_1^1 \bigcirc \zeta_5) \text{ and}$$
$$\bar{\beta}_6 = \beta_6/[\beta_5 + (\xi_1^1 \bigcirc \eta_4) \wedge \xi_1^1 + (\xi_2^2 \bigcirc \xi_3) \wedge \xi_1^1],$$

we get the map $\bar{\sigma}_2 : \bar{\alpha}_6 \to \bar{\beta}_6$ induced by σ_2. Here $\bar{\alpha}_6$ and $\bar{\beta}_6$ are two and one dimensional, respectively.

We have the projective bundle $\mathbb{P}(\bar{\alpha}_6)$, and the tautological line bundle $\bar{\alpha}_6^1$ over it. Write ρ for the projection of $\mathbb{P}(\bar{\alpha}_6)$ onto $\tilde{\Sigma}_5'$. The map $\bar{\sigma}_2$ induces a section Ψ of $\mathrm{Hom}(\bar{\alpha}_6^1, \bar{\beta}_6)$ over $\mathbb{P}(\bar{\alpha}_6)$. Let $\tilde{\Sigma}_5''$ be the zeros of the section Ψ.

$$\begin{array}{ccc} & & \mathrm{Hom}(\bar{\alpha}_6^1, \bar{\beta}_6) \\ & & \downarrow\upharpoonright \Psi \\ \tilde{\Sigma}_5'' & \hookrightarrow & \mathbb{P}(\bar{\alpha}_6) \\ & & \downarrow \rho \\ & & \tilde{\Sigma}_5' \end{array}$$

Set $\tilde{S}_5'' = (\pi_2 \circ \pi_4 \circ \pi_5' \circ \rho|_{\tilde{\Sigma}_5''})^{-1}(S)$.

THEOREM 5.6. *The restriction of $\pi_2 \circ \pi_4 \circ \pi_5' \circ \rho$ to $\tilde{\Sigma}_5''$ is also a desingularization of $\overline{\Sigma_5}$.*

For the proof, see §6.5.

5.6.3 The desingularization of $\overline{\Sigma_6}$

We now work over $\tilde{\Sigma}_5''$. Then $T_5(f_t)(\omega_5) = \{0\}$.

We have the line bundle $\bar{\alpha}_6^1$. Let α be the inverse image of $\bar{\alpha}_6^1$ under the quotient map $\alpha_6 \to \bar{\alpha}_6$, and let ω_6 be the inverse image of α under σ (cf. §7.5). Then ω_6 is a $(4n+2)$ dimensional subbundle of $T_6\mathbb{C}^n$ (see §6.6), and satisfies $\omega_5 \subset \omega_6$.

Since $\xi_1^1 \subset \mathrm{Ker}T_1(f_t)$, $\xi_2^2 \subset \mathrm{Ker}T_2(f_t)$, and $\xi_3^3 \subset \mathrm{Ker}T_3(f_t)$, by Lemma 4.1, $T_6(f_t)$ maps ω_6 into ι_1. So let $\bar{T}_6(f_t) : \omega_6/\omega_5 \to \iota_1$ be the map induced by $T_6(f_t)$. We have a section, say Φ_6, of $\mathrm{Hom}(\omega_6/\omega_5, \iota_1)$ over $\tilde{\Sigma}_5''$, induced by $\bar{T}_6(f_t)$. Let $\tilde{\Sigma}_6$ be the zeros of the section Φ_6.

$$\begin{array}{ccc} & & \mathrm{Hom}(\omega_6/\omega_5, \iota_1) \\ & & \downarrow\upharpoonright \Phi_6 \\ \tilde{\Sigma}_6 & \hookrightarrow & \tilde{\Sigma}_5'' \end{array}$$

We have $\Sigma_6 = \overline{\Sigma_6} \setminus S$, and set $\tilde{S}_6 = (\pi_2 \circ \pi_4 \circ \pi_5' \circ \rho|_{\tilde{\Sigma}_6})^{-1}(S)$.

THEOREM 5.7. *The restriction of $\pi_2 \circ \pi_4 \circ \pi_5' \circ \rho$ to $\tilde{\Sigma}_6$ is a desingularization of $\overline{\Sigma_6}$.*

The proof is similar to that of Theorem 5.1. See §7.5 for the relation $\pi_2 \circ \pi_4 \circ \pi_5' \circ \rho(\tilde{\Sigma}_6 \setminus \tilde{S}_6) = \Sigma_6$, the existence of the inverse map of $\pi_2 \circ \pi_4 \circ \pi_5' \circ \rho|_{\tilde{\Sigma}_6 \setminus \tilde{S}_6} : \tilde{\Sigma}_6 \setminus \tilde{S}_6 \to \Sigma_6$, and the density of $\tilde{\Sigma}_6 \setminus \tilde{S}_6$ in $\tilde{\Sigma}_6$.

6. Well-definedness of the vector bundles and transversality of the sections

In this section we will prove the results stated in §5; the inverse images $\zeta_4, \eta_4, \zeta_5, \omega_5, \omega_6$ are certainly vector bundles, and the sections $\Phi_2, \ldots, \Phi_5, \Phi_5', \Psi$, Φ_6 are transverse to the zero sections. We do these by choosing local coordinates.

6.1 Transversality of $\Phi_2, \ldots, \Phi_5, \Phi_5', \Phi_6$

We prove the transversality only for Φ_2 as the others except Ψ are similar. The transversality of Ψ will be proved in §6.5.

Over $\tilde{\Sigma}_1$, we have the projective bundle $\mathbb{P}(\xi_1)$ and the map $\bar{T}_2(f_t) : \eta_2/\xi_1 \to \iota_1$. This map induces the section $\Phi_2 : \mathbb{P}(\xi_1) \to \mathrm{Hom}(\eta_2/\xi_1, \iota_1)$. Take any point P of $\tilde{\Sigma}_2 \cap \pi_2^{-1}(0,0)$. Choose a small open neighbourhood U of P in $\mathbb{P}(\xi_1)$ such that over U, there exists the local projection pr of $\mathrm{Hom}(\eta_2/\xi_1, \iota_1)$ onto its fibre $\mathrm{Hom}_{fib}(\eta_2/\xi_1, \iota_1)$ along the zero section. We want to show that $pr \circ \Phi_2|_U : U \to \mathrm{Hom}_{fib}(\eta_2/\xi_1, \iota_1)$ is submersive at P.

Replacing U by a smaller neighbourhood, we choose local coordinates $(x, t, \xi_1^1) \in U$ with $\xi_1^1 = \langle v_1 \rangle$ $(0 \neq v_1 \in \xi_1)$. Then we can write $\eta_2 = \xi_1 \oplus \langle v_1 \rangle \circ \xi_1$. Identifying $\mathrm{Hom}_{fib}(\eta_2/\xi_1, \iota_1)$ with $\mathrm{Hom}(\mathbb{C} \circ \mathbb{C}^n, \mathbb{C})$, we have $pr \circ \Phi_2|_U = (f_t)_2(v_1, -)|_U$, where $(f_t)_2$ is the second derivative of f_t with respect to x. Set $P = (0, 0, \langle v_1^\circ \rangle)$ and define $U(v_1^\circ) = U \cap \{(x, t, \xi_1^1) | \xi_1^1 = \langle v_1^\circ \rangle\}$. It is enough to show that $(f_t)_2(v_1^\circ, -)|_{U(v_1^\circ)} : U(v_1^\circ) \to \mathrm{Hom}(\mathbb{C} \circ \mathbb{C}^n, \mathbb{C})$ is submersive at P.

Assume that $v_1^\circ = \frac{\partial}{\partial x_1}$ and define an unfolding $G : (\mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}^n, (0,0,0)) \to (\mathbb{C} \times \mathbb{C}^p \times \mathbb{C}^n, (0,0,0))$ of f_0 by

$$G(x, t, u) = (g(x, t, u), t, u), \; g(x, t, u) = f(x, t) + \sum_{i=1}^{n} u_i x_1 x_i.$$

Let $(\tilde{\Sigma}_1)_G$ denote the singular set of G in $\mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}^n$. Then $(\tilde{\Sigma}_1)_G \cap \{(x, t, u) | u = 0\} = \tilde{\Sigma}_1$ and $\{(x, t, u) | x = t = 0\} \subset (\tilde{\Sigma}_1)_G$.

By the versality of F, the unfolding G is equivalent to a constant unfolding of F (as unfoldings of f_0). Namely, there exist a holomorphic unfolding $H : (\mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}^n, (0,0,0)) \to (\mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}^n, (0,0,0))$, $H(x,t,u) = (h(x,t,u), \phi(t,u), \psi(t,u))$ of $id_{\mathbb{C}^n}$ and a holomorphic function germ $a : (\mathbb{C}^p \times \mathbb{C}^n, (0,0)) \to (\mathbb{C}, 0)$ such that H is biholomorphic and

$$g(H(x,t,u)) + a(t,u) = f(x,t).$$

By definition of G, we can take H to satisfy $H(x,t,0) = (x,t,0)$.

We have $g_{\phi_{t,u},\psi_{t,u}}(h_{t,u}(x)) + a_{t,u} = f_t(x)$, where $g_{t,u}(x) = g(x,t,u)$, $h_{t,u}(x) = h(x,t,u)$, $\phi_{t,u} = \phi(t,u)$, $\psi_{t,u} = \psi(t,u)$ and $a_{t,u} = a(t,u)$. Hence over $\mathbb{C}^n \times \mathbb{C}^p \times \mathbb{C}^n$,

$$T_2(g_{\phi_{t,u},\psi_{t,u}}) \circ T_2(h_{t,u}) = T_2(f_t), \tag{6. 1}$$

where T_2 is Turnbull's second derivative with respect to x.

We now work over $(\tilde{\Sigma}_1)_G$. The same procedure as for Φ_2 and $\tilde{\Sigma}_2$ gives the section $(\Phi_2)_G : \mathbb{P}(\xi_1) \to \mathrm{Hom}(\eta_2/\xi_1, \iota_1)$ over $(\tilde{\Sigma}_1)_G$ and its zeros $(\tilde{\Sigma}_2)_G$. Then $(\Phi_2)_G|_{\{(x,t,u,\xi_1^1)|u=0\}} = \Phi_2$ and $(\tilde{\Sigma}_2)_G \cap \{(x,t,u,\xi_1^1)|u=0\} = \tilde{\Sigma}_2$. Again replacing U by a smaller neighbourhood, we have an open neighbourhood U_G of P in $\mathbb{P}(\xi_1)$ over $(\tilde{\Sigma}_1)_G$ such that (i) $U_G \cap \{(x,t,u,\xi_1^1)|u=0\} = U$, and (ii) over U_G there exists the local projection $pr : \mathrm{Hom}(\eta_2/\xi_1, \iota_1) \to \mathrm{Hom}_{fib}(\eta_2/\xi_1, \iota_1)$. Define $U_G(v_1^\circ) = U_G \cap \{(x,t,u,\xi_1^1)|\xi_1^1 = \langle v_1^\circ \rangle\}$. Then $pr \circ (\Phi_2)_G|_{U_G(v_1^\circ)} = (g_{t,u})_2(v_1^\circ, -)|_{U_G(v_1^\circ)}$, and by definition of g this is submersive at P.

By (6. 1) we get that

$$\left((g_{\phi_{t,u},\psi_{t,u}})_2 \circ h_{t,u}\right)((h_{t,u})_1(v_1^\circ), (h_{t,u})_1(-))|_{U_G(v_1^\circ)} = (f_t)_2(v_1^\circ, -)|_{U_G(v_1^\circ)}.$$

Since $\psi : (\mathbb{C}^p \times \mathbb{C}^n, (0,0)) \to (\mathbb{C}^n, 0)$ is submersive and $h_{t,u} : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is biholomorphic, we see that $(f_t)_2(v_1^\circ, -)|_{U_G(v_1^\circ)} : U_G(v_1^\circ) \to \mathrm{Hom}(\mathbb{C} \bigcirc \mathbb{C}^n, \mathbb{C})$ is submersive at P. Since this map is independent of u, its restriction to $\tilde{\Sigma}_1$, $(f_t)_2(v_1^\circ, -)|_{U(v_1^\circ)} : U(v_1^\circ) \to \mathrm{Hom}(\mathbb{C} \bigcirc \mathbb{C}^n, \mathbb{C})$ is submersive at P. Therefore $pr \circ \Phi_2|_U$ is submersive at P, from the start. $\square$

6.2 Well-definedness of ζ_4

In §5.4, we defined ζ_4 as the inverse image of $\xi_1^1 \bigcirc \eta_3 + \xi_2^2 \bigcirc \xi_2$ under σ. By definition ζ_4 is independent of coordinates. But to see that ζ_4 is a vector bundle, we must prove that the fibres of ζ_4 are equidimensional.

The space $\xi_1^1 \bigcirc \eta_3 + \xi_2^2 \bigcirc \xi_2$ contains $\xi_1^1 \bigcirc \xi_2$, and we already know that $\eta_3(= \sigma^{-1}(\xi_1^1 \bigcirc \xi_2))$ is a vector bundle. So let $\bar{\sigma} : T_r\mathbb{C}^n/\eta_3 \to (T_r\mathbb{C}^n \bigcirc T_r\mathbb{C}^n)/(\xi_1^1 \bigcirc \xi_2)$ be the linear map induced by σ. Since $\mathrm{Ker}\sigma \subset \eta_3$, this is injective.

By choosing local coordinates, we can write $(x, t, \xi_1^1, \bar{\xi}_2^1) \in \mathbb{P}(\bar{\xi}_2)$ and

$$\begin{aligned} &\xi_1^1 = \langle v_1 \rangle,\ \eta_2 = \xi_1 \oplus \langle v_1 \rangle \circ \xi_1,\ \xi_2 = \xi_1 \oplus \langle v_1^2 \rangle, \\ &\eta_3 = \xi_1 \oplus \langle v_1 \rangle \circ \xi_1 \oplus \langle v_1^3 \rangle,\ \xi_2^2 = \langle v_1 \rangle \oplus \langle v_2 + b_1 v_1^2 \rangle, \end{aligned} \tag{6.2}$$

where $0 \neq v_1 \in \xi_1, 0 \neq v_2 + b_1 v_1^2 \in \xi_2$ $(b_1 \in \mathbb{C})$.

Then $\bar{\xi}_2^1 (= \xi_2^2 / \langle v_1 \rangle)$ can be written in the form $\bar{\xi}_2^1 = \langle \bar{v}_2 + b_1 v_1^2 \rangle$, where $\bar{v}_2$ is the equivalent class of v_2 in $\xi_1 / \langle v_1 \rangle$. But by abuse of notation, only in §6 we will write v_2 also for $\bar{v}_2$.

Write $\bar{\xi}_1$ for $\xi_1 / \langle v_1 \rangle$. Then the quotient space $(\xi_1^1 \bigcirc \eta_3 + \xi_2^2 \bigcirc \xi_2)/(\xi_1^1 \bigcirc \xi_2)$ can be written in the form

$$\langle v_1 \rangle \bigcirc (\langle v_1 \rangle \circ \bar{\xi}_1 \oplus \langle v_1^3 \rangle) + \langle v_2 + b_1 v_1^2 \rangle \bigcirc (\bar{\xi}_1 \oplus \langle v_1^2 \rangle).$$

Write B for this sum of bundles, B_1 for the first bundle, and B_2 for the second bundle. These have the properties that $\dim B_1 = \dim B_2 = n$ and $B_1 \cap B_2 = \{0\}$.

Consider the subbundle

$$\langle v_2 + b_1 v_1^2 \rangle \circ \bar{\xi}_1 \oplus \langle 3 v_1^2 v_2 + b_1 v_1^4 \rangle$$

of $T_r \mathbb{C}^n / \eta_3$. Write A for this direct sum of bundles, A_1 for the first bundle, and A_2 for the second bundle. Then $\dim A = n$.

We want to show that A is independent of coordinates. Since so is B, we will prove that $A = \bar{\sigma}^{-1}(B)$. For this, the following two claims are useful.

Claim 6.1. $\bar{\sigma}(A) \subset B$.

Proof. An element $(v_2 + b_1 v_1^2)x$ $(x \in \bar{\xi}_1)$ of A_1 is mapped by $\bar{\sigma}$ to $2 b_1 v_1 \bigcirc v_1 x + (v_2 + b_1 v_1^2) \bigcirc x$, and an element $3 v_1^2 v_2 + b_1 v_1^4$ of A_2 is mapped by $\bar{\sigma}$ to $2 v_1 \bigcirc (3 v_1 v_2 + 2 b_1 v_1^3) + 3 (v_2 + b_1 v_1^2) \bigcirc v_1^2$. These imply that $\bar{\sigma}(A) \subset B$. □

Claim 6.2. $\dim(\mathrm{Im}\bar{\sigma} \cap B) \leq n$.

Proof. By the calculations in the proof of Claim 6.1, we see that there are no nonzero elements of B_1 that have the inverse image under $\bar{\sigma}$, by itself without elements of B_2. Hence $\mathrm{Im}\bar{\sigma} \cap B_1 = \{0\}$. (Note that if $b_1 = 0$ then $\mathrm{Im}\bar{\sigma} \cap B_2 \neq \{0\}$.) From the properties of B_1 and B_2, it follows that $\dim(\mathrm{Im}\bar{\sigma} \cap B) \leq n$. □

By Claim 6.1 we have $A \subset \bar{\sigma}^{-1}(B)$, and by Claim 6.2 we have $\dim \bar{\sigma}^{-1}(B) \leq n$. Since $\dim A = n$, we get that $A = \bar{\sigma}^{-1}(B)$.

Consequently ζ_4 can be written in the form

$$\zeta_4 = \eta_3 \oplus \langle v_2 + b_1 v_1^2 \rangle \circ \bar{\xi}_1 \oplus \langle 3 v_1^2 v_2 + b_1 v_1^4 \rangle. \tag{6.3}$$

It follows from this that ζ_4 has the equidimensional fibres and is a $(3n+1)$ dimensional vector bundle.

Note 6.1. The inverse image A of B will appear clearly in Y_4 of §7.2.

6.3 Well-definedness of η_4, ζ_5

We will show that η_4 and ζ_5 defined in §5.5 are certainly vector bundles, that is, have the equidimensional fibres, respectively.

We first consider η_4, which was defined as the inverse image of $\xi_1^1 \circ \xi_3 + \xi_2^2 \bigcirc \xi_2^2$ under σ. The space $\xi_1^1 \bigcirc \xi_3 + \xi_2^2 \bigcirc \xi_2^2$ contains $\xi_1^1 \bigcirc \xi_2$, and we already know that $\eta_3 (= \sigma^{-1}(\xi_1^1 \bigcirc \xi_2))$ is a vector bundle. So we use the same map $\bar{\sigma}$ as in the previous section.

By choosing local coordinates, we can write $(x, t, \xi_1^1, \bar{\xi}_2^1) \in \tilde{\Sigma}_4$ and, in addition to (6.2) and (6.3),

$$\xi_3 = \xi_1 \oplus \langle v_1^2 \rangle \oplus \langle 3v_1v_2 + b_1v_1^3 \rangle. \tag{6.4}$$

By (6.2), (6.4), the quotient space $(\xi_1^1 \bigcirc \xi_3 + \xi_2^2 \bigcirc \xi_2^2)/(\xi_1^1 \bigcirc \xi_2)$ can be written in the form

$$\langle v_1 \rangle \bigcirc \langle 3v_1v_2 + b_1v_1^3 \rangle + \langle v_2 + b_1v_1^2 \rangle \bigcirc \langle v_2 + b_1v_1^2 \rangle.$$

Write B for this sum of bundles, B_1 for the first bundle, and B_2 for the second bundle. These have the properties that $\dim B_1 = \dim B_2 = 1$ and $B_1 \cap B_2 = \{0\}$.

Consider the subbundle

$$\langle 3v_2^2 + 6b_1v_1^2v_2 + b_1^2v_1^4 \rangle$$

of $T_r\mathbb{C}^n/\eta_3$, and write A for this. Then $\dim A = 1$.

An element $3v_2^2+6b_1v_1^2v_2+b_1^2v_1^4$ of A is mapped by $\bar{\sigma}$ to $4b_1v_1 \bigcirc (3v_1v_2+b_1v_1^3)+3(v_2+b_1v_1^2) \bigcirc (v_2+b_1v_1^2)$. So $\bar{\sigma}(A) \subset B$. From this calculation and the properties of B_1 and B_2, as in the proof of Claim 6.2, it follows that $\dim(\mathrm{Im}\bar{\sigma} \cap B) \leq 1$. Hence as in the previous section, we get that $A = \bar{\sigma}^{-1}(B)$. Therefore η_4 can be written in the form

$$\eta_4 = \eta_3 \oplus \langle 3v_2^2 + 6b_1v_1^2v_2 + b_1^2v_1^4 \rangle \tag{6.5}$$

and is a $(2n+2)$ dimensional vector bundle.

We next consider ζ_5, which was defined as the inverse image of $\xi_1^1 \bigcirc \eta_4 + \xi_2^2 \bigcirc \xi_3$ under σ. The space $\xi_1^1 \bigcirc \eta_4 + \xi_2^2 \bigcirc \xi_3$ contains $\xi_1^1 \bigcirc \eta_3 + \xi_2^2 \bigcirc \xi_2$, and we have already seen that $\zeta_4 (= \sigma^{-1}(\xi_1^1 \bigcirc \eta_3 + \xi_2^2 \bigcirc \xi_2))$ is a vector bundle. So let $\bar{\bar{\sigma}} : T_r\mathbb{C}^n/\zeta_4 \to (T_r\mathbb{C}^n \bigcirc T_r\mathbb{C}^n)/(\xi_1^1 \bigcirc \eta_3 + \xi_2^2 \bigcirc \xi_2)$ be the linear map induced by σ. This is injective.

By using (6.2), (6.4), (6.5), we can write the quotient space $(\xi_1^1 \bigcirc \eta_4 + \xi_2^2 \bigcirc \xi_3)/(\xi_1^1 \bigcirc \eta_3 + \xi_2^2 \bigcirc \xi_2)$ in the form

$$\langle v_1 \rangle \bigcirc \langle 3v_2^2 + 6b_1v_1^2v_2 + b_1^2v_1^4 \rangle + \langle v_2 + b_1v_1^2 \rangle \bigcirc \langle 3v_1v_2 + b_1v_1^3 \rangle.$$

Write B for this.

Consider the subbundle

$$\langle 15v_1v_2^2 + 10b_1v_1^3v_2 + b_1^2v_1^5\rangle$$

of $T_r\mathbb{C}^n/\zeta_4$, and write A for this.

An element $15v_1v_2^2 + 10b_1v_1^3v_2 + b_1^2v_1^5$ of A is mapped by $\bar{\bar{\sigma}}$ to $5v_1 \bigcirc (3v_2^2 + 6b_1v_1^2v_2 + b_1^2v_1^4) + 10(v_2 + b_1v_1^2) \bigcirc (3v_1v_2 + b_1v_1^3)$. From this, in the same way as before, it follows that $A = \bar{\bar{\sigma}}^{-1}(B)$. Therefore ζ_5 can be written in the form

$$\zeta_5 = \zeta_4 \oplus \langle 15v_1v_2^2 + 10b_1v_1^3v_2 + b_1^2v_1^5\rangle \tag{6.6}$$

and is a $(3n+2)$ dimensional vector bundle.

6.4 Well-definedness of ω_5

In §5.6.1 we defined ω_5 as the inverse image of $\xi_1^1 \bigcirc \zeta_4 + \xi_2^2 \bigcirc \eta_3 + \xi_3^3 \bigcirc \xi_2$ under σ. The space $\xi_1^1 \bigcirc \zeta_4 + \xi_2^2 \bigcirc \eta_3 + \xi_3^3 \bigcirc \xi_2$ contains $\xi_1^1 \bigcirc \eta_4 + \xi_2^2 \bigcirc \xi_3$, and we have already seen that $\zeta_5(= \sigma^{-1}(\xi_1^1 \bigcirc \eta_4 + \xi_2^2 \bigcirc \xi_3))$ is a vector bundle. So let $\bar{\sigma}^{(3)} : T_r\mathbb{C}^n/\zeta_5 \to (T_r\mathbb{C}^n \bigcirc T_r\mathbb{C}^n)/(\xi_1^1 \bigcirc \eta_4 + \xi_2^2 \bigcirc \xi_3)$ be the linear map induced by σ. This is injective. It is enough to show that the inverse image of $(\xi_1^1 \bigcirc \zeta_4 + \xi_2^2 \bigcirc \eta_3 + \xi_3^3 \bigcirc \xi_2)/(\xi_1^1 \bigcirc \eta_4 + \xi_2^2 \bigcirc \xi_3)$ under $\bar{\sigma}^{(3)}$ has the equidimensional fibres.

We choose local coordinates $(x, t, \xi_1^1, \bar{\xi}_2^1, \bar{\xi}_3^1) \in \mathbb{P}(\bar{\xi}_3)$ satisfying (6.2)-(6.6). Then

$$\xi_2^2 = \langle v_1\rangle \oplus \langle v_2 + b_1v_1^2\rangle,$$

where $0 \neq v_1 \in \xi_1, 0 \neq v_2 + b_1v_1^2 \in \xi_2$ $(b_1 \in \mathbb{C})$. Since $v_2 + b_1v_1^2 \neq 0$, there are the two cases with $v_2 \neq 0$ or $b_1 \neq 0$. If $b_1 \neq 0$ then by a linear change of the coordinates v_2, we can assume that $b_1 = 1$. Then

$$\bar{\xi}_3 = \xi_3/\xi_2^2 = \begin{cases} (\xi_1/\langle v_1, v_2\rangle) \oplus \langle v_1^2\rangle \oplus \langle 3v_1v_2 + b_1v_1^3\rangle & (v_2 \neq 0), \\ \bar{\xi}_1 \oplus \langle 3v_1v_2 + v_1^3\rangle & (b_1 \neq 0), \end{cases}$$

where v_1, v_2 are linearly independent in the case $v_2 \neq 0$. So we can write

$$\xi_3^3 = \begin{cases} \langle v_1\rangle \oplus \langle v_2 + b_1v_1^2\rangle \oplus \langle v_3 + b_3v_1^2 + b_2(3v_1v_2 + b_1v_1^3)\rangle & (v_2 \neq 0), \\ \langle v_1\rangle \oplus \langle v_2 + v_1^2\rangle \oplus \langle v_3 + b_2(3v_1v_2 + v_1^3)\rangle & (b_1 \neq 0), \end{cases} \tag{6.7}$$

where $0 \neq v_3 + b_3v_1^2 + b_2(3v_1v_2 + b_1v_1^3) \in \xi_3$ or $0 \neq v_3 + b_2(3v_1v_2 + v_1^3) \in \xi_3$ $(b_2, b_3 \in \mathbb{C})$.

For the expression of ξ_3^3 in the case $v_2 \neq 0$, if $b_1 \neq 0$ then by linear changes of the coordinates v_2, v_3, we can assume that $b_1 = 1$ and $b_3 = 0$, that is, we get the expression of ξ_3^3 in the case $b_1 \neq 0$. For this reason, in the rest of §6 we describe the calculations only for the expression of ξ_3^3 in the case $v_2 \neq 0$, and the results for both cases $v_2 \neq 0$ and $b_1 \neq 0$.

Write $\bar{\bar{\xi}}_1$ for $\xi_1/\langle v_1, v_2\rangle$. By using (6. 2)-(6. 5),(6. 7), the quotient space $(\xi_1^1 \circ \zeta_4 + \xi_2^2 \circ \eta_3 + \xi_3^3 \circ \xi_2)/(\xi_1^1 \circ \eta_4 + \xi_2^2 \circ \xi_3)$ can be written in the form

$$\begin{aligned}
&\langle v_1\rangle \bigcirc \left(\langle v_2 + b_1 v_1^2\rangle \circ \bar{\bar{\xi}}_1 \oplus \langle 3v_1^2 v_2 + b_1 v_1^4\rangle\right) \\
&+ \langle v_2 + b_1 v_1^2\rangle \bigcirc \left(\langle v_1\rangle \circ \bar{\bar{\xi}}_1 \oplus \langle v_1^3\rangle\right) \\
&+ \langle v_3 + b_3 v_1^2 + b_2(3v_1 v_2 + b_1 v_1^3)\rangle \bigcirc \left(\bar{\bar{\xi}}_1 \oplus \langle v_1^2\rangle\right).
\end{aligned}$$

Here we again write v_3 for its equivalent class.

In almost the same way as in the previous sections, we are able to prove that ω_5 can be written in the form

$$\begin{aligned}
\omega_5 = \zeta_5 &\oplus \langle v_3 + b_3 v_1^2 + b_2(3v_1 v_2 + b_1 v_1^3)\rangle \circ \bar{\bar{\xi}}_1 \\
&\oplus \langle 15 v_1^2 v_3 + 5 b_3 v_1^4 + 3 b_2 (5 v_1^3 v_2 + b_1 v_1^5)\rangle.
\end{aligned}$$

Indeed, this is shown by the following two calculations; an element $(v_3 + b_3 v_1^2 + b_2(3v_1 v_2 + b_1 v_1^3))x$ ($x \in \bar{\bar{\xi}}_1$) of ω_5/ζ_5 is mapped by $\bar{\sigma}^{(3)}$ to $2b_3 v_1 \bigcirc v_1 x + 3b_2 v_1 \bigcirc (v_2 + b_1 v_1^2)x + 3b_2(v_2 + b_1 v_1^2) \bigcirc v_1 x + (v_3 + b_3 v_1^2 + b_2(3v_1 v_2 + b_1 v_1^3)) \bigcirc x$, where the first term $2b_3 v_1 \bigcirc v_1 x$ is zero modulo $\xi_1^1 \bigcirc \eta_4 + \xi_2^2 \bigcirc \xi_3$; an element $15 v_1^2 v_3 + 5 b_3 v_1^4 + 3 b_2 (5 v_1^3 v_2 + b_1 v_1^5)$ of ω_5/ζ_5 is mapped by $\bar{\sigma}^{(3)}$ to $10 v_1 \bigcirc (3 v_1 v_3 + 2 b_3 v_1^3) + 15 b_2 (3 v_1^2 v_2 + b_1 v_1^4) \bigcirc v_1 + 15 b_2 (v_2 + b_1 v_1^2) \bigcirc v_1^3 + 15 (v_3 + b_3 v_1^2 + b_2 (3 v_1 v_2 + b_1 v_1^3)) \bigcirc v_1^2$, where the first term $10 v_1 \bigcirc (3 v_1 v_3 + 2 b_3 v_1^3)$ is zero modulo $\xi_1^1 \bigcirc \eta_4 + \xi_2^2 \bigcirc \xi_3$.

Together with the case $b_1 \neq 0$, we get that

$$\omega_5 = \zeta_5 \oplus \begin{cases} \begin{array}{l} \langle v_3 + b_3 v_1^2 + b_2(3v_1 v_2 + b_1 v_1^3)\rangle \circ \bar{\bar{\xi}}_1 \\ \qquad \oplus \langle 15 v_1^2 v_3 + 5 b_3 v_1^4 + 3 b_2 (5 v_1^3 v_2 + b_1 v_1^5)\rangle \end{array} & (v_2 \neq 0), \\ \langle v_3 + b_2(3 v_1 v_2 + v_1^3)\rangle \circ \bar{\xi}_1 & (b_1 \neq 0). \end{cases} \tag{6. 8}$$

Therefore ω_5 is a $(4n+1)$ dimensional vector bundle.

6.5 Transversality of Ψ

This is essentially Turnbull's result (see [11, §3.2]). Instead of the proof, we will write out the bundle $\bar{\alpha}_6^1$ and the section Ψ in local coordinates.

We choose local coordinates $(x, t, \xi_1^1, \bar{\xi}_2^1, \bar{\xi}_3^1, \bar{\alpha}_6^1) \in \mathbb{P}(\bar{\alpha}_6)$ satisfying (6. 2)-(6. 8). We again describe the calculations only for the expression (6. 7) of ξ_3^3 in the case $v_2 \neq 0$.

The quotient space $\bar{\alpha}_6(= (\xi_1^1 \bigcirc \zeta_5 + \xi_2^2 \bigcirc \eta_4 + \xi_3^3 \bigcirc \xi_3)/(\xi_1^1 \bigcirc \zeta_4 + \xi_2^2 \bigcirc \eta_3 + \xi_3^3 \bigcirc \xi_2 + \xi_1^1 \bigcirc \zeta_5))$ can be written in the form

$$\begin{aligned}\bar{\alpha}_6 = &\langle v_2 + b_1 v_1^2\rangle \bigcirc \langle 3v_2^2 + 6b_1 v_1^2 v_2 + b_1^2 v_1^4\rangle \\ &+ \langle v_3 + b_3 v_1^2 + b_2(3v_1v_2 + b_1v_1^3)\rangle \bigcirc \langle 3v_1v_2 + b_1v_1^3\rangle.\end{aligned}$$

Let $X = (v_2 + b_1v_1^2) \bigcirc (3v_2^2 + 6b_1v_1^2v_2 + b_1^2v_1^4)$ and $Y = (v_3 + b_3v_1^2 + b_2(3v_1v_2 + b_1v_1^3)) \bigcirc (3v_1v_2 + b_1v_1^3)$. Then we can write the tautological line bundle $\bar{\alpha}_6^1$ in the form

$$\bar{\alpha}_6^1 = \langle 3\lambda X + 2\mu Y\rangle \ ((\lambda, \mu) \neq (0, 0)).$$

The section Ψ is induced by $\bar{\sigma}_2$, and can be written in the form

$$\begin{aligned}&\bar{\sigma}_2(3\lambda X + 2\mu Y)\\ &= 3\lambda[b_1(v_1 \bigcirc v_1) \wedge (3v_2^2 + 6b_1v_1^2v_2 + b_1^2v_1^4)\\ &\qquad + 4b_1(v_1 \bigcirc (3v_1v_2 + b_1v_1^3)) \wedge (v_2 + b_1v_1^2)]\\ &\quad + 2\mu[b_3(v_1 \bigcirc v_1) \wedge (3v_1v_2 + b_1v_1^3)\\ &\qquad + 3b_2(v_1 \bigcirc (v_2 + b_1v_1^2)) \wedge (3v_1v_2 + b_1v_1^3)\\ &\qquad + 3(v_1 \bigcirc (v_2 + b_1v_1^2)) \wedge (v_3 + b_3v_1^2 + b_2(3v_1v_2 + b_1v_1^3))]\\ &\equiv 12\lambda b_1[v_1 \bigcirc (3v_1v_2 + b_1v_1^3)] \wedge (v_2 + b_1v_1^2)\\ &\quad + 12\mu b_2[v_1 \bigcirc (v_2 + b_1v_1^2)] \wedge (3v_1v_2 + b_1v_1^3)\\ &\equiv 12(-\lambda b_1 + \mu b_2)[v_1 \bigcirc (v_2 + b_1v_1^2)] \wedge (3v_1v_2 + b_1v_1^3),\end{aligned}$$

modulo $[(\xi_1^1 \bigcirc \eta_3) \wedge \xi_1^1 + (\xi_1^1 \bigcirc \xi_2) \wedge \xi_2^2 + (\xi_2^2 \bigcirc \xi_2) \wedge \xi_1^1] + [(\xi_1^1 \bigcirc \eta_4) \wedge \xi_1^1 + (\xi_2^2 \bigcirc \xi_3) \wedge \xi_1^1]$. If $\bar{\sigma}_2(3\lambda X + 2\mu Y) = 0$ then

$$\lambda b_1 = \mu b_2.$$

Hence the independence of b_1, b_2 implies the transversality of Ψ, and the nonsingularity of $\tilde{\Sigma}_5''$.

For later use we describe the expression of $\bar{\alpha}_6^1$ together with the case $b_1 \neq 0$:

$$(6.9) \quad \bar{\alpha}_6^1 = \begin{cases} \begin{aligned}&\langle 3\lambda(v_2 + b_1v_1^2) \bigcirc (3v_2^2 + 6b_1v_1^2v_2 + b_1^2v_1^4)\\ &+ 2\mu[v_3 + b_3v_1^2 + b_2(3v_1v_2 + b_1v_1^3)] \bigcirc (3v_1v_2 + b_1v_1^3)\rangle\end{aligned} & (v_2 \neq 0),\\ \begin{aligned}&\langle 3\lambda(v_2 + v_1^2) \bigcirc (3v_2^2 + 6v_1^2v_2 + v_1^4)\\ &+ 2\mu[v_3 + b_2(3v_1v_2 + v_1^3)] \bigcirc (3v_1v_2 + v_1^3)\rangle\end{aligned} & (b_1 \neq 0),\end{cases}$$

where (i) $(\lambda, \mu) \neq (0, 0)$, and (ii) over $\tilde{\Sigma}_5''$ we have $\lambda b_1 = \mu b_2$.

6.6 Well-definedness of ω_6

In §5.6.3 we defined ω_6 as the inverse image of $\alpha(\subset \alpha_6 = \xi_1^1 \bigcirc \zeta_5 + \xi_2^2 \bigcirc \eta_4 + \xi_3^3 \bigcirc \xi_3)$ under σ. The space α contains $\alpha_5(= \xi_1^1 \bigcirc \zeta_4 + \xi_2^2 \bigcirc \eta_3 + \xi_3^3 \bigcirc \xi_2)$, and we have already seen that $\omega_5(= \sigma^{-1}(\alpha_5))$ is a vector bundle. So let $\bar{\sigma}^{(4)} : T_r\mathbb{C}^n/\omega_5 \to (T_r\mathbb{C}^n \bigcirc T_r\mathbb{C}^n)/\alpha_5$ be the linear map induced by σ. This is injective.

We choose local coordinates $(x, t, \xi_1^1, \bar{\xi}_2^1, \bar{\xi}_3^1, \bar{\alpha}_6^1) \in \tilde{\Sigma}_5''$ satisfying (6. 2)-(6. 9). We again describe the calculations only for the expression (6. 7) of ξ_3^3 in the case $v_2 \neq 0$.

The quotient space α/α_5 can be written in the form

$$\begin{aligned}\langle v_1\rangle \bigcirc \langle 15v_1v_2^2 + 10b_1v_1^3v_2 + b_1^2v_1^5\rangle \\ + \langle 3\lambda(v_2 + b_1v_1^2) \bigcirc (3v_2^2 + 6b_1v_1^2v_2 + b_1^2v_1^4) \\ + 2\mu[v_3 + b_3v_1^2 + b_2(3v_1v_2 + b_1v_1^3)] \bigcirc (3v_1v_2 + b_1v_1^3)\rangle.\end{aligned}$$

In the same way as in the previous sections, we are able to prove that ω_6 can be written in the form

$$\begin{aligned}\omega_6 = \omega_5 \oplus \langle 10\mu(3v_1v_2v_3 + b_1v_1^3v_3) + 2\mu b_3(5v_1^3v_2 + b_1v_1^5) \\ + \lambda(15v_2^3 + 45b_1v_1^2v_2^2 + 15b_1^2v_1^4v_2 + b_1^3v_1^6)\rangle,\end{aligned}$$

where $(\lambda, \mu) \neq (0, 0)$ and $\lambda b_1 = \mu b_2$.

Indeed, this is shown by the following calculation; an element $10\mu(3v_1v_2v_3 + b_1v_1^3v_3) + 2\mu b_3(5v_1^3v_2 + b_1v_1^5) + \lambda(15v_2^3 + 45b_1v_1^2v_2^2 + 15b_1^2v_1^4v_2 + b_1^3v_1^6)$ of ω_6/ω_5 is mapped by $\bar{\sigma}^{(4)}$ to $10\mu v_1 \bigcirc [3(v_2 + b_1v_1^2)v_3 + b_3(3v_1^2v_2 + b_1v_1^4)] + 10\mu(v_2 + b_1v_1^2) \bigcirc (3v_1v_3 + b_3v_1^3) + 6\lambda b_1v_1 \bigcirc (15v_1v_2^2 + 10b_1v_1^3v_2 + b_1^2v_1^5) + 15\lambda(v_2 + b_1v_1^2) \bigcirc (3v_2^2 + 6b_1v_1^2v_2 + b_1^2v_1^4) + 10\mu[v_3 + b_3v_1^2 + b_2(3v_1v_2 + b_1v_1^3)] \bigcirc (3v_1v_2 + b_1v_1^3)$, where the first two terms $10\mu v_1 \bigcirc [3(v_2 + b_1v_1^2)v_3 + b_3(3v_1^2v_2 + b_1v_1^4)]$, $10\mu(v_2 + b_1v_1^2) \bigcirc (3v_1v_3 + b_3v_1^3)$ are zero modulo $\xi_1^1 \bigcirc \zeta_4 + \xi_2^2 \bigcirc \eta_3 + \xi_3^3 \bigcirc \xi_2$. (Note that $\lambda b_1 = \mu b_2$.)

Together with the case $b_1 \neq 0$, we get that

$$(6.\,10) \quad \omega_6 = \begin{cases} \omega_5 \oplus \langle 10\mu(3v_1v_2v_3 + b_1v_1^3v_3) + 2\mu b_3(5v_1^3v_2 + b_1v_1^5) \\ \quad +\lambda(15v_2^3 + 45b_1v_1^2v_2^2 + 15b_1^2v_1^4v_2 + b_1^3v_1^6)\rangle & (v_2 \neq 0), \\ \omega_5 \oplus \langle 10\mu(3v_1v_2v_3 + v_1^3v_3) \\ \quad +\lambda(15v_2^3 + 45v_1^2v_2^2 + 15v_1^4v_2 + v_1^6)\rangle & (b_1 \neq 0), \end{cases}$$

where $(\lambda, \mu) \neq (0, 0)$ and $\lambda b_1 = \mu b_2$. Therefore ω_6 is a $(4n + 2)$ dimensional vector bundle.

NOTE 6.2. Without the extra blowup, when $b_1 = b_2 = b_3 = 0$, each of $(v_2 + b_1v_1^2) \bigcirc (3v_2^2 + 6b_1v_1^2v_2 + b_1^2v_1^4)$ and $[v_3 + b_3v_1^2 + b_2(3v_1v_2 + b_1v_1^3)] \bigcirc (3v_1v_2 + b_1v_1^3)$ in $\bar{\alpha}_6$ has the inverse image under the composite of $\bar{\sigma}^{(4)}$ and the quotient map $\alpha_6/\alpha_5 \to \bar{\alpha}_6$, separately.

7. The relationship between the defining equations and the vector bundles

In this section, we will establish the relationship between the equations defining Σ_k and the vector bundles used to construct $\tilde{\Sigma}_k$. Then we give proofs of the existence and density results stated in §5. We use the same local coordinates as in §6. Since the cases $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are straightforward, we begin with the case $\tilde{\Sigma}_3$.

7.1 The case $\tilde{\Sigma}_3$

The points of Σ_3 in $\Sigma_2(=\Sigma^{n,1}F)$ are characterized by the equations (3. 2), (3. 3), and the subset $\tilde{\Sigma}_3$ of $\tilde{\Sigma}_2$ is defined by the zeros of the section $\Phi_3 : \tilde{\Sigma}_2 \to \mathrm{Hom}(\eta_3/\eta_2, \iota_1)$. To clarify the relationship between Σ_3 and $\tilde{\Sigma}_3$, we will prove that $\pi_2(\tilde{\Sigma}_3 \setminus \tilde{S}_3) = \Sigma_3$, where $\tilde{S}_3 = (\pi_2|_{\tilde{\Sigma}_3})^{-1}(S)$ and $S = \cup_{i\geq 2}\Sigma^{n,i}F$. For this, we shall examine a necessary and sufficient condition for a point of $\tilde{\Sigma}_2 \setminus \tilde{S}_2$ to lie above Σ_3, where $\tilde{S}_2 = (\pi_2|_{\tilde{\Sigma}_2})^{-1}(S)$. Note that $\pi_2|_{\tilde{\Sigma}_2\setminus\tilde{S}_2} : \tilde{\Sigma}_2 \setminus \tilde{S}_2 \to \Sigma_2$ is biholomorphic and its restriction over Σ_3 is a one sheeted cover of Σ_3.

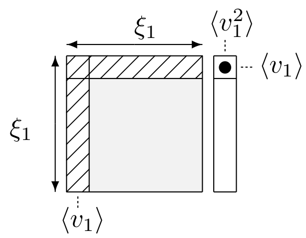
Let W be a small open neighbourhood of $(0, 0)$ in $\mathbb{C}^n \times \mathbb{C}^p$ and choose the local coordinates $(x, t, \xi_1^1) \in (\pi_2|_{\tilde{\Sigma}_2})^{-1}(W)$ with $\xi_1^1 = \langle v_1\rangle$. Take a point P of $(\pi_2|_{\tilde{\Sigma}_2})^{-1}(W)\setminus\tilde{S}_2$ and let U be an open neighbourhood of P in $(\pi_2|_{\tilde{\Sigma}_2})^{-1}(W)\setminus\tilde{S}_2$. By definition of $\tilde{\Sigma}_2$, the vector v_1 at P satisfies the equation (3. 2). Hence the point P lies above Σ_3 if and only if there exists a vector $v_2 \in \xi_1$ at P satisfying the equation (3. 3). Consider the map

$$(f_t)_2 + (f_t)_3 : (\xi_1 \oplus \langle v_1^2\rangle)\circ\xi_1 \to \iota_1 \tag{7. 1}$$

over U. The condition for the vector v_2 at P means that the restriction of (7. 1) to the subspace $\langle v_2 + v_1^2\rangle\circ\xi_1$ vanishes at P.

As a subspace of the tensor product $(\xi_1 \oplus \langle v_1^2\rangle) \otimes \xi_1$, the source space $(\xi_1 \oplus \langle v_1^2\rangle)\circ\xi_1$ of (7. 1) can be visualized by a diagram (Figure 1).

Figure 1 The diagram for Σ_3



The diagram represents the space $(\xi_1 \oplus \langle v_1^2\rangle)\circ\xi_1$ by the coordinates with respect to the standard basis. Each row represents $\xi_1 \oplus \langle v_1^2\rangle$, and each column represents ξ_1. In the diagram, we assume that the subspace $\langle v_1\rangle$ of ξ_1 is represented by the first coordinate. The symbol $\bullet$ indicates the quotient space $\langle v_1^3\rangle = [(\xi_1 \oplus \langle v_1^2\rangle)\bigcirc\langle v_1\rangle]\,/(\xi_1\bigcirc\langle v_1\rangle)$ considered as the subspace of $(\xi_1 \oplus \langle v_1^2\rangle)\bigcirc\xi_1$.

Since U is contained in $\tilde{\Sigma}_2$, the hatched areas $\xi_1\bigcirc\langle v_1\rangle, \langle v_1\rangle\bigcirc\xi_1$ in Figure 1 are mapped by (7.1) to zero. Hence (7.1) (more precisely $(f_t)_2$) induces the map Δ from $\bar{\xi}_1\bigcirc\bar{\xi}_1$ to ι_1, where $\bar{\xi}_1 = \xi_1/\langle v_1\rangle$. By definition of U, this Δ is a nondegenerate bilinear form over U. The space $\bar{\xi}_1\bigcirc\bar{\xi}_1$ can be considered as the shaded box in Figure 1.

Consider the map ϵ_3 from $\bullet$ $(= \langle v_1^3\rangle)$ to ι_1 induced by (7.1). Since Δ is nondegenerate at P, the condition for v_2 at P mentioned above is equivalent to the condition that ϵ_3 vanishes at P. By the expression (6.2), we see that ϵ_3 is $\bar{T}_3(f_t) : \eta_3/\eta_2 \to \iota_1$, which induces the section Φ_3. Hence the point P lies above Σ_3 if and only if it is in $\tilde{\Sigma}_3$. Since P is any point of $\tilde{\Sigma}_2 \setminus \tilde{S}_2$, we get that $\pi_2(\tilde{\Sigma}_3 \setminus \tilde{S}_3) = \Sigma_3$. $\square$

The map $\pi_2|_{\tilde{\Sigma}_2\setminus\tilde{S}_2} : \tilde{\Sigma}_2 \setminus \tilde{S}_2 \to \Sigma_2$ has the inverse map, and the map $\pi_2|_{\tilde{\Sigma}_3\setminus\tilde{S}_3} : \tilde{\Sigma}_3 \setminus \tilde{S}_3 \to \Sigma_3$ is the restriction of $\pi_2|_{\tilde{\Sigma}_2\setminus\tilde{S}_2}$ to $\tilde{\Sigma}_3 \setminus \tilde{S}_3$. Hence there exists the inverse map of $\pi_2|_{\tilde{\Sigma}_3\setminus\tilde{S}_3} : \tilde{\Sigma}_3 \setminus \tilde{S}_3 \to \Sigma_3$.

Referring the map Δ, we consider the map $\overline{\Delta} : \bar{\xi}_1\bigcirc\bar{\xi}_1 \to \iota_1$ over $(\pi_2|_{\tilde{\Sigma}_3})^{-1}(W)$ induced by $(f_t)_2$. Then $\tilde{S}_3 \cap (\pi_2|_{\tilde{\Sigma}_3})^{-1}(W)$ consists of the points of $(\pi_2|_{\tilde{\Sigma}_3})^{-1}(W)$ at which $\overline{\Delta}$ is degenerate. Since $\tilde{\Sigma}_3$ is nonsingular and $\overline{\Delta}$ is not identically zero (on each connected component of $\tilde{\Sigma}_3$), it follows that $\tilde{S}_3 \cap (\pi_2|_{\tilde{\Sigma}_3})^{-1}(W)$ is a nowhere dense analytic subset of $(\pi_2|_{\tilde{\Sigma}_3})^{-1}(W)$. Hence $\tilde{\Sigma}_3 \setminus \tilde{S}_3$ is dense in $\tilde{\Sigma}_3$.

7.2 The case $\tilde{\Sigma}_4$

We will prove that $\pi_2 \circ \pi_4(\tilde{\Sigma}_4 \setminus \tilde{S}_4) = \Sigma_4$, where $\tilde{S}_4 = (\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4})^{-1}(S)$. For this, we shall examine a sufficient condition for a point of $\mathbb{P}(\bar{\xi}_2)$ to lie above Σ_4. Note that $\pi_2|_{\tilde{\Sigma}_3\setminus\tilde{S}_3} : \tilde{\Sigma}_3 \setminus \tilde{S}_3 \to \Sigma_3$ is biholomorphic and the restriction of $\mathbb{P}(\bar{\xi}_2)$ over $(\pi_2|_{\tilde{\Sigma}_3\setminus\tilde{S}_3})^{-1}(\Sigma_4)$ is a fibre bundle.

Let W be a small open neighbourhood of $(0,0)$ in $\mathbb{C}^n\times\mathbb{C}^p$ and choose the local coordinates $(x, t, \xi_1^1, \bar{\xi}_2^1) \in (\pi_2 \circ \pi_4)^{-1}(W)$ $(\subset \mathbb{P}(\bar{\xi}_2))$ with $\xi_2^2 = \langle v_1\rangle \oplus \langle v_2 + b_1 v_1^2\rangle$. Take a point P of $(\pi_2 \circ \pi_4)^{-1}(W \setminus S)$ and let U be an open neighbourhood of P in $(\pi_2 \circ \pi_4)^{-1}(W \setminus S)$.

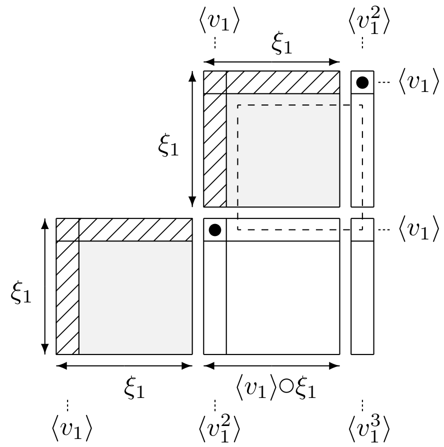
The vector v_1 at P satisfies the equation (3.2), and hence P lies above Σ_4 if the vector v_2 at P satisfies the equation (3.3) and there exists a vector $v_3 \in \xi_1$ at P satisfying the equation (3.4). Consider the maps

$$\begin{aligned} (f_t)_2 + (f_t)_3 \quad &: (\xi_1 \oplus \langle v_1^2\rangle)\bigcirc\xi_1 \to \iota_1, \\ (f_t)_2 + (f_t)_3 + (f_t)_4 \quad &: (\xi_1 \oplus \langle v_1\rangle\bigcirc\xi_1 \oplus \langle v_1^3\rangle)\bigcirc\xi_1 \to \iota_1 \end{aligned} \tag{7.2}$$

over U. The condition for the vectors v_2, v_3 at P means that the restrictions of (7. 2) to the subspaces $\langle v_2+v_1^2\rangle \circ \xi_1$, $\langle v_3+3v_1v_2+v_1^3\rangle \circ \xi_1$ vanish at P, respectively.

As in the previous section, the source spaces of (7. 2) can be visualized by a diagram (Figure 2).

Figure 2 The diagram for Σ_4



Since U lies above $\tilde{\Sigma}_3$, the symbols $\bullet$ $(= \langle v_1^3\rangle)$ in Figure 2 are mapped by (7. 2) to zero. The other symbols except the dashed box are the same as in Figure 1. The dashed box is the space

$$\left[(\bar{\xi}_1 \oplus \langle v_1^2\rangle)\circ\bar{\xi}_1\right] \oplus \left[(\langle v_1\rangle\circ\bar{\xi}_1 \oplus \langle v_1^3\rangle)\circ\langle v_1\rangle\right]$$

over U.

Consider the subspace

$$K_4 = \left[\langle \bar{v}_2 + v_1^2\rangle\circ\bar{\xi}_1\right] \oplus \left[\langle 3v_1\bar{v}_2 + v_1^3\rangle\circ\langle v_1\rangle\right]$$

of the dashed box, and let δ_4 be the map from K_4 to $\iota_1 \oplus \iota_1$ induced by (7. 2). Over U, the map Δ (on the lower left shaded box in Figure 2), which is the pullback of Δ of §7.1, is nondegenerate. Hence the condition for v_2, v_3 at P mentioned above is equivalent to the condition that δ_4 vanishes at P. Let ϵ_4 be the map from K_4 to ι_1 induced by $\sum_{i=2}^{4}(f_t)_i$. Then the condition for δ_4 at P is equivalent to the condition that ϵ_4 vanishes at P.

By the condition for ϵ_4 at P, we perform the following operations. At each point of U, choose two vectors in K_4 and let $(v_2+v_1^2)V_2$, $(3v_1v_2+v_1^3)v_1$ be their representatives, where $V_2 \in \xi_1$ is some vector. Insert these representatives to the third and fourth rows in Table 2, where $V_3, V_4 \in \xi_1$ are some vectors.

Table 2 The table for Y_4

1	V_4
v_1	$3V_3$
$v_2+v_1^2$	$3V_2$
$V_3+3v_1v_2+v_1^3$	v_1

Applying Lemma 2.1 (i) to Table 2, we get the vector

$$Y_4 = V_4 + 4v_1V_3 + 3(v_2+v_1^2)V_2 + (3v_1^2v_2+v_1^4).$$

Then the point P lies above Σ_4 if Y_4 at P is mapped by $\sum_{i=1}^4 (f_t)_i$ to zero for all $V_2, V_3, V_4 \in \xi_1$.

At a point of U with $b_1 \neq 0$, we can consider Y_4 as a representative of a basis element of ζ_4/η_3 in ζ_4, and also $\sum_{i=1}^4 (f_t)_i$ on Y_4 as $T_4(f_t)$ on Y_4. Indeed, by the expression (6.3), the space ζ_4/η_3 has a basis of the form $3(\bar v_2+b_1v_1^2)\bar V_2^{(i)}+(3v_1^2\bar v_2+b_1v_1^4)$ ($\bar V_2^{(i)} \in \bar\xi_1, 1 \le i \le n$), and an element $3(\bar v_2+b_1v_1^2)\bar V_2+(3v_1^2\bar v_2+b_1v_1^4)$ of the basis has a representative $V_4+4v_1V_3+3(v_2+b_1v_1^2)V_2+(3v_1^2v_2+b_1v_1^4)$ ($V_2, V_3, V_4 \in \xi_1$) in ζ_4. At a point of U with $b_1 \neq 0$, by a linear change of the coordinates v_2, we can assume that $b_1 = 1$ in the representative. Then the representative is Y_4, and $T_4(f_t)$ on Y_4 is essentially $\sum_{i=1}^4 (f_t)_i$ on Y_4.

Then the following claim is useful.

Claim 7.1. *At every point of $\tilde\Sigma_4 \cap U$, we have $b_1 \neq 0$.*

Proof. At a point of $\tilde\Sigma_4 \cap U$, the map $\bar T_4(f_t) : \zeta_4/\eta_3 \to \iota_1$ vanishes, and then, by the expression (6.3), so is the map from $\langle \bar v_2+b_1v_1^2\rangle \circ \bar\xi_1$ to ι_1 induced by $\sum_{i=2}^3 (f_t)_i$.

At the point, if $b_1 = 0$ then the restriction of the map Δ to $\langle \bar v_2\rangle \circ \bar\xi_1$ vanishes. This contradicts the fact that Δ is nondegenerate over U. The claim is proved. □

Because $\tilde\Sigma_4$ is the zeros of Φ_4 induced by $\bar T_4(f_t)$, the above argument shows that if the point P is in $\tilde\Sigma_4$ then it lies above Σ_4. Since P is any point of $(\pi_2\circ\pi_4)^{-1}(W \setminus S)$, we see that $\pi_2\circ\pi_4(\tilde\Sigma_4 \setminus \tilde S_4) \subset \Sigma_4$.

The converse also holds. Indeed, for each point Q of $\Sigma_4 \cap W$ there exist vectors $v_1, v_2 \in \xi_1$ at Q satisfying (3.2),(3.3). These determine a point P of $(\pi_2\circ\pi_4)^{-1}(W\setminus S)$ ($\subset \mathbb{P}(\bar\xi_2)$); for such Q, v_1, v_2, the point P is given by $(Q, \xi_1^1, \bar\xi_2^1)$ with $\xi_2^2 = \langle v_1\rangle \oplus \langle v_2+v_1^2\rangle$. By definition of $\tilde\Sigma_4$, this P is in $\tilde\Sigma_4$.

Thus we get that $\pi_2\circ\pi_4(\tilde\Sigma_4 \setminus \tilde S_4) = \Sigma_4$. □

For any point Q of $\Sigma_4 \cap W$, a point P of $\tilde\Sigma_4 \setminus \tilde S_4$ is given as above. By the nondegeneracy of Δ, such a point P is unique; indeed, there exists a unique vector $v_1 \in \xi_1$ at Q satisfying (3. 2) and there exists a unique vector $\bar{v}_2 \in \bar{\xi}_1$ at Q such that ϵ_4 vanishes at P. This implies the existence of the inverse map of $\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4 \setminus \tilde{S}_4} : \tilde{\Sigma}_4 \setminus \tilde{S}_4 \to \Sigma_4$.

Consider the map $\overline{\Delta}$ over $(\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4})^{-1}(W)$, which is the pullback of $\overline{\Delta}$ of §7.1. Then $\tilde{S}_4 \cap (\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4})^{-1}(W)$ consists of the points of $(\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4})^{-1}(W)$ at which $\overline{\Delta}$ is degenerate. As in §7.1, we see that $\tilde{\Sigma}_4 \setminus \tilde{S}_4$ is dense in $\tilde{\Sigma}_4$.

NOTE 7.1. Applying Lemma 4.2 to Y_4, we get that

$$\sigma(Y_4) = 4v_1 \bigcirc (V_3 + \frac{3}{2}v_1 V_2 + \frac{3}{2}v_1 v_2 + v_1^3) + 3(v_2 + v_1^2) \bigcirc (V_2 + v_1^2),$$

which is a representative of an element of $(\xi_1^1 \bigcirc \eta_3 + \xi_2^2 \bigcirc \xi_2)/(\xi_1^1 \bigcirc \xi_2)$. Following this recipe, we defined ζ_4 as in §5.4.

7.3 The case $\tilde{\Sigma}_5$

We will prove that $\pi_2 \circ \pi_4(\tilde{\Sigma}_5 \setminus \tilde{S}_5) = \Sigma_5$, where $\tilde{S}_5 = (\pi_2 \circ \pi_4|_{\tilde{\Sigma}_5})^{-1}(S)$. For this, we shall examine a necessary and sufficient condition for a point of $\tilde{\Sigma}_4 \setminus \tilde{S}_4$ to lie above Σ_5. Note that $\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4 \setminus \tilde{S}_4} : \tilde{\Sigma}_4 \setminus \tilde{S}_4 \to \Sigma_4$ is biholomorphic and its restriction over Σ_5 is a one sheeted cover of Σ_5.

Let W be a small open neighbourhood of $(0, 0)$ in $\mathbb{C}^n \times \mathbb{C}^p$, and choose the local coordinates $(x, t, \xi_1^1, \bar{\xi}_2^1) \in (\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4})^{-1}(W)$ with $\xi_2^2 = \langle v_1 \rangle \oplus \langle v_2 + b_1 v_1^2 \rangle$. Take a point P of $(\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4})^{-1}(W) \setminus \tilde{S}_4$ and let U be an open neighbourhood of P in $(\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4})^{-1}(W) \setminus \tilde{S}_4$. Applying Claim 7.1 to this U, by a linear change of the coordinates v_2, we can assume that $b_1 = 1$ on U.

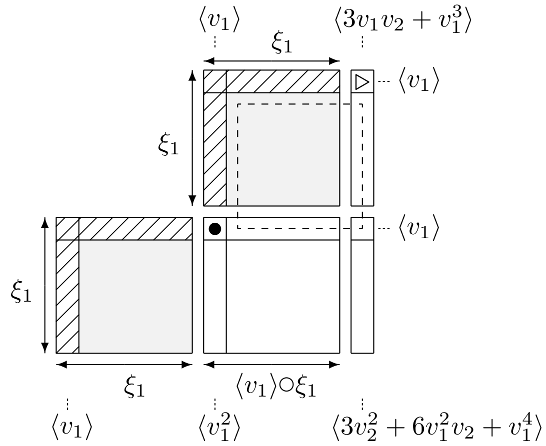
The vectors v_1, v_2 at P satisfy the equations (3. 2), (3. 3), and hence P lies above Σ_5 if and only if there exist vectors $v_3, v_4 \in \xi_1$ at P satisfying the equations (3. 4), (3. 5). Consider the maps

$$(7.3) \qquad \begin{array}{ll} \sum_{i=2}^4 (f_t)_i & : (\xi_1 \oplus \langle 3v_1 v_2 + v_1^3 \rangle) \circ \xi_1 \to \iota_1, \\ \sum_{i=2}^5 (f_t)_i & : (\xi_1 \oplus \langle v_1 \rangle \circ \xi_1 \oplus \langle 3v_2^2 + 6v_1^2 v_2 + v_1^4 \rangle) \circ \xi_1 \to \iota_1 \end{array}$$

over U. The condition for the vectors v_3, v_4 at P means that the restrictions of (7. 3) to the subspaces $\langle v_3 + 3v_1 v_2 + v_1^3 \rangle \circ \xi_1$, $\langle v_4 + 4v_1 v_3 + 3v_2^2 + 6v_1^2 v_2 + v_1^4 \rangle \circ \xi_1$ vanish at P, respectively.

As in the previous sections, the source spaces of (7. 3) can be visualized by a diagram (Figure 3).

Figure 3 The diagram for Σ_5



Since U is contained in $\tilde{\Sigma}_4$, the symbol $\triangleright$ $(= \langle 3v_1^2v_2 + v_1^4\rangle)$ in Figure 3 is mapped by (7. 3) to zero. The other symbols except the dashed box are the same as in Figure 2. The dashed box is the space

$$\left[(\bar{\xi}_1 \oplus \langle 3v_1v_2 + v_1^3\rangle)\circ\bar{\xi}_1\right] \oplus \left[(\langle v_1\rangle\circ\bar{\xi}_1 \oplus \langle 3v_2^2 + 6v_1^2v_2 + v_1^4\rangle)\circ\langle v_1\rangle\right]$$

over U.

Consider the subspace

$$K_5 = \left[\langle \bar{v}_3 + 3v_1v_2 + v_1^3\rangle\circ\bar{\xi}_1\right] \oplus \left[\langle 4v_1\bar{v}_3 + 3v_2^2 + 6v_1^2v_2 + v_1^4\rangle\circ\langle v_1\rangle\right]$$

of the dashed box, and let δ_5 be the map from K_5 to $\iota_1 \oplus \iota_1$ induced by (7. 3). Over U, the map Δ (on the lower left shaded box in Figure 3) is nondegenerate. Hence the condition for v_3, v_4 at P mentioned above is equivalent to the condition that δ_5 vanishes at P. Let ϵ_5 be the map from K_5 to ι_1 induced by $\sum_{i=2}^{5}(f_t)_i$. Then the condition for δ_5 at P is equivalent to the condition that ϵ_5 vanishes at P.

We can consider the space $\bar{\xi}_1\circ\langle \bar{v}_2 + v_1^2\rangle$ as a subspace of the dashed box, and since U is contained in $\tilde{\Sigma}_4$, this space is mapped to zero under the map induced by $\sum_{i=2}^{3}(f_t)_i$. On the other hand, over U the map Δ (on the shaded box in the dashed box) is nondegenerate. Bearing these two facts in mind, by the condition for ϵ_5 at P, we perform the following operations (cf. the argument in §7.1). At each point of U, choose two vectors in K_5 and let $(V_3 + 3v_1v_2 + v_1^3)v_2, (4v_1V_3 + 3v_2^2 + 6v_1^2v_2 + v_1^4)v_1$ be their representatives, where $V_3 \in \xi_1$ is some vector. Insert these representatives to the fourth and fifth rows in Table 3, where $V_4, V_5 \in \xi_1$ are some vectors.

Table 3 The table for Y_5

1	V_5
v_1	$4V_4$
$v_2 + v_1^2$	$6V_3$
$V_3 + 3v_1v_2 + v_1^3$	$4v_2$
$V_4 + 4v_1V_3 + 3v_2^2 + 6v_1^2v_2 + v_1^4$	v_1

Applying Lemma 2.1 (i) to Table 3, we get the vector

$$Y_5 = V_5 + 5v_1V_4 + 10(v_2 + v_1^2)V_3 + (15v_1v_2^2 + 10v_1^3v_2 + v_1^5).$$

Then the point P lies above Σ_5 if and only if Y_5 at P is mapped by $\sum_{i=1}^{5}(f_t)_i$ to zero for all $V_3, V_4, V_5 \in \xi_1$.

By Claim 7.1 we have $b_1 \neq 0$ at every point of U, and hence we can consider Y_5 as a representative of a basis element of ζ_5/ζ_4 in ζ_5 (for the expression (6.6), substitute $b_1 = 1$), and also $\sum_{i=1}^{5}(f_t)_i$ on Y_5 as $T_5(f_t)$ on Y_5. Hence the point P lies above Σ_5 if and only if it is in $\tilde{\Sigma}_5$. Since P is any point of $\tilde{\Sigma}_4 \setminus \tilde{S}_4$, we get that $\pi_2 \circ \pi_4(\tilde{\Sigma}_5 \setminus \tilde{S}_5) = \Sigma_5$. $\square$

Since $\pi_2 \circ \pi_4|_{\tilde{\Sigma}_4 \setminus \tilde{S}_4} : \tilde{\Sigma}_4 \setminus \tilde{S}_4 \to \Sigma_4$ has the inverse map, as in §7.1, there exists the inverse map of $\pi_2 \circ \pi_4|_{\tilde{\Sigma}_5 \setminus \tilde{S}_5} : \tilde{\Sigma}_5 \setminus \tilde{S}_5 \to \Sigma_5$. The density of $\tilde{\Sigma}_5 \setminus \tilde{S}_5$ in $\tilde{\Sigma}_5$ follows from the same argument as in §7.2.

7.4 The case $\tilde{\Sigma}'_5$

In this section, we will prove that $\pi_2 \circ \pi_4 \circ \pi'_5(\tilde{\Sigma}'_5 \setminus \tilde{S}'_5) = \Sigma_5$, where $\tilde{S}'_5 = (\pi_2 \circ \pi_4 \circ \pi'_5|_{\tilde{\Sigma}'_5})^{-1}(S)$. For this, we shall examine a sufficient condition for a point of $\mathbb{P}(\bar{\xi}_3)$ to lie above Σ_5. Note that $\pi_2 \circ \pi_4|_{\tilde{\Sigma}_5 \setminus \tilde{S}_5} : \tilde{\Sigma}_5 \setminus \tilde{S}_5 \to \Sigma_5$ is biholomorphic and the restriction of $\mathbb{P}(\bar{\xi}_3)$ over $\tilde{\Sigma}_5 \setminus \tilde{S}_5$ is a fibre bundle.

Let W be a small open neighbourhood of $(0,0)$ in $\mathbb{C}^n \times \mathbb{C}^p$, and choose the local coordinates $(x, t, \xi_1^1, \bar{\xi}_2^1, \bar{\xi}_3^1) \in (\pi_2 \circ \pi_4 \circ \pi'_5)^{-1}(W)$ $(\subset \mathbb{P}(\bar{\xi}_3))$ with $\xi_3^3 = \langle v_1 \rangle \oplus \langle v_2 + b_1v_1^2 \rangle \oplus \langle v_3 + b_3v_1^2 + b_2(3v_1v_2 + b_1v_1^3) \rangle$. Take a point P of $(\pi_2 \circ \pi_4 \circ \pi'_5)^{-1}(W \setminus S)$ and let U be an open neighbourhood of P in $(\pi_2 \circ \pi_4 \circ \pi'_5)^{-1}(W \setminus S)$. Applying Claim 7.1 to this U, by linear changes of the coordinates v_2, v_3, we can assume that $b_1 = 1, b_3 = 0$ on U.

The vectors v_1, v_2 at P satisfy the equations (3.2), (3.3), and hence P lies above Σ_5 if the vector v_3 at P satisfies the equation (3.4) and there exists a vector $v_4 \in \xi_1$ at P satisfying the equation (3.5).

We again consider the maps and the spaces in the previous section, but over U of this section. The condition for the vectors v_3, v_4 at P is equivalent to the condition that ϵ_5 vanishes at P. Note that the space $\left[\bar{\xi}_1 \circ \langle 4(\bar{v}_2 + v_1^2)\rangle\right] \oplus \langle 4(3v_1v_2 + v_1^3)\bar{v}_2 + 3v_1v_2^2 + 6v_1^3v_2 + v_1^5\rangle$ can be considered as a subspace of the dashed box in Figure 3, and since U lies above $\tilde{\Sigma}_5$, this space is mapped to zero under the map induced by $\sum_{i=2}^{5}(f_t)_i$.

By the condition for ϵ_5 at P, we perform the following operations. At each point of U, choose two vectors in K_5 and let $(v_3 + 3v_1v_2 + v_1^3)V_2, (4v_1v_3 + 3v_2^2 + 6v_1^2v_2 + v_1^4)v_1$ be their representatives, where $V_2 \in \xi_1$ is some vector. Insert these representatives to the fourth and fifth rows in Table 4, where $V_3, V_4, V_5 \in \xi_1$ are some vectors.

Table 4 The table for Y_5'

1	V_5
v_1	$4V_4$
$v_2 + v_1^2$	$6V_3$
$v_3 + 3v_1v_2 + v_1^3$	$4V_2$
$V_4 + 4v_1v_3 + 3v_2^2 + 6v_1^2v_2 + v_1^4$	v_1

Applying Lemma 2.1 (i), we get the vector

$$\begin{aligned} Y_5' &= V_5 + 5v_1V_4 + 6(v_2 + v_1^2)V_3 + 4(v_3 + 3v_1v_2 + v_1^3)V_2 \\ &\quad + (4v_1^2v_3 + 3v_1v_2^2 + 6v_1^3v_2 + v_1^5). \end{aligned}$$

By eliminating the term $3v_1v_2^2$ (see Note 7.2), we rewrite Y_5' in the simpler form

$$\begin{aligned} Y_5'' &= V_5 + 5v_1V_4 + 5(v_2 + v_1^2)V_3' + 5(v_3 + 3v_1v_2 + v_1^3)V_2' + (v_2 + v_1^2)v_3 \\ &\quad - (v_3 + 3v_1v_2 + v_1^3)v_2 + (4v_1^2v_3 + 3v_1v_2^2 + 6v_1^3v_2 + v_1^5) \\ &= V_5 + 5v_1V_4 + 5(v_2 + v_1^2)V_3' + 5(v_3 + 3v_1v_2 + v_1^3)V_2' \\ &\quad + (5v_1^2v_3 + 5v_1^3v_2 + v_1^5), \end{aligned}$$

where $4V_2 = 5V_2' - v_2, 6V_3 = 5V_3' + v_3$. Then the point P lies above Σ_5 if Y_5'' at P is mapped by $\sum_{i=1}^{5}(f_t)_i$ to zero for all $V_2', V_3', V_4, V_5 \in \xi_1$.

By an argument similar to the proof of Claim 7.1, we can prove the following claim.

Claim 7.2. *At every point of $\tilde{\Sigma}_5' \cap U$, we have $b_1b_2 \neq 0$.*

By Claim 7.2, at every point of $\tilde{\Sigma}'_5 \cap U$ we can consider Y''_5 as a representative of a basis element of ω_5/ζ_5 in ω_5 (for the expression (6.8) in the case $v_2 \neq 0$, substitute $b_1 = b_2 = 1, b_3 = 0$), and also $\sum_{i=1}^{5}(f_t)_i$ on Y''_5 as $T_5(f_t)$ on Y''_5. It follows that if the point P is in $\tilde{\Sigma}'_5$ then it lies above Σ_5. Since P is any point of $(\pi_2 \circ \pi_4 \circ \pi'_5)^{-1}(W \setminus S)$, we see that $\pi_2 \circ \pi_4 \circ \pi'_5(\tilde{\Sigma}'_5 \setminus \tilde{S}'_5) \subset \Sigma_5$.

By the same argument as in §7.2, the converse also holds. Thus we get that $\pi_2 \circ \pi_4 \circ \pi'_5(\tilde{\Sigma}'_5 \setminus \tilde{S}'_5) = \Sigma_5$. □

By the nondegeneracy of Δ, as in §7.2, there exists the inverse map of $\pi_2 \circ \pi_4 \circ \pi'_5|_{\tilde{\Sigma}'_5 \setminus \tilde{S}'_5} : \tilde{\Sigma}'_5 \setminus \tilde{S}'_5 \to \Sigma_5$. The density of $\tilde{\Sigma}'_5 \setminus \tilde{S}'_5$ in $\tilde{\Sigma}'_5$ follows from the same argument as in §7.2.

Note 7.2. Based on Y'_5, we can make another expression of ω_5/ζ_5 similar to (6.8), but the calculations to get (6.8) fail. For this, we eliminated the term $3v_1v_2^2$ in Y'_5 as a representative of an element of ω_5/ζ_5 in ω_5.

Note 7.3. When we consider Y''_5 as a representative of a basis element of ω_5/ζ_5 in ω_5, we used not the case $b_1 \neq 0$, but the case $v_2 \neq 0$ of the expression (6.8). The reason for this is that Y''_5 is an element of not a quotient space of ω_5 but ω_5, and the case $v_2 \neq 0$ of the expression (6.8) is closer to ω_5.

7.5 The case $\tilde{\Sigma}_6$

We will prove that $\pi_2 \circ \pi_4 \circ \pi'_5 \circ \rho(\tilde{\Sigma}_6 \setminus \tilde{S}_6) = \Sigma_6$, where $\tilde{S}_6 = (\pi_2 \circ \pi_4 \circ \pi'_5 \circ \rho|_{\tilde{\Sigma}_6})^{-1}(S)$. The map $\pi_2 \circ \pi_4 \circ \pi'_5|_{\tilde{\Sigma}'_5 \setminus \tilde{S}'_5} : \tilde{\Sigma}'_5 \setminus \tilde{S}'_5 \to \Sigma_5$ is biholomorphic. By Claim 7.2, the center of the extra blowup ρ is contained in $\tilde{S}'_5$. (This is seen by using the local coordinates of $\mathbb{P}(\bar{\alpha}_6)$ given in §6.5.) Hence $\pi_2 \circ \pi_4 \circ \pi'_5 \circ \rho|_{\tilde{\Sigma}''_5 \setminus \tilde{S}''_5} : \tilde{\Sigma}''_5 \setminus \tilde{S}''_5 \to \Sigma_5$ is biholomorphic, where $\tilde{S}''_5 = (\pi_2 \circ \pi_4 \circ \pi'_5 \circ \rho|_{\tilde{\Sigma}''_5})^{-1}(S)$, and its restriction over Σ_6 is a one sheeted cover of Σ_6. So we shall examine a necessary and sufficient condition for a point of $\tilde{\Sigma}''_5 \setminus \tilde{S}''_5$ to lie above Σ_6.

Let W be a small open neighbourhood of $(0,0)$ in $\mathbb{C}^n \times \mathbb{C}^p$, and choose the local coordinates $(x, t, \xi_1^1, \bar{\xi}_2^1, \bar{\xi}_3^1, \bar{\alpha}_6^1) \in (\pi_2 \circ \pi_4 \circ \pi'_5 \circ \rho|_{\tilde{\Sigma}''_5})^{-1}(W)$ with $\xi_3^3 = \langle v_1 \rangle \oplus \langle v_2 + b_1 v_1^2 \rangle \oplus \langle v_3 + b_3 v_1^2 + b_2(3v_1v_2 + b_1 v_1^3) \rangle, \bar{\alpha}_6^1 = \langle 3\lambda \bar{X} + 2\mu \bar{Y} \rangle$, where $X = (v_2 + b_1 v_1^2) \bigcirc (3v_2^2 + 6b_1 v_1^2 v_2 + b_1^2 v_1^4), Y = (v_3 + b_3 v_1^2 + b_2(3v_1v_2 + b_1 v_1^3)) \bigcirc (3v_1v_2 + b_1 v_1^3)$, and $\bar{X}, \bar{Y} \in \bar{\alpha}_6$ are the equivalent classes of $X, Y \in \alpha_6$, respectively. Take a point P of $(\pi_2 \circ \pi_4 \circ \pi'_5 \circ \rho|_{\tilde{\Sigma}''_5})^{-1}(W) \setminus \tilde{S}''_5$ and let U be an open neighbourhood of P in $(\pi_2 \circ \pi_4 \circ \pi'_5 \circ \rho|_{\tilde{\Sigma}''_5})^{-1}(W) \setminus \tilde{S}''_5$. Applying Claim 7.2 to this U, by linear changes of the coordinates v_2, v_3, we can assume that $b_1 = b_2 = 1, b_3 = 0$ on U.

The vectors v_1, v_2, v_3 at P satisfy the equations (3.2), (3.3), (3.4), and hence P lies above Σ_6 if and only if there exist vectors $v_4, v_5 \in \xi_1$ at P satisfying the equations (3. 5), (3. 6). Consider the maps

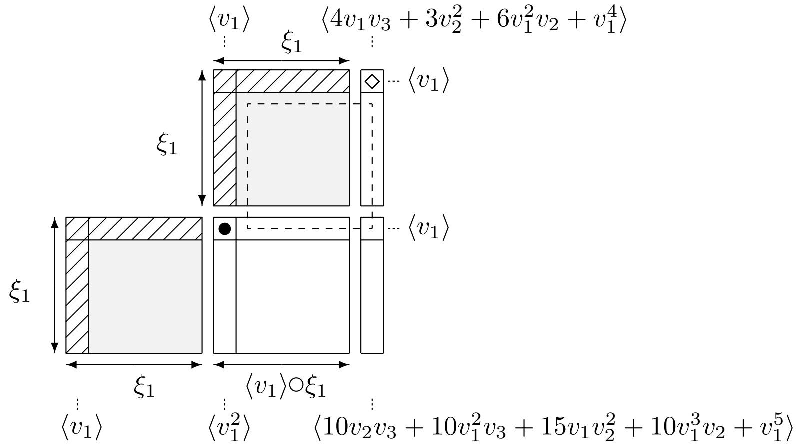
$$\textstyle\sum_{i=2}^{5}(f_t)_i : (\xi_1 \oplus \langle 4v_1v_3 + 3v_2^2 + 6v_1^2v_2 + v_1^4\rangle)\circ\xi_1 \to \iota_1,$$

$$\textstyle\sum_{i=2}^{6}(f_t)_i : \quad (\xi_1 \oplus \langle v_1\rangle\circ\xi_1 \oplus \langle 10v_2v_3 + 10v_1^2v_3 + 15v_1v_2^2 + 10v_1^3v_2 + v_1^5\rangle)\circ\xi_1 \to \iota_1 \tag{7.4}$$

over U. The condition for the vectors v_4, v_5 at P means that the restrictions of (7. 4) to the subspaces $\langle v_4 + 4v_1v_3 + 3v_2^2 + 6v_1^2v_2 + v_1^4\rangle\circ\xi_1$, $\langle v_5 + 5v_1v_4 + 10v_2v_3 + 10v_1^2v_3 + 15v_1v_2^2 + 10v_1^3v_2 + v_1^5\rangle\circ\xi_1$ vanish at P, respectively.

As in the previous sections, the source spaces of (7. 4) can be visualized by a diagram (Figure 4).

Figure 4 The diagram for Σ_6



Since U lies above $\tilde{\Sigma}_5'$, the symbol $\diamond$ $(= \langle 4v_1^2v_3 + 3v_1v_2^2 + 6v_1^3v_2 + v_1^5\rangle)$ in Figure 4 is mapped by (7. 4) to zero (cf. the vector Y_5' in §7.4). The other symbols except the dashed box are the same as in Figure 3. The dashed box is the space

$$\begin{aligned}&\big[(\bar{\xi}_1 \oplus \langle 4v_1v_3 + 3v_2^2 + 6v_1^2v_2 + v_1^4\rangle)\circ\bar{\xi}_1\big]\\ &\quad\oplus \big[(\langle v_1\rangle\circ\bar{\xi}_1 \oplus \langle 10v_2v_3 + 10v_1^2v_3 + 15v_1v_2^2 + 10v_1^3v_2 + v_1^5\rangle)\circ\langle v_1\rangle\big]\end{aligned}$$

over U.

Consider the subspace

$$\begin{aligned}K_6 = &\big[\langle \bar{v}_4 + 4v_1v_3 + 3v_2^2 + 6v_1^2v_2 + v_1^4\rangle\circ\bar{\xi}_1\big]\\ &\quad\oplus \big[\langle 5v_1\bar{v}_4 + 10v_2v_3 + 10v_1^2v_3 + 15v_1v_2^2 + 10v_1^3v_2 + v_1^5\rangle\circ\langle v_1\rangle\big]\end{aligned}$$

of the dashed box, and let δ_6 be the map from K_6 to $\iota_1 \oplus \iota_1$ induced by (7. 4). Over U, the map Δ (on the lower left shaded box in Figure 4) is nondegenerate.

Hence the condition for v_4, v_5 at P mentioned above is equivalent to the condition that δ_6 vanishes at P. Let ϵ_6 be the map from K_6 to ι_1 induced by $\sum_{i=2}^{6}(f_t)_i$. Then the condition for δ_6 at P is equivalent to the condition that ϵ_6 vanishes at P.

For the same reason as in §7.3, we perform the following operations. At each point of U, choose two vectors in K_6 and let $(V_4+4v_1v_3+3v_2^2+6v_1^2v_2+v_1^4)v_2, (5v_1V_4+10v_2v_3+10v_1^2v_3+15v_1v_2^2+10v_1^3v_2+v_1^5)v_1$ be their representatives, where $V_4 \in \xi_1$ is some vector. Insert these representatives to the fifth and sixth rows in Table 5, where $V_3, V_5, V_6 \in \xi_1$ are some vectors.

Table 5 The table for Y_6

1	V_6
v_1	$5V_5$
$v_2+v_1^2$	$10V_4$
$v_3+3v_1v_2+v_1^3$	$10V_3$
$V_4+4v_1v_3+3v_2^2+6v_1^2v_2+v_1^4$	$5v_2$
$V_5+5v_1V_4+10v_2v_3+10v_1^2v_3+15v_1v_2^2+10v_1^3v_2+v_1^5$	v_1

Applying Lemma 2.1 (i) to Table 5, we get the vector

$$\begin{aligned} Y_6 = {} & V_6+6v_1V_5+15(v_2+v_1^2)V_4+10(v_3+3v_1v_2+v_1^3)V_3 \\ & +(30v_1v_2v_3+10v_1^3v_3+15v_2^3+45v_1^2v_2^2+15v_1^4v_2+v_1^6). \end{aligned}$$

Then the point P lies above Σ_6 if and only if Y_6 at P is mapped by $\sum_{i=1}^{6}(f_t)_i$ to zero for all $V_3, V_4, V_5, V_6 \in \xi_1$.

By Claim 7.2, we have $b_1b_2 \neq 0$ at every point of U. On the other hand, since U is contained in $\tilde{\Sigma}_5''$, over U we have $(\lambda,\mu) \neq (0,0)$ and $\lambda b_1 = \mu b_2$. Hence we can consider Y_6 as a representative of a basis element of ω_6/ω_5 in ω_6 (for the expression (6. 10) in the case $v_2 \neq 0$, substitute $b_1=b_2=1, b_3=0, \lambda=\mu=1$), and also $\sum_{i=1}^{6}(f_t)_i$ on Y_6 as $T_6(f_t)$ on Y_6. Hence the point P lies above Σ_6 if and only if it is in $\tilde{\Sigma}_6$. Since P is any point of $\tilde{\Sigma}_5'' \setminus \tilde{S}_5''$, we get that $\pi_2 \circ \pi_4 \circ \pi_5' \circ \rho(\tilde{\Sigma}_6 \setminus \tilde{S}_6) = \Sigma_6$. $\square$

Since $\pi_2 \circ \pi_4 \circ \pi_5' \circ \rho|_{\tilde{\Sigma}_5'' \setminus \tilde{S}_5''} : \tilde{\Sigma}_5'' \setminus \tilde{S}_5'' \to \Sigma_5$ has the inverse map, as in §7.1, there exists the inverse map of $\pi_2 \circ \pi_4 \circ \pi_5' \circ \rho|_{\tilde{\Sigma}_6 \setminus \tilde{S}_6} : \tilde{\Sigma}_6 \setminus \tilde{S}_6 \to \Sigma_6$. The density of $\tilde{\Sigma}_6 \setminus \tilde{S}_6$ in $\tilde{\Sigma}_6$ follows from the same argument as in §7.2.

NOTE 7.4. The center $\{b_1=b_2=0\} \subset \tilde{\Sigma}_5'$ of the extra blowup ρ is the locus corresponding to $\overline{E_7}$; the locus $\{b_1=0\} \subset \tilde{\Sigma}_5'$ is projected onto $\overline{E_6}$ and the locus

$\{b_2 = 0\} \subset \tilde{\Sigma}'_5$ is projected onto $\overline{D_6}$. (For more details, see the next section.) It is worth comparing our locus $b_1 = b_2 = 0$ with Gaffney's locus $B_1 = B_2 = 0$ in [4, Proposition 1.6].

8. Appendix

Desingularizations of the closures $\overline{D_5}, \overline{E_6}, \overline{D_6}, \overline{E_7}, \overline{E_8}, \overline{X_9}$ are given by restricting the desingularizations $\tilde{\Sigma}_4, \tilde{\Sigma}_5, \tilde{\Sigma}'_5, \tilde{\Sigma}_6$ to some loci. In this appendix we will explain these facts.

In the local coordinates of $\mathbb{P}(\bar{\xi}_2)$ given in §6.2, the locus $\{b_1 = 0\} \subset \tilde{\Sigma}_4$ gives a desingularization of $\overline{D_5}$. Indeed, by Claim 7.1, for every point $P \in \tilde{\Sigma}_4$ with $b_1 = 0$ its image $\pi_2 \circ \pi_4(P)$ is in S $(= \cup_{i\geq 2}\Sigma^{n,i}F)$. Moreover, the following claim holds.

Claim 8.1. *For every point $P \in \tilde{\Sigma}_4$ with $b_1 = 0$ its image $\pi_2 \circ \pi_4(P)$ is in $\overline{D_5}$.*

Proof. At a point $P \in \tilde{\Sigma}_4$ with $b_1 = 0$, the map $T_4(f_t)$ vanishes on

$$\zeta_4 = \xi_1 \oplus \langle v_1 \rangle \circ \xi_1 \oplus \langle v_1^3 \rangle \oplus \langle v_2 \rangle \circ \bar{\xi}_1 \oplus \langle 3v_1^2 v_2 \rangle$$

(see the expression (6. 3)). Replace the local coordinates (x, t) of $\mathbb{C}^n \times \mathbb{C}^p$ by the local coordinates centered at $\pi_2 \circ \pi_4(P)$ with $v_1 = \frac{\partial}{\partial x_1}, v_2 = \frac{\partial}{\partial x_2}$. Then at $\pi_2 \circ \pi_4(P)$, the function f_t does not have the terms involving monomials

$$x_i \ (i = 1, \dots, n),\ x_1 x_i \ (i = 1, \dots, n),\ x_1^3,\ x_2 x_i \ (i = 2, \dots, n),\ x_1^2 x_2,$$

that is, it is written in the form

$$f_t(x) = a_2 x_1 x_2^2 + a_3 x_2^3 + b_{11} x_1^4 + \sum_{i=3}^{n} 2b_{1i} x_1^2 x_i + \sum_{i,j=3}^{n} b_{ij} x_i x_j + \cdots,$$

where $a_i, b_{ij} \in \mathbb{C}\{t\}$ (cf. [5, Proof of Lemma 2.3.2]). It follows from this that $\pi_2 \circ \pi_4(P)$ is in $\overline{D_5}$. □

Then the almost same argument as for Theorem 5.3 shows that

(i) in the local coordinates of $\mathbb{P}(\bar{\xi}_2)$ given in §6.2, the restriction of $\pi_2 \circ \pi_4$ to the locus $\{b_1 = 0\} \cap \tilde{\Sigma}_4$ is a desingularization of $\overline{D_5}$.

Similarly, we can show the following facts:

(ii) In the local coordinates of $\tilde{\Sigma}_4$ given in §6.3, the restriction of $\pi_2 \circ \pi_4$ to the locus $\{b_1 = 0\} \cap \tilde{\Sigma}_5$ is a desingularization of $\overline{E_6}$.

(iii) In the local coordinates of $\mathbb{P}(\bar{\xi}_3)$ given in §6.4, the restriction of $\pi_2 \circ \pi_4 \circ \pi'_5$ to the locus $\{b_2 = 0\} \cap \tilde{\Sigma}'_5$ is a desingularization of $\overline{D_6}$, and its restriction to the locus $\{b_1 = b_2 = 0\} \cap \tilde{\Sigma}'_5$ is a desingularization of $\overline{E_7}$.

(iv) In the local coordinates of $\tilde{\Sigma}''_5$ given in §6.6, the restriction of $\pi_2 \circ \pi_4 \circ \pi'_5 \circ \rho$ to the locus $\{b_1 = \lambda/\mu = 0\} \cap \tilde{\Sigma}_6$ is a desingularization of $\overline{E_8}$, and to the locus $\{b_2 = \mu/\lambda = 0\} \cap \tilde{\Sigma}_6$ is a desingularization of $\overline{X_9}$.

These loci are defined in terms of local coordinates. However all can be defined independently of coordinates.

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