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A PROBLEM DUAL TO SOLVING THE LINEAR DIFFERENTIAL EQUATION

By

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Abstract. A problem which is dual to solving the linear ordinary differential equation is considered. That is, given a function $f(x)$, a generalized differential operator $\varphi(d/dx)$ is found so as to satisfy $\varphi(d/dx)f(x)\theta(x) = \delta(x)$, where $\theta(x)$ and $\delta(x)$ stand for the Heaviside function and the Dirac measure, respectively. The general formulae are presented for $\varphi(d/dx)$ and for its representation in terms of the convolution integral, which is what was previously proposed by modifying the framework of Mikusiński's operational calculus. Various explicit examples are considered. Especially, the case of $f(x) = (\log x)^m$, m being a positive integer, is analyzed completely, that is, the convolution representation formula for $\varphi(d/dx)$ is explicitly calculated and it is confirmed that $\varphi(d/dx)(\log x)^m\theta(x) = \delta(x)$ is satisfied in terms of the convolution integral.

1. Introduction

To solve a linear ordinary differential equation, $\varphi(D, x)f(x) = 0 \quad (D \equiv d/dx)$, amounts to find $f(x)$ for a given differential operator $\varphi(D, x)$. We wish to consider a problem which is dual to this problem, that is, to find such a "differential operator" that the corresponding equation is satisfied by a *given* function $f(x)$. But, in order for our problem to be well-posed, the setup of the problem needs some arrangements.

If we admit the "differential operator" which depends not only on D but also on x explicitly, there always exists a trivial solution and independent solutions are of formidable variety. Thus this situation is uninteresting. Hence we exclude such a general case, that is, we restrict ourselves to considering the situation in which the "differential operator" is independent of x. Then, if the "differential operator" $\varphi(D)$ is restricted to a genuine (local) differential operator, the solution will not exist for most cases. Hence, we should admit more general functions of D, such as constant coefficient pseudo-differential operators, that is, we suppose

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that the function φ is arbitrary in principle. We call $\varphi(D)$ for an arbitrary function φ [1] a generalized differential operator. If we set up the problem in the sense of $\varphi(D)f(x) = 0$, it is inconvenient because if a solution $\varphi(D)$ is found, $\phi(D)\varphi(D)$ is also a solution for any well-defined $\phi(D)$. Hence, we normalize the solution by requiring $\varphi(D)f(x) = \delta(x)$, where $\delta(x)$ stands for the Dirac measure. Furthermore, in order to avoid the ambiguity caused by adding solutions of the corresponding homogeneous equation, we restrict the function considered to the one whose support is included in $\mathbb{R}_+ \equiv \{x | x \geqq 0\}$. That is, we consider $f(x)\theta(x)$ rather than $f(x)$, where $\theta(x) = 1$ for $x \geqq 0$, $\theta(x) = 0$ for $x < 0$.

From the above consideration, we set up our problem in the following way. [2]

PROBLEM 1.1. *Given a well-behaved function $f(x)$, find the generalized differential operator $\varphi(D)$ satisfying*

$$\varphi(D)f(x)\theta(x) = \delta(x). \tag{1.1}$$

EXAMPLE. If $f(x) = e^{ax}$, then $\varphi(D) = D - a$ is the solution to (1.1). Note that the right-hand side of (1.1) arises from $D\theta(x) = \delta(x)$.

As a concrete framework of representing the generalized differential operator, we adopt our previous formulation based on the convolution [3], [3] which is a slight modification of the framework of Mikusiński's operational calculus [2]. That is, for an arbitrary C^1-class function $F(x)$ whose support is included in $\mathbb{R}_+$, we represent $\varphi(D)$ in the following way:

$$\varphi(D)F(x) \equiv \int_{-0}^{x+0} dy\, \Omega(x-y)F'(y). \tag{1.2}$$

Here, Ω is, in general, a distribution in the sense of Schwartz [4]. But, in particular, if it is a C^1-class function whose support is included in $\mathbb{R}_+$ and if $\Omega(+0) = 0$, then (1.2) reduces to Mikusiński's formula,

$$\varphi(D)F(x) = \int_0^x dy\, \Phi(x-y)F(y), \tag{1.3}$$

by setting $\Phi = \Omega'$.

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[1] The function φ which we actually consider is such that the corresponding operator $\varphi(D)$ is definable by (1.2).

[2] Extension of (1.1) to the case of several-variables is straightforward:
For $x = (x_1, x_2, \cdots, x_n)$ and $D = (D_1, D_2, \cdots D_n)$, we have only to set $\theta(x) \equiv \prod_j \theta(x_j)$ and $\delta(x) \equiv \prod_j \delta(x_j)$.

[3] Hereafter, we quote Ref. [3] as I.

Example. (Riemann-Liouville)

$$D^{-\alpha}F(x) = \int_0^x dy\, \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)} F(y) \quad (\Re\alpha > 0). \tag{1.4}$$

As is shown in I, it is convenient to introduce Schwartz's pseudo-function [4]:[4)]

$$Y_\lambda(x) = \begin{cases} \mathrm{Pf.}\dfrac{x^{\lambda-1}}{\Gamma(\lambda)}\theta(x) & \text{for } \lambda \neq 0, -1, -2, \cdots \\ \delta^{(n)}(x) & \text{for } \lambda = -n = 0, -1, -2, \cdots . \end{cases}$$

If we introduce a test function $g(x)$ according to the definition of the distribution, $\int dx\ g(x)Y_\lambda(x)$ is an entire function of λ. Therefore, the Riemann-Liouville formula (1.4) can be extended to an arbitrary complex value of α. That is, corresponding to (1.2), we obtain

$$D^{-\alpha}F(x) = \int_{-0}^{x+0} dy\, Y_{\alpha+1}(x-y)F'(y). \tag{1.5}$$

The main results of the present paper are as follows. In §2, we show that the solution to the above problem is given by

$$\varphi(D) = \frac{1}{\mathcal{L}[f](D)},$$

where $\mathcal{L}[\cdot]$ stands for the Laplace transform. Furthermore, it is shown that this solution can be represented in the form of (1.2), that is, we obtain the following identity:

$$\int_{-0}^{x+0} dy\, \mathcal{L}^{-1}\Big[\frac{p^{-1}}{\mathcal{L}[f](p)}\Big](x-y)\theta(x-y)\frac{d}{dy}[f(y)\theta(y)] = \delta(x),$$

where $\mathcal{L}^{-1}[\cdot]$ stands for the inverse Laplace transform. In §3, we discuss the implication of the results presented in §2 by various concrete examples. In §4, we explicitly demonstrate that the above identity indeed holds for $f(x) = (\log x)^m$, m being a positive integer.

2. General consideration

The Laplace transform of $F(x)$ is defined by

$$G(p) = \mathcal{L}[F](p) \equiv \int_0^\infty dx\, e^{-px}F(x) \quad (\Re p > 0), \tag{2.1}$$

[4)] Pf. means Hadamard's finite part; see Section II-2 of Ref. [4]. We omit writing Pf. in what follows.

and the inverse Laplace transform of $G(p)$ is given by

$$F(x) = \mathcal{L}^{-1}[G](x) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dp\, e^{px} G(p). \tag{2.2}$$

Theorem 1. *The formal solution to the problem (1.1) is given by* $\varphi(D) = \dfrac{1}{\mathcal{L}[f](D)}$, *that is,*

$$\frac{1}{\mathcal{L}[f](D)} f(x)\theta(x) = \delta(x). \tag{2.3}$$

Proof. We see that (2.1) implies $\mathcal{L}[F(x)\theta(x)](p) = \mathcal{L}[F(x)](p)$; (2.2) yields $\varphi(D)\mathcal{L}^{-1}[G(p)](x) = \mathcal{L}^{-1}[\varphi(p)G(p)](x)$, and $\delta(x) = \mathcal{L}^{-1}[1](x)$. Hence, (1.1) can be rewritten as

$$\varphi(D)\mathcal{L}^{-1}[\mathcal{L}[f\theta](p)](x) = \mathcal{L}^{-1}[\varphi(p)\mathcal{L}[f](p)](x) = \mathcal{L}^{-1}[1](x). \tag{2.4}$$

Taking the Laplace transforms of both sides, we find

$$\varphi(p)\mathcal{L}[f](p) = 1. \tag{2.5}$$

From (2.5), we obtain

$$\varphi(D) = \frac{1}{\mathcal{L}[f](D)}.$$

This is the formal solution to (1.1). □

Theorem 2. *The convolution representation of (2.3) is given by*

$$\int_{-0}^{x+0} dy\, \mathcal{L}^{-1}\Big[\frac{p^{-1}}{\mathcal{L}[f](p)}\Big](x-y)\theta(x-y)\frac{d}{dy}[f(y)\theta(y)] = \delta(x). \tag{2.6}$$

Proof. Multiplying both sides of (2.3) by $\mathcal{L}[f](D)$ and using $\delta(x) = D\theta(x)$, we have

$$D\mathcal{L}[f](D)\theta(x) = f(x)\theta(x). \tag{2.7}$$

Since this formula holds for an arbitrary $f(x)$, we may set

$$f(x) = \mathcal{L}^{-1}[p^{-1}\varphi(p)](x).$$

Then (2.7) becomes

$$\varphi(D)\theta(x) = \mathcal{L}^{-1}[p^{-1}\varphi(p)](x)\theta(x). \tag{2.8}$$

Using (2.5), we rewrite (2.8) as

$$\varphi(D)\theta(x) = \mathcal{L}^{-1}\Big[\frac{p^{-1}}{\mathcal{L}[f](p)}\Big](x)\theta(x). \tag{2.9}$$

On the other hand, (1.2) with $F(x) = \theta(x)$ yields

$$\Omega(x) = \varphi(D)\theta(x). \tag{2.10}$$

Substituting (2.9) into (2.10), we obtain

$$\Omega(x) = \mathcal{L}^{-1}\Big[\frac{p^{-1}}{\mathcal{L}[f](p)}\Big](x)\theta(x). \tag{2.11}$$

By setting $F(x) = f(x)\theta(x)$ in (1.2), (1.1) and (2.11) yield (2.6). □

Example. For $f(x) = e^{ax}$, we have $\mathcal{L}[e^{ax}](p) = (p-a)^{-1}$. Then (2.11) implies $\Omega(x) = \mathcal{L}^{-1}[1 - ap^{-1}](x)\theta(x) = \delta(x) - a\theta(x)$. Hence

$$\int_{-0}^{x+0} dy\;[\delta(x-y) - a\theta(x-y)]\frac{d}{dy}[e^{ay}\theta(y)] = ae^{ax}\theta(x) + e^{ax}\delta(x) - ae^{ax}\theta(x) = \delta(x).$$

Thus (2.6) is indeed satisfied.

In the above example, the genuine Laplace transform of e^{ax} exists only for $\Re a < \Re p$. But we can remove this restriction by analytic continuation with respect to a. We always make such extension of the Laplace transform as long as possible.

Our problem is nonlinear in the sense that the solution for $f(x) = c_1 f_1(x) + c_2 f_2(x)$ is not a linear combination of those for $f_1(x)$ and $f_2(x)$.

Example. For $f(x) = \cosh ax$, we have $\mathcal{L}[\cosh ax](p) = p(p^2 - a^2)^{-1}$. Then (2.11) implies $\Omega(x) = \mathcal{L}^{-1}[1 - a^2p^{-2}](x)\theta(x) = \delta(x) - a^2 x\theta(x)$. Hence

$$\int_{-0}^{x+0} dy\;[\delta(x-y) - a^2(x-y)\theta(x-y)]\frac{d}{dy}[e^{\pm ay}\theta(y)] = \delta(x) \pm a\theta(x),$$

so that we have $\delta(x)$ for putting $\cosh ay$ in place of $e^{\pm ay}$ in the integrand.

3. Concrete examples of Theorem 1

The above consideration is heavily based on the Laplace transforms. From the well-known table of Laplace transforms [1],[5] we quote the following formulae:

$$\mathcal{L}[x^\alpha](p) = \Gamma(\alpha+1)p^{-\alpha-1} \quad (\Re\alpha > -1,\ \Re p > 0); \tag{3.1}$$

$$\mathcal{L}[\log x](p) = -p^{-1}\log(e^\gamma p) \quad (\Re p > 0), \tag{3.2}$$

$$\mathcal{L}[(\log x)^2](p) = p^{-1}[\pi^2/6 + \{\log(e^\gamma p)\}^2] \quad (\Re p > 0); \tag{3.3}$$

$$\mathcal{L}[\mu(x, \alpha-1)](p) = \Gamma(\alpha)p^{-1}(\log p)^{-\alpha} \quad (\Re\alpha > 0,\ \Re p > 1), \tag{3.4}$$

$$\mathcal{L}[\nu(x,\alpha)](p) = p^{-\alpha-1}(\log p)^{-1} \quad (\Re\alpha > -1,\ \Re p > 1), \tag{3.5}$$

where

$$\mu(x,\alpha) \equiv \int_0^\infty ds\, \frac{x^s s^\alpha}{\Gamma(s+1)}, \tag{3.6}$$

$$\nu(x,\alpha) \equiv \int_0^\infty ds\, \frac{x^{s+\alpha}}{\Gamma(s+\alpha+1)}, \tag{3.7}$$

$$\mu(x,0) = \nu(x,0) = \nu(x) \equiv \int_0^\infty ds\, \frac{x^s}{\Gamma(s+1)}.$$

First, applying (3.1) to (2.3), we obtain the well-known result,

$$D^{\alpha+1}Y_{\alpha+1}(x) = \delta(x),$$

which follows from (1.6) with $F(x) = \theta(x)$.

Next, applying (3.2) and (3.3) to (2.7), we obtain

$$-(\gamma + \log D)\theta(x) = (\log x)\,\theta(x), \tag{3.8}$$

$$[\zeta(2) + (\gamma+\log D)^2]\theta(x) = (\log x)^2\theta(x), \tag{3.9}$$

respectively, where $\pi^2/6 = \zeta(2)$ has been used. Replacing x by $e^\gamma x$, writing $\gamma + \log x = L(x)$ and then solving (3.8) and (3.9) with respect to $\log D$, we obtain [I, (3.2)] and [I, (3.3)], respectively.

Conversely, therefore, we solve the general formula for $(\log D)^m\theta(x)$ presented in [I, (3.7)], namely,

$$(\log D)^m\theta(x)$$
$$= \Big(\sum_{l=0}^{[m/2]} \frac{1}{l!} \sum_{k_1\geqq 2,\cdots,k_l\geqq 2} (-1)^{m-|k|-l} \frac{m! \prod_{j=1}^l \zeta(k_j)}{\prod_{j=1}^l k_j \cdot (m-|k|)!} L^{m-|k|}(x)\Big)\theta(x), \tag{3.10}$$

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[5] See p.137, p.148, p.149, p.225, p.225 of Ref.[1], respectively. Note that our "γ" stands for the Euler constant $\gamma = \lim_{n\to\infty}(\sum_{k=1}^n \frac{1}{k} - \log n)$ but not its exponentiated one. See also p.388 of Ref.[1].

where $|k| \equiv \sum_{j=1}^{l} k_j$, with respect to $L(x)$. Then the solution is seen to be

$$L^m(x)\theta(x) = (-1)^m \Big(\sum_{l=0}^{[m/2]} \frac{1}{l!} \sum_{k_1 \geqq 2, \cdots, k_l \geqq 2} \frac{m! \prod_{j=1}^{l} \zeta(k_j)}{\prod_{j=1}^{l} k_j \cdot (m-|k|)!} (\log D)^{m-|k|} \Big) \theta(x). \tag{3.11}$$

This can be shown as follows.

Substituting (3.10) with replacement of m by $m-|k|$ into the right-hand side of (3.11) and then canceling the factors $(-1)^m$ and $(m-|k|)!$, we find[6)]

$$L^m(x) \stackrel{?}{=} \sum_{l=0}^{[m/2]} \frac{1}{l!} \sum_{k_j \geqq 2;\, 1 \leqq j \leqq l} \frac{m! \prod_{j=1}^{l} \zeta(k_j)}{\prod_{j=1}^{l} k_j} \cdot \sum_{l'=0}^{[(m-|k|)/2]} \frac{(-1)^{|k|+|k'|+l'}}{l'!} \sum_{k'_{j'} \geqq 2;\, 1 \leqq j' \leqq l'} \frac{\prod_{j'=1}^{l'} \zeta(k'_{j'})}{\prod_{j'=1}^{l'} k'_{j'} \cdot (m-|k|-|k'|)!} L^{m-|k|-|k'|}(x), \tag{3.12}$$

where $\theta(x)$ on both sides has been omitted. By setting $\tilde{l} = l + l'$, $\tilde{k}_j = k_j$ $(1 \leqq j \leqq l)$ and $\tilde{k}_{l+j'} = k'_{j'}$ $(1 \leqq j' \leqq l')$, toghther with $|\tilde{k}| = |k| + |k'|$, (3.12) becomes

$$L^m(x) \stackrel{?}{=} \sum_{\tilde{l}=0}^{[m/2]} \sum_{l'=0}^{\tilde{l}} \frac{(-1)^{l'}}{(\tilde{l}-l')!\, l'!} \sum_{\tilde{k}_j \geqq 2;\, 1 \leqq j \leqq \tilde{l}} \frac{(-1)^{|\tilde{k}|} m! \prod_{j=1}^{\tilde{l}} \zeta(\tilde{k}_j)}{\prod_{j=1}^{\tilde{l}} \tilde{k}_j \cdot (m-|\tilde{k}|)!} L^{m-|\tilde{k}|}(x). \tag{3.13}$$

Since $\sum_{l'=0}^{\tilde{l}} \frac{(-1)^{l'}}{(\tilde{l}-l')!\, l'!} = \frac{(1-1)^{\tilde{l}}}{\tilde{l}!} = \delta_{\tilde{l}\,0}$, we see that the right-hand side of (3.13) is indeed equal to $L^m(x)$. □

On the right-hand side of (3.11), if we replace $\log D$ by $\gamma + \log D$, the formula for $(\log x)^m$ is obtained. Removing $\theta(x)$, replacing D by p and then multiplying the resultant by p^{-1}, we should obtain the Laplace transform of $(\log x)^m$ owing to (2.7). That is, we should have

$$\mathcal{L}[(\log x)^m](p) = (-1)^m p^{-1} \sum_{l=0}^{[m/2]} \frac{1}{l!} \sum_{k_1 \geqq 2, \cdots, k_l \geqq 2} \frac{m! \prod_{j=1}^{l} \zeta(k_j)}{\prod_{j=1}^{l} k_j \cdot (m-|k|)!} (\gamma + \log p)^{m-|k|}. \tag{3.14}$$

This is indeed the case, as is shown in the following way.

6) The symbol $\stackrel{?}{=}$ means the equality to be established.

We consider the generating functions of both sides of (3.14). The generating function of the left-hand side is

$$\mathcal{L}\Big[\sum_{m=0}^{\infty}\frac{\alpha^m}{m!}(\log x)^m\Big](p)=\mathcal{L}[x^\alpha](p)=\Gamma(\alpha+1)p^{-\alpha-1}.$$

On the other hand, the generating function of the right-hand side of (3.14), which is denoted by

$$K(p;\alpha)\equiv\sum_{m=0}^{\infty}\frac{\alpha^m}{m!}[\text{r.h.s. of (3.14)}],$$

becomes as follows. Setting $m-|k|=n$, we change the summation over m into that over n, and then perform calculation similar to the one done in I:

$$\begin{aligned}K(p;\alpha)&=p^{-1}\sum_{n=0}^{\infty}\frac{[-\alpha(\gamma+\log p)]^n}{n!}\sum_{l=0}^{\infty}\frac{1}{l!}\Big(\sum_{k\geqq 2}(-1)^k\frac{\zeta(k)}{k}\alpha^k\Big)^l\\&=p^{-1}\exp\Big(-\alpha(\gamma+\log p)+\sum_{k\geqq 2}\frac{\zeta(k)}{k}(-\alpha)^k\Big)\\&=p^{-1}\exp[-\alpha\log p+\log\Gamma(\alpha+1)]\\&=\Gamma(\alpha+1)p^{-\alpha-1}.\end{aligned}$$

Thus the generating functions of both sides coincide. □

Digression. A similar calculation based on the generating function yields

$$\sum_{l=0}^{[m/2]}\frac{1}{l!}\sum_{k_1\geqq 2,\cdots,k_l\geqq 2}\frac{1}{\prod_{j=1}^{l}k_j\cdot(m-|k|)!}=1.$$

Finally, applying (3.4) and (3.5) to (2.7), we obtain

$$(\log D)^{-\alpha}\theta(x)=\frac{\mu(x,\alpha-1)}{\Gamma(\alpha)}\theta(x), \tag{3.15}$$

$$D^{-\alpha}(\log D)^{-1}\theta(x)=\nu(x,\alpha)\theta(x), \tag{3.16}$$

respectively. Substitutions of (3.6) into (3.15) and of (3.7) into (3.16) yield

$$\begin{aligned}(\log D)^{-\alpha}\theta(x)&=\int_0^\infty dt\ Y_\alpha(t)Y_{t+1}(x),\\D^{-\alpha}(\log D)^{-1}\theta(x)&=\int_0^\infty dt\ Y_{t+\alpha+1}(x),\end{aligned}$$

respectively, where use has been made of $Y_\lambda(x)$ defined in (1.5). They are nothing but two particular cases of [I, (5.10)], namely,

$$D^\alpha(\log D)^\beta\theta(x) = \int_0^\infty dt\, Y_{-\beta}(t)Y_{-\alpha+t+1}(x).$$

4. Concrete examples of Theorem 2

In this section, we directly confirm (2.6) for $f(x) = (\log x)^m$.

4.1 The case of $\log x$

From (3.2), we have

$$\frac{p^{-1}}{\mathcal{L}[\log x](p)} = -\frac{1}{\gamma + \log p}. \tag{4.1}$$

On the other hand, the inverse Laplace transform of (3.5) with $\alpha = 0$ yields

$$\mathcal{L}^{-1}\Big[\frac{1}{p\log p}\Big](x) = \int_0^\infty ds\, \frac{x^s}{\Gamma(s+1)}. \tag{4.2}$$

Extending (4.2) in the sense of the distribution and multiplying it by D, we have

$$\mathcal{L}^{-1}\Big[\frac{1}{\log p}\Big](x)\theta(x) = \int_0^\infty ds\, Y_s(x). \tag{4.3}$$

By scale transformation, (4.3) becomes

$$\mathcal{L}^{-1}\Big[\frac{1}{\gamma + \log p}\Big](x)\theta(x) = e^{-\gamma}\int_0^\infty ds\, Y_s(e^{-\gamma}x). \tag{4.4}$$

Thus, from (4.1) and (4.4), we obtain[7)]

$$\Omega_{\log}(x) \equiv \mathcal{L}^{-1}\Big[\frac{p^{-1}}{\mathcal{L}[\log](p)}\Big](x)\theta(x) = -e^{-\gamma}\int_0^\infty ds\, Y_s(e^{-\gamma}x). \tag{4.5}$$

Now, from (2.6), what we should calculate is

$$H_{\log}(x) \equiv \int_{-0}^{x+0} dy\, \Omega_{\log}(x-y)\frac{d}{dy}(\log y\,\theta(y)). \tag{4.6}$$

[7)] Note that since $\Omega_{\log}(+0)$ does not exist, we cannot construct the corresponding $\Phi_{\log}(x)$ of the Mikusiński theory (see (1.3)).

Substituting (4.5) into (4.6), we have

$$H_{\log}(x) = -\int_0^\infty ds\, \frac{e^{-\gamma s}}{\Gamma(s)} I_1(x;s), \tag{4. 7}$$

where

$$I_1(x;s) \equiv \int_{-0}^{x+0} dy\, (x-y)^{s-1}\theta(x-y)\frac{d}{dy}(\log y\, \theta(y)). \tag{4. 8}$$

By setting $y = xt$, (4.8) becomes

$$I_1(x;s) = x^{s-1}(\log x + J_1(s))\theta(x) \tag{4. 9}$$

with

$$\begin{aligned} J_1(s) &\equiv \int_{-0}^{1} dt\, (1-t)^{s-1}\frac{d}{dt}(\log t\, \theta(t)) \\ &= \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{-0}^{1} dt\, (1-t)^{s-1}\frac{d}{dt}(t^\epsilon\, \theta(t)). \end{aligned}$$

After integration by parts, we encouter a beta-function integral. In this way, we obtain

$$J_1(s) = -\gamma - \psi(s), \tag{4. 10}$$

where $\psi(s) \equiv \Gamma'(s)/\Gamma(s)$ is the digamma function. Substituting (4.9) with (4.10) into (4.7), we have

$$H_{\log}(x) = -\int_0^\infty ds\, \frac{e^{-\gamma s}x^{s-1}}{\Gamma(s)}(\log x - \gamma - \psi(s))\theta(x).$$

This integral is easily calculated as follows:

$$H_{\log}(x) = -e^{-\gamma}\int_0^\infty ds\, \frac{\partial}{\partial s}Y_s(e^{-\gamma}x) = e^{-\gamma}Y_0(e^{-\gamma}x) = \delta(x).$$

Thus the right-hand side of (2.6) for $f(x) = \log x$ is reproduced.

4.2 The case of $(\log x)^2$

From (3.3), we have

$$\begin{aligned} \frac{p^{-1}}{\mathcal{L}[(\log x)^2](p)} &= \frac{1}{(\log e^\gamma p)^2 + \zeta(2)} \\ &= -\frac{1}{\alpha - \overline{\alpha}}\Big(\frac{1}{\log e^\alpha p} - \frac{1}{\log e^{\overline{\alpha}} p}\Big), \end{aligned}$$

where $\alpha = \gamma + i\sqrt{\zeta(2)}$. Hence, in the same way as in the case of $\log x$, we obtain

$$\Omega_{\log^2}(x) \equiv \mathcal{L}^{-1}\Big[\frac{p^{-1}}{\mathcal{L}[\log^2](p)}\Big](x)\theta(x) = -\frac{1}{\alpha - \overline{\alpha}}\int_0^\infty ds\ (e^{-\alpha s} - e^{-\overline{\alpha}s})Y_s(x).$$

What we should calculate is

$$\begin{aligned} H_{\log^2}(x) &\equiv \int_{-0}^{x+0} dy\ \Omega_{\log^2}(x-y)\frac{d}{dy}((\log y)^2\theta(y)) \\ &= -\frac{1}{\alpha - \overline{\alpha}}\int_0^\infty ds\ \frac{e^{-\alpha s} - e^{-\overline{\alpha}s}}{\Gamma(s)} I_2(x;s), \end{aligned} \tag{4.11}$$

where

$$I_2(x;s) \equiv \int_{-0}^{x+0} dy\ (x-y)^{s-1}\theta(x-y)\frac{d}{dy}((\log y)^2\ \theta(y)). \tag{4.12}$$

By setting $y = xt$, (4.12) becomes

$$I_2(x;s) = x^{s-1}((\log x)^2 + 2(\log x)J_1(s) + J_2(s))\theta(x), \tag{4.13}$$

where $J_1(s)$ is given by (4.10) and

$$\begin{aligned} J_2(s) &\equiv \int_{-0}^{1} dt\ (1-t)^{s-1}\frac{d}{dt}((\log t)^2\ \theta(t)) \\ &= \Gamma(s)\lim_{\epsilon\to 0}\Big(\frac{\partial}{\partial\epsilon}\Big)^2\frac{\Gamma(1+\epsilon)}{\Gamma(s+\epsilon)} \\ &= (\psi(s))^2 + 2\gamma\psi(s) - \psi'(s) + \gamma^2 + \zeta(2) \\ &= (\alpha + \psi(s))(\overline{\alpha} + \psi(s)) - \psi'(s). \end{aligned} \tag{4.14}$$

Substituting (4.10) and (4.14) into (4.13), we obtain

$$I_2(x;s) = x^{s-1}[(\log x - \alpha - \psi(s))(\log x - \overline{\alpha} - \psi(s)) - \psi'(s)]\theta(x). \tag{4.15}$$

Substituting (4.15) into (4.11), we find

$$\begin{aligned} H_{\log^2}(x) = &-\frac{1}{\alpha - \overline{\alpha}}\int_0^\infty ds\ \frac{\partial}{\partial s}\Big[\frac{x^{s-1}e^{-\alpha s}}{\Gamma(s)}(\log x - \overline{\alpha} - \psi(s)) \\ &\qquad - \frac{x^{s-1}e^{-\overline{\alpha}s}}{\Gamma(s)}(\log x - \alpha - \psi(s))\Big]\theta(x) \\ = &\ \frac{1}{\alpha - \overline{\alpha}}\lim_{s\to 0} Y_s(x)[e^{-\alpha s}(\log x - \overline{\alpha} - \psi(s)) - e^{-\overline{\alpha}s}(\log x - \alpha - \psi(s))]. \end{aligned} \tag{4.16}$$

If $\psi(s)$ had no singularity at $s = 0$, this limit would be $Y_0(x) = \delta(x)$. Actually, however, we know that $\psi(s)$ behaves like $-1/s$ near $s = 0$; hence this reasoning is wrong. But if this singularity were naively taken into account, we would encounter an extra contribution of $-\delta(x)$, so that (4.16) would become equal to 0 (incorrectly!). The problem essentially arises in the following way: If one applies the usual rule for a limit of a product (i.e., $\lim AB = \lim A \cdot \lim B$ if both $\lim A$ and $\lim B$ exist) to

$$\lim_{s\to 0} s\frac{\partial}{\partial s}Y_s(x) = \lim_{s\to 0} s(\log x - \psi(s)) \cdot Y_s(x), \tag{4.17}$$

then one finds that (4.17) would be equal to $\delta(x)$. But this rule is no longer applicable to the present case, because $\log x$ is undefined at $x = 0$. The correct way of calculating (4.17) is to return to the original definition of the Schwartz distribution, that is, before taking the limit, we should multiply a test function $g(x)$ and integrate the product over x. We assume that $g(x)$ may be restricted to the function that has its inverse Laplace transform, that is, we set

$$g(x) = \int_0^\infty dp\, e^{-px}\mathcal{L}^{-1}[g](p).$$

Then to show $F(x) = \delta(x)$ is equivalent to showing $\int dx\, e^{-px}F(x) = 1$ for any $p > 0$. Applying this procedure to (4.17), we find

$$\lim_{s\to 0} s\frac{\partial}{\partial s}\int dx\, e^{-px}Y_s(x) = \lim_{s\to 0} s\frac{\partial}{\partial s}p^{-s} = 0.$$

Thus we see that (4.17) equals not $\delta(x)$ but 0.

We return to the original problem of evaluating (4.16). Applying the above procedure to it, we obtain

$$\begin{aligned}&\int dx\, e^{-px}H_{\log^2}(x)\\ &= \frac{1}{\alpha-\overline{\alpha}}\lim_{s\to 0}\Big[\Big(\frac{\partial}{\partial s}+\alpha-\overline{\alpha}\Big)(p^{-s}e^{-\alpha s}) - \Big(\frac{\partial}{\partial s}+\overline{\alpha}-\alpha\Big)(p^{-s}e^{-\overline{\alpha}s})\Big]\\ &= 1,\end{aligned}$$

that is, $H_{\log^2}(x) = \delta(x)$.

4.3 The case of $(\log x)^m$

As is seen in (3.14), $p\mathcal{L}[(\log x)^m](p)$ is a polynomial in $\log p$ of order m. It has m zero points $-\alpha_k$ $(k = 1, ..., m)$; it is natural to suppose that all of them are different from each other[8)]. Then we expand its inverse into partial fractions:

$$\frac{p^{-1}}{\mathcal{L}[(\log x)^m](p)} = \sum_{k=1}^m \frac{h_k}{\alpha_k + \log p}, \tag{4.18}$$

8) If not, we have only to change the expression infinitesimally. At the final step of the proof, we make this change tend to zero.

where h_k's are certain constants. Then, as before, we have

$$\Omega_{\log^m}(x) \equiv \mathcal{L}^{-1}\Big[\frac{p^{-1}}{\mathcal{L}[\log^m](p)}\Big](x)\theta(x) = \sum_{k=1}^{m} h_k \int_0^\infty ds\ e^{-\alpha_k s} Y_s(x).$$

Hence what we should calculate is

$$\begin{aligned} H_{\log^m}(x) &\equiv \int_{-0}^{x+0} dy\ \Omega_{\log^m}(x-y)\frac{d}{dy}((\log y)^m \theta(y)) \\ &= \sum_{k=1}^{m} h_k \int_0^\infty ds\ \frac{e^{-\alpha_k s}}{\Gamma(s)} I_m(x;s), \end{aligned} \tag{4.19}$$

where

$$I_m(x;s) \equiv \int_{-0}^{x+0} dy\ (x-y)^{s-1}\theta(x-y)\frac{d}{dy}((\log y)^m\ \theta(y)).$$

As before, we write

$$I_m(x;s) = x^{s-1}\sum_{j=0}^{m} \frac{m!}{j!(m-j)!}(\log x)^{m-j} J_j(s),$$

where

$$J_j(s) \equiv \int_{-0}^{1} dt\ (1-t)^{s-1}\frac{d}{dt}((\log t)^j\ \theta(t)). \tag{4.20}$$

It is convenient to rewrite (4.20) in the following way:

$$\begin{aligned} J_j(s) &= \Gamma(s)\lim_{\epsilon\to 0}\Big(\frac{\partial}{\partial\epsilon}\Big)^j \frac{\Gamma(1+\epsilon)}{\Gamma(s+\epsilon)} \\ &= \Gamma(s)\lim_{\epsilon\to 0}\Big(\frac{\partial}{\partial\epsilon}\Big)^j \int_0^\infty dv\ \frac{e^{-v}v^{\epsilon}}{\Gamma(s+\epsilon)} \\ &= \Gamma(s)\int_0^\infty dv\ v^{-s}\lim_{\lambda\to s}\Big(\frac{\partial}{\partial\lambda}\Big)^j \frac{e^{-v}v^{\lambda}}{\Gamma(\lambda)} \\ &= \Gamma(s)\int_0^\infty dv\ e^{-v}v^{-s}\Big(\frac{\partial}{\partial s}\Big)^j \frac{v^s}{\Gamma(s)}. \end{aligned}$$

Thus we have

$$\begin{aligned} I_m(x;s) &= x^{s-1}\Gamma(s)\int_0^\infty dv\ e^{-v}v^{-s}\Big(\log x + \frac{\partial}{\partial s}\Big)^m \frac{v^s}{\Gamma(s)} \\ &= \Gamma(s)\int_0^\infty dv\ e^{-v}v^{-s}\Big(\frac{\partial}{\partial s}\Big)^m \frac{x^{s-1}v^s}{\Gamma(s)}. \end{aligned}$$

As was discussed in the case of $(\log x)^2$, it is necessary to introduce a test function e^{-px}. With $v = pt$, we have

$$\begin{aligned}\int dx\, e^{-px} I_m(x;s) &= \Gamma(s) \int_0^\infty dv\, e^{-v} v^{-s} \Big(\frac{\partial}{\partial s}\Big)^m (p^{-s} v^s) \\ &= \Gamma(s) p^{-s+1} \int_0^\infty dt\, e^{-pt} t^{-s} \Big(\frac{\partial}{\partial s}\Big)^m t^s \\ &= \Gamma(s) p^{-s+1} \mathcal{L}[\log^m](p). \qquad (4.21)\end{aligned}$$

From (4.19) and (4.21), we obtain

$$\begin{aligned}\int dx\, e^{-px} H_{\log^m}(x) &= \sum_{k=1}^m h_k \int_0^\infty ds\, e^{-\alpha_k s} p^{-s+1} \mathcal{L}[\log^m](p) \\ &= \sum_{k=1}^m \frac{h_k}{\alpha_k + \log p} \cdot p\mathcal{L}[\log^m](p). \qquad (4.22)\end{aligned}$$

Owing to (4.18), the right-hand side of (4.22) equals 1. Thus we have confirmed

$$H_{\log^m}(x) = \delta(x).$$

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