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CANONICAL FORM OF ELEMENT OF JORDAN ALGEBRA BY GROUP $Spin(9)$

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Abstract. We give the canonical form an element of the exceptional Jordan algebras and explicit classification of orbits in those Jordan algebras under the action of the spinor group $Spin(9)$.

Introduction

It is well known that any element of the exceptional Jordan algebra $\mathfrak{J}$ can be uniquely transformed to a diagonal form by the natural action of some element of the exceptional Lie group F_4 ([1], [5]).

In the present paper, we shall first give the canonical form of an element of $\mathfrak{J}$ by the natural action of some element of the group $Spin(9)$ which is a maximal subgroup of the compact exceptional Lie group F_4 with the maximal rank (Theorem 2.1). In almost the same way as Proof of Theorem 2.1, we shall next give the canonical form of an element of another exceptional Jordan algebra $\mathfrak{J}_1$ by the natural action of some element of the group $Spin(9)$ which is a maximal compact subgroup of the noncompact exceptional Lie group $F_{4(-20)}$ (Theorem 3.1).

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1. The Jordan algebra $\mathfrak{J}$ and the group F_4 ([1], [4], [5])

Let $\mathfrak{C}$ be the Cayley algebra with the canonical $\boldsymbol{R}$-basis $e_0 = 1, e_1, \cdots, e_7$ and $\mathfrak{C}_0 = \{\sum_{k=1}^7 x_k e_k \in \mathfrak{C} \,|\, x_k \in \boldsymbol{R}\}$ be the $\boldsymbol{R}$-vector subspace of $\mathfrak{C}$ consisting of pure Cayley numbers. In $\mathfrak{C}$, the conjugate $\overline{x}$ and the length $|x|$ of $x = \sum_{k=0}^7 x_k e_k (x_k \in \boldsymbol{R})$ are defined as $\overline{x_0 + \sum_{k=1}^7 x_k e_k} = x_0 - \sum_{k=1}^7 x_k e_k$ and $|x| = \sqrt{x\overline{x}}$, respectively.

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Now, let $\mathfrak{J}$ be the exceptional Jordan algebra:

$$\mathfrak{J} = \mathfrak{J}(3, \mathfrak{C}) = \{X \in M(3, \mathfrak{C}) \,|\, X^* = X\}$$

with the Jordan multiplication $X \circ Y = \dfrac{1}{2}(XY + YX)$. In $\mathfrak{J}$, the Freudenthal multiplication $X \times Y$, an inner product (X, Y), a trilinear form (X, Y, Z) and the determinant $\det X$ are defined by

$$X \times Y = \frac{1}{2}(2X \circ Y - \mathrm{tr}(X)Y - \mathrm{tr}(Y)X + (\mathrm{tr}(X)\mathrm{tr}(Y) - \mathrm{tr}(X \circ Y))E),$$
$$(X, Y) = \mathrm{tr}(X \circ Y), \quad (X, Y, Z) = (X \times Y, Z), \quad \det X = \frac{1}{3}(X, X, X),$$

respectively, where tr(X) denotes the trace of X and E is the unit matrix. The group F_4 is defined as the automorphism group of the Jordan algebra $\mathfrak{J}$:

$$\begin{aligned} F_4 &= \{\alpha \in \mathrm{Iso}_R(\mathfrak{J}) \,|\, \alpha(X \circ Y) = \alpha X \circ \alpha Y\} \\ &= \{\alpha \in \mathrm{Iso}_R(\mathfrak{J}) \,|\, \alpha(X \times Y) = \alpha X \times \alpha Y\}, \end{aligned}$$

which is a connected compact exceptional Lie group of type F_4. The group F_4 leaves the inner product (X, Y), the trilinear form (X, Y, Z) and the determinant $\det X$ invariant, that is, for $\alpha \in F_4$, we have

$$(\alpha X, \alpha Y) = (X, Y), \quad (\alpha X, \alpha Y, \alpha Z) = (X, Y, Z), \quad \det \alpha X = \det X.$$

The group F_4 has the following series of subgroups:

$$F_4 \supset Spin(9) \supset Spin(8) \supset G_2.$$

These subgroups are constructed as follows.

The group $Spin(9)$ is defined as

$$Spin(9) = \{\alpha \in F_4 \,|\, \alpha E_1 = E_1\},$$

where $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. For $A \in O(2) = \{A \in M(2, \boldsymbol{R}) \,|\, {}^tAA = E\}$, let $\alpha(A)$ be the $\boldsymbol{R}$-linear transformation of $\mathfrak{J}$ defined by

$$\alpha(A)X = \widetilde{A}X\widetilde{A}^{-1}, \ X \in \mathfrak{J}, \quad \widetilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Then $\alpha(A) \in F_4$, furthermore $\alpha(A) \in Spin(9)$.

The group $Spin(8)$ is defined as

$$Spin(8) = \{\alpha \in Spin(9) \,|\, \alpha E_k = E_k, k = 2, 3\},$$

where $E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. For $a \in \mathfrak{C}, |a| = 1$, let $\beta_1(a)$ be the $\boldsymbol{R}$-linear transformation of $\mathfrak{J}$ defined by

$$\beta_1(a)X = \beta_1(a)\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \overline{a}x_3\overline{a} & \overline{a}\,\overline{x}_2 \\ a\overline{x}_3 a & \xi_2 & ax_1 \\ x_2 a & \overline{x}_1\overline{a} & \xi_3 \end{pmatrix}.$$

Then we see that $\beta_1(a) \in Spin(8)$ by using the Moufang formula $(ax)(ya) = a(xy)a, x, y \in \mathfrak{C}$. Similarly, for $a \in \mathfrak{C}, |a| = 1$, $\boldsymbol{R}$-linear transformations $\beta_2(a)$ and $\beta_3(a)$ of $\mathfrak{J}$ defined by

$$\beta_2(a)X = \begin{pmatrix} \xi_1 & x_3 a & \overline{x}_2\overline{a} \\ \overline{a}\,\overline{x}_3 & \xi_2 & \overline{a}x_1\overline{a} \\ ax_2 & a\overline{x}_1 a & \xi_3 \end{pmatrix}, \quad \beta_3(a)X = \begin{pmatrix} \xi_1 & ax_3 & a\overline{x}_2 a \\ \overline{x}_3\overline{a} & \xi_2 & x_1 a \\ \overline{a}x_2\overline{a} & \overline{a}\,\overline{x}_1 & \xi_3 \end{pmatrix}$$

are elements of $Spin(8)$.

The group G_2 is defined as the automorphism group of the Cayley algebra $\mathfrak{C}$:

$$G_2 = \{\alpha \in \mathrm{Iso}_R(\mathfrak{C}) \,|\, \alpha(xy) = (\alpha x)(\alpha y),\, x, y \in \mathfrak{C}\}.$$

Let $S^6 = \{x \in \mathfrak{C}_0 \,|\, |x| = 1\}$. Then it is known that the group G_2 acts transitively on S^6. Next, for $\alpha \in G_2$, let $\widetilde{\alpha}$ be the $\boldsymbol{R}$-linear transformation of $\mathfrak{J}$ defined by

$$\widetilde{\alpha}\begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \alpha x_3 & \overline{\alpha x_2} \\ \overline{\alpha x_3} & \xi_2 & \alpha x_1 \\ \alpha x_2 & \overline{\alpha x_1} & \xi_3 \end{pmatrix}.$$

Then $\widetilde{\alpha} \in Spin(8)$, and α will be identified with $\widetilde{\alpha}$. Thus the group G_2 is regarded as a subgroup of $Spin(8)$:

$$G_2 = \{\widetilde{\alpha} \in Spin(8) \,|\, \alpha \in G_2\}.$$

2. Canonical form of X of $\mathfrak{J}$ under the action of of maximal compact subgroup $Spin(9)$ of $F_{4(-20)}$

For an element $X = \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}$ of $\mathfrak{J}$, we define the discriminant $\delta(X)$ of X by

$$\begin{aligned}\delta(X) &= (E - E_1, X)^2 - 4(E_1, X, X) \\ &= (\xi_2 + \xi_3)^2 - 4(\xi_2\xi_3 - |x_1|^2) = (\xi_2 - \xi_3)^2 + 4|x_1|^2.\end{aligned}$$

Note that $\delta(X)$ is left invariant under the action of $Spin(9)$, that is, we have

$$\delta(\alpha X) = \delta(X), \quad \alpha \in Spin(9).$$

Theorem 2.1. *An element X of $\mathfrak{J}$ can be uniquely transformed to the following form by some element α of the group $Spin(9)$ (the $(1,1)$-entry ξ_1 of X is invariant).*

(1) *Case* $\delta(X) \neq 0$. $\quad \alpha X = \begin{pmatrix} \xi_1 & s_3 & s_2 \\ s_3 & r_2 & 0 \\ s_2 & 0 & r_3 \end{pmatrix}, \quad \begin{array}{l} r_2, r_3, s_2, s_3 \in \boldsymbol{R}, \\ r_2 > r_3,\ s_2 \geq 0, s_3 \geq 0. \end{array}$

(2) *Case* $\delta(X) = 0$. $\quad \alpha X = \begin{pmatrix} \xi_1 & 0 & s_2 \\ 0 & r_2 & 0 \\ s_2 & 0 & r_2 \end{pmatrix}, \quad \begin{array}{l} r_2, s_2 \in \boldsymbol{R}, \\ s_2 \geq 0. \end{array}$

All these orbits are distinct, and the union of all these orbits is the whole space $\mathfrak{J}$.

Proof. Let X be a given element of $\mathfrak{J}$. First we shall show that the x_1-entry of X can be transformed to 0 by the action of $Spin(9)$. Suppose $x_1 \neq 0$, and let $a = \dfrac{x_1}{|x_1|}$. By the action $\beta_1(\overline{a})$, x_1 can be transformed to a real number t_1, that is, $\beta_1(\overline{a})X = \begin{pmatrix} \xi_1 & {x_3}' & {\overline{x}_2}' \\ {\overline{x}_3}' & {\xi_2}' & t_1 \\ {x_2}' & t_1 & {\xi_3}' \end{pmatrix} = X'$. Since the matrix $X_1 = \begin{pmatrix} {\xi_2}' & t_1 \\ t_1 & {\xi_3}' \end{pmatrix}$ is real symmetric, this can be transformed to a diagonal form by some matrix $A \in O(2)$: $AX_1A^{-1} = \begin{pmatrix} r_2 & 0 \\ 0 & r_3 \end{pmatrix}, r_2 \geq r_3$. Act $\alpha(A) \in Spin(9)$ on X'. Then $\alpha(A)X'$ is of the form $\begin{pmatrix} \xi_1 & {x_3}'' & {\overline{x}_2}'' \\ {\overline{x}_3}'' & r_2 & 0 \\ {x_2}'' & 0 & r_3 \end{pmatrix} = X''$. We shall show that $r_2 > r_3$. Indeed, $(r_2 - r_3)^2 = \delta(X'') = \delta(X) \neq 0$, so $r_2 \neq r_3$, that is, $r_2 > r_3$.

We may assume that the x_2-entry of X'' is a non-negative real number s_2, applying $\beta_1(\dfrac{{\overline{x}_2}''}{|{x_2}''|}) \in Spin(8)$ to X'' if necessary. We shall show that ${x_3}''$ is transformed to a real number by the action of $Spin(8)$. Let ${x_3}'' = u + vt$, $u, v \in \boldsymbol{R}, t \in S^6$. Since the group G_2 acts transitively on S^6, t is transformed to e_1 by some element of G_2. Hence ${x_3}''$ is transformed to the form $u + ve_1$ (the other entries are left invariant). Next, act $\beta_3(e_2)$ on this. Then this is transformed to the form $\begin{pmatrix} \xi_1 & p_3 & -s_2 \\ \overline{p}_3 & r_2 & 0 \\ -s_2 & 0 & r_3 \end{pmatrix}$, where $p_3 = ue_2 - ve_3$, that is, p_3 is a pure Cayley number. If $p_3 \neq 0$, then let $b = \dfrac{p_3}{|p_3|}$ and act $\beta_3(\overline{b})$ on this. Then this is transformed to $\begin{pmatrix} \xi_1 & s_3 & -s_2 \\ s_3 & r_2 & 0 \\ -s_2 & 0 & r_3 \end{pmatrix}$, where $s_3 = |p_3|$. Therefore, since the sign of the x_2-entry is exchanged by the action of $\beta_1(-1)$, we have shown that the element X of $\mathfrak{J}$ can be transformed to $\begin{pmatrix} \xi_1 & s_3 & s_2 \\ s_3 & r_2 & 0 \\ s_2 & 0 & r_3 \end{pmatrix}$, $r_2 > r_3,\ s_2 \geq 0, s_3 \geq 0$.

Finally, We shall show the uniqueness of r_2, r_3, s_2 and s_3. Since r_2 and r_3 satisfy

$$\begin{aligned} r_2 + r_3 &= (E - E_1, \alpha X) = (\alpha(E - E_1), \alpha X) = (E - E_1, X), \\ r_2 r_3 &= (E_1, \alpha X, \alpha X) = (\alpha E_1, \alpha X, \alpha X) = (E_1, X, X), \end{aligned}$$

r_2 and r_3 are uniquely determined as solutions of the equation

$$t^2 - (E - E_1, X)t + (E_1, X, X) = 0$$

under the condition $r_2 > r_3$. Similarly, since s_2 and s_3 satisfy

$$\begin{aligned} (X, X) = (\alpha X, \alpha X) &= {\xi_1}^2 + {r_2}^2 + {r_3}^2 + 2{s_2}^2 + 2{s_3}^2, \\ \det X = \det(\alpha X) &= \xi_1 r_2 r_3 - r_2 {s_2}^2 - r_3 {s_3}^2, \end{aligned}$$

that is, ${s_2}^2$ and ${s_3}^2$ are solutions of the following simultaneous equation,

$$\begin{aligned} {s_2}^2 + {s_3}^2 &= \frac{1}{2}((X, X) - {\xi_1}^2 - {r_2}^2 - {r_3}^2), \\ r_2 {s_2}^2 + r_3 {s_3}^2 &= \xi_1 r_2 r_3 - \det X. \end{aligned}$$

From $r_2 \neq r_3$, ${s_2}^2$ and ${s_3}^2$ are uniquely determined (and depend on only the given X). Thus s_2 and s_3 are also uniquely determined, since $s_2 \geq 0, s_3 \geq 0$.

Case (2) As in Case (1), X can be transformed to the form $\alpha X = \begin{pmatrix} \xi_1 & t_3 & t_2 \\ t_3 & r_2 & 0 \\ t_2 & 0 & r_2 \end{pmatrix} = X'$, $r_2 \in \boldsymbol{R}, t_2 \geq 0, t_3 \geq 0$ by some element $\alpha \in Spin(9)$. Suppose $t_3 \neq 0$. Let $T = \dfrac{1}{\sqrt{{t_2}^2 + {t_3}^2}} \begin{pmatrix} t_2 & -t_3 \\ t_3 & t_2 \end{pmatrix} \in O(2)$ and act $\alpha(T)$ on X'. Then $\alpha(T)X'$ is of the form

$$\begin{pmatrix} \xi_1 & 0 & s_2 \\ 0 & r_2 & 0 \\ s_2 & 0 & r_2 \end{pmatrix}, \quad s_2 = \sqrt{{t_2}^2 + {t_3}^2} \geq 0,$$

which is the required form.

We shall show the uniqueness of r_2 and s_2. As similar to Cace (1), r_2 and s_2 satisfy

$$2r_2 = (E - E_1, X), \quad 2{s_2}^2 = (X, X) - {\xi_1}^2 - 2{r_2}^2.$$

Hence $r_2, {s_2}^2$ (so s_2 also) are uniquely determined as solutions of equations above. We have thus completed the proof of the theorem. □

3. Canonical form of element X of $\mathfrak{J}_1$ under the action of maximal compact subgroup $Spin(9)$ of $F_{4(-20)}$ ([2], [3], [4], [5])

Another exceptional Jordan algebra $\mathfrak{J}_1$ is consisting of all

$$X = \begin{pmatrix} \xi_1 & ix_3 & i\overline{x}_2 \\ i\overline{x}_3 & \xi_2 & x_1 \\ ix_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \quad \xi_k \in \boldsymbol{R}, x_k \in \mathfrak{C},\ i^2 = -1,$$

in which the Jordan multiplication is defined as $X \circ Y = \dfrac{1}{2}(XY + YX)$. In $\mathfrak{J}_1$, the Freudenthal multiplication $X \times Y$, the inner product (X, Y), the trilinear form (X, Y, Z), the determinant $\det X$ and the discriminant $\delta(X)$ are defined as similar to those of $\mathfrak{J}$. Further the group $F_{4(-20)}$ defined by

$$\begin{aligned} F_{4(-20)} &= \{\alpha \in \mathrm{Iso}_R(\mathfrak{J}_1) \,|\, \alpha(X \circ Y) = \alpha X \circ \alpha Y\} \\ &= \{\alpha \in \mathrm{Iso}_R(\mathfrak{J}_1) \,|\, \alpha(X \times Y) = \alpha X \times \alpha Y\} \end{aligned}$$

is a connected noncompact exceptional Lie group of type $F_{4(-20)}$. The following subgroup $Spin(9)$ of $F_{4(-20)}$:

$$Spin(9) = \{\alpha \in F_{4(-20)} \,|\, \alpha E_1 = E_1\}$$

is a maximal compact subgroup of $F_{4(-20)}$. Then, in almost the same way as Proof of Theorem 2.1 (when using the map β_k for ix_k, we have to use $\beta_k(\dfrac{x_k}{|x_k|})$ not $\beta_k(\dfrac{ix_k}{|ix_k|}) \notin Spin(8), k = 1, 2, 3)$, we obtain the following theorem.

Theorem 3.1. *An element X of $\mathfrak{J}_1$ can be uniquely transformed to the following form by some element α of the group $Spin(9)$ (the $(1,1)$-entry ξ_1 of X is invariant).*

(1) *Case $\delta(X) \neq 0$.* $$\alpha X = \begin{pmatrix} \xi_1 & is_3 & is_2 \\ is_3 & r_2 & 0 \\ is_2 & 0 & r_3 \end{pmatrix}, \quad \begin{array}{l} r_2, r_3,\ s_2, s_3 \in \boldsymbol{R}, \\ r_2 > r_3,\ s_2 \geq 0, s_3 \geq 0. \end{array}$$

(2) *Case* $\delta(X) = 0$. $\quad \alpha X = \begin{pmatrix} \xi_1 & 0 & is_2 \\ 0 & r_2 & 0 \\ is_2 & 0 & r_2 \end{pmatrix}, \quad r_2, s_2 \in \boldsymbol{R}, \ s_2 \geq 0.$

All these orbits are distinct, and the union of all these orbits is the whole space $\mathfrak{J}_1$.

References

[1] H. Freudenthal, Oktaven, Ausnahmegruppen und Oktavengeometry, *Math. Inst. Rijkuniv.* te Utrecht, 1951.

[2] A. Nishio, The Freudenthal product and orbits in the Jordan algebra over the exceptional Lie group of type $F_{4(-20)}$, in preparation.

[3] I. Yokota, Realization of involutive automorphisms σ of exceptional Lie groups G, Part I, $G = G_2, F_4$ and E_6, *Tsukuba J. Math.* **14** (1990), 185–223.

[4] I. Yokota, On a non compact simple Lie group $F_{4,1}$ of type F_4, *J. Fac. Sci, Shinshu univ.* **10-2** (1975), 71–80.

[5] I. Yokota, *Exceptional Lie groups*, arXiv:0902.0431.

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