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ORBIT EQUIVALENCE OF ONE-SIDED SUBSHIFTS AND THE ASSOCIATED C^*-ALGEBRAS

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Abstract. A λ-graph system $\mathfrak{L}$ is a generalization of a finite labeled graph and presents a subshift. We will prove that the topological dynamical systems $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ for λ-graph systems $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are continuously orbit equivalent if and only if there exists an isomorphism between the associated C^*-algebras $\mathcal{O}_{\mathfrak{L}_1}$ and $\mathcal{O}_{\mathfrak{L}_2}$ keeping their commutative C^*-subalgebras $C(X_{\mathfrak{L}_1})$ and $C(X_{\mathfrak{L}_2})$. It is also equivalent to the condition that there exists a homeomorphism from $X_{\mathfrak{L}_1}$ to $X_{\mathfrak{L}_2}$ intertwining their topological full inverse semigroups. In particular, one-sided subshifts X_{Λ_1} and X_{Λ_2} are λ-continuously orbit equivalent if and only if there exists an isomorphism between the associated C^*-algebras $\mathcal{O}_{\Lambda_1}$ and $\mathcal{O}_{\Lambda_2}$ keeping their commutative C^*-subalgebras $C(X_{\Lambda_1})$ and $C(X_{\Lambda_2})$.

1. Introduction

H. Dye has initiated to study of orbit equivalence of ergodic finite measure preserving transformations, who proved that any two such transformations are orbit equivalent ([13], [14]). W. Krieger [21] has proved that two ergodic non-singular transformations are orbit equivalent if and only if the associated von Neumann crossed produtcs are isomorphic. In topological setting, Giordano-Putnam-Skau [15], [16] (cf. [19]) have proved that two Cantor minimal systems are strong orbit equivalent if and only if the associated C^*-crossed products are isomorphic. In more general setting, J. Tomiyama [34] (cf. [2], [35]) has proved that two topological free homeomorphisms (X, ϕ) and (Y, ψ) on compact Hausdorff spaces are continuously orbit equivalent if and only if there exists an isomorphism between the associated C^*-crossed products keeping their commutative C^*-subalgebras $C(X)$ and $C(Y)$. He also proved that it is equivalent to the condition that there exists a homeomorphism $h : X \to Y$ such that h preserves their topological full groups. Orbit equivalence of continuous maps on compact Hausdorff spaces that are not homeomorphisms are not covered by the above

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Tomiyama's setting. The class of one-sided subshifts is an important class of topological dynamical systems on Cantor sets with continuous surjections that are not homeomorphisms. The one-sided topological Markov shifts is a subclass of the class. The associated C^*-algebras to the topological Markov shifts are known to be the Cuntz-Krieger algebras. In the recent paper [30], the author has shown that similar results to the Tomiyama's results hold for one-sided topological Markov shifts. He has proved that one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) for matrices A and B with entries in $\{0, 1\}$ are continuously orbit equivalent if and only if there exists an isomorphism between the Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_B$ keeping their commutative C^*-subalgebras $C(X_A)$ and $C(X_B)$ (Note that the term "topological "orbit equivalence has been used in [30] instead of "continuous "orbit equivalence). It is also equivalent to the condition that there exists a homeomorphism from X_A to X_B intertwining their topological full groups $[\sigma_A]_c$ and $[\sigma_B]_c$.

In this paper we will extend the above results for one-sided topological Markov shifts to the class of general one-sided subshifts. A λ-graph system $\mathfrak{L}$ is a generalization of a finite labeled graph and presents a subshift. It yields a topological dynamical system $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ of a zero-dimensional compact Hausdorff space $X_{\mathfrak{L}}$ with shift transformation $\sigma_{\mathfrak{L}}$, that is a continuous surjection and not a homeomorphism. The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is associated with the dynamical system $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ such that $C(X_{\mathfrak{L}})$ is naturally embedded into $\mathcal{O}_{\mathfrak{L}}$ as a diagonal algebra of the canonical AF-algebra $\mathcal{F}_{\mathfrak{L}}$ inside of $\mathcal{O}_{\mathfrak{L}}$. We will prove that the topological dynamical systems $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ for λ-graph systems $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are continuously orbit equivalent if and only if there exists an isomorphism between the associated C^*-algebras $\mathcal{O}_{\mathfrak{L}_1}$ and $\mathcal{O}_{\mathfrak{L}_2}$ keeping their commutative C^*-subalgebras $C(X_{\mathfrak{L}_1})$ and $C(X_{\mathfrak{L}_2})$. It is also equivalent to the condition that there exists a homeomorphism from $X_{\mathfrak{L}_1}$ to $X_{\mathfrak{L}_2}$ intertwining their topological full inverse semigroups $[\sigma_{\mathfrak{L}_1}]_{sc}$ and $[\sigma_{\mathfrak{L}_1}]_{sc}$. Let X_{Λ_1} and X_{Λ_2} be the right one-sided subshifts for two-sided subshifts Λ_1 and Λ_2 respectively. We in particular show that two one-sided subshifts X_{Λ_1} and X_{Λ_2} are λ-continuously orbit equivalent if and only if there exists an isomorphism between the associated C^*-algebras $\mathcal{O}_{\Lambda_1}$ and $\mathcal{O}_{\Lambda_2}$ keeping their commutative C^*-subalgebras $C(X_{\Lambda_1})$ and $C(X_{\Lambda_2})$, where $\mathcal{O}_{\Lambda_1}$ and $\mathcal{O}_{\Lambda_2}$ are the C^*-algebras associated with subshifts ([25], cf. [3]).

Let $[\sigma_{\mathfrak{L}}]_c$ be the topological full group of $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ whose elements consist of homeomorphisms τ on $X_{\mathfrak{L}}$ such that $\tau(x)$ is contained in the orbit $orb_{\sigma_{\mathfrak{L}}}(x)$ of x under $\sigma_{\mathfrak{L}}$ for $x \in X_{\mathfrak{L}}$, and its orbit cocycles are continuous. If $\mathfrak{L}$ comes from a finite directed graph and hence $X_{\mathfrak{L}}$ is a topological Markov shift, then the topological full group is large enough to cover orbits of $x \in X_{\mathfrak{L}}$. However if $\mathfrak{L}$ does not come from a finite graph, the topological full group is not necessarily large enough to cover orbits of $X_{\mathfrak{L}}$. To obtain enough informations of orbit structure of $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$, we need to enlarge $[\sigma_{\mathfrak{L}}]_c$ to topological inverse semigroup $[\sigma_{\mathfrak{L}}]_{sc}$ whose elements consist of partial homeomorphisms τ on $X_{\mathfrak{L}}$ such that $\tau(x)$ is contained in $orb_{\sigma_{\mathfrak{L}}}(x)$ for each x in the domain of τ. Let us denote by $\mathcal{D}_{\mathfrak{L}}$ the commutative C^*-subalgebra $C(X_{\mathfrak{L}})$ of $\mathcal{O}_{\mathfrak{L}}$. The corresponding object to the inverse semigroup $[\sigma_{\mathfrak{L}}]_{sc}$ is the normalizer semigroup $N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$ of $\mathcal{D}_{\mathfrak{L}}$ in $\mathcal{O}_{\mathfrak{L}}$ whose elements consist of partial isometries v of $\mathcal{O}_{\mathfrak{L}}$ such that $v\mathcal{D}_{\mathfrak{L}}v^* \subset \mathcal{D}_{\mathfrak{L}}$ and $v^*\mathcal{D}_{\mathfrak{L}}v \subset \mathcal{D}_{\mathfrak{L}}$. Then we will show that the exact sequence

$$1 \longrightarrow \mathcal{U}(\mathcal{D}_{\mathfrak{L}}) \longrightarrow N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}}) \longrightarrow [\sigma_{\mathfrak{L}}]_{sc} \longrightarrow 1$$

of semigroups holds so that the following theorem will be proved:

Theorem 1.1. (Theorem 5.7) *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be λ-graph systems satisfying condition (I). The following are equivalent:*

(1) *There exists an isomorphism $\Psi : \mathcal{O}_{\mathfrak{L}_1} \to \mathcal{O}_{\mathfrak{L}_2}$ such that $\Psi(\mathcal{D}_{\mathfrak{L}_1}) = \mathcal{D}_{\mathfrak{L}_2}$.*
(2) *$(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ are continuously orbit equivalent.*
(3) *There exists a homeomorphism $h : X_{\mathfrak{L}_1} \to X_{\mathfrak{L}_2}$ such that $h \circ [\sigma_{\mathfrak{L}_1}]_{sc} \circ h^{-1} = [\sigma_{\mathfrak{L}_2}]_{sc}$.*

Let Λ be the subshift presented by a λ-graph system $\mathfrak{L}$ and $(X_\Lambda, \sigma_\Lambda)$ the right one-sided subshift for Λ. There exists a natural factor map $\pi_\Lambda^{\mathfrak{L}} : (X_{\mathfrak{L}}, \sigma_{\mathfrak{L}}) \longrightarrow (X_\Lambda, \sigma_\Lambda)$. It induces an inclusion $C(X_\Lambda) \hookrightarrow C(X_{\mathfrak{L}})$. We regard the algebra $C(X_\Lambda)$ as a subalgebra $\mathfrak{D}_\Lambda$ of $\mathcal{D}_{\mathfrak{L}}$ and of $\mathcal{O}_{\mathfrak{L}}$. We say that two factor maps $\pi_{\Lambda_1}^{\mathfrak{L}_1}$ and $\pi_{\Lambda_2}^{\mathfrak{L}_2}$ are continuously orbit equivalent if there exist homeomorphisms $h_{\mathfrak{L}} : X_{\mathfrak{L}_1} \longrightarrow X_{\mathfrak{L}_2}$ and $h_\Lambda : X_{\Lambda_1} \longrightarrow X_{\Lambda_2}$ such that $\pi_{\Lambda_2}^{\mathfrak{L}_2} \circ h_{\mathfrak{L}} = h_\Lambda \circ \pi_{\Lambda_1}^{\mathfrak{L}_1}$ and there exist continuous functions $k_1, l_1 : X_{\mathfrak{L}_1} \longrightarrow \mathbb{Z}_+$ and $k_2, l_2 : X_{\mathfrak{L}_2} \longrightarrow \mathbb{Z}_+$ such that

$$\begin{aligned} \sigma_{\mathfrak{L}_2}^{k_1(x)}(h_{\mathfrak{L}} \circ \sigma_{\mathfrak{L}_1}(x)) &= \sigma_{\mathfrak{L}_2}^{l_1(x)}(h_{\mathfrak{L}}(x)), \quad x \in X_{\mathfrak{L}_1}, \\ \sigma_{\mathfrak{L}_1}^{k_2(y)}(h_{\mathfrak{L}}^{-1} \circ \sigma_{\mathfrak{L}_2}(y)) &= \sigma_{\mathfrak{L}_1}^{l_2(x)}(h_{\mathfrak{L}}^{-1}(y)), \quad y \in X_{\mathfrak{L}_2}. \end{aligned}$$

Then we will prove

Theorem 1.2. (Theorem 6.6) *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be λ-graph systems satisfying condition (I) and Λ_1 and Λ_2 their respect subshifts. The following are equivalent:*

(1) *There exists an isomorphism $\Psi : \mathcal{O}_{\mathfrak{L}_1} \longrightarrow \mathcal{O}_{\mathfrak{L}_2}$ such that $\Psi(\mathfrak{D}_{\Lambda_1}) = \mathfrak{D}_{\Lambda_2}$.*
(2) *The factor maps $\pi_{\Lambda_1}^{\mathfrak{L}_1}$ and $\pi_{\Lambda_2}^{\mathfrak{L}_2}$ are continuously orbit equivalent.*
(3) *There exist homeomorphisms $h_{\mathfrak{L}} : X_{\mathfrak{L}_1} \longrightarrow X_{\mathfrak{L}_2}$ and $h_\Lambda : X_{\Lambda_1} \longrightarrow X_{\Lambda_2}$ such that $\pi_{\Lambda_2}^{\mathfrak{L}_2} \circ h_{\mathfrak{L}} = h_\Lambda \circ \pi_{\Lambda_1}^{\mathfrak{L}_1}$ and $h_{\mathfrak{L}} \circ [\sigma_{\mathfrak{L}_1}]_{sc} \circ h_{\mathfrak{L}}^{-1} = [\sigma_{\mathfrak{L}_2}]_{sc}$.*

Let $\mathfrak{L}^\Lambda$ be the canonical λ-graph system for Λ (see [26]). Then the C^*-algebra $\mathcal{O}_\Lambda$ coincides with the algebra $\mathcal{O}_{\mathfrak{L}^\Lambda}$. The natural inclusion $\iota : X_\Lambda \hookrightarrow X_{\mathfrak{L}^\Lambda}$

induces a new topology on X_Λ. The topological space is denoted by $\widetilde{X}_\Lambda$. Two subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are said to be λ-*continuously orbit equivalent* if there exist a homeomorphism $h : X_{\Lambda_1} \longrightarrow X_{\Lambda_2}$, and continuous functions $k_1, l_1 : \widetilde{X}_{\Lambda_1} \longrightarrow \mathbb{Z}_+$ and $k_2, l_2 : \widetilde{X}_{\Lambda_2} \longrightarrow \mathbb{Z}_+$ such that h is also homeomorphic from $\widetilde{X}_{\Lambda_1}$ onto $\widetilde{X}_{\Lambda_2}$ such that

$$\begin{aligned} \sigma_{\Lambda_2}^{k_1(a)}(h \circ \sigma_{\Lambda_1}(a)) &= \sigma_{\Lambda_2}^{l_1(a)}(h(a)), \qquad a \in X_{\Lambda_1}, \\ \sigma_{\Lambda_1}^{k_2(b)}(h^{-1} \circ \sigma_{\Lambda_2}(b)) &= \sigma_{\Lambda_1}^{l_2(b)}(h^{-1}(b)), \qquad b \in X_{\Lambda_2}. \end{aligned}$$

Then we will prove

Theorem 1.3. (Theorem 7.5) *Let Λ_1 and Λ_2 be subshifts satisfying condition (I). The following are equivalent:*

(1) *There exists an isomorphism $\Psi : \mathcal{O}_{\mathfrak{L}_1} \longrightarrow \mathcal{O}_{\Lambda_2}$ such that $\Psi(\mathfrak{D}_{\Lambda_1}) = \mathfrak{D}_{\Lambda_2}$.*
(2) *The subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are λ-continuously orbit equivalent.*

The theorem is a generalization of a result in [30] for topological Markov shifts. Throughout the paper, we denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of nonnegative integers and the set of positive integers respectively.

2. Preliminaries

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ with vertex set $V = \cup_{l \in \mathbb{Z}_+} V_l$ and edge set $E = \cup_{l \in \mathbb{Z}_+} E_{l,l+1}$ that is labeled with symbols in Σ by a map $\lambda : E \to \Sigma$, and that is supplied with surjective maps $\iota (= \iota_{l,l+1}) : V_{l+1} \to V_l$ for $l \in \mathbb{Z}_+$. Here the vertex sets $V_l, l \in \mathbb{Z}_+$ are finite disjoint sets. Also $E_{l,l+1}, l \in \mathbb{Z}_+$ are finite disjoint sets. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l and its terminal vertex $t(e)$ in V_{l+1} respectively. Every vertex in V has a successor and every vertex in V_l for $l \in \mathbb{N}$ has a predecessor. It is then required that there exists an edge in $E_{l,l+1}$ with label α and its terminal is $v \in V_{l+1}$ if and only if there exists an edge in $E_{l-1,l}$ with label α and its terminal is $\iota(v) \in V_l$. For $u \in V_{l-1}$ and $v \in V_{l+1}$, put

$$\begin{aligned} E^\iota(u, v) &= \{e \in E_{l,l+1} \mid t(e) = v, \iota(s(e)) = u\}, \\ E_\iota(u, v) &= \{e \in E_{l-1,l} \mid s(e) = u, t(e) = \iota(v)\}. \end{aligned}$$

Then we require a bijective correspondence between $E^\iota(u, v)$ and $E_\iota(u, v)$ that preserves labels for each pair of vertices u, v. We call this property the local property of $\mathfrak{L}$. We henceforth assume that $\mathfrak{L}$ is left-resolving, which means that $t(e) \neq t(f)$ whenever $\lambda(e) = \lambda(f)$ for $e, f \in E$.

Let $\Omega_{\mathfrak{L}}$ be the compact Hausdorff space of the projective limit of the system $\iota_{l,l+1} : V_{l+1} \to V_l, l \in \mathbb{Z}_+$, that is defined by

$$\Omega_{\mathfrak{L}} = \{(v^l)_{l\in\mathbb{Z}_+} \in \prod_{l\in\mathbb{Z}_+} V_l \mid \iota_{l,l+1}(v^{l+1}) = v^l, l \in \mathbb{Z}_+\}.$$

An element v in $\Omega_{\mathfrak{L}}$ is called an ι-orbit or also a vertex. Let $E_{\mathfrak{L}}$ be the set of all triplets $(u, \alpha, v) \in \Omega_{\mathfrak{L}} \times \Sigma \times \Omega_{\mathfrak{L}}$, where $u = (u^l)_{l\in\mathbb{Z}_+}, v = (v^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ such that for each $l \in \mathbb{Z}_+$, there exists $e_{l,l+1} \in E_{l,l+1}$ satisfying

$$u^l = s(e_{l,l+1}), \quad v^{l+1} = t(e_{l,l+1}) \quad \text{and} \quad \alpha = \lambda(e_{l,l+1}).$$

Then the set $E_{\mathfrak{L}} \subset \Omega_{\mathfrak{L}} \times \Sigma \times \Omega_{\mathfrak{L}}$ is a zero-dimensional continuous graph in the sense of Deaconu ([28, Proposition 2.1], [9], [10], [11], [12]). It has been also studied in [23] as a Shannon graph. Following Deaconu [10] and Krieger [22], we consider the set $X_{\mathfrak{L}}$ of all one-sided paths of $E_{\mathfrak{L}}$:

$$X_{\mathfrak{L}} = \{(\alpha_n, u_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} (\Sigma \times \Omega_{\mathfrak{L}}) \mid (u_n, \alpha_{n+1}, u_{n+1}) \in E_{\mathfrak{L}} \text{ for all } n \in \mathbb{N} \\ \text{and } (u_0, \alpha_1, u_1) \in E_{\mathfrak{L}} \text{ for some } u_0 \in \Omega_{\mathfrak{L}}\}.$$

The set $X_{\mathfrak{L}}$ becomes a zero-dimensional compact Hausdorff space under the relative topology from the infinite product topology of $\Sigma \times \Omega_{\mathfrak{L}}$. For $x = (\alpha_n, u_n)_{n\in\mathbb{N}} \in X_{\mathfrak{L}}$, the vertex $u_0 \in \Omega_{\mathfrak{L}}$ satisfying $(u_0, \alpha_1, u_1) \in E_{\mathfrak{L}}$ is unique because $\mathfrak{L}$ is left-resolving. We denote it by $u_0(x)$. The shift map $\sigma_{\mathfrak{L}} : (\alpha_n, u_n)_{n\in\mathbb{N}} \in X_{\mathfrak{L}} \to (\alpha_{n+1}, u_{n+1})_{n\in\mathbb{N}} \in X_{\mathfrak{L}}$ is a local homeomorphism by [28, Lemma 2.2]. We have a topological dynamical system $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ of a compact Hausdorff space $X_{\mathfrak{L}}$ with a continuous surjection $\sigma_{\mathfrak{L}}$. The set

$$X_\Lambda = \{(\alpha_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}} \mid (\alpha_n, u_n)_{n\in\mathbb{N}} \in X_{\mathfrak{L}}\}$$

becomes the right one-sided subshift for the subshift Λ presented by $\mathfrak{L}$ with shift transformation σ_Λ defined by

$$\sigma_\Lambda((\alpha_n)_{n\in\mathbb{N}}) = (\alpha_{n+1})_{n\in\mathbb{N}}, \qquad (\alpha_n)_{n\in\mathbb{N}} \in X_\Lambda.$$

The factor map

$$\pi^{\mathfrak{L}}_\Lambda : (\alpha_n, u_n)_{n\in\mathbb{N}} \in X_{\mathfrak{L}} \to (\alpha_n)_{n\in\mathbb{N}} \in X_\Lambda$$

is a continuous surjective map satisfying

$$\pi^{\mathfrak{L}}_\Lambda \circ \sigma_{\mathfrak{L}} = \sigma_\Lambda \circ \pi^{\mathfrak{L}}_\Lambda.$$

A word $\mu = \mu_1 \cdots \mu_k$ for $\mu_i \in \Sigma$ is said to be admissible for X_Λ if μ appears in somewhere in some element a in X_Λ. We denote by $B_k(X_\Lambda)$ the set of all admissible words of length $k \in \mathbb{Z}_+$, where $B_0(X_\Lambda)$ means the empty word $\emptyset$. We set $B_*(X_\Lambda) = \cup_{k=0}^{\infty} B_k(X_\Lambda)$. For $a = (a_n)_{n\in\mathbb{N}} \in X_\Lambda$ and positive integers k, l with $k \le l$, we put the word $a_{[k,l]} = (a_k, a_{k+1}, \dots, a_l) \in B_{l-k+1}(X_\Lambda)$ and the right infinite sequence $a_{[k,\infty)} = (a_k, a_{k+1}, \dots) \in X_\Lambda$. Similarly we use the notations $B_k(X_\mathfrak{L})$ defined by the set $\{(\alpha_n, u_n)_{n=1}^k \mid (\alpha_n, u_n)_{n\in\mathbb{N}} \in X_\mathfrak{L}\}$ and $x_{[k,l]} = (x_k, \dots, x_l)$ for $x = (x_n)_{n\in\mathbb{N}} \in X_\mathfrak{L}$.

Let us now briefly review the C^*-algebra $\mathcal{O}_\mathfrak{L}$ associated with λ-graph system $\mathfrak{L}$. The C^*-algebras $\mathcal{O}_\mathfrak{L}$ are generalization of the C^*-algebras associated with subshifts ([28], cf. [3]). We denote by $\{v_1^l, \dots, v_{m(l)}^l\}$ the vertex set V_l. Define the transition matrices $A_{l,l+1}, I_{l,l+1}$ of $\mathfrak{L}$ by setting for $i = 1, 2, \dots, m(l)$, $j = 1, 2, \dots, m(l+1)$, $\alpha \in \Sigma$,

$$A_{l,l+1}(i,\alpha,j) = \begin{cases} 1 & \text{if } \ s(e) = v_i^l, \lambda(e) = \alpha, t(e) = v_j^{l+1} \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } \ \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise.} \end{cases}$$

The C^*-algebra $\mathcal{O}_\mathfrak{L}$ is realized as the universal unital C^*-algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and projections $E_i^l, i = 1, 2, \dots, m(l), l \in \mathbb{Z}_+$ subject to the following operator relations called $(\mathfrak{L})$:

$$\sum_{\alpha\in\Sigma} S_\alpha S_\alpha^* = 1, \tag{2.1}$$

$$\sum_{i=1}^{m(l)} E_i^l = 1, \qquad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) E_j^{l+1}, \tag{2.2}$$

$$S_\beta S_\beta^* E_i^l = E_i^l S_\beta S_\beta^*, \tag{2.3}$$

$$S_\beta^* E_i^l S_\beta = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\beta,j) E_j^{l+1}, \tag{2.4}$$

for $\beta \in \Sigma$, $i = 1, 2, \dots, m(l), l \in \mathbb{Z}_+$. It is nuclear ([28, Proposition 5.6]). For a word $\mu = \mu_1 \cdots \mu_k \in B_k(X_\Lambda)$, we set $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$. The algebra of all finite linear combinations of the elements of the form

$$S_\mu E_i^l S_\nu^* \quad \text{for} \quad \mu, \nu \in B_*(X_\Lambda), \quad i = 1, \dots, m(l), \quad l \in \mathbb{Z}_+$$

is a dense $*$-subalgebra of $\mathcal{O}_\mathfrak{L}$. Let us denote by $\mathcal{A}_\mathfrak{L}$ the C^*-subalgebra of $\mathcal{O}_\mathfrak{L}$ generated by the projections $E_i^l, i = 1, \dots, m(l), l \in \mathbb{Z}_+$. By the universality of the algebra $\mathcal{O}_\mathfrak{L}$ the algebra $\mathcal{A}_\mathfrak{L}$ is isomorphic to the commutative C^*-algebra $C(\Omega_\mathfrak{L})$ of all complex valued continuous functions on $\Omega_\mathfrak{L}$. We define

C^*-subalgebra $\mathcal{F}_k^l$ with $k \le l$, that is a finite dimensional algebra generated by $S_\mu E_i^l S_\nu^*, \mu, \nu \in B_k(X_\Lambda), i = 1, \dots, m(l)$. Denote by $\mathcal{F}_{\mathfrak{L}}$ the AF-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by $\cup_{k,l}\mathcal{F}_k^l$. For a vertex $v_i^l \in V_l$, put

$$\Gamma^+(v_i^l) = \{(\alpha_1, \alpha_2, \dots,) \in \Sigma^{\mathbb{N}} \mid \text{ there exists an edge } e_{n,n+1} \in E_{n,n+1} \text{ for } n \ge l \\ \text{such that } v_i^l = s(e_{l,l+1}), \quad t(e_{n,n+1}) = s(e_{n+1,n+2}),\ \lambda(e_{n,n+1}) = \alpha_{n-l+1}\}$$

the set of all label sequences in $\mathfrak{L}$ starting at v_i^l. We say that $\mathfrak{L}$ satisfies condition (I) if for each $v_i^l \in V$, the set $\Gamma^+(v_i^l)$ contains at least two distinct sequences. Under the condition (I), the algebra $\mathcal{O}_{\mathfrak{L}}$ can be realized as the unique C^*-algebra subject to the relations ($\mathfrak{L}$) ([28, Theorem 4.3]). A λ-graph system $\mathfrak{L}$ is said to be *irreducible* if for a vertex $v \in V_l$ and an ι-orbit $x = (x_i)_{i \in \mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$, there exists a λ-path starting at v and terminating at x_{l+N} for some $N \in \mathbb{N}$. If $\mathfrak{L}$ is irreducible with condition (I), the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple ([28, Theorem 4.7]).

Let $\mathcal{D}_{\mathfrak{L}}$ be the C^*-subalgebra of $\mathcal{F}_{\mathfrak{L}}$ generated by $S_\mu E_i^l S_\mu^*,\ \mu \in B_*(X_\Lambda), i = 1, \dots, m(l), l \in \mathbb{Z}_+$ and $\mathfrak{D}_\Lambda$ the C^*-subalgebra of $\mathcal{D}_{\mathfrak{L}}$ generated by $S_\mu S_\mu^*,\ \mu \in B_*(X_\Lambda)$. For $\mu = \mu_1 \cdots \mu_k \in B_k(X_\Lambda)$ and $v_i^l \in V_l$, we set the cylinder set

$$U_{\mu, v_i^l} = \{(\alpha_n, u_n) \in X_{\mathfrak{L}} \mid \alpha_1 = \mu_1, \dots, \alpha_1 = \mu_k, u_k^l = v_i^l\}$$

of $X_{\mathfrak{L}}$ where $u_k = (u_k^l)_{l \in \mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$. Let $\chi_{U_{\mu, v_i^l}}$ denote the chracteristic function on $X_{\mathfrak{L}}$ for the cylinder set U_{μ, v_i^l}. Then the correspondence $S_\mu E_i^l S_\mu^* \in \mathcal{D}_{\mathfrak{L}} \longleftrightarrow \chi_{U_{\mu, v_i^l}} \in C(X_{\mathfrak{L}})$ yields an isomorphism between $\mathcal{D}_{\mathfrak{L}}$ and $C(X_{\mathfrak{L}})$. Similarly let $U_\mu = \{(a_n)_{n \in \mathbb{N}} \in X_\Lambda \mid a_1 = \mu_1, \dots, a_k = \mu_k\}$ be the cylinder set of X_Λ. The correspondence $S_\mu S_\mu^* \in \mathfrak{D}_\Lambda \longleftrightarrow \chi_\mu \in C(X_\Lambda)$ yields an isomorphism between $\mathfrak{D}_\Lambda$ and $C(X_\Lambda)$.

By the universality for the relations ($\mathfrak{L}$), the correspondence $S_\alpha \longrightarrow e^{\sqrt{-1}t}S_\alpha$, $\alpha \in \Sigma$, $E_i^l \longrightarrow E_i^l, i = 1, \dots, m(l)$, $l \in \mathbb{Z}_+$ for $e^{\sqrt{-1}t} \in \mathbb{T} = \{e^{\sqrt{-1}t} \mid t \in [0, 2\pi]\}$ gives rise to an action $\rho : \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_{\mathfrak{L}})$ called gauge action. The fixed point algebra of $\mathcal{O}_{\mathfrak{L}}$ under ρ is the AF-algebra $\mathcal{F}_{\mathfrak{L}}$. We denote by $E : \mathcal{O}_{\mathfrak{L}} \to \mathcal{F}_{\mathfrak{L}}$ the conditional expectation defined by $E(a) = \int_{\mathbb{T}} \rho_t(a)dt$ for $a \in \mathcal{O}_{\mathfrak{L}}$.

The following lemma is basic in our further discussions.

Lemma 2.1. ([27, Proposition 3.3], cf.[8, Remark 2.18]) *Suppose that $\mathfrak{L}$ satisfies condition (I). Then we have $\mathfrak{D}'_\Lambda \cap \mathcal{O}_{\mathfrak{L}} = \mathcal{D}_{\mathfrak{L}}$ and hence $\mathcal{D}'_{\mathfrak{L}} \cap \mathcal{O}_{\mathfrak{L}} = \mathcal{D}_{\mathfrak{L}}$.*

This means that the algebra $\mathcal{D}_{\mathfrak{L}}$ is maximal abelian in $\mathcal{O}_{\mathfrak{L}}$.

Proof. The proof of $\mathfrak{D}'_\Lambda \cap \mathcal{O}_{\mathfrak{L}} = \mathcal{D}_{\mathfrak{L}}$ is completely similar to the proof of [27, Proposition 3.3]. Since $\mathcal{D}_{\mathfrak{L}} \subset \mathcal{D}'_{\mathfrak{L}} \cap \mathcal{O}_{\mathfrak{L}} \subset \mathfrak{D}_\Lambda \cap \mathcal{O}_{\mathfrak{L}}$, we have $\mathcal{D}'_{\mathfrak{L}} \cap \mathcal{O}_{\mathfrak{L}} = \mathcal{D}_{\mathfrak{L}}$. □

In [30], a representation of the Cuntz-Krieger algebra $\mathcal{O}_A$ on a Hilbert space having the shift space X_A as a complete orthonormal basis has been used. Let us generalize the representation to the C^*-algebras $\mathcal{O}_{\mathfrak{L}}$ as in the following way. Let $\mathfrak{H}_{\mathfrak{L}}$ be the Hilbert space with its complete orthonormal system $e_x, x \in X_{\mathfrak{L}}$. The Hilbert space is not separable. Consider the partial isometries $T_\alpha : \mathfrak{H}_{\mathfrak{L}} \to \mathfrak{H}_{\mathfrak{L}}, \alpha \in \Sigma$ and projections $P_i^l : \mathfrak{H}_{\mathfrak{L}} \to \mathfrak{H}_{\mathfrak{L}}, i = 1, \dots, m(l)$ defined by

$$T_\alpha e_x = \begin{cases} e_y & \text{if there exists an } \iota\text{-orbit } u_{-1} \in \Omega_{\mathfrak{L}}; (u_{-1}, \alpha, u_0(x)) \in E_{\mathfrak{L}}, \\ 0 & \text{otherwise} \end{cases}$$

where $y = ((\alpha, u_0(x)), (\alpha_1, u_1), (\alpha_2, u_2), \dots) \in X_{\mathfrak{L}}$ for $x = ((\alpha_1, u_1), (\alpha_2, u_2), \dots) \in X_{\mathfrak{L}}$ and

$$P_i^l e_x = \begin{cases} e_x & \text{if } u_0(x)^l = v_i^l, \\ 0 & \text{otherwise} \end{cases}$$

where $u_0(x) = (u_0(x)^l)_{l \in \mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$.

Lemma 2.2. *The partial isometries $T_\alpha, \alpha \in \Sigma$ and the projections $P_i^l, i = 1, \dots, m(l)$ on the Hilbert space $\mathfrak{H}_{\mathfrak{L}}$ satisfy the relation $(\mathfrak{L})$. Hence if $\mathfrak{L}$ satisfies condition (I), the correspondence $S_\alpha \to T_\alpha$ and $E_i^l \to P_i^l$ gives rise to a faithful representation of the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ on $\mathfrak{H}_{\mathfrak{L}}$.*

We call it the universal shift representation of $\mathcal{O}_{\mathfrak{L}}$ on $\mathfrak{H}_{\mathfrak{L}}$. In what follows, we assume that $\mathfrak{L}$ satisfies condition (I) and regard the algebra $\mathcal{O}_{\mathfrak{L}}$ as the C^*-algebra generated by $T_\alpha, \alpha \in \Sigma$ and $P_i^l, i = 1, \dots, m(l)$ on the Hilbert space $\mathfrak{H}_{\mathfrak{L}}$.

3. Topological full inverse semigroups

For $x = (x_n)_{n \in \mathbb{N}} \in X_{\mathfrak{L}}$, the orbit $orb_{\sigma_{\mathfrak{L}}}(x)$ of x is defined by

$$orb_{\sigma_{\mathfrak{L}}}(x) = \cup_{k=0}^{\infty} \cup_{l=0}^{\infty} \sigma_{\mathfrak{L}}^{-k}(\sigma_{\mathfrak{L}}^{l}(x)) \subset X_{\mathfrak{L}}.$$

Hence $y = (y_n)_{n \in \mathbb{N}} \in X_{\mathfrak{L}}$ belongs to $orb_{\sigma_{\mathfrak{L}}}(x)$ if and only if there exists a a finite sequence $z_1 \cdots z_k \in B_k(X_{\mathfrak{L}})$ such that

$$y = (z_1, \dots, z_k, x_{l+1}, x_{l+2}, \dots) \qquad \text{for some } k, l \in \mathbb{Z}_+.$$

We denote by $\mathrm{Homeo}(X_{\mathfrak{L}})$ the group of all homeomorphisms on $X_{\mathfrak{L}}$. We define the full group $[\sigma_{\mathfrak{L}}]$ and the topological full group $[\sigma_{\mathfrak{L}}]_c$ for $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ as in the following way.

Definition. Let $[\sigma_{\mathfrak{L}}]$ be the set of all homeomorphism $\tau \in \mathrm{Homeo}(X_{\mathfrak{L}})$ such that $\tau(x) \in orb_{\sigma_{\mathfrak{L}}}(x)$ for all $x \in X_{\mathfrak{L}}$. We call $[\sigma_{\mathfrak{L}}]$ the full group of $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$.

Let $[\sigma_{\mathfrak{L}}]_c$ be the set of all τ in $[\sigma_{\mathfrak{L}}]$ such that there exist continuous functions $k, l : X_{\mathfrak{L}} \to \mathbb{Z}_+$ such that

$$\sigma_{\mathfrak{L}}^{k(x)}(\tau(x)) = \sigma_{\mathfrak{L}}^{l(x)}(x) \quad \text{for all } x \in X_{\mathfrak{L}}. \tag{3.1}$$

We call $[\sigma_{\mathfrak{L}}]_c$ the topological full group for $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$.

If a subshift is not a sofic shift, the full groups are not necessarily large enough to cover the orbit structure. Hence to study of orbit structure of general subshifts, we will extend the notion of full groups to full inverse semigroups as in the following way. Let $\tau : U \to V$ be a homeomorphism from a clopen set $U \subset X_{\mathfrak{L}}$ onto a clopen set $V \subset X_{\mathfrak{L}}$. We call τ a partial homeomorphism. Let us denote by X_τ and Y_τ the clopen sets U and V respectively. We denote by $PH(X_{\mathfrak{L}})$ the set of all partial homeomorphisms of $X_{\mathfrak{L}}$. Then $PH(X_{\mathfrak{L}})$ has a natural structure of inverse semigroup (cf. [31]). We define the full inverse semigroup $[\sigma_{\mathfrak{L}}]_s$ and the topological full inverse semigroup $[\sigma_{\mathfrak{L}}]_{sc}$ for $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ as in the following way.

Definition. Let $[\sigma_{\mathfrak{L}}]_s$ be the set of all partial homeomorphisms $\tau \in PH(X_{\mathfrak{L}})$ such that $\tau(x) \in orb_{\sigma_{\mathfrak{L}}}(x)$ for all $x \in X_\tau$. We call $[\sigma_{\mathfrak{L}}]_s$ the full inverse semigroup of $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$. Let $[\sigma_{\mathfrak{L}}]_{sc}$ be the set of all τ in $[\sigma_{\mathfrak{L}}]_s$ such that there exist continuous functions $k, l : X_\tau \to \mathbb{Z}_+$ such that

$$\sigma_{\mathfrak{L}}^{k(x)}(\tau(x)) = \sigma_{\mathfrak{L}}^{l(x)}(x) \quad \text{for all } x \in X_\tau. \tag{3.2}$$

We call $[\sigma_{\mathfrak{L}}]_{sc}$ the topological full inverse semigroup for $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$. The maps k, l above are called orbit cocycles for τ, and sometimes written as k_τ, l_τ respectively. We remark that the orbit cocyles are not necessarily uniquely determined for τ. It is clear that $[\sigma_{\mathfrak{L}}]_s$ is a subsemigroup of $PH(X_{\mathfrak{L}})$ and $[\sigma_{\mathfrak{L}}]_{sc}$ is a subsemigroup of $[\sigma_{\mathfrak{L}}]_c$. Although $\sigma_{\mathfrak{L}}$ does not belong to $[\sigma_{\mathfrak{L}}]_{sc}$, the following lemma shows that $\sigma_{\mathfrak{L}}$ locally belongs to $[\sigma_{\mathfrak{L}}]_{sc}$, and that $[\sigma_{\mathfrak{L}}]_{sc}$ is not trivial in any case.

Lemma 3.1. *For any $\mu = (\mu_1, \dots, \mu_k) \in B_k(X_\Lambda)$ and $v_i^l \in V_l$ with $2 \le k \le l$ and $U_{\mu, v_i^l} \ne \emptyset$, there exists $\tau_{\mu, v_i^l} \in [\sigma_{\mathfrak{L}}]_{sc}$ such that*

$$\tau_{\mu, v_i^l}(x) = \sigma_{\mathfrak{L}}(x) \qquad \textit{for } x \in U_{\mu, v_i^l}. \tag{3.3}$$

Proof. Put $\nu = (\mu_2, \dots, \mu_k) \in B_{k-1}(X_\Lambda)$. Then the map $\tau_{\mu, v_i^l} : U_{\mu, v_i^l} \longrightarrow U_{\nu, v_i^l}$ defined by $\tau_{\mu, v_i^l}(x) = \sigma_{\mathfrak{L}}(x)$ for $x \in U_{\mu, v_i^l}$ is a partial homeomorphism, and it belongs to $[\sigma_{\mathfrak{L}}]_{sc}$. □

Lemma 3.2. *For $x = (x_n)_{n \in \mathbb{N}} \in X_{\mathfrak{L}}$ with $x_n = (\alpha_n, u_n), n \in \mathbb{N}$, put $u_0 = u_0(x) \in \Omega_{\mathfrak{L}}$. Let $\alpha_0 \in \Sigma$ be a symbol such that $(\alpha_{n-1}, u_{n-1})_{n \in \mathbb{N}} \in X_{\mathfrak{L}}$. Then there exists $\tau \in [\sigma_{\mathfrak{L}}]_{sc}$ with a clopen set $X_\tau \subset X_{\mathfrak{L}}$ such that $x \in X_\tau$ and $\tau(y) = (y_{n-1})_{n \in \mathbb{N}}$ for all $y = (y_n)_{n \in \mathbb{N}} \in X_\tau$, where $y_0 = (\alpha_0, u_0(y))$.*

Proof. Let X_τ be the clopen set U_{μ,v_i^l} for $\mu = \alpha_1\alpha_2 \in B_2(X_\Lambda)$ and $v_i^2 = u_2^2 \in V_2$, where $u_2 = (u_2^l)_{l\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$, so that x belongs to X_τ. One has $(y_{n-1})_{n\in\mathbb{N}} \in X_{\mathfrak{L}}$ for $(y_n)_{n\in\mathbb{N}} \in X_\tau$, where $y_0 = (\alpha_0, u_0(y))$. By setting $\tau(y) = (y_{n-1})_{n\in\mathbb{N}}$ for $y = (y_n)_{n\in\mathbb{N}} \in X_{\mathfrak{L}}$, we have $\sigma_{\mathfrak{L}}(\tau(y)) = y$ for $y \in X_\tau$ so that $\tau \in [\sigma_{\mathfrak{L}}]_{sc}$. $\square$

For $x \in X_{\mathfrak{L}}$, put $[\sigma_{\mathfrak{L}}]_{sc}(x) = \{\tau(x) \in X_{\mathfrak{L}} \mid \tau \in [\sigma_{\mathfrak{L}}]_{sc} \text{ with } X_\tau \ni x\}$.

Lemma 3.3. $[\sigma_{\mathfrak{L}}]_{sc}(x) = orb_{\sigma_{\mathfrak{L}}}(x)$.

Proof. For any $\tau \in [\sigma_{\mathfrak{L}}]_{sc}$ with $X_\tau \ni x$, one sees $\tau(x) \in orb_{\sigma_{\mathfrak{L}}}(x)$ and hence $[\sigma_{\mathfrak{L}}]_{sc}(x) \subset orb_{\sigma_{\mathfrak{L}}}(x)$. For the other inclusion relation, by the previous lemmas, for $x = (x_n)_{n\in\mathbb{N}} \in X_{\mathfrak{L}}$ and $x_0 = (\alpha_0, u_0(x)) \in \Sigma \times \Omega_{\mathfrak{L}}$, there exist $\tau_1, \tau_2 \in [\sigma_{\mathfrak{L}}]_{sc}$ such that

$$\tau_1(x) = (x_{n-1})_{n\in\mathbb{N}}, \qquad \tau_2(x) = (x_{n+1})_{n\in\mathbb{N}} \in X_{\mathfrak{L}}$$

so that both $(x_{n-1})_{n\in\mathbb{N}}$ and $(x_{n+1})_{n\in\mathbb{N}}$ belong to $[\sigma_{\mathfrak{L}}]_{sc}(x)$. Since $[\sigma_{\mathfrak{L}}]_{sc}$ is a semigroup, one sees that

$$[\sigma_{\mathfrak{L}}]_{sc}(x) \ni (x_{-k}, \dots, x_{-1}, x_0, x_{l+1}, x_{l+2}, \dots)$$

for all $k, l \in \mathbb{Z}_+$ with $(x_{-k}, \dots, x_{-1}, x_0, x_{l+1}, x_{l+2}, \dots) \in X_{\mathfrak{L}}$. Hence $[\sigma_{\mathfrak{L}}]_{sc}(x) \supset orb_{\sigma_{\mathfrak{L}}}(x)$. $\square$

4. Full inverse semigroups and normalizers

Let us denote by $\mathcal{U}(\mathcal{O}_{\mathfrak{L}})$ the group of unitaries of $\mathcal{O}_{\mathfrak{L}}$ and $\mathcal{U}(\mathcal{D}_{\mathfrak{L}})$ the group of unitaries of $\mathcal{D}_{\mathfrak{L}}$ respectively. As in [30], the topological full group $[\sigma_{\mathfrak{L}}]_c$ will correspond to the normalizer $N(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$ of $\mathcal{D}_{\mathfrak{L}}$ in $\mathcal{O}_{\mathfrak{L}}$ defined by

$$N(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}}) = \{v \in \mathcal{U}(\mathcal{O}_{\mathfrak{L}}) \mid v\mathcal{D}_{\mathfrak{L}}v^* = \mathcal{D}_{\mathfrak{L}}\}.$$

For the topological full inverse semigroup $[\sigma_{\mathfrak{L}}]_{sc}$, we will define the normalizer $N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$ of partial isometries as in the following way:

$$N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}}) = \{v \in \mathcal{O}_{\mathfrak{L}} \mid v \text{ is a partial isometry}; v\mathcal{D}_{\mathfrak{L}}v^* \subset \mathcal{D}_{\mathfrak{L}}, v^*\mathcal{D}_{\mathfrak{L}}v \subset \mathcal{D}_{\mathfrak{L}}\}.$$

It is easy to see that $N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$ has a natural structure of inverse semigroup. We will identify the subalgebra $\mathcal{D}_{\mathfrak{L}}$ of $\mathcal{O}_{\mathfrak{L}}$ with the algebra $C(X_{\mathfrak{L}})$. For a partial isometry $v \in \mathcal{O}_{\mathfrak{L}}$, put $Ad(v)(x) = vxv^*$ for $x \in \mathcal{O}_{\mathfrak{L}}$. The following proposition holds.

PROPOSITION 4.1. *For $\tau \in [\sigma_{\mathfrak{L}}]_{sc}$, there exists a partial isometry $u_\tau \in N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$ such that*

$$Ad(u_\tau)(f) = f \circ \tau^{-1} \quad \text{for } f \in C(X_\tau), \qquad Ad(u_\tau^*)(g) = g \circ \tau \quad \text{for } g \in C(Y_\tau),$$

and the correspondence $\tau \in [\sigma_{\mathfrak{L}}]_{sc} \longrightarrow u_\tau \in N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$ is a homomorphism of inverse semigroup. If in particular $\tau \in [\sigma_{\mathfrak{L}}]_c$, the partial isometry u_τ is a unitary so that $u_\tau \in N(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$.

Proof. Let the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ be represented on the Hilbert space $\mathfrak{H}_{\mathfrak{L}}$ with complete orthonormal basis $\{e_x \mid x \in X_{\mathfrak{L}}\}$. Put the subspaces

$$\mathfrak{H}_{X_\tau} = \operatorname{span}\{e_x \mid x \in X_\tau\}, \qquad \mathfrak{H}_{Y_\tau} = \operatorname{span}\{e_x \mid x \in Y_\tau\}.$$

Since $\tau : X_\tau \longrightarrow Y_\tau$ is a homeomorphism, the operator $u_\tau : \mathfrak{H}_{X_\tau} \longrightarrow \mathfrak{H}_{Y_\tau}$ defined by $u_\tau(e_x) = e_{\tau(x)}$ for $x \in X_\tau$ yields a partial isometry on $\mathfrak{H}_{\mathfrak{L}}$. By a similar manner to the proof of [30, Proposition 4.1], one knows that u_τ belongs to $N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$. $\square$

For $v \in N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$, put the projections $p_v = v^*v, q_v = vv^*$ in $\mathcal{D}_{\mathfrak{L}}$, and the clopen subsets $X_v = \operatorname{supp}(p_v), Y_v = \operatorname{supp}(q_v)$ of $X_{\mathfrak{L}}$. Then $Ad(v) : \mathcal{D}_{\mathfrak{L}} p_v \longrightarrow \mathcal{D}_{\mathfrak{L}} q_v$ is an isomorphism and induces a partial homeomorphism $\tau_v : X_v \longrightarrow Y_v$ such that

$$Ad(v)(f) = f \circ \tau_v^{-1} \quad \text{for } f \in C(X_v), \qquad Ad(v^*)(g) = g \circ \tau_v \quad \text{for } g \in C(Y_v).$$

We will prove that τ_v gives rise to an element of $[\sigma_{\mathfrak{L}}]_{sc}$. Since the proof basically follows a line of the proof of [30, Proposition 4.7], we will give a sketch of the proof. Fix $v \in N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$ for a while.

LEMMA 4.2.

(i) *There exists a family $v_m, m \in \mathbb{Z}$ of partial isometries in $\mathcal{O}_{\mathfrak{L}}$ such that all but finitely many $v_m, m \in \mathbb{Z}$ are zero, and*
   (1) $v = \sum_{m \in \mathbb{Z}} v_m$ *: finite sum.*
   (2) $v_m^* v_m, v_m v_m^*$ *are projections in $\mathcal{D}_{\mathfrak{L}}$ for $m \in \mathbb{Z}$.*
   (3) $v_m \mathcal{D}_{\mathfrak{L}} v_m^* \subset \mathcal{D}_{\mathfrak{L}}$ *and* $v_m^* \mathcal{D}_{\mathfrak{L}} v_m \subset \mathcal{D}_{\mathfrak{L}}$ *for $m \in \mathbb{Z}$.*
   (4) $v_m^* v_{m'} = v_m v_{m'}^* = 0$ *for $m \neq m'$.*
   (5) $v_0 \in \mathcal{F}_{\mathfrak{L}}$.

(ii) *For a fixed $n \in \mathbb{N}$, there exist partial isometries $v_\mu, v_{-\mu} \in \mathcal{F}_{\mathfrak{L}}$ for each $\mu \in B_n(X_\Lambda)$ satisfying the following conditions:*
   (1) $v_n = \sum_{\mu \in B_n(X_\Lambda)} S_\mu v_\mu$ *and* $v_{-n} = \sum_{\mu \in B_n(X_\Lambda)} v_{-\mu} S_\mu^*$.

(2) *$v_\mu^* v_\mu$, $S_\mu v_\mu v_\mu^* S_\mu^*$, $S_\mu v_{-\mu}^* v_{-\mu} S_\mu^*$ and $v_{-\mu} v_{-\mu}^*$ are projections in $\mathcal{D}_{\mathfrak{L}}$ such that*

$$v_n^* v_n = \sum_{\mu \in B_n(X_\Lambda)} v_\mu^* v_\mu, \qquad v_n v_n^* = \sum_{\mu \in B_n(X_\Lambda)} S_\mu v_\mu v_\mu^* S_\mu^*,$$
$$v_{-n}^* v_{-n} = \sum_{\mu \in B_n(X_\Lambda)} S_\mu v_{-\mu}^* v_{-\mu} S_\mu^*, \qquad v_{-n} v_{-n}^* = \sum_{\mu \in B_n(X_\Lambda)} v_{-\mu} v_{-\mu}^*.$$

(3) *$v_\mu v_\nu^* = v_{-\mu}^* v_{-\nu} = 0$ for $\mu, \nu \in B_n(X_\Lambda)$ with $\mu \neq \nu$.*

(4) *The algebras $v_\mu \mathcal{D}_{\mathfrak{L}} v_\mu^*, v_\mu^* \mathcal{D}_{\mathfrak{L}} v_\mu, v_{-\mu} \mathcal{D}_{\mathfrak{L}} v_{-\mu}^*$ and $v_{-\mu}^* \mathcal{D}_{\mathfrak{L}} v_{-\mu}$ are contained in $\mathcal{D}_{\mathfrak{L}}$.*

Proof. (i) Put a partial isometry $g(t) = v^* \rho_t(v) \in \mathcal{O}_{\mathfrak{L}}$ for $t \in \mathbb{T}$. For $f \in \mathcal{D}_{\mathfrak{L}}$, it follows that $\rho_t(v) f \rho_t(v)^* = \rho_t(v f v^*) = v f v^*$ and hence

$$g(t) f = v^* \rho_t(v) f \rho_t(v^*) \rho_t(v) = v^* v f v^* \rho_t(v) = f g(t)$$

so that $g(t)$ commutes with each element of $\mathcal{D}_{\mathfrak{L}}$. By Lemma 2.1, $g(t)$ belongs to the algebra $\mathcal{D}_{\mathfrak{L}}$. Since $g(t)^* = g(-t)$ and $g(t+s) = g(t)g(s)$, by putting

$$v_m = \int_{\mathbb{T}} \rho_t(v) e^{-\sqrt{-1}mt} dt, \qquad \hat{g}(m) = \int_{\mathbb{T}} g(t) e^{-\sqrt{-1}mt} dt \quad \text{ for } m \in \mathbb{Z}.$$

one has $v_m = v\hat{g}(m)$. By a similar argument to the proof of [30, Lemma 4.2], one has the assertions (1),(2),(3), (4) and (5).

(ii) Put for $\mu \in B_n(X_{\mathfrak{L}})$,

$$v_\mu = E(S_\mu^* v), \qquad v_{-\mu} = E(v S_\mu).$$

By a similar argument to the proof of [30, Lemma 4.3], one has the assertions (1),(2),(3) and (4). □

For $u \in N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$, let $\tau_u : X_u \to Y_u$ be the induced homeomorphism.

Lemma 4.3. *Keep the above notation. For $x = (x_n)_{n \in \mathbb{N}} \in X_u$ with $x_n = (\alpha_n, u_n(x))$, $u_n(x) = (u_n^l(x))_{l \in \mathbb{Z}_+}$, put $y = (y_n)_{n \in \mathbb{N}} = \tau_u(x) \in Y_u$, where $y_n = (\beta_n, u_n(y))$, $u_n(y) = (u_n^l(y))_{l \in \mathbb{Z}_+}$. For a fixed integer $l \in \mathbb{Z}_+$, take $i(x_n) \in \{1, \dots, m(l)\}$ and $i(y_n) \in \{1, \dots, m(l)\}$ such that $v_{i(x_n)}^l = u_n^l(x)$ and $v_{i(y_n)}^l = u_n^l(y)$ respectively. Then we have*

$$\| E_{i(y_n)}^l S_{\beta_1 \cdots \beta_n}^* u S_{\alpha_1 \cdots \alpha_n} E_{i(x_n)}^l \| = 1 \quad \textit{for all } n \in \mathbb{N}.$$

Proof. It suffices to show that $E^l_{i(y_n)}S^*_{\beta_1\cdots\beta_n}uS_{\alpha_1\cdots\alpha_n}E^l_{i(x_n)} \neq 0$. Since $v^l_{i(y_n)} = u^l_n(y)$, one sees that $E^l_{i(y_n)}e_{\sigma_{\mathfrak{L}}{}^n(y)} = e_{\sigma_{\mathfrak{L}}{}^n(y)}$ so that

$$\begin{aligned}
&(E^l_{i(y_n)}S^*_{\beta_1\cdots\beta_n}uS_{\alpha_1\cdots\alpha_n}E^l_{i(x_n)}S^*_{\alpha_1\cdots\alpha_n}u^*S_{\beta_1\cdots\beta_n}E^l_{i(y_n)}e_{\sigma_{\mathfrak{L}}{}^n(y)} \mid e_{\sigma_{\mathfrak{L}}{}^n(y)})\\
&= (Ad(u)(S_{\alpha_1\cdots\alpha_n}E^l_{i(x_n)}S^*_{\alpha_1\cdots\alpha_n})S_{\beta_1\cdots\beta_n}e_{\sigma_{\mathfrak{L}}{}^n(y)} \mid S_{\beta_1\cdots\beta_n}e_{\sigma_{\mathfrak{L}}{}^n(y)})\\
&= (Ad(u)(S_{\alpha_1\cdots\alpha_n}E^l_{i(x_n)}S^*_{\alpha_1\cdots\alpha_n})e_y \mid e_y).
\end{aligned}$$

Consider the cylinder set

$$U_{\alpha_1\cdots\alpha_n,v^l_{i(x_n)}} = \{(\gamma_m,u_m)_{m\in\mathbb{N}} \in X_{\mathfrak{L}} \mid \gamma_1 = \alpha_1,\dots,\gamma_n = \alpha_n, u^l_n = v^l_{i(x_n)}\}$$

of $X_{\mathfrak{L}}$. As $S_{\alpha_1\cdots\alpha_n}E^l_{i(x_n)}S^*_{\alpha_1\cdots\alpha_n} = \chi_{U_{\alpha_1\cdots\alpha_n,v^l_{i(x_n)}}}$ and

$$\begin{aligned}
&Ad(u)(\chi_{U_{\alpha_1\cdots\alpha_n,v^l_{i(x_n)}}})e_y\\
&= (\chi_{U_{\alpha_1\cdots\alpha_n,v^l_{i(x_n)}}}\circ\tau_u^{-1})(y)e_y = \chi_{U_{\alpha_1\cdots\alpha_n,v^l_{i(x_n)}}}(x)e_y = e_y,
\end{aligned}$$

we have

$$\begin{aligned}
&(E^l_{i(y_n)}S^*_{\beta_1\cdots\beta_n}uS_{\alpha_1\cdots\alpha_n}E^l_{i(x_n)}S^*_{\alpha_1\cdots\alpha_n}u^*S_{\beta_1\cdots\beta_n}E^l_{i(y_n)}e_{\sigma_{\mathfrak{L}}{}^n(y)} \mid e_{\sigma_{\mathfrak{L}}{}^n(y)})\\
&= (e_y \mid e_y) = 1
\end{aligned}$$

so that $E^l_{i(y_n)}S^*_{\beta_1\cdots\beta_n}uS_{\alpha_1\cdots\alpha_n}E^l_{i(x_n)} \neq 0$. □

Lemma 4.4. *Keep the above situation. Assume in particular that $u \in \mathcal{F}_{\mathfrak{L}}$. Then there exists $k \in \mathbb{N}$ such that for all $x = (x_n)_{n\in\mathbb{N}} \in X_u$*

$$\tau_u(x)_n = x_n \qquad \textit{for all } n > k$$

where $\tau_u(x) = (\tau_u(x)_n)_{n\in\mathbb{N}}$.

Proof. Suppose that for any $k \in \mathbb{N}$ there exist $x \in X_u$ and $N > k$ such that $\tau_u(x)_N \neq x_N$. Put $y_n = \tau_u(x)_n, n \in \mathbb{N}$. Now $u \in \mathcal{F}_{\mathfrak{L}}$ so that take $u_0 \in \mathcal{F}^{k_0}_{l_0}$ for some $k_0 \leq l_0$ such that $\|u - u_0\| < \frac{1}{2}$. Take $x \in X_u$ and $N_0 > k_0$ such as $y_{N_0} \neq x_{N_0}$. Since $x_{N_0} = (\alpha_{N_0}, u_{N_0}(x)), y_{N_0} = (\beta_{N_0}, u_{N_0}(y))$ and $u_{N_0}(x) = (u^l_{N_0}(x))_{l\in\mathbb{N}}, u_{N_0}(y) = (u^l_{N_0}(y))_{l\in\mathbb{N}} \in \Omega_{\mathfrak{L}}$, one has $\alpha_{N_0} \neq \beta_{N_0}$ or there exists l_1 such that $u^l_{N_0}(x) \neq u^l_{N_0}(y)$ fo all $l \geq l_1$. As $u^l_{N_0}(x) = v^l_{i(x_{N_0})}, u^l_{N_0}(y) = v^l_{i(y_{N_0})}$, the later condition is equivalent to the condition that $E^l_{i(x_{N_0})} \neq E^l_{i(y_{N_0})}$ fo all $l \geq l_1$. Now $u_0 \in \mathcal{F}^{k_0}_{l_0} \subset \mathcal{F}^{N_0-1}_{l'_0}$, where $l'_0 = l_0 + N_0 - 1 - k_0$, it is written as

$$u_0 = \sum_{\xi,\eta\in B_{N_0-1}(X_\Lambda),j=1,\dots,m(l'_0)} c_{\xi,j,\eta}S_\xi E^{l'_0}_j S^*_\eta \in \mathcal{F}^{N_0-1}_{l'_0} \quad \text{for some } c_{\xi,j,\eta} \in \mathbb{C}.$$

Hence we have

$$
\begin{aligned}
& S^*_{\beta_1\cdots\beta_{N_0-1}} u_0 S_{\alpha_1\cdots\alpha_{N_0-1}} \\
= & \sum_{j=1}^{m(l'_0)} c_{\beta_1\cdots\beta_{N_0-1},j,\alpha_1\cdots\alpha_{N_0-1}} S^*_{\beta_1\cdots\beta_{N_0-1}} S_{\beta_1\cdots\beta_{N_0-1}} E_j^{l'_0} S^*_{\alpha_1\cdots\alpha_{N_0-1}} S_{\alpha_1\cdots\alpha_{N_0-1}}.
\end{aligned}
$$

Take an integer l'_1 such that $l'_1 \ge \max\{l_1, l'_0\}$ and hence the condition $\alpha_{N_0} \ne \beta_{N_0}$ or $E^{l'_1}_{i(x_{N_0})} \cdot E^{l'_1}_{i(y_{N_0})} = 0$ holds. It follows that

$$
\begin{aligned}
& E^{l'_1}_{i(y_{N_0})} S^*_{\beta_1\cdots\beta_{N_0}} u_0 S_{\alpha_1\cdots\alpha_{N_0}} E^{l'_1}_{i(x_{N_0})} = \\
& \sum_{j=1}^{m(l'_0)} c_{\beta_1\cdots\beta_{N_0-1},j,\alpha_1\cdots\alpha_{N_0-1}} E^{l'_1}_{i(y_{N_0})} S^*_{\beta_1\cdots\beta_{N_0}} S_{\beta_1\cdots\beta_{N_0-1}} E_j^{l'_0} S^*_{\alpha_1\cdots\alpha_{N_0-1}} S_{\alpha_1\cdots\alpha_{N_0}} E^{l'_1}_{i(x_{N_0})}.
\end{aligned}
$$

Since $S^*_{\beta_1\cdots\beta_{N_0-1}} S_{\beta_1\cdots\beta_{N_0-1}} E_j^{l'_0} S^*_{\alpha_1\cdots\alpha_{N_0-1}} S_{\alpha_1\cdots\alpha_{N_0-1}}$ belongs to $\mathcal{D}_{\mathfrak{L}}$, one has

$$
E^{l'_1}_{i(y_{N_0})} S^*_{\beta_1\cdots\beta_{N_0}} S_{\beta_1\cdots\beta_{N_0-1}} E_j^{l'_0} S^*_{\alpha_1\cdots\alpha_{N_0-1}} S_{\alpha_1\cdots\alpha_{N_0}} E^{l'_1}_{i(x_{N_0})} = 0, \quad j = 1, \dots, m(l'_0)
$$

because $\alpha_{N_0} \ne \beta_{N_0}$ or $E^{l'_1}_{i(x_{n_0})} \cdot E^{l'_1}_{i(y_{n_0})} = 0$. This implies that

$$
E^{l'_1}_{i(y_{N_0})} S^*_{\beta_1\cdots\beta_{N_0}} u_0 S_{\alpha_1\cdots\alpha_{N_0}} E^{l'_1}_{i(x_{N_0})} = 0
$$

so that

$$
E^{l'_1}_{i(y_{N_0})} S^*_{\beta_1\cdots\beta_{N_0}} u S_{\alpha_1\cdots\alpha_{N_0}} E^{l'_1}_{i(x_{N_0})} = 0
$$

a contradiction to the preceding lemma. □

Thus we have

Lemma 4.5. *For a partial isometry $u \in \mathcal{F}_{\mathfrak{L}}$ satisfying*

$$
u\mathcal{D}_{\mathfrak{L}}u^* \subset \mathcal{D}_{\mathfrak{L}}, \qquad u^*\mathcal{D}_{\mathfrak{L}}u \subset \mathcal{D}_{\mathfrak{L}},
$$

let $\tau_u : \mathrm{supp}(u^*u) \to \mathrm{supp}(uu^*)$ be the homeomorphism defined by $Ad(u)(g) = g \circ \tau_u^{-1}$ for $g \in \mathcal{D}_{\mathfrak{L}}u^*u$. Then there exists $k_u \in \mathbb{N}$ such that

$$
\sigma_{\mathfrak{L}}^{k_u}(\tau_u(x)) = \sigma_{\mathfrak{L}}^{k_u}(x) \qquad \text{for } x \in \mathrm{supp}(u^*u).
$$

Therefore by Lemma 4.2 and Lemma 4.5 we have

Proposition 4.6. *For any $v \in N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$, the partial homomorphism τ_v induced by $Ad(v)$ on $\mathcal{D}_{\mathfrak{L}}$ gives rise to an element of the topological full inverse semigroup $[\sigma_{\mathfrak{L}}]_{sc}$. If in particular v belongs to $N(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$, then τ_v belongs to $[\sigma_{\mathfrak{L}}]_c$.*

Proof. The argument of the proof is the same as that of [30, Proposition 4.7]. □

The unitaries $\mathcal{U}(\mathcal{D}_{\mathfrak{L}})$ are naturally embedded into $N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$. We denote the embedding by id. For $v \in N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$, the induced partial homemorphism τ_v on $X_{\mathfrak{L}}$ gives rise to an element of $[\sigma_{\mathfrak{L}}]_{sc}$ by the above proposition. We then have

THEOREM 4.7. *The diagrams*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{U}(\mathcal{D}_{\mathfrak{L}}) & \xrightarrow{\text{id}} & N(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}}) & \xrightarrow{\tau} & [\sigma_{\mathfrak{L}}]_c & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \\
1 & \longrightarrow & \mathcal{U}(\mathcal{D}_{\mathfrak{L}}) & \xrightarrow{\text{id}} & N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}}) & \xrightarrow{\tau} & [\sigma_{\mathfrak{L}}]_{sc} & \longrightarrow & 1.
\end{array}$$

are all commutative, where two vertical arrows denoted by ι are inclusions. The first row sequence is exact and splits as group, and the second row sequence is exact and splits as inverse semigroup.

Proof. By Proposition 4.6, the map $\tau : v \in N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}}) \longrightarrow \tau_v \in [\sigma_{\mathfrak{L}}]_{sc}$ defines a homomorphism as inverse semigroup such that $\tau(N(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})) = [\sigma_{\mathfrak{L}}]_c$. It is surjective by Proposition 4.1. Suppose that $\tau_v = \text{id}$ on $X_{\mathfrak{L}}$ for some $v \in N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$. This means that $Ad(v) = \text{id}$ on $\mathcal{D}_{\mathfrak{L}}$. Hence v commutes with all of elements of $\mathcal{D}_{\mathfrak{L}}$. By Lemma 2.1, v belongs to $\mathcal{D}_{\mathfrak{L}}$. Therefore the second row sequence is exact. Similarly, the first row sequence is exact. As in Proposition 4.1, the partial isometry u_τ for $\tau \in [\sigma_{\mathfrak{L}}]_{sc}$ defined by $u_\tau e_x = e_{\tau(x)}$, $x \in X_\tau \subset X_{\mathfrak{L}}$ gives rise to sections of the both exact sequences. Hence the both row sequences split. The commutativity of the diagrams is clear. □

5. Orbit equivalence of $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$

In this section, we will study orbit equivalence between two dynamical systems $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ defined by λ-graph systems $\mathfrak{L}_1$ and $\mathfrak{L}_2$ respectively.

DEFINITION. For λ-graph systems $\mathfrak{L}_1$ and $\mathfrak{L}_2$, if there exists a homeomorphism $h : X_{\mathfrak{L}_1} \to X_{\mathfrak{L}_2}$ such that $h(orb_{\sigma_{\mathfrak{L}_1}}(x)) = orb_{\sigma_{\mathfrak{L}_2}}(h(x))$ for $x \in X_{\mathfrak{L}_1}$, then $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ are said to be topologically orbit equivalent. In this case, there exist functions $k_1, l_1 : X_{\mathfrak{L}_1} \to \mathbb{Z}_+$ and $k_2, l_2 : X_{\mathfrak{L}_2} \to \mathbb{Z}_+$ satisfying

$$\begin{cases} \sigma_{\mathfrak{L}_2}^{k_1(x)}(h(\sigma_{\mathfrak{L}_1}(x))) = & \sigma_{\mathfrak{L}_2}^{l_1(x)}(h(x)) \quad \text{for } x \in X_{\mathfrak{L}_1}, \\ \sigma_{\mathfrak{L}_1}^{k_2(y)}(h^{-1}(\sigma_{\mathfrak{L}_2}(y))) = & \sigma_{\mathfrak{L}_1}^{l_2(y)}(h^{-1}(y)) \quad \text{for } y \in X_{\mathfrak{L}_2}. \end{cases} \tag{5.1}$$

We say that $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ are *continuously orbit equivalent* if there exist continuous functions k_1, $l_1 : X_{\mathfrak{L}_1} \to \mathbb{Z}_+$ and k_2, $l_2 : X_{\mathfrak{L}_2} \to \mathbb{Z}_+$ satisfying the equalities (5.1).

The following lemma is straightforward.

Lemma 5.1. *If $h : X_{\mathfrak{L}_1} \to X_{\mathfrak{L}_2}$ is a homeomorphism satisfying $\sigma_{\mathfrak{L}_2}^{k(x)}(h(\sigma_{\mathfrak{L}_1}(x))) = \sigma_{\mathfrak{L}_2}^{l(x)}(h(x)), x \in X_{\mathfrak{L}_1}$ for some functions $k, l : X_{\mathfrak{L}_1} \to \mathbb{Z}_+$, then by putting*

$$k^n(x) = \sum_{i=0}^{n-1} k(\sigma_{\mathfrak{L}_1}^i(x)), \qquad l^n(x) = \sum_{i=0}^{n-1} l(\sigma_{\mathfrak{L}_1}^i(x)), \qquad n \in \mathbb{N}$$

we have

$$\sigma_{\mathfrak{L}_2}^{k^n(x)}(h(\sigma_{\mathfrak{L}_1}^n(x))) = \sigma_{\mathfrak{L}_2}^{l^n(x)}(h(x)), \qquad x \in X_{\mathfrak{L}_1}.$$

Lemma 5.2. *If $h : X_{\mathfrak{L}_1} \to X_{\mathfrak{L}_2}$ is a homeomorphism satisfying (5.1), then it satisfies*

$$h(orb_{\sigma_{\mathfrak{L}_1}}(x)) = orb_{\sigma_{\mathfrak{L}_2}}(h(x)) \qquad \text{for } x \in X_{\mathfrak{L}_1}.$$

Hence continuous orbit equivalence implies topological orbit equivalence.

Proof. By the preceding lemma, one has

$$h(\sigma_{\mathfrak{L}_1}^n(x)) \subset \sigma_{\mathfrak{L}_2}^{-k^n(x)}(\sigma_{\mathfrak{L}_2}^{l^n(x)}(h(x))), \qquad x \in X_{\mathfrak{L}_1}, n \in \mathbb{N}$$

so that $h(\sigma_{\mathfrak{L}_1}^n(x)) \subset orb_{\sigma_{\mathfrak{L}_2}}(h(x))$. For $(z_1, \dots, z_m, x_1, x_2, \dots) \in \sigma_{\mathfrak{L}_1}^{-m}(x)$, where $x = (x_n)_{n\in\mathbb{N}}$, one has $\sigma^m(z_1, \dots, z_m, x_1, x_2, \dots) = x$ and hence $h(z_1, \dots, z_m, x_1, x_2, \dots) \in \sigma_{\mathfrak{L}_2}^{-l_1^m(x)} \sigma_{\mathfrak{L}_2}^{-k_1^m(x)}(h(x))$. This implies that $h(orb_{\sigma_{\mathfrak{L}_1}}(x)) \subset orb_{\sigma_{\mathfrak{L}_2}}(h(x))$.

One similarly has the inclusion relation $h^{-1}(orb_{\sigma_{\mathfrak{L}_2}}(y)) \subset orb_{\sigma_{\mathfrak{L}_1}}(h^{-1}(y))$ for $y \in X_{\mathfrak{L}_2}$ by considering h^{-1} as h in the above discussion. This implies that $orb_{\sigma_{\mathfrak{L}_2}}(h(x)) \subset h(orb_{\sigma_{\mathfrak{L}_1}}(x))$ for $x \in X_{\mathfrak{L}_1}$ so that $h(orb_{\sigma_{\mathfrak{L}_1}}(x)) = orb_{\sigma_{\mathfrak{L}_2}}(h(x))$. □

Proposition 5.3. *If there exists a homeomorphism $h : X_{\mathfrak{L}_1} \longrightarrow X_{\mathfrak{L}_2}$ such that $h \circ [\sigma_{\mathfrak{L}_1}]_{sc} \circ h^{-1} = [\sigma_{\mathfrak{L}_2}]_{sc}$, then $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ are continuously orbit equivalent.*

Proof. Let us denote by $\{v_1^2, \dots, v_{m(2)}^2\}$ the vertex set V_2. For $i = 1, \dots, m(2)$, let $B_2(v_i^2)$ be the set of all admissible words of length 2 terminating at v_i^2. That is

$$\begin{aligned} B_2(v_i^2) = \{(\mu_1, \mu_2) \in B_2(X_\Lambda) \mid &\text{there exist } e_1 \in E_{0,1}, e_2 \in E_{1,2}; \\ &\lambda(e_1) = \mu_1, \lambda(e_2) = \mu_2, t(e_1) = s(e_2), t(e_2) = v_i^2\}. \end{aligned}$$

For $\mu \in B_2(v_i^2)$, by Lemma 3.1, there exists $\tau_\mu \in [\sigma_{\mathfrak{L}_1}]_{sc}$ such that $\tau_\mu(x) = \sigma_{\mathfrak{L}}(x)$ for $x \in U_{\mu,v_i^2}$. Put $\tau_{h,\mu} = h \circ \tau_\mu \circ h^{-1} \in h \circ [\sigma_{\mathfrak{L}_1}]_{sc} \circ h^{-1} = [\sigma_{\mathfrak{L}_2}]_{sc}$. There exist continuous functions $k_{\tau_{h,\mu}}, l_{\tau_{h,\mu}} : h(U_{\mu,v_i^2}) \to \mathbb{Z}_+$ such that

$$\sigma_{\mathfrak{L}_2}^{k_{\tau_{h,\mu}}(y)}(\tau_{h,\mu}(y)) = \sigma_{\mathfrak{L}_2}^{l_{\tau_{h,\mu}}(y)}(y), \qquad y \in h(U_{\mu,v_i^2}).$$

For $x \in U_{\mu,v_i^2}$, one has $\tau_{h,\mu}(h(x)) = h \circ \tau_\mu(x) = h \circ \sigma_{\mathfrak{L}_1}(x)$ so that

$$\sigma_{\mathfrak{L}_2}^{k_{\tau_{h,\mu}}(h(x))}(h \circ \sigma_{\mathfrak{L}_1}(x)) = \sigma_{\mathfrak{L}_2}^{l_{\tau_{h,\mu}}(h(x))}(h(x)), \qquad x \in U_{\mu,v_i^2}.$$

Since $X_{\mathfrak{L}_1}$ is a disjoint union $\cup_{i=1}^{m(2)} \cup_{\mu \in B_2(v_i^2)} U_{\mu,v_i^2}$, by putting

$$k_1(x) = k_{\tau_{h,\mu}}(h(x)), \quad l_1(x) = l_{\tau_{h,\mu}}(h(x)) \qquad \text{for } x \in U_{\mu,v_i^2},$$

we have continuous functions $k_1, l_1 : X_{\mathfrak{L}_1} \longrightarrow \mathbb{Z}_+$ satisfying

$$\sigma_{\mathfrak{L}_2}^{k_1(x)}(h \circ \sigma_{\mathfrak{L}_1}(x)) = \sigma_{\mathfrak{L}_2}^{l_1(x)}(h(x)), \quad x \in X_{\mathfrak{L}_1}.$$

We similarly have continuous functions $k_2, l_2 : X_{\mathfrak{L}_2} \longrightarrow \mathbb{Z}_+$ satisfying

$$\sigma_{\mathfrak{L}_1}^{k_2(y)}(h^{-1} \circ \sigma_{\mathfrak{L}_2}(y)) = \sigma_{\mathfrak{L}_1}^{l_2(x)}(h^{-1}(y)), \quad y \in X_{\mathfrak{L}_2}.$$

Hence $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ are continuously orbit equivalent. □

Conversely we have

PROPOSITION 5.4. *If $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ are continuously orbit equivalent, then there exists a homeomorphism $h : X_{\mathfrak{L}_1} \longrightarrow X_{\mathfrak{L}_2}$ such that $h \circ [\sigma_{\mathfrak{L}_1}]_{sc} \circ h^{-1} = [\sigma_{\mathfrak{L}_2}]_{sc}$.*

Proof. Suppose that there exist a homeomorphism $h : X_{\mathfrak{L}_1} \to X_{\mathfrak{L}_2}$ and continuous functions $k_1, l_1 : X_{\mathfrak{L}_1} \to \mathbb{Z}_+$ and $k_2, l_2 : X_{\mathfrak{L}_2} \to \mathbb{Z}_+$ satisfying (5.1). For $n \in \mathbb{N}$, let $k_1^n, l_1^n : X_{\mathfrak{L}_1} \longrightarrow \mathbb{Z}_+$ and $k_2^n, l_2^n : X_{\mathfrak{L}_2} \longrightarrow \mathbb{Z}_+$ be continuous functions as in Lemma 5.1 such that

$$\sigma_{\mathfrak{L}_2}^{k_1^n(x)}(h(\sigma_{\mathfrak{L}_1}^n(x)) = \sigma_{\mathfrak{L}_2}^{l_1^n(x)}(h(x)), \quad \sigma_{\mathfrak{L}_1}^{k_2^n(y)}(h^{-1}(\sigma_{\mathfrak{L}_2}^n(y)) = \sigma_{\mathfrak{L}_1}^{l_2^n(y)}(h^{-1}(y)) \qquad (5.2)$$

for $x \in X_{\mathfrak{L}_1}$ and $y \in X_{\mathfrak{L}_2}$. For any $\tau \in [\sigma_{\mathfrak{L}_1}]_{sc}$, there exist continuous functions: $k_\tau, l_\tau : X_\tau \longrightarrow \mathbb{Z}_+$ such that

$$\sigma_{\mathfrak{L}_1}^{k_\tau(x)}(\tau(x)) = \sigma_{\mathfrak{L}_1}^{l_\tau(x)}(x), \qquad x \in X_\tau. \qquad (5.3)$$

For $y \in h(X_\tau)$, set $x = h^{-1}(y) \in X_\tau$. Put $m = k_\tau(x)$. By (5.2) and (5.3), one has

$$\sigma_{\mathfrak{L}_2}^{l_1^m(\tau(x))}(h(\tau(x)) = \sigma_{\mathfrak{L}_2}^{k_1^m(\tau(x))}(h(\sigma_{\mathfrak{L}_1}^m(\tau(x))) = \sigma_{\mathfrak{L}_2}^{k_1^m(\tau(x))}(h(\sigma_{\mathfrak{L}_1}^{l_\tau(x)}(x))$$

Put $n = l_\tau(x) \in \mathbb{N}$. By applying $\sigma_{\mathfrak{L}_2}^{k_1^n(x)}$ to the above equalities, one has by (5.2)

$$\begin{aligned}&\sigma_{\mathfrak{L}_2}^{k_1^n(x)+l_1^m(\tau(x))}(h(\tau(x))\\ =&\sigma_{\mathfrak{L}_2}^{k_1^m(\tau(x))}\sigma_{\mathfrak{L}_2}^{k_1^n(x)}(h(\sigma_{\mathfrak{L}_1}^n(x))) = \sigma_{\mathfrak{L}_2}^{k_1^m(\tau(x))}\sigma_{\mathfrak{L}_2}^{l_1^n(x)}(h(x)) = \sigma_{\mathfrak{L}_2}^{k_1^m(\tau(x))+l_1^n(x)}(h(x))\end{aligned}$$

and hence

$$\sigma_{\mathfrak{L}_2}^{k_1^n(x)+l_1^m(\tau(x))}(h\circ\tau\circ h^{-1}(y)) = \sigma_{\mathfrak{L}_2}^{k_1^m(\tau(x))+l_1^n(x)}(y).$$

By setting for $y \in h(X_\tau)$,

$$\begin{aligned}k_\tau^h(y) &= k_1^n(x) + l_1^m(\tau(x)) = k_1^{l_\tau(h^{-1}(y))}(h^{-1}(y)) + l_1^{k_\tau(h^{-1}(y))}(\tau(h^{-1}(y))),\\ l_\tau^h(y) &= k_1^m(\tau(x)) + l_1^n(x) = k_1^{k_\tau(h^{-1}(y))}(\tau(h^{-1}(y))) + l_1^{l_\tau(h^{-1}(y))}(h^{-1}(y)),\end{aligned}$$

one has

$$\sigma_{\mathfrak{L}_2}^{k_\tau^h(y)}(h\circ\tau\circ h^{-1}(y)) = \sigma_{\mathfrak{L}_2}^{l_\tau^h(y)}(y) \qquad \text{for } y \in h(X_\tau)$$

so that $h\circ\tau\circ h^{-1} \in [\sigma_{\mathfrak{L}_2}]_{sc}$ and hence $h\circ[\sigma_{\mathfrak{L}_1}]_{sc}\circ h^{-1} \subset [\sigma_{\mathfrak{L}_2}]_{sc}$. Similarly one has $h^{-1}\circ[\sigma_{\mathfrak{L}_2}]_{sc}\circ h \subset [\sigma_{\mathfrak{L}_1}]_{sc}$ and concludes $h\circ[\sigma_{\mathfrak{L}_1}]_{sc}\circ h^{-1} = [\sigma_{\mathfrak{L}_2}]_{sc}$. $\square$

Proposition 5.5. *If there exists an isomorphism $\Psi : \mathcal{O}_{\mathfrak{L}_1} \longrightarrow \mathcal{O}_{\mathfrak{L}_2}$ such that $\Psi(\mathcal{D}_{\mathfrak{L}_1}) = \mathcal{D}_{\mathfrak{L}_2}$, then there exists a homeomorphism $h : X_{\mathfrak{L}_1} \longrightarrow X_{\mathfrak{L}_2}$ such that $h\circ[\sigma_{\mathfrak{L}_1}]_{sc}\circ h^{-1} = [\sigma_{\mathfrak{L}_2}]_{sc}$.*

Proof. Suppose that there exists an isomorphism $\Psi : \mathcal{O}_{\mathfrak{L}_1} \longrightarrow \mathcal{O}_{\mathfrak{L}_2}$ such that $\Psi(\mathcal{D}_{\mathfrak{L}_1}) = \mathcal{D}_{\mathfrak{L}_2}$. By the split exact sequences

$$1 \longrightarrow \mathcal{U}(\mathcal{D}_{\mathfrak{L}_i}) \longrightarrow N_s(\mathcal{O}_{\mathfrak{L}_i},\mathcal{D}_{\mathfrak{L}_i}) \longrightarrow [\sigma_{\mathfrak{L}_i}]_{sc} \longrightarrow 1, \qquad i = 1,2$$

of inverse semigroups, one may find an isomorphism $\widetilde{\Psi} : [\sigma_{\mathfrak{L}_1}]_{sc} \longrightarrow [\sigma_{\mathfrak{L}_2}]_{sc}$ of inverse semigroup such that the following diagrams are commutative:

$$\begin{array}{ccccccccc}1 & \longrightarrow & \mathcal{U}(\mathcal{D}_{\mathfrak{L}_1}) & \xrightarrow{\text{id}} & N_s(\mathcal{O}_{\mathfrak{L}_1},\mathcal{D}_{\mathfrak{L}_1}) & \xrightarrow{\tau} & [\sigma_{\mathfrak{L}_1}]_{sc} & \longrightarrow & 1\\ & & \Big\downarrow{\scriptstyle \Psi|_{\mathcal{U}(\mathcal{D}_{\mathfrak{L}_1})}} & & \Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle\widetilde{\Psi}} & & \\ 1 & \longrightarrow & \mathcal{U}(\mathcal{D}_{\mathfrak{L}_2}) & \xrightarrow{\text{id}} & N_s(\mathcal{O}_{\mathfrak{L}_2},\mathcal{D}_{\mathfrak{L}_2}) & \xrightarrow{\tau} & [\sigma_{\mathfrak{L}_2}]_{sc} & \longrightarrow & 1.\end{array}$$

Let $h : X_{\mathfrak{L}_1} \longrightarrow X_{\mathfrak{L}_2}$ be the homeomorphism satisfying $\Psi(f) = f\circ h^{-1}$ for $f \in C(X_{\mathfrak{L}_1})$. For $v \in N_s(\mathcal{O}_{\mathfrak{L}_1},\mathcal{D}_{\mathfrak{L}_1})$, take the partial homeomorphism $\tau_v : X_v \longrightarrow Y_v$ satisfying $Ad(v)(f) = f\circ\tau_v^{-1}$ for $f \in C(X_v)$. For $g \in C(h(X_v))$, we have

$$\Psi\circ Ad(v)\circ\Psi^{-1}(g) = g\circ h\circ\tau_v^{-1}\circ h^{-1}, \quad \text{ and } \quad Ad(\Psi(v))(g) = g\circ\tau_{\Psi(v)}^{-1}.$$

By the identity $\Psi \circ Ad(v) \circ \Psi^{-1} = Ad(\Psi(v))$, one has

$$g \circ h \circ \tau_v^{-1} \circ h^{-1} = g \circ \tau_{\Psi(v))}^{-1} \quad \text{for } g \in C(h(X_v)).$$

Hence $h \circ \tau_v \circ h^{-1} = \tau_{\Psi(v)}$. As $[\sigma_{\mathfrak{L}_i}]_{sc} = \{\tau_v \mid v \in N_s(\mathcal{O}_{\mathfrak{L}_i}, \mathcal{D}_{\mathfrak{L}_i})\}, i = 1, 2$, one sees that $h \circ [\sigma_{\mathfrak{L}_1}]_{sc} \circ h^{-1} = [\sigma_{\mathfrak{L}_2}]_{sc}$. □

Proposition 5.6. *If $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ are continuously orbit equivalent, then there exists an isomorphism $\Psi : \mathcal{O}_{\mathfrak{L}_1} \longrightarrow \mathcal{O}_{\mathfrak{L}_2}$ such that $\Psi(\mathcal{D}_{\mathfrak{L}_1}) = \mathcal{D}_{\mathfrak{L}_2}$.*

Proof. The proof is essentially same as the proof of Proposition 4.1 and [30, Proposition 5.5]. We omit its proof. □

Therefore we have

Theorem 5.7. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be λ-graph systems satisfying condition (I). The following are equivalent:*

1. *There exists an isomorphism $\Psi : \mathcal{O}_{\mathfrak{L}_1} \to \mathcal{O}_{\mathfrak{L}_2}$ such that $\Psi(\mathcal{D}_{\mathfrak{L}_1}) = \mathcal{D}_{\mathfrak{L}_2}$.*
2. *$(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ are continuously orbit equivalent.*
3. *There exists a homeomorphism $h : X_{\mathfrak{L}_1} \to X_{\mathfrak{L}_2}$ such that $h \circ [\sigma_{\mathfrak{L}_1}]_{sc} \circ h^{-1} = [\sigma_{\mathfrak{L}_2}]_{sc}$.*

Example. Let $G = (V, E)$ be a finite directed graph with $V = \{v_1, v_2\}$ and $E = \{e, f, g\}$ such that

$$s(e) = t(e) = s(f) = t(g) = v_1, \qquad t(f) = s(g) = v_2.$$

Put the alphabet sets $\Sigma_1 = \{\mathbf{1}, \mathbf{2}\}$ and $\Sigma_2 = \{\alpha, \beta\}$. Define two labeling maps $\lambda_i : E \longrightarrow \Sigma_i, i = 1, 2$ by setting

$$\lambda_1(e) = \lambda_1(f) = \mathbf{1}, \ \ \lambda_1(g) = \mathbf{2}, \qquad \lambda_2(e) = \alpha, \ \ \lambda_2(f) = \lambda_2(g) = \beta.$$

Let us denote by $\mathcal{G}_i$ the labeled graph (G, λ_i) over Σ_i for $i = 1, 2$. Hence their underlying directed graphs are both G. The labeled graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have its adjacency matrices as

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{2} & 0 \end{bmatrix}, \qquad \begin{bmatrix} \alpha & \beta \\ \beta & 0 \end{bmatrix}$$

respectively. Let $\mathfrak{L}_i = (V^{(i)}, E^{(i)}, \lambda^{(i)}, \Sigma_i)$ be the λ-graph systems associated to the labeled graphs $\mathcal{G}_i$ for $i = 1, 2$ respectively. They are defined by setting

$$V_{l,l+1}^{(i)} = V, \quad E_{l,l+1}^{(i)} = E, \quad \lambda^{(i)} = \lambda_i$$

for all $l \in \mathbb{Z}_+$ and $i = 1,2$. We then have $\Omega_{\mathfrak{L}_i} = V = \{v_1, v_2\}, i = 1,2$. The correspondence:

$$(\mathbf{1}, v_1) \to (\alpha, v_1), \quad (\mathbf{1}, v_2) \to (\beta, v_2), \quad (\mathbf{2}, v_1) \to (\beta, v_1)$$

yields a homeomorphism $h : X_{\mathfrak{L}_1} \longrightarrow X_{\mathfrak{L}_2}$ that gives rise to a continuous orbit equivalence between $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$. One indeed sees that the C^*-algebras $\mathcal{O}_{\mathfrak{L}_1}$ and $\mathcal{O}_{\mathfrak{L}_2}$ are both isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_F$ where $F = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, although the subshift presented by the λ-graph system $\mathfrak{L}_2$ is the even shift that is not a Markov shift.

6. Orbit equivalence of the factor map $\pi_\Lambda^{\mathfrak{L}} : X_{\mathfrak{L}} \longrightarrow X_\Lambda$

For a λ-graph system $\mathfrak{L}$ over Σ, let Λ be the subshift presented by $\mathfrak{L}$. Then we have a factor map $\pi_\Lambda^{\mathfrak{L}} : (X_{\mathfrak{L}}, \sigma_{\mathfrak{L}}) \longrightarrow (X_\Lambda, \sigma_\Lambda)$. In this section, we will study orbit structure between two dynamical systems $(X_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ and $(X_\Lambda, \sigma_\Lambda)$ through the factor map $\pi_\Lambda^{\mathfrak{L}}$.

Lemma 6.1. *$\pi_\Lambda^{\mathfrak{L}}(orb_{\sigma_{\mathfrak{L}}}(x)) = orb_{\sigma_\Lambda}(\pi_\Lambda^{\mathfrak{L}}(x))$ for $x \in X_{\mathfrak{L}}$.*

Proof. Take an arbitrary element $x = (x_n)_{n\in\mathbb{N}} \in X_{\mathfrak{L}}$. For $w \in orb_{\sigma_{\mathfrak{L}}}(x)$, we have $w = (z_1, \dots, z_k, x_{l+1}, x_{l+2}, \dots) \in X_{\mathfrak{L}}$ for some $z_1 \cdots z_k \in B_k(X_{\mathfrak{L}})$ and $l \in \mathbb{Z}_+$. It is easy to see that

$$\pi_\Lambda^{\mathfrak{L}}(w) \in \sigma_\Lambda^{-k}(\sigma_\Lambda^l(\pi_\Lambda^{\mathfrak{L}}(x))) \subset orb_{\sigma_\Lambda}(\pi_\Lambda^{\mathfrak{L}}(x)).$$

Conversely, put $(\alpha_n)_{n\in\mathbb{N}} = \pi_\Lambda^{\mathfrak{L}}(x)$. Each element $a \in orb_{\sigma_\Lambda}(\pi_\Lambda^{\mathfrak{L}}(x))$ has of the form $a = (\gamma_1, \dots, \gamma_k, \alpha_{l+1}, \alpha_{l+2}, \dots) \in X_\Lambda$ for some $\gamma_1 \cdots \gamma_k \in B_k(X_\Lambda)$ and $l \in \mathbb{Z}_+$. Put $v_0 = v_0(\sigma_{\mathfrak{L}}^l(x)) \in \Omega_{\mathfrak{L}}$. Since $\mathfrak{L}$ is left-resolving, there uniquely exists $v_{-1} \in \Omega_{\mathfrak{L}}$ such that $(v_{-1}, \gamma_k, v_0) \in E_{\mathfrak{L}}$. Inductively there uniquely exist $v_{-2}, v_{-3}, \dots, v_{-k} \in \Omega_{\mathfrak{L}}$ such that $(v_{-i}, \gamma_{k-(i-1)}, v_{-(i-1)}) \in E_{\mathfrak{L}}$ for $i = 1, 2, \dots, k$. Put $z_{k-(i-1)} = (\gamma_{k-(i-1)}, v_{-(i-1)})$ for $i = 1, 2, \dots, k$ so that $w = (z_1, \dots, z_k, x_{l+1}, x_{l+2}, \dots) \in X_{\mathfrak{L}}$ and $\pi_\Lambda^{\mathfrak{L}}(w) = a$. Since $w \in \sigma_{\mathfrak{L}}^{-k}(\sigma_{\mathfrak{L}}^l(x)) \subset orb_{\sigma_{\mathfrak{L}}}(x)$, one has $a \in \pi_\Lambda^{\mathfrak{L}}(orb_{\sigma_{\mathfrak{L}}}(x))$. □

For λ-graph systems $\mathfrak{L}_1$ and $\mathfrak{L}_2$, let Λ_1 and Λ_2 be the subshifts presented by $\mathfrak{L}_1$ and $\mathfrak{L}_2$ respectively.

Definition. Two factor maps $\pi_{\Lambda_1}^{\mathfrak{L}_1}$ and $\pi_{\Lambda_2}^{\mathfrak{L}_2}$ are said to be continuously orbit equivalent if there exist homeomorphisms $h_{\mathfrak{L}} : X_{\mathfrak{L}_1} \longrightarrow X_{\mathfrak{L}_2}$, $h_\Lambda : X_{\Lambda_1} \longrightarrow X_{\Lambda_2}$ such that $\pi_{\Lambda_2}^{\mathfrak{L}_2} \circ h_{\mathfrak{L}} = h_\Lambda \circ \pi_{\Lambda_1}^{\mathfrak{L}_1}$ and continuous functions $k_1, l_1 : X_{\mathfrak{L}_1} \longrightarrow \mathbb{Z}_+$ and

$k_2, l_2 : X_{\mathfrak{L}_2} \longrightarrow \mathbb{Z}_+$ such that

$$\sigma_{\mathfrak{L}_2}^{k_1(x)}(h_{\mathfrak{L}} \circ \sigma_{\mathfrak{L}_1}(x)) = \sigma_{\mathfrak{L}_2}^{l_1(x)}(h_{\mathfrak{L}}(x)), \quad x \in X_{\mathfrak{L}_1}, \tag{6.1}$$

$$\sigma_{\mathfrak{L}_1}^{k_2(y)}(h_{\mathfrak{L}}^{-1} \circ \sigma_{\mathfrak{L}_2}(y)) = \sigma_{\mathfrak{L}_1}^{l_2(x)}(h_{\mathfrak{L}}^{-1}(y)), \quad y \in X_{\mathfrak{L}_2}. \tag{6.2}$$

We note that the equalities (6.1) and (6.2) imply

$$h_{\mathfrak{L}}(orb_{\sigma_{\mathfrak{L}_1}}(x)) = orb_{\sigma_{\mathfrak{L}_2}}(h_{\mathfrak{L}}(x)) \qquad \text{for } x \in X_{\mathfrak{L}_1}. \tag{6.3}$$

Lemma 6.2. *Suppose that two factor maps $\pi_{\Lambda_1}^{\mathfrak{L}_1}$ and $\pi_{\Lambda_2}^{\mathfrak{L}_2}$ are continuously orbit equivalent and keep the above notation. Then we have*

(i)

$$\sigma_{\Lambda_2}^{k_1(x)}(h_\Lambda \circ \sigma_{\Lambda_1}(\pi_{\Lambda_1}^{\mathfrak{L}_1}(x)) = \sigma_{\Lambda_2}^{l_1(x)}(h_\Lambda(\pi_{\Lambda_1}^{\mathfrak{L}_1}(x)), \quad x \in X_{\mathfrak{L}_1},$$
$$\sigma_{\Lambda_1}^{k_2(y)}(h_\Lambda^{-1} \circ \sigma_{\Lambda_2}(\pi_{\Lambda_2}^{\mathfrak{L}_2}(y)) = \sigma_{\Lambda_1}^{l_2(y)}(h_\Lambda^{-1}(\pi_{\Lambda_2}^{\mathfrak{L}_2}(y)), \quad y \in X_{\mathfrak{L}_2}.$$

(ii)

$$h_\Lambda(orb_{\sigma_{\Lambda_1}}(a)) = orb_{\sigma_{\Lambda_2}}(h_\Lambda(a)) \qquad \textit{for } a \in X_{\Lambda_1}.$$

Proof. (i) follows from (6.1) and (6.2), and (ii) follows from (6.3). □

The following lemma is direct.

Lemma 6.3. *Two factor maps $\pi_{\Lambda_1}^{\mathfrak{L}_1}$ and $\pi_{\Lambda_2}^{\mathfrak{L}_2}$ are continuously orbit equivalent if and only if there exists a homeomorphism $h_{\mathfrak{L}} : X_{\mathfrak{L}_1} \longrightarrow X_{\mathfrak{L}_2}$ that yields a continuously orbit equivalence between $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ and there exists a homemorphism $h_\Lambda : X_{\Lambda_1} \longrightarrow X_{\Lambda_2}$ such that $\pi_{\Lambda_2}^{\mathfrak{L}_2} \circ h_{\mathfrak{L}} = h_\Lambda \circ \pi_{\Lambda_1}^{\mathfrak{L}_1}$.*

We note that the factor map $\pi_\Lambda^{\mathfrak{L}} : X_{\mathfrak{L}} \longrightarrow X_\Lambda$ induces an embedding of $C(X_\Lambda)$ into $C(X_{\mathfrak{L}})$, that corresponds to the natural embedding of $\mathfrak{D}_\Lambda$ into $\mathcal{D}_{\mathfrak{L}}$. Let $N_s(\mathcal{O}_{\mathfrak{L}}, \mathfrak{D}_\Lambda)$be the set of all partial isometries $v \in \mathcal{O}_{\mathfrak{L}}$ such that $v\mathfrak{D}_\Lambda v^* \subset \mathfrak{D}_\Lambda$ and $v^*\mathfrak{D}_\Lambda v \subset \mathfrak{D}_\Lambda$.

Lemma 6.4. $N_s(\mathcal{O}_{\mathfrak{L}}, \mathfrak{D}_\Lambda) \subset N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$.

Proof. For $v \in N_s(\mathcal{O}_{\mathfrak{L}}, \mathfrak{D}_\Lambda)$, and $x \in \mathcal{D}_{\mathfrak{L}}, a \in \mathfrak{D}_\Lambda$, we have

$$vxv^*a = vxv^*avv^* = vv^*avxv^* = avxv^*$$

so that $vxv^* \in \mathfrak{D}_\Lambda' \cap \mathcal{O}_{\mathfrak{L}} = \mathcal{D}_{\mathfrak{L}}$. Hence $v\mathcal{D}_{\mathfrak{L}}v^* \subset \mathcal{D}_{\mathfrak{L}}$, and similarly $v^*\mathcal{D}_{\mathfrak{L}}v \subset \mathcal{D}_{\mathfrak{L}}$. This implies that $v \in N_s(\mathcal{O}_{\mathfrak{L}}, \mathcal{D}_{\mathfrak{L}})$. □

Suppose that both λ-graph systems $\mathfrak{L}_1$ and $\mathfrak{L}_2$ satisfy condition (I).

Lemma 6.5. *If there exists an isomorphism $\Psi : \mathcal{O}_{\mathfrak{L}_1} \longrightarrow \mathcal{O}_{\mathfrak{L}_2}$ such that $\Psi(\mathfrak{D}_{\Lambda_1}) = \mathfrak{D}_{\Lambda_2}$, then $\Psi(\mathcal{D}_{\mathfrak{L}_1}) = \mathcal{D}_{\mathfrak{L}_2}$.*

Proof. Suppose that $\Psi(\mathfrak{D}_{\Lambda_1}) = \mathfrak{D}_{\Lambda_2}$. For $x \in \mathcal{D}_{\mathfrak{L}_1}$ and $b \in \mathfrak{D}_{\Lambda_2}$, take $a \in \mathfrak{D}_{\Lambda_1}$ such that $\Psi(a) = b$. It then follows that

$$\Psi(x)b = \Psi(xa) = \Psi(a)\Psi(x) = b\Psi(x)$$

so that $\Psi(x)$ commutes with all elements of $\mathfrak{D}_{\Lambda_2}$, and hence $\Psi(x) \in \mathcal{D}_{\mathfrak{L}_2}$. This implies that $\Psi(\mathcal{D}_{\mathfrak{L}_1}) \subset \mathcal{D}_{\mathfrak{L}_2}$. Similarly we have $\Psi^{-1}(\mathcal{D}_{\mathfrak{L}_2}) \subset \mathcal{D}_{\mathfrak{L}_1}$ so that $\Psi(\mathcal{D}_{\mathfrak{L}_1}) = \mathcal{D}_{\mathfrak{L}_2}$. □

Theorem 6.6. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be λ-graph systems satisfying condition (I). Let X_{Λ_1} and X_{Λ_2} be their respect right one-sided subshifts. The following are equivalent:*

1. *There exists an isomorphism $\Psi : \mathcal{O}_{\mathfrak{L}_1} \longrightarrow \mathcal{O}_{\mathfrak{L}_2}$ such that $\Psi(\mathfrak{D}_{\Lambda_1}) = \mathfrak{D}_{\Lambda_2}$.*
2. *The factor maps $\pi^{\mathfrak{L}_1}_{\Lambda_1}$ and $\pi^{\mathfrak{L}_2}_{\Lambda_2}$ are continuously orbit equivalent.*
3. *There exist homeomorphisms $h_{\mathfrak{L}} : X_{\mathfrak{L}_1} \longrightarrow X_{\mathfrak{L}_2}$ and $h_\Lambda : X_{\Lambda_1} \longrightarrow X_{\Lambda_2}$ such that $\pi^{\mathfrak{L}_2}_{\Lambda_2} \circ h_{\mathfrak{L}} = h_\Lambda \circ \pi^{\mathfrak{L}_1}_{\Lambda_1}$ and $h_{\mathfrak{L}} \circ [\sigma_{\mathfrak{L}_1}]_{sc} \circ h_{\mathfrak{L}}^{-1} = [\sigma_{\mathfrak{L}_2}]_{sc}$.*

Proof. (2)⇔ (3): The equivalence between (2) and (3) comes from Lemma 6.3.

(1)⇒(3): Suppose that there exists an isomorphism $\Psi : \mathcal{O}_{\mathfrak{L}_1} \longrightarrow \mathcal{O}_{\mathfrak{L}_2}$ such that $\Psi(\mathfrak{D}_{\Lambda_1}) = \mathfrak{D}_{\Lambda_2}$. By Lemma 6.5, one has $\Psi(\mathcal{D}_{\mathfrak{L}_1}) = \mathcal{D}_{\mathfrak{L}_2}$. Let $h_{\mathfrak{L}} : X_{\mathfrak{L}_1} \to X_{\mathfrak{L}_2}$ be the homeomorphism induced by $\Psi : \mathcal{D}_{\mathfrak{L}_1} \longrightarrow \mathcal{D}_{\mathfrak{L}_2}$ such that $\Psi(f) = f \circ h^{-1}$ for $f \in \mathcal{D}_{\mathfrak{L}_1}$. Then $h_{\mathfrak{L}}$ satisfies $h \circ [\sigma_{\mathfrak{L}_1}]_{sc} \circ h^{-1} = [\sigma_{\mathfrak{L}_2}]_{sc}$ by Proposition 5.5. Since $\Psi(\mathfrak{D}_{\Lambda_1}) = \mathfrak{D}_{\Lambda_2}$, there exists a homeomorphism $h_\Lambda : X_{\Lambda_1} \longrightarrow X_{\Lambda_2}$ such that $h_\Lambda \circ \pi^{\mathfrak{L}_1}_{\Lambda_1} = \pi^{\mathfrak{L}_2}_{\Lambda_2} \circ h_{\mathfrak{L}}$.

(2)⇒(1): Suppose that the factor maps $\pi^{\mathfrak{L}_1}_{\Lambda_1}$ and $\pi^{\mathfrak{L}_2}_{\Lambda_2}$ are continuously orbit equivalent. Since $(X_{\mathfrak{L}_1}, \sigma_{\mathfrak{L}_1})$ and $(X_{\mathfrak{L}_2}, \sigma_{\mathfrak{L}_2})$ are continuously orbit equivalent, by Proposition 5.6 there exists an isomorphism $\Psi : \mathcal{O}_{\mathfrak{L}_1} \longrightarrow \mathcal{O}_{\mathfrak{L}_2}$ such that $\Psi(\mathcal{D}_{\mathfrak{L}_1}) = \mathcal{D}_{\mathfrak{L}_2}$ and $\Psi(f) = f \circ h_{\mathfrak{L}}^{-1}$ for $f \in \mathcal{D}_{\mathfrak{L}_1}$. For $g \in \mathfrak{D}_{\Lambda_1}$, one sees that $g \circ \pi^{\mathfrak{L}_1}_{\Lambda_1} \in \mathcal{D}_{\mathfrak{L}_1}$ so that

$$\Psi(g \circ \pi^{\mathfrak{L}_1}_{\Lambda_1}) = g \circ \pi^{\mathfrak{L}_1}_{\Lambda_1} \circ h_{\mathfrak{L}}^{-1} = g \circ h_\Lambda^{-1} \circ \pi^{\mathfrak{L}_2}_{\Lambda_2}$$

This means that $\Psi(\mathfrak{D}_{\Lambda_1}) \subset \mathfrak{D}_{\Lambda_2}$, and similarly $\Psi^{-1}(\mathfrak{D}_{\Lambda_2}) \subset \mathfrak{D}_{\Lambda_1}$. Therefore we conclude that $\Psi(\mathfrak{D}_{\Lambda_1}) = \mathfrak{D}_{\Lambda_2}$. □

7. Orbit equivalence of one-sided subshifts

Let Λ be a two-sided subshift over Σ and X_Λ its right one-sided subshift. The canonical λ-graph system $\mathfrak{L}^\Lambda$ for Λ is defined as in the following way ([26]). For

$a = (a_n)_{n\in\mathbb{N}} \in X_\Lambda$ and $l \in \mathbb{Z}_+$, denote by $P_l(a)$ the predecessor set of a of length l, that is

$$P_l(a) = \{(\mu_1, \dots, \mu_l) \in B_l(X_\Lambda) \mid (\mu_1, \dots, \mu_l, a_1, a_2, \dots) \in X_\Lambda\}.$$

Two sequences $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ in X_Λ are said to be l-past equivalent if $P_l(a) = P_l(b)$, and written as $a \underset{l}{\sim} b$. The equivalence class of a in $X_\Lambda/\underset{l}{\sim}$ is denoted by $[a]_l$. The vertex set V_l of the λ-graph system is the set $X_\Lambda/\underset{l}{\sim}$. We set $v^l(a) = [a]_l$. Then $(v^l(a))_{l\in\mathbb{Z}_+}$ defines an ι-orbit of $\Omega_{\mathfrak{L}^\Lambda}$, denoted by $v(a)$. An edge labeled α from $v^l(a)$ to $v^{l+1}(b)$ is defined if $a \underset{l}{\sim} (\alpha, b_1, b_2, \dots)$, where $b = (b_n)_{n\in\mathbb{N}}$.

Lemma 7.1. *For $a = (a_n)_{n\in\mathbb{N}} \in X_\Lambda$, $(a_n, v_n(a))_{n\in\mathbb{N}}$ defines an element of $X_{\mathfrak{L}^\Lambda}$.*

Proof. For each $n \in \mathbb{N}$ and $l \in \mathbb{Z}_+$, there is a unique edge from $[(a_n, a_{n+1}, \dots)]_l \in V_l$ to $[(a_{n+1}, a_{n+2}, \dots)]_{l+1} \in V_{l+1}$ labeled a_n. Hence $(v_{n-1}(a), a_n, v_n(a))$ belongs to $E_{\mathfrak{L}^\Lambda}$ for all $n \in \mathbb{N}$, so that $(a_n, v_n(a))_{n\in\mathbb{N}}$ defines an element of $X_{\mathfrak{L}^\Lambda}$. □

We put the embedding of X_Λ into $X_{\mathfrak{L}^\Lambda}$:

$$\iota_\Lambda : a = (a_n)_{n\in\mathbb{N}} \in X_\Lambda \longrightarrow (a_n, v_n(a))_{n\in\mathbb{N}} \in X_{\mathfrak{L}^\Lambda}.$$

It is straightforward to see that the following lemma holds:

Lemma 7.2. *The map $\iota_\Lambda : X_\Lambda \longrightarrow X_{\mathfrak{L}^\Lambda}$ is injective and $\iota_\Lambda(X_\Lambda)$ is dense in $X_{\mathfrak{L}^\Lambda}$.*

We endow X_Λ with a new topology induced by the injection $\iota_\Lambda : X_\Lambda \longrightarrow X_{\mathfrak{L}^\Lambda}$, which is the weakest topology for which ι_Λ is continuous. Denote by $\widetilde{X}_\Lambda$ the topological space X_Λ with the topology. If Λ is a topological Markov shift, the induced topology of $\widetilde{X}_\Lambda$ coincides with the original topology of X_Λ.

Lemma 7.3. *The topological space $\widetilde{X}_\Lambda$ is generated by the clopen sets of the form $U_\mu \cap \sigma_\Lambda^{-k}(\sigma_\Lambda^l(U_\nu))$ for $\mu \in B_k(X_\Lambda), \nu \in B_l(X_\Lambda)$ with $k \le l$. Hence the correspondence $\chi_{U_\mu \cap \sigma_\Lambda^{-k}(\sigma_\Lambda^l(U_\nu))} \longleftrightarrow S_\mu S_\nu^* S_\nu S_\mu^*$ yields an isomorphism between $C(\widetilde{X}_\Lambda)$ and $\mathcal{D}_{\mathfrak{L}^\Lambda}$.*

By the above lemma, we know that $C(\widetilde{X}_\Lambda)$ is isomorphic to $C(X_{\mathfrak{L}^\Lambda})$.

Let Λ_1 and Λ_2 be subshifts, and X_{Λ_1} and X_{Λ_2} their right one-sided subshifts.

Definition. The subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are said to be *λ-continuously orbit equivalent* if there exists a homeomorphism $h : X_{\Lambda_1} \longrightarrow X_{\Lambda_2}$, that is also homeomorphic from $\widetilde{X}_{\Lambda_1} \longrightarrow \widetilde{X}_{\Lambda_2}$ and there exist continuous functions $k_1, l_1 : \widetilde{X}_{\Lambda_1} \longrightarrow \mathbb{Z}_+$ and $k_2, l_2 : \widetilde{X}_{\Lambda_2} \longrightarrow \mathbb{Z}_+$ such that

$$\sigma_{\Lambda_2}^{k_1(a)}(h \circ \sigma_{\Lambda_1}(a)) = \sigma_{\Lambda_2}^{l_1(a)}(h(a)) \quad \text{for } a \in X_{\Lambda_1}, \tag{7.1}$$
$$\sigma_{\Lambda_1}^{k_2(b)}(h^{-1} \circ \sigma_{\Lambda_2}(b)) = \sigma_{\Lambda_1}^{l_2(b)}(h^{-1}(b)) \quad \text{for } b \in X_{\Lambda_2}. \tag{7.2}$$

We note that the conditions (7.1) and (7.2) imply that

$$h(orb_{\sigma_{\Lambda_1}}(a)) = orb_{\sigma_{\Lambda_2}}(h(a)), \quad h^{-1}(orb_{\sigma_{\Lambda_2}}(b)) = orb_{\sigma_{\Lambda_1}}(h^{-1}(b))$$

for $a \in X_{\Lambda_1}, b \in X_{\Lambda_2}$.

Lemma 7.4. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be the canonical λ-graph systems for Λ_1 and Λ_2 respectively. The following are equivalent:*

(1) *The subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are λ-continuously orbit equivalent.*
(2) *The factor maps $\pi_{\Lambda_1}^{\mathfrak{L}_1}$ and $\pi_{\Lambda_2}^{\mathfrak{L}_2}$ are continuously orbit equivalent.*

Proof. (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2): It suffices to show the equalities

$$\sigma_{\mathfrak{L}_2}^{k_1(x)}(h(\sigma_{\mathfrak{L}_1}(x))) = \sigma_{\mathfrak{L}_2}^{l_1(x)}(h(x)), \quad \text{for } x \in X_{\mathfrak{L}_1},$$
$$\sigma_{\mathfrak{L}_1}^{k_2(y)}(h^{-1}(\sigma_{\mathfrak{L}_2}(y))) = \sigma_{\mathfrak{L}_1}^{l_2(y)}(h^{-1}(y)), \quad \text{for } y \in X_{\mathfrak{L}_2}.$$

For $x \in X_{\mathfrak{L}_1}$, put $k = k_1(x), l = l_1(x)$. Since $k_1, l_1 : X_{\mathfrak{L}_1} \longrightarrow \mathbb{Z}_+$ are continuous, the set $U = \{z \in X_{\mathfrak{L}_1} \mid k_1(z) = k, l_1(z) = l\}$ is a clopen set in $X_{\mathfrak{L}_1}$. Since X_{Λ_1} is dense in $X_{\mathfrak{L}_1}$ through ι_{Λ_1}, one sees $x \in U$ with $U \cap X_{\Lambda_1} \neq \emptyset$ and the equality

$$\sigma_{\mathfrak{L}_2}^{k_1(x)}(h\sigma_{\mathfrak{L}_1}(x)) = \sigma_{\mathfrak{L}_2}^{l_1(x)}(h(x)) \quad \text{for } x \in X_{\mathfrak{L}_1}$$

holds because the equality holds for elements of X_{Λ_1}. We similarly have the equality

$$\sigma_{\mathfrak{L}_1}^{k_2(y)}(h^{-1}\sigma_{\mathfrak{L}_2}(y)) = \sigma_{\mathfrak{L}_1}^{l_2(y)}(h^{-1}(y)) \quad \text{for } y \in X_{\mathfrak{L}_2}.$$

Hence the factor maps $\pi_{\Lambda_1}^{\mathfrak{L}_1}$ and $\pi_{\Lambda_2}^{\mathfrak{L}_2}$ are continuously orbit equivalent. □

Therefore we conclude:

Theorem 7.5. *Let Λ_1 and Λ_2 be subshifts satisfying condition (I). The following are equivalent:*

(1) *There exists an isomorphism $\Psi : \mathcal{O}_{\Lambda_1} \longrightarrow \mathcal{O}_{\Lambda_2}$ such that $\Psi(\mathfrak{D}_{\Lambda_1}) = \mathfrak{D}_{\Lambda_2}$.*
(2) *The subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are λ-continuously orbit equivalent.*

Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries in $\{0,1\}$. The Cuntz-Krieger algebra $\mathcal{O}_A$ is generated by partial isometries $S_1, \dots, S_N$ satisfying $\sum_{j=1}^N S_j S_j^* = 1, S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, i = 1, \dots, N$. The C^*-subalgebra generated by projections $S_{\mu_n}^* \cdots S_{\mu_1}^* S_{\mu_1} \cdots S_{\mu_n}, \mu_1, \dots, \mu_n \in \{1, \dots, N\}$ is canonically isomorphic to the commutative C^*-algebra $C(X_A)$, that is denoted by $\mathfrak{D}_A$.

Corollary 7.6. ([30], cf. [29]) *Let A and B be square matrices with entries in $\{0,1\}$ satisfying condition (I) in [8]. Then the following are equivalent:*

(1) *There exists an isomorphism $\Psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\Psi(\mathfrak{D}_A) = \mathfrak{D}_B$.*
(2) *(X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*

Proof. For a topological Markov shift (X_A, σ_A), the topology on $\widetilde{X}_A$ coincides with the original topology on X_A. Let Λ_A be the two-sided topological Markov shift for the matrix A. Then $X_{\Lambda_A} = X_A$ and $\mathcal{O}_{\Lambda_A} = \mathcal{O}_A$ so that the assertion holds. □

Two one-sided subshifts $(X_{\Lambda_1}, \sigma_{\Lambda_1})$ and $(X_{\Lambda_2}, \sigma_{\Lambda_2})$ are said to be topologically conjugate if there exists a homeomorphism $h : X_{\Lambda_1} \longrightarrow X_{\Lambda_2}$ such that $\sigma_{\Lambda_2} \circ h = h \circ \sigma_{\Lambda_1}$, and the homeomorphism h is called a topological conjugacy. One can prove that topological conjugacy gives rise to a λ-continuous orbit equivalence. Hence we have.

Corollary 7.7. ([27]) *Suppose that both subshifts Λ_1 and Λ_2 satisfy condition (I). Let $h : (X_{\Lambda_1}, \sigma_{\Lambda_1}) \to (X_{\Lambda_2}, \sigma_{\Lambda_2})$ be a topological conjugacy of one-sided subshifts. Then there exists an isomorphism $\Psi : \mathcal{O}_{\Lambda_1} \to \mathcal{O}_{\Lambda_2}$ such that $\Psi(\mathfrak{D}_{\Lambda_1}) = \mathfrak{D}_{\Lambda_2}$.*

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