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MORITA EQUIVALENCE OF TWISTED POISSON MANIFOLDS

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Abstract. This paper is devoted to the study of Morita equivalence for twisted Poisson manifolds. We prove that integrable twisted Poisson manifolds which are gauge equivalent are Morita equivalent. Moreover, we introduce the notion of weak Morita equivalence and show that there exists a one-to-one correspondence between their twisted symplectic leaves if two twisted Poisson manifolds are weak Morita equivalent.

1. Introduction

Geometric Morita theory is one of the interesting topics in Poisson geometry. The geometric notion of Morita equivalence was introduced by Xu, P. ([20],[21] and [22]) on the basis of algebraic Morita equivalence. Morita equivalence is first introduced by Morita, K. in [13]. He gave a necessary and sufficient condition for representation categories of two rings to be equivalent: two rings have equivalent categories of left modules if and only if there exists an equivalence bimodule for rings. Ring theoretical Morita equivalence is generalized to the theory of C^*-algebras by Rieffel, M. [17],[18]. Morita equivalence of C^*-algebra is useful in studying some C^*-algebras. Also, Morita equivalent C^*-algebras share many properties, such as equivalent categories of Hermitian left modules, isomorphic K-group, and so on. C^*-algebras are the quantum objects; in contrast, Poisson manifolds are the classical one. Morita equivalence for integrable Poisson manifolds and (quasi-) symplectic groupoids ware introduced by Xu as the classical analogue of this equivalence relation. Geometric Morita equivalence plays an important role in Poisson geometry as Morita equivalence of C^*-algebras does. There exist some invariants under Morita equivalence such as the representation categories of symplectic realizations, fundamental groups and the first Poisson cohomology groups (see Ginzburg, V. L. and Lu, J.-H. [8] and [21]). And furthermore, the theory of geometric Morita equivalence is related to momentum map

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theory. Since Morita equivalence establishes an equivalence of representation categories, we are provided with the notion of equivalence for momentum map theories. It is shown that some known correspondence of momentum map theories can be described by Morita equivalences [22]. On the basis of Xu's work, the author introduced the notion of Morita equivalence for integrable twisted Poisson manifolds [9], [10]. As for Poisson manifolds, Morita equivalent twisted Poisson manifolds have isomorphic fundamental groups, isomorphic first cohomology groups and equivalent categories of modules. Morita equivalence is applied only for integrable (twisted) Poisson manifolds. To remedy this defect, we will introduce refined version of Morita equivalence and discuss it in this paper.

The paper is organized as follows: In Section 2 we study the basic properties of twisted Poisson manifolds and discuss the relation with Lie algebroids. Section 3 begins with the review of Morita equivalence discussed in [9] and [10]. The latter part of this section deals with Dirac structures and a gauge transformation. After that, we will prove that gauge equivalence of integrable twisted Poisson manifolds implies Morita equivalence. In Section 4, we introduce the notion of weak Morita equivalence of (twisted) Poisson manifolds and show that Morita equivalence implies weak Morita equivalence. Furthermore, we define a bijective correspondence between the twisted symplectic leaves of P_1 and those of P_2 when twisted Poisson manifolds P_1 and P_2 are weak Morita equivalent.

Finally, we note that smooth manifolds appeared in this paper are assumed to be connected. We denote by $\Gamma(E)$ the set of smooth sections of a vector bundle $E \to M$.

2. Preliminaries

2.1 Twisted Poisson manifolds

Twisted Poisson manifolds first appeared in the study of string theory by Park. J.-S. [15] and Klimčík, C. and Strobl, T. [11], and treated mathematically by Ševera, P. and Weinstein, A. [19]. We start by recalling the definition of a twisted Poisson manifold.

A twisted Poisson manifold is a smooth manifold P equipped with a bivector field Π and a closed 3-form ϕ on P which satisfy the following equation:

$$\frac{1}{2}[\Pi,\, \Pi] \,=\, \wedge^3\Pi^\sharp(\phi), \tag{2.1}$$

where, $[\cdot,\, \cdot]$ means a Schouten-Nijenhuis bracket and $\wedge^3\Pi^\sharp(\phi)$ is a linear map from $\Gamma(\wedge^3 T^*P)$ to $\Gamma(\wedge^3 TP)$ induced from the natural homomorphism $\Pi^\sharp : T^*M \to TM$ given by $\beta\big(\Pi^\sharp(\alpha)\big) = \langle \Pi,\, \beta\wedge\alpha\rangle$. Namely, for any $\alpha,\, \beta,\, \gamma \in \Gamma(T^*P)$, $\wedge^3\Pi^\sharp(\phi)$

is defined as

$$\wedge^3\Pi^\sharp(\phi)(\alpha,\, \beta,\, \gamma) := \phi\big(\Pi^\sharp(\alpha),\, \Pi^\sharp(\beta),\, \Pi^\sharp(\gamma)\big).$$

The bivector field Π is called a twisted Poisson bivector. We give typical examples of twisted Poisson manifolds:

Example 2.1. (Poisson manifolds) Let $(P,\, \Pi)$ be a Poisson manifold. For a closed 3-form ϕ on P such that $\wedge^3\Pi^\sharp(\phi) = 0$, it holds that $[\Pi,\, \Pi] = 0 = \wedge^3\Pi^\sharp(\phi)$. Therefore, $(P,\, \Pi,\, \phi)$ is a twisted Poisson manifold.

Example 2.2. Let A be the set of elements $\{x_1,\, x_2,\, x_3,\, x_4\} \subset \mathbb{R}^4$ which satisfy $x_1 = 0$ or $x_3 = 0$. The closed 3-form $\phi = \big((1/x_3^2)dx_2 - (1/x_1^2)dx_4\big) \wedge dx_1 \wedge dx_3$ on $\mathbb{R}^4 \setminus A$ and the bivector $\Pi = x_3(\partial/\partial x_1) \wedge (\partial/\partial x_2) + x_1(\partial/\partial x_3) \wedge (\partial/\partial x_4)$ satisfy the condition (2. 1). In other words, $(\mathbb{R}^4 \setminus A,\, \pi,\, \phi)$ is a twisted Poisson manifold.

Given a ϕ-twisted Poisson manifold $(P,\, \Pi)$, one can define a bilinear skew-symmetric map $\{\cdot,\, \cdot\}$ on $C^\infty(P)$ and a vector field on P by

$$\{f,\, g\} := \langle \Pi,\, df \wedge dg\rangle, \qquad H_f := \Pi^\sharp(df), \quad \big(\forall f,\, g \in C^\infty(P)\big).$$

The vector field H_f determined by $f \in C^\infty(P)$ is called the Hamiltonian vector field of f. It is easy to verify that the map $\{\cdot,\, \cdot\}$ satisfies the Leibniz identity. By using the bracket and the Hamiltonian vector fields, the formula (2. 1) can be written as

$$\big\{\{f,\, g\},\, h\big\} + \big\{\{g,\, h\},\, f\big\} + \big\{\{h,\, f\},\, g\big\} + \phi(H_f,\, H_g,\, H_h) = 0. \tag{2. 2}$$

Conversely, if a bilinear skew-symmetric map $\{\cdot,\, \cdot\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ and a closed 3-form $\phi \in \Gamma(\wedge^3 T^*P)$ satisfy the Leibniz identity and

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = \big\langle \{f,\, \cdot\} \wedge \{g,\, \cdot\} \wedge \{h,\, \cdot\},\, \phi\big\rangle, \tag{2. 3}$$

then $\{\cdot,\, \cdot\}$ arises from a 2-vector field Π given by

$$\langle \Pi,\, df \wedge dg\rangle = \{f,\, g\}, \quad (\forall f,\, g \in C^\infty(P)).$$

Furthermore, it can be verified that Π and ϕ satisfy the formula (2. 1). In consequence, we can define a twisted Poisson manifold as a smooth manifold P together with a closed 3-form $\phi \in \Gamma(\wedge^3 T^*P)$ and a bilinear skew-symmetric map $\{\cdot,\, \cdot\} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ satisfy the equation (2. 3) and the Leibniz identity.

Definition 2.1. For a closed 3-form ψ on a smooth manifold S, ψ-twisted symplectic form is a non-degenerate 2-form $\omega \in \Gamma(\wedge^2 T^*S)$ such that $d\omega = \psi$. A smooth manifold equipped with a ψ-twisted symplectic form is called a ψ-twisted symplectic manifold.

The non-degeneracy of a ψ-twisted symplectic form ω implies that the natural homomorphism $\omega^\flat : TS \to T^*S, \quad X \mapsto i_X\omega$ is an isomorphism, where $i_X\omega$ means the contraction of ω by X. Therefore, given a smooth function f on S, we can define its Hamiltonian vector field H_f by $i_{H_f}\omega = df$. Moreover, as for symplectic manifolds, we can define a bracket on S as $\{f,\, g\} := \omega(H_f,\, H_g)$. Then it is verified that the bracket $\{\cdot,\, \cdot\}$ obtained from ω and ψ satisfy the equation (2.2) and the Leibniz identity. That is, a twisted symplectic manifold is a twisted Poisson manifold.

Let $(P_i,\, \Pi_i,\, \phi_i)$ $(i = 1, 2)$ be twisted Poisson manifolds and $J : P_1 \to P_2$ a smooth map. The smooth map J is called a twisted Poisson map if, for any $x \in P_1$, the following formula holds:

$$(\Pi_2^\sharp)_{J(x)} \;=\; (dJ)_x \circ \Pi_1^\sharp \circ (dJ)_x^*. \tag{2.4}$$

By using the bracket, a twisted Poisson map $J : P_1 \to P_2$ can be written as

$$\{f,\, g\}_2 \circ J \;=\; \{J^*f,\, J^*g\}_1, \quad \bigl(\forall f,\, g \in C^\infty(P_2)\bigr), \tag{2.5}$$

where $\{\cdot,\, \cdot\}_i$ $(i = 1, 2)$ mean the brackets induced from Π_i.

Definition 2.2. A twisted symplectic realization (t.s.realization for short) of a twisted Poisson manifold P is a twisted symplectic manifold S together with a twisted Poisson map $J : S \to P$.

Analogously, we define an anti-twisted symplectic realization (anti-t.s.realization, for short) of a twisted Poisson manifold $(P,\, \Pi,\, \phi)$ as a twisted symplectic manifold S together with a twisted Poisson map $J' : S \to \overline{P}$, where $\overline{P}$ means a twisted Poisson manifold $(P,\, -\Pi,\, -\phi)$.

2.2 Lie algebroids of twisted Poisson manifolds

If $(P,\, \Pi,\, \phi)$ is a twisted Poisson manifold, then the cotangent bundle $T^*P \to P$ carries a Lie algebroid structure whose anchor map is the natural anchor map $\Pi^\sharp : T^*P \to TP$, $\beta(\Pi^\sharp\alpha) = \langle \Pi,\, \beta \wedge \alpha \rangle$ and whose Lie bracket is

$$[\alpha,\, \beta]_\phi := \mathcal{L}_{\sharp\alpha}\beta \;-\; \mathcal{L}_{\sharp\beta}\alpha \;+\; d\bigl(\Pi(\alpha,\, \beta)\bigr) \;-\; \phi(\sharp\alpha,\, \sharp\beta,\, \cdot\,) \tag{2.6}$$

where we denote by $\mathcal{L}_X\omega$ the Lie derivative of ω by X.

Definition 2.3. Let $A \to M$ be a Lie algebroid with anchor map $\sharp : A \to TM$. A left(right) action of A on a smooth manifold N consists of a smooth map $J : N \to M$ called the moment map, and a Lie algebra (anti)homomorphism $\varrho_N : \Gamma(A) \to \Gamma(TN)$ which satisfy the following conditions:

(1) $dJ \circ \varrho_N(\alpha) = \sharp\alpha$;
(2) $\varrho_N(f\alpha) = (J^*f)\varrho_N(\alpha)$,

for any $f \in C^\infty(M)$ and $\alpha \in \Gamma(A)$.

If (P, Π_P, ϕ_P), (Q, Π_Q, ϕ_Q) are twisted Poisson manifolds, then we can easily show that any twisted Poisson map $J : Q \to P$ induces a Lie algebroid action of T^*P on Q by

$$\varrho_Q : \Gamma(T^*P) \longrightarrow \Gamma(TQ), \quad \alpha \longmapsto \Pi_Q^\sharp(J^*\alpha).$$

As is well known, given a Lie groupoid $\Gamma \rightrightarrows M$, one can construct the Lie algebroid over M denoted by $\mathcal{A}(\Gamma)$. For a full discussion of the Lie algebroid of the Lie groupoid, we refer to Crainic, M. and Fernandes, R.-L. [7]. A Lie algebroid $A \to M$ is said to be integrable if there exists a Lie groupoid $\Gamma \rightrightarrows M$ such that $\mathcal{A}(\Gamma)$ is isomorphic to A.

Definition 2.4. A twisted Poisson manifold is said to be integrable if its cotangent bundle is integrable as Lie algebroid.

The integrability problem of Lie algebroids was studied by many people, for instance, Pradines, J. [16], Mackenzie, K. [12] and Crainic, M. and Fernandes, R.-L. [6]. The solution of integrability problem of twisted Poisson manifolds was given by Cattaneo, A. and Xu, P. ([5]). They proved the following result:

Theorem 2.5. (Cattaneo,A. and Xu,P.) *There is a bijection between integrable twisted Poisson structures and twisted symplectic groupoids which are source-simply connected.*

That is, twisted Poisson manifolds may be integrated to twisted symplectic groupoids. For an integrable twisted Poisson manifold P, we denote by $\mathcal{G}(P)$ the twisted symplectic groupoid associated with P in the above theorem. We refer to Definition 2.1 in [5] for a twisted symplectic groupoid.

3. Geometric Morita equivalence

3.1 Morita invariants

First, we will review the notion of Morita equivalence of twisted Poisson manifolds and exhibit some examples.

Definition 3.1. ([9],[10]) Let P_i be integrable ϕ_i-twisted Poisson manifolds $(i=1,2)$. P_1 and P_2 are said to be (strong) Morita equivalent if there exist a smooth manifold S equipped with a non-degenerate 2-form ω_S and surjective submersions $J_i : S \to P_i$ such that

(1) $(S,\, \omega_S)$ is a $(J_1^*\phi_1 - J_2^*\phi_2)$-twisted symplectic manifold;
(2) J_1 is a complete t.s.realization, and J_2 is a complete anti-t.s.realization;
(3) Each J_i-fiber $(i = 1, 2)$ is connected, and simply-connected;
(4) The subspaces $\ker(dJ_1)_x$, $\ker(dJ_2)_x$ of T_xS $(\forall x \in S)$ are symplectically orthogonal to one another:

$$\big(\ker(dJ_1)_x\big)^{\perp} = \ker(dJ_2)_x \quad \text{and} \quad \big(\ker(dJ_2)_x\big)^{\perp} = \ker(dJ_1)_x,$$

where

$$\big(\ker(dJ_i)_x\big)^{\perp} = \big\{\, \boldsymbol{u} \in T_xS \;\big|\; \omega_S(\boldsymbol{u},\, \boldsymbol{v}) = \boldsymbol{0} \ \big(\forall \boldsymbol{v} \in \ker(dJ_i)_x\big) \,\big\}, \quad (i = 1, 2).$$

A twisted symplectic manifold S in Definition 3.1 is called a $(P_1,\, P_2)$-equivalence bimodule (or an equivalence bimodule for short), and denoted by $P_1 \overset{J_1}{\leftarrow} S \overset{J_2}{\rightarrow} P_2$.

The following examples may help us understand Morita equivalence for integrable twisted Poisson manifolds:

Example 3.1. An integrable twisted Poisson manifold is Morita equivalent to itself with an equivalence bimodule $\mathcal{G}(P)$.

Example 3.2. (Example 2.1 in [21]) Let S be a connected and simply-connected symplectic manifold, and M a connected smooth manifold with a trivial Poisson structure:$\{\cdot,\, \cdot\} \equiv 0$. Then, $S \times M$ is Morita equivalent to M with a equivalence bimodule $S \times T^*M$.

Example 3.3. Let P_i and Q_i be twisted Poisson manifolds $(i = 1, 2)$. Assume that P_1 and Q_1 are Morita equivalent to P_2 and Q_2 respectively, with equivalence bimodules $P_1 \overset{J_1}{\leftarrow} X \overset{J_2}{\rightarrow} P_2$ and $Q_1 \overset{J_1'}{\leftarrow} Y \overset{J_2'}{\rightarrow} Q_2$. Then, $P_1 \times Q_1$ and $P_2 \times Q_2$ are Morita equivalent: $P_1 \times Q_1 \overset{J_1 \times J_1'}{\longleftarrow} X \times Y \overset{J_2 \times J_2'}{\longrightarrow} P_2 \times Q_2$.

Example 3.4. Two simply-connected twisted symplectic manifolds $(S_i,\, \omega_i,\, \psi_i)$ $(i = 1, 2)$ are Morita equivalent with a equivalence bimodule $S_1 \times S_2$.

Morita equivalence is indeed an equivalence relation among twisted Poisson manifolds: As for the transitivity, suppose that $(S_1,\, \omega_1)$ is a $(P_1,\, P_2)$-equivalence bimodule with moment maps $P_1 \overset{J_1}{\leftarrow} S_1 \overset{J_2}{\rightarrow} P_2$ and $(S_2,\, \omega_2)$ is a $(P_2,\, P_3)$-equivalence bimodule with moment maps $P_2 \overset{J_2'}{\leftarrow} S_2 \overset{J_3}{\rightarrow} P_3$. We define a smooth manifold $S_1 \otimes S_2$ to be the quotient of the fiber product by its characteristic foliation:

$$S_1 \otimes S_2 = (S_1 \times_{P_2}^{J_2, J_2'} S_2) \big/ \ker\big(\iota^*(\omega_1 \oplus \omega_2)\big), \tag{3. 1}$$

where $\iota : S_1 \times_{P_2}^{J_2, J_2'} S_2 \hookrightarrow S_1 \times S_2$ is the canonical embedding map. In addition, we define a non-degenerate 2-form $\tilde{\omega}$ on $S_1 \otimes S_2$ by

$$\tilde{\omega}_{\pi(p)}(\pi_* u,\, \pi_* v) = (\omega_1 \oplus \omega_2)_p(\iota_* u,\, \iota_* v),$$

where $\pi : S_1 \times_{P_2}^{J_2, J_2'} S_2 \to S_1 \otimes S_2$ is the natural projection. It is verified that the 2-form $\tilde{\omega}$ is well-defined in a way similar to [22]. Then $(S_1 \otimes S_2,\, \tilde{\omega})$ is a $(P_1,\, P_3)$-equivalence bimodule with the moment maps $\tilde{J}_1 : S_1 \otimes S_2 \to P_1$ and $\tilde{J}_3 : S_1 \otimes S_2 \to P_3$ given by $\tilde{J}_1([x,\, y]) := J_1(x)$ and $\tilde{J}_3([x,\, y]) := J_3(y)$, respectively.

Remark. We note that the tensor product in (3. 1) is not associative, but just associative up to a bimodule isomorphism. For any integrable twisted Poisson manifolds P_1 and P_2, we denote by $\mathcal{B}(P_1,\, P_2)$ the set of all $(P_1,\, P_2)$-equivalence bimodules. It is verified that $\mathcal{B}(P_1,\, P_2)$ forms a category whose morphisms are complete twisted Poisson maps f between equivalence bimodules $P_1 \overset{J_1}{\leftarrow} S_1 \overset{J_2}{\rightarrow} P_2$ and $P_1 \overset{K_1}{\leftarrow} S_2 \overset{K_2}{\rightarrow} P_2$ which satisfies $J_1 = K_1 \circ f$ and $J_2 = K_2 \circ f$. Then, we have the bicategory $\mathsf{twPoiss}$ which has integrable twisted Poisson manifolds as 0-cells, equivalence bimodules as 1-cells and the tensor product as compositions. Two integrable twisted Poisson manifolds are Morita equivalent if and only if they are isomorphic object in $\mathsf{twPoiss}$. For the definition of a bicategory, we refer to [1].

3.2 Gauge equivalence

Let ϕ be a closed 3-form on a smooth manifold M. A ϕ-twisted Dirac structure on M is a subbundle $L_M \subset \mathbb{T}M := TM \oplus T^*M$ which is maximal isotropic with respect to the symmetric paring $\langle \cdot,\, \cdot \rangle$ and whose the set of sections $\Gamma(L_M)$ is closed under the bracket $[\![\cdot,\, \cdot]\!]$, where $\langle \cdot,\, \cdot \rangle$ and $[\![\cdot,\, \cdot]\!]$ are defined as follows:

(1) $\langle \cdot,\, \cdot \rangle : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to C^\infty(M), \quad \langle (X,\, \xi),\, (Y,\, \eta) \rangle := \eta(X) + \xi(Y);$
(2) $[\![\cdot,\, \cdot]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M), \quad [\![(X,\, \xi),\, (Y,\, \eta)]\!] := \big([X,\, Y],\, \mathcal{L}_X \eta - i_Y d\xi + i_X i_Y \phi \big).$

For a full discussion of Dirac structures, we refer to [2], [3] and Bursztyn, H. and Radko, O. [4].

Example 3.5. (Twisted Poisson manifolds) Let ϕ be a closed 3-form on a smooth manifold P,and Π a bivector on P. The graph L_Π of $\Pi^\sharp : T^*P \to TP$ is a ϕ-twisted Dirac structure if and only if Π and ϕ satisfy the formula (2. 1) , that is, P is a ϕ-twisted Poisson manifold.

Example 3.6. (Twisted symplectic manifolds) If ω is a non-degenerate 2-form on S, then the graph L_ω of $\omega^\flat : TS \to T^*S$ is a ψ-twisted Dirac structure if and only if ω is a ψ-twisted symplectic form.

Let $(M,\, L_M,\, \phi_M)$ and $(N,\, L_N,\, \phi_N)$ be twisted Dirac manifolds. A smooth map $J : M \to N$ is said to be a forward Dirac map if

$$(L_N)_{J(x)} \;=\; \big\{\big((dJ)_x V,\, \alpha\big) \mid V \in T_xM,\, \alpha \in T_{J(x)}N,\, (V,\, (dJ)^*_x\alpha) \in (L_M)_x\big\}$$

for all $x \in M$. We write $\big(\mathfrak{F}(J)\big)(L_M)$ for the right-hand side in the above formula. As verified easily, if L_M and L_N are associated with twisted Poisson structures, then a forward Dirac map is equivalent to a twisted Poisson map.

Now we recall gauge transformations on (ϕ-)twisted Dirac structures. Let L_M be a ϕ-twisted Dirac structure on M and $B \in \Gamma(\wedge^2 T^*M)$. We set

$$\tau_B(L_M) := \big\{\big(X,\, \xi + B^\flat(X)\big) \;\big|\; (X,\, \xi) \in L_M\big\}\,.$$

The subbundle $\tau_B(L_M)$ defines a $(\phi - dB)$-twisted Dirac structure on M. The operation $L_M \mapsto \tau_B(L_M)$ is called a gauge transformation of L_M associated with B. Especially, if L_Π is a ϕ-Dirac structure associated with a twisted Poisson manifold P, the gauge transformation associated with B is given by

$$L_\Pi \longmapsto \tau_B(L_\Pi) = \big\{\big(\Pi^\sharp(\alpha),\, \alpha + B^\flat(\Pi^\sharp(\alpha))\big) \;\big|\; \alpha \in T^*P\big\}\,.$$

As discussed in [19], $\tau_B(L_\Pi)$ may fail to be induced from a twisted Poisson bivector. $\tau_B(L_\Pi)$ is associated with a $(\phi - dB)$-twisted Poisson manifold if and only if $1 + B^\flat\Pi^\sharp : T^*P \to T^*P$ is invertible. Two twisted Poisson manifold $(P,\, \Pi,\, \phi)$ and $(P,\, \Pi',\, \phi')$ are said to be gauge equivalent if there exists a 2-form B on P such that

$$\Pi' = \Pi \circ (1 + B^\flat\Pi^\sharp)^{-1} \quad \text{and} \quad \phi - \phi' = dB.$$

For a twisted Poisson bivector which is gauge equivalent to P with respect to B, we write $\tau_B(\Pi)$.

THEOREM 3.2. *Let* $(P,\,\Pi,\,\phi)$ *be an integrable twisted Poisson manifold and* $(\mathcal{G}(P),\,\omega)$ *the associated twisted symplectic groupoid. Then, for any 2-form* $B \in \Gamma(\wedge^2 T^*P)$ *such that* $(1 + B^\flat \Pi^\sharp)$ *is invertible,* $(P,\,\Pi,\,\phi)$ *and* $(P,\,\tau_B(\Pi),\,\phi - dB)$ *are Morita equivalent with a equivalence bimodule* $(\mathcal{G}(P),\,\hat{\omega})$*, where* $\hat{\omega} := \omega - \mathbf{s}^* B$*.*

Proof. For any $x \in \mathcal{G}(P)$, we set $V = T_x\big(\mathcal{G}(P)\big)$, $W = T_{\mathbf{s}(x)}P$. We define H_1 and H_2 by

$$H_1 := \tau_{\mathbf{s}^*B}(L_\omega) = \{\,(v,\, i_v(\omega + \mathbf{s}^*B)) \mid v \in V\,\},$$
$$H_2 := \big(\mathfrak{F}(\mathbf{s})\big)(L_\omega) = \{\,(\mathbf{s}_* v,\, \eta) \mid v \in V,\, \eta \in W^*,\, \eta \circ \mathbf{s} = i_v\omega\,\}.$$

Then,

$$\big(\mathfrak{F}(\mathbf{s})\big)(H_1) = \{\,(d\mathbf{s}(v),\, \eta) \mid v \in V,\, \eta \in W^*,\, \eta \circ \mathbf{s} = i_v(\omega + \mathbf{s}^*B)\,\}$$
$$\tau_B(H_2) = \{\,(d\mathbf{s}(v),\, \xi + i_{\mathbf{s}_*v}B) \mid v \in V,\, \xi \in W^*,\, \mathbf{s}^*\xi = i_v\omega\,\}$$
$$= \{\,(d\mathbf{s}(v),\, \eta) \mid v \in V,\, \eta \in W^*,\, \mathbf{s}^*(\eta - i_{\mathbf{s}_*v}B) = i_v\omega\,\}.$$

Since $i_v(\mathbf{s}^*B) = \mathbf{s}^*\big(i_{\mathbf{s}_*v}B\big)$, we have $\mathbf{s}^*\eta = i_v(\omega + \mathbf{s}^*B)$. Therefore, $\big(\mathfrak{F}(\mathbf{s})\big)(H_1) = \tau_B(H_2)$. From the fact that $\mathbf{s} : \mathcal{G}(P) \to (P,\, -\Pi)$ is a twisted Poisson map ([5]), it follows that $\big(\mathfrak{F}(\mathbf{s})\big)(H_1) = \tau_B(-\Pi)$, that is, $\mathbf{s} : (\mathcal{G}(P),\, \widehat{\omega}) \to (P,\, \tau_B(\Pi))$ is an anti-t.s.realization. It also can be shown that $\mathbf{t} : (\mathcal{G}(P),\, \widehat{\omega}) \to (P,\, \Pi)$ is a t.s.realization in a similar way. Moreover, using $(\ker d\mathbf{s})^\omega = \ker d\mathbf{t}$, we have

$$(\ker d\mathbf{s})^{\widehat{\omega}} = \{\, v \in V \mid \widehat{\omega}(v,\, w) = 0\ (\forall w \in \ker d\mathbf{s})\,\}$$
$$= \{\, v \in V \mid \omega(v,\, w) = 0\ (\forall w \in \ker d\mathbf{s})\,\} = (\ker d\mathbf{s})^\omega$$
$$= \ker d\mathbf{t}.$$

Similarly, we can prove $(\ker d\mathbf{t})^{\widehat{\omega}} = \ker d\mathbf{s}$.

In what follows, we prove that $\mathbf{t}$ and $\mathbf{s}$ are complete. Let $\hat{H}_\bullet$ and $\hat{X}_\bullet$ denote the Hamiltonian vector field with regard to $\hat{\omega}$ and ω', respectively. Then, from assumption we have $\mathbf{s}^*B(H_{\mathbf{t}^*f}) = \mathbf{0}$ $(f \in C^\infty(P))$. Therefore, $\hat{\omega}(H_{\mathbf{t}^*f},\, \cdot) = \omega(H_{\mathbf{t}^*f},\, \cdot) = d(\mathbf{t}^*f)(\cdot)$. This implies that $\hat{H}_{\mathbf{t}^*f} = H^*_{\mathbf{t}}f$. From the completeness of $\mathbf{t} : (\mathcal{G}(P),\, \omega) \to P$, we can conclude that $\mathbf{t} : (\mathcal{G}(P),\, \hat{\omega}) \to P$ is complete. The completeness of $\mathbf{s}$ can be proved similarly (see [4]). □

4. Weak Morita equivalence

Let $A_i \to M_i$ $(i = 1, 2)$ be Lie algebroids. Assume that A_1 and A_2 act on X from the left and right, respectively. If the actions ϱ_1, ϱ_2 commute i.e., $[\varrho_1(\xi),\, \varrho_2(\eta)] = 0$ for any $\xi \in \Gamma(A_1)$, $\eta \in \Gamma(A_2)$, and the moment maps are surjective submersions, then we call X an $(A_1,\, A_2)$-algebroid bimodule.

DEFINITION 4.1. Let (P_i, Π_i, ϕ_i) $(i = 1, 2)$ be twisted Poisson manifolds. P_1 and P_2 are said to be (weak) Morita equivalent if there exists a (T^*P_1, T^*P_2)-algebroid bimodule $P_1 \overset{J_1}{\leftarrow} M \overset{J_2}{\rightarrow} P_2$ which satisfies

(1) Each J_i-fiber $(i = 1, 2)$ is connected and simply-connected;
(2) For any $x \in M$,

$$T_x\big(J_1^{-1}(J_1(x))\big) = \left\{ \varrho_2(\eta)_x \;\middle|\; \eta \in \Gamma(T^*P_2) \right\} \quad \text{and}$$
$$T_x\big(J_2^{-1}(J_2(x))\big) = \left\{ \varrho_1(\xi)_x \;\middle|\; \xi \in \Gamma(T^*P_1) \right\},$$

where $\varrho_1 : \Gamma(T^*P_1) \to \Gamma(TM)$ and $\varrho_2 : \Gamma(T^*P_2) \to \Gamma(TM)$ mean the Lie algebroid actions of T^*P_1 and T^*P_2, respectively.

THEOREM 4.2. *Weak Morita equivalence is an equivalence relation for (twisted) Poisson manifolds.*

Proof. First, we verify the reflectivity. If P is a twisted Poisson manifold, its cotangent bundle $T^*P \overset{\pi}{\rightarrow} P$ is a (T^*P, T^*P)-algebroid bimodule under the left action $\varrho_L : \Gamma(T^*P) \to \Gamma\big(T(T^*P)\big)$, $\alpha \mapsto \Pi_C^\sharp(\pi^*\alpha)$ and the right action $\varrho_R : \Gamma(T^*P) \to \Gamma\big(T(T^*P)\big)$, $\alpha \mapsto -\Pi_C(\pi^*\alpha)$, where Π_C means a Poisson bivector induced from a canonical symplectic structure on T^*P. From $\pi_*\Pi_C^\sharp(\pi^*\alpha) = 0$, we have $T_u\big(\pi^{-1}(\pi(u))\big) = \rho_L\big(\Gamma(T^*P)\big)_u = \rho_R\big(\Gamma(T^*P)\big)_u$ $(\forall u \in T^*P)$. Therefore, P is weak Morita equivalent to itself.

As for the symmetry, we suppose that P_1 is weak Morita equivalent to P_2 with an algebroid bimodule $P_1 \overset{J_1}{\leftarrow} M \overset{J_2}{\rightarrow} P_2$. Then, a smooth manifold $P_2 \overset{J_2}{\leftarrow} M \overset{J_1}{\rightarrow} P_1$ with the reversed actions $\varrho'_L := -\varrho_R$, $\varrho'_R := -\varrho_L$ is a (T^*P_2, T^*P_1)-algebroid bimodule. It follows that P_2 is weak Morita equivalent to P_1.

The transitivity will be shown in what follows. Suppose that $P_1 \overset{J_1}{\leftarrow} M \overset{J_2}{\rightarrow} P_2$ is a (T^*P_1, T^*P_2)-algebroid bimodule and $P_2 \overset{J'_2}{\leftarrow} N \overset{J_3}{\rightarrow} P_3$ is a (T^*P_2, T^*P_3)-algebroid bimodule. We define the left and right actions on the fiber product $L := M \times_{P_2}^{J_2, J'_2} N$ by

$$\tilde{\varrho}_1 : \Gamma(T^*P_1) \to \Gamma(TL),\ \alpha \mapsto \big(\varrho_1(\alpha),\, \mathbf{0}\big) \quad \text{and}$$
$$\tilde{\varrho}_3 : \Gamma(T^*P_3) \to \Gamma(TL),\ \beta \mapsto \big(\mathbf{0},\, \varrho_3(\beta)\big),$$

respectively. Then, L is a (T^*P_1, T^*P_3)-algebroid bimodule with the moment maps $P_1 \overset{\rho}{\leftarrow} L \overset{\sigma}{\rightarrow} P_3$, where $\rho(m, n) := J_1(m)$ and $\sigma(m.n) := J_3(n)$. From assumption, we have

$$T_{(m,n)}\left(\sigma^{-1}\big(\sigma(m, n)\big)\right) = T_{(m.n)}\left(J_2^{-1}\big(J'_2(n)\big) \times \{n\}\right) = \tilde{\varrho}_1\big(\Gamma(T^*P_1)\big)_{(m,n)}.$$

Similarly, $T_{(m,n)}\left(\rho^{-1}\big(\rho(m, n)\big)\right) = \tilde{\varrho}_3\big(\Gamma(T^*P_3)\big)_{(m,n)}$. Obviously, each fiber is connected and simply-connected. Hence, P_1 and P_3 is weak Morita equivalent. □

Proposition 4.3. *Strong Morita equivalence implies weak Morita equivalence.*

Proof. Assume that integrable twisted Poisson manifolds P_1 and P_2 are strong Morita equivalent with an equivalence bimodule $P_1 \overset{J_1}{\leftarrow} (S,\, \omega_S) \overset{J_2}{\rightarrow} P_2$. As discussed in Section 2, the moment maps J_1 and J_2 induce Lie algebroid actions $\varrho_1(\alpha) := \Pi_S^\sharp(J_1^*\alpha)$ and $\varrho_2(\beta) := \Pi_S^\sharp(J_2^*\beta)$, respectively, where Π_S means the bivector field induced from ω_S. Using (2. 2), we have $[\varrho_1(\alpha),\, \varrho_2(\beta)] = 0$ for any $\alpha \in \Gamma(T^*P_1)$, $\beta \in \Gamma(T^*P_2)$. This implies that S is a $(T^*P_1,\, T^*P_2)$-algebroid bimodule. From assumption, it follows that, for any $x \in S$,

$$T_x\left(J_1^{-1}\big(J_1(x)\big)\right) = \ker(dJ_1)_x = \big(\ker(dJ_2)_x\big)^{\perp} = \Pi_S^\sharp\big(\ker(dJ_2)_x^\circ\big) = \varrho_2\big(\Gamma(T^*P_2)\big)_x,$$

where $\ker(dJ_2)_x^\circ$ denotes the annihilator of $\ker(dJ_2)_x$.
Analogously, we have $T_x\left(J_2^{-1}\big(J_2(x)\big)\right) = \varrho_1\big(\Gamma(T^*P_1)\big)_x$. Therefore, P_1 and P_2 are weak Morita equivalent. □

Weak Morita equivalence induces one-to-one correspondence between twisted symplectic leaves. The following theorem can be shown in a way similar to Theorem 11.1.9 in Ortega, J. and Ratiu, T. [14].

Theorem 4.4. *Suppose that P_1 and P_2 are weak Morita equivalent with a algebroid bimodule $P_1 \overset{J_1}{\leftarrow} M \overset{J_2}{\rightarrow} P_2$. Let $M/\mathcal{D}$ be the leaf space of the distribution $\mathcal{D}$ defined by $\mathcal{D}_m := \ker(dJ_1)_m + \ker(dJ_2)_m\, (\forall m \in M)$ and $\mathcal{L}(P_i)$ $(i = 1, 2)$ the spaces of twisted symplectic leaves of P_i, respectively. Then,*

1. *The distribution $\mathcal{D} = \ker(dJ_1) + \ker(dJ_2)$ is integrable.*
2. *$M/\mathcal{D} \to \mathcal{L}(P_i)$ $(i = 1, 2)$ are bijections. In particular, the map $\mathcal{L}(P_1) \to \mathcal{L}(P_2)$, $L \mapsto J_2\big(J_1^{-1}(L)\big)$ is the bijective correspondence between the leaves of P_1 and the leaves of P_2.*

Proof. (1) From assumption, $\ker(dJ_1)$ and $\ker(dJ_2)$ can be considered as the distributions

$$V_1 = \{\, \varrho_2(\beta) \,|\, \beta \in \Gamma(T^*P_2)\,\} \quad \text{and} \quad V_2 = \{\, \varrho_1(\alpha) \,|\, \alpha \in \Gamma(T^*P_1)\,\},$$

respectively. The distribution $\mathcal{D}$ is spanned by $V = V_1 \cup V_2$. Let θ_t and η_t be the flows of $\varrho_1(\alpha)$ and $\varrho_2(\beta)$, respectively. We will show that

$$(d\theta_t)_m\big(\varrho_2(\beta)_m\big) \in \mathcal{D}\big(\theta_t(m)\big) \quad \text{and} \quad (d\eta_t)_m\big(\varrho_1(\alpha)_m\big) \in \mathcal{D}\big(\eta_t(m)\big).$$

Since θ_t is a diffeomorphism, we can define the pull-back of $X \in \Gamma(TM)$ by $\theta_t^*X := (d\theta_{-t}) \circ X \circ \theta_t$. Then, we have the following formula (see [14]):

$$\frac{d}{dt}\theta_t^*\big(\varrho_2(\beta)\big) = \theta_t^*[\varrho_1(\alpha),\, \varrho_2(\beta)].$$

By assumption that the two Lie algebroid actions commute, the right-hand side in the above formula is equal to $\mathbf{0}$. Therefore, $\theta_t^*\varrho_2(\beta) = \theta_0^*\varrho_2(\beta) = \varrho_2(\beta)$. It follows that

$$(d\theta_t)_m\big(\varrho_2(\beta)_m\big) = (d\theta_t)_m \circ (d\theta_{-t})_{\theta_t(m)}\big(\varrho_2(\beta)\big)_{\theta_t(m)} = \big(\varrho_2(\beta)\big)_{\theta_t(m)} \in \mathcal{D}\big(\theta_t(m)\big).$$

We can show that $(d\eta_t)_m\big(\varrho_1(\alpha)_m\big) \in \mathcal{D}\big(\eta_t(m)\big)$ in similar way.

(2) We will denote by Φ_V, Φ_{V_1} and Φ_{V_2} the pseudogroups of local transformations generated by the flows of elements in V, V_1 and V_2, respectively. For full discussion of pseudogroups, we refer to [14]. Let $N \subset M$ be the integrable manifold of $\mathcal{D}$ containing a given point $m \in M$. We note that L coincides with the Φ_V-orbit of m:

$$N = \Phi_V \cdot m = \{\, \varphi(m) \mid \varphi \in \Phi_V \,\}.$$

Since the two Lie algebroid actions commute, N can be written as

$$N = \Phi_V \cdot m = \Phi_{V_1}\big(\Phi_{V_2} \cdot m\big).$$

From assumption, the J_i-fibers ($i = 1, 2$) are preserved by the elements in Φ_{V_i}, respectively. Accordingly,

$$J_1(N) = J_1\big(\Phi_{V_1}(\Phi_{V_2} \cdot m)\big) = J_1(\Phi_{V_2} \cdot m).$$

Any element $\theta \in \Phi_{V_2}$ can be represented as $\theta = \theta^1_{t_1} \circ \cdots \circ \theta^n_{t_n}$, where $\theta^j_{t_j}$ ($j = 1, \cdots, n$) mean the flows of a vector fields $\varrho_1(dJ_1^* f_j)$, $f_j \in C^\infty(P_1)$. Accordingly,

$$J_1\big(\theta(m)\big) = J_1\big((\theta^1_{t_1} \circ \cdots \circ \theta^n_{t_n})(m)\big) = \big(\xi^1_{t_1} \circ \cdots \circ \xi^n_{t_n}\big)\big(J_1(m)\big),$$

where $\xi^j_{t_j}$ ($j = 1, \cdots, n$) are the flows of Hamiltonian vector fields H_{f_j}. $\xi^1_{t_1} \circ \cdots \circ \xi^n_{t_n}$ is the element of the pseudogroup Φ_H of local transformations generated by the flows of Hamiltonian vector fields on P_1. Moreover, the twisted symplectic leaf of P_1 is the maximal integral manifold of the distribution spanned by the Hamiltonian vector fields on P_1. Therefore, we have $J_1(N) = J_1(\Phi_{V_2} \cdot m) = L_{J_1(m)}$, where $L_{J_1(m)}$ means the leaf of P_1 containing $J_1(m)$. Consequently, we can define the map $\Psi : M/\mathcal{D} \to \mathcal{L}(P_1)$ by

$$\Psi : M/\mathcal{D} \longrightarrow \mathcal{L}(P_1), \quad N = \Phi_V \cdot m \longmapsto J_1(N) = \Phi_H \cdot J_1(m).$$

To show the bijectivity of Ψ, we will prove that the map Ψ' defined by $J_1(N) \mapsto J_1^{-1}\big(J_1(N)\big)$ is an inverse of Ψ. From $J_1(N) = J_1(\Phi_{V_2} \cdot m)$, it follows that

$$J_1^{-1}\big(J_1(N)\big) = \bigcup_{\theta \in \Phi_{V_2}} J_1^{-1}\big(J_1(\theta(m))\big).$$

Here, since the elements in Φ_{V_1} preserve each J_1-fiber, we have $J_1^{-1}\big(J_1(\theta(m))\big) = \Phi_{V_1} \cdot \theta(m)$ for any $\theta \in \Phi_{V_2}$. Therefore,

$$J_1^{-1}\big(J_1(N)\big) = \bigcup_{\theta \in \Phi_{V_2}} \Phi_{V_1} \cdot \theta(m) = \Phi_{V_1} \cdot (\Phi_{V_2} \cdot m) = \Phi_V \cdot m = N.$$

This leads us to the conclusion that Ψ is bijective. Similarly, we can construct the map $M/\mathcal{D} \to \mathcal{L}(P_2)$ and show that this map is bijective. □

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