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ALMOST STABILITY OF ITERATIVE SCHEMES INVOLVING A UNIFORMLY HEMI-CONTRACTIVE SET-VALUED MAPPING IN A BANACH SPACE

By

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Abstract. Let E be an arbitrary Banach space and let $T : E \to CB(E)$ be a uniformly hemi-contractive set-valued mapping, where $CB(E)$ is the set of non-empty closed and bounded subsets of E. For T, we consider an iterative scheme $\{f_n(T, \cdot, \cdot)\}_n$ which is defined as follows: For a sequence $\{v_n\}$ in E, any $n \geq 0$ and any $x \in E$,

$$f_n(T, v_n, x) = t_n T v_n + (1 - t_n)x + u_n,$$

where $\{t_n\}$ is a coefficient sequence in $[0, 1]$ and $\{u_n\}$ is an error term sequence in E. In the present paper, we prove almost stability of the iterative scheme $\{f_n(T, v_n, \cdot)\}_n$, and show this result implies strong convergence theorems of generalized Mann and Ishikawa iterative schemes for the set-valued mapping.

1. Introduction

Let E be an arbitrary Banach space with a norm $\|\cdot\|$ and let $T : E \to E$ be a mapping such that the set $F(T)$ of fixed points of T is nonempty. A family $\{f_n(T, \cdot)\}_n$ is said to be an iterative scheme when $\{f_n(T, \cdot)\}_n$ is considered as a procedure which yields a sequence of points $\{x_n\} \subset E$ defined by $x_{n+1} = f_n(T, x_n)$ for $n \geq 1$, where $x_1 \in E$ is given. The notation of $x_n \to p$ means that the sequence $\{x_n\}$ converges strongly to p. If $\lim_{n\to\infty} \|y_{n+1} - f_n(T, y_n)\| = 0$ implies $y_n \to p \in F(T)$, then the iterative scheme $\{f_n(T, \cdot)\}_n$ is said to be stable with respect to T (see [4]). We say that the iterative scheme $\{f_n(T, \cdot)\}_n$ is almost stable with respect to T if $\sum_{n=1}^{\infty} \|y_{n+1} - f_n(T, y_n)\| < \infty$ implies $y_n \to p \in F(T)$ (cf.[10], [15]). Clearly, the iterative scheme $\{f_n(T, \cdot)\}_n$ which is stable is almost stable. In [10] Osilike gave an example of iterative scheme which is almost stable, but not stable. In [4], Harder and Hicks pointed out the importance of the stability of iterative schemes from the view point of practical use of iterations, and

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gave some results. Recently, the stability of the iterative schemes for nonlinear mappings has been investigated by several authors (cf. [8], [9], [10], [11], [12], [13], [15], [16], [17], [18]). In [18], an iterative scheme $\{f_n(T,\cdot,\cdot)\}$ with a sequence $\{v_n\}$ in E was defined as follows: For any $x_0 \in E$ and $n \geq 0$,

$$x_{n+1} = f_n(T, v_n, x_n) = t_n T v_n + (1 - t_n)x_n + u_n,$$

where $\{t_n\}$ and $\{u_n\}$ are a coefficient sequence in $[0, 1]$ and an error term sequence in E, respectively. The result of almost stability of $\{f_n(T, v_n, \cdot)\}$ was proved.

In this paper, we treat iterative schemes involving a set-valued mapping $T : D(T) \subset E \to CB(E)$, where $CB(E)$ is the set of non-empty closed and bounded subsets of E. Let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|_*^2\},$$

where E^* denotes the dual space of E with a norm $\|\cdot\|_*$ and $\langle\cdot,\cdot\rangle$ denotes the duality pairing on $E \times E^*$. Let $\Omega = \{\psi : [0,\infty) \to [0,\infty) :$ strictly increasing with $\psi(0) = 0\}$. A mapping $T : D(T) \subset E \to CB(E)$ is called a uniform pseudo-contraction with $\psi \in \Omega$ if for all $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that for any $\theta_x \in Tx, \theta_y \in Ty$,

$$\langle \theta_x - \theta_y, j(x - y)\rangle \leq \|x - y\|^2 - \psi(\|x - y\|).$$

As well-known, if $\psi(t) = kt^2$ for some $k \in (0, 1)$, then T is strongly pseudo-contractive. If $\psi(t) = \phi(t)t$ for some $\phi \in \Omega$, then T is said to be ϕ-strongly pseudo-contractive. For the uniform pseudo-contraction T with $\psi \in \Omega$, let $\omega(t) = \min\{\psi(t), t^2\}$ on $[0,\infty)$. Then the following inequality holds: For any $x, y \in D(T)$ and $\theta_x \in Tx, \theta_y \in Ty$,

$$\begin{aligned}\langle \theta_x - \theta_y, j(x - y)\rangle &\leq \|x - y\|^2 - \psi(\|x - y\|)\\ &\leq \|x - y\|^2 - \omega(\|x - y\|).\end{aligned}$$

Thus, we can assume that $\psi(t) \leq t^2$ on $[0,\infty)$ when T is uniformly pseudo-contractive with $\psi \in \Omega$, without any loss of generality. The class of uniform pseudo-contraction seems to have been first introduced by Alber in [1] under the name “weakly contractive” mappings, and C. E. Chidume and C. O. Chidume also called it a “generalized Φ-pseudo-contractive” mapping in [3]. A mapping $T : D(T) \subset E \to CB(E)$ is called uniformly hemi-contractive with $\omega \in \Omega$ if $F(T) = \{p : p \in Tp\} \neq \emptyset$ and for any $x \in D(T)$ and any $p \in F(T)$, there exists $j(x - p) \in J(x - p)$ such that for any $\theta_x \in Tx$ and $\theta_p \in Tp$,

$$\langle \theta_x - \theta_p, j(x - p)\rangle \leq \|x - p\|^2 - \omega(\|x - p\|).$$

Since we have that for $p, q \in F(T)$,

$$\|p-q\|^2 = \langle p-q, j(p-q)\rangle \le \|p-q\|^2 - \omega(\|p-q\|),$$

the uniformly hemi-contractive mapping T can have at most one fixed point p. For the set-valued mapping T, we consider an iterative scheme $\{f_n(T,\cdot,\cdot)\}_n$ with a coefficient sequence $\{t_n\}$ in $[0,1]$ and an error term sequence $\{u_n\}$ in E as follows: For a sequence $\{v_n\}$ in E, any $n \ge 0$ and any $x \in E$,

$$f_n(T, v_n, x) = t_n T v_n + (1-t_n)x + u_n. \tag{1. 1}$$

This iterative scheme gives generalized Mann iterative sequence $\{x_n\}$ by taking $x_{n+1} \in f_n(T, x_n, x_n)$ for $x_0 \in E$ and any $n \ge 0$, and also generalized Ishikawa iterative sequence $\{w_n\}$ by taking $w_{n+1} \in f_n(T, w_n^{(1)}, w_n)$ with $w_n^{(1)} \in t_n^{(1)} T w_n + (1-t_n^{(1)})w_n + u_n^{(1)}$ for $w_0 \in E$ any $n \ge 0$, where $\{t_n^{(1)}\}$ and $\{u_n^{(1)}\}$ are a coefficient sequence in $[0,1]$ and an error term sequence in E, respectively. C. E. Chidume and C. O. Chidume [2] showed strong convergence theorems of generalized Mann iterative sequence involving a generalized Lipschitz continuous and uniformly hemi-contractive set-valued mapping T in a uniformly smooth Banach space. Moore and Nnouli [7] showed a strong convergence theorem of generalized Ishikawa iterative sequence involving a uniformly continuous and uniformly hemi-contractive set-valued mapping T in a real normed space. For $y \in f_n(T, v_n, x)$ defined by (1. 1), there exist $\theta_n \in T v_n$ such that $y = f_n(T, v_n, x)(\theta_n)$, which is denoted by

$$f_n(T, v_n, x)(\theta_n) = t_n \theta_n + (1-t_n)x + u_n. \tag{1. 2}$$

Then, we give the definition that an iterative scheme $\{f_n(T, v_n, \cdot)\}_n$ involving a set-valued mapping T is called almost stable if a sequence $\{y_n\}$, which satisfies $\sum_{n\ge 1} \|y_{n+1} - f_n(T, v_n, y_n)(\theta_n)\| < \infty$ for some $\{\theta_n\}$ with $\theta_n \in T v_n$ for $n \ge 1$, converges strongly to $p \in F(T)$, and prove the theorem of almost stability of the $\{f_n(T, v_n, \cdot)\}_n$ defined by (1. 1) for a sequence $\{v_n\}$ and the set-valued mapping T which is uniformly continuous and uniformly hemi-contractive in an arbitrary Banach space. This result concerning almost stability implies the strong convergence theorems of generalized Mann and Ishikawa iterative sequences with weaker assumptions than that in [2] and [7].

2. Preliminaries

We shall show some crucial lemmas in order to present our statements and to prove the main theorem.

Lemma 1. ([6]) *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be non-negative real number sequences satisfying the difference inequality*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n. \tag{2.1}$$

Suppose

$$\{t_n\} \subseteq [0, 1], \quad \sum_{n=1}^{\infty} t_n = \infty, \quad b_n = o(t_n) \text{ and } \sum_{n=1}^{\infty} c_n < \infty .$$

Then

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 2. ([17]) *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be non-negative real number sequences satisfying the difference inequality (2.1). Suppose $\{t_n\} \subseteq [0, 1]$, $\sum_{n=1}^{\infty} t_n = \infty$, $b_n = Kt_n$ for some K, and $\sum_{n=1}^{\infty} c_n < \infty$. Then $\{a_n\}$ is bounded.*

Lemma 3. ([17]) *Let $\{a_n\}, \{b_n\}, \{t_n\}, \{\delta_n\}$ and $\{\rho_n\}$ be non-negative real number sequences satisfying the following conditions :*

(a) $$a_{n+1} \leq \left(1 - t_n \frac{f_1(b_n)}{f_2(b_n)}\right) a_n + t_n\delta_n + \rho_n,$$

where f_1 and f_2 are non-negative increasing functions on $[0, \infty)$ and $f_2(0) > 0$,

(b) $$\{t_n\} \subset [0, 1] \quad \text{and} \quad \lim_{n\to\infty} t_n = 0,$$

(c) $$\lim_{n\to\infty} \delta_n = 0,$$

(d) $$\sum_{n=1}^{\infty} \rho_n < \infty,$$

(e) $$\{a_n\} \text{ is bounded} \quad \text{and} \quad \liminf_{n\to\infty} a_n = 0,$$

(f) $$\lim_{n\to\infty} (a_{n+1} - a_n) = 0,$$

(g) $$\lim_{n\to\infty} (a_n - b_n) = 0.$$

Then

$$\lim_{n\to\infty} a_n = 0.$$

Lemma 4. *Let E be a Banach space and let $T : E \to CB(E)$ be uniformly hemi-contractive with $\omega \in \Omega$ satisfying $\omega(t) \leq t^2$on $[0, \infty)$. Suppose that p denotes a unique fixed point of T. Then we obtain for any $x \in E$ and any $\theta_x \in Tx$,*

$$\|x - p\| \leq \|x - p + \alpha\{(1 - \gamma_x)(x - p) - (\theta_x - p)\}\| \text{ for any } \alpha > 0,$$

where

$$\gamma_x = \frac{\omega(\|x - p\|)}{\|x - p\|^2 + 1}.$$

Proof. Since T is a uniformly hemi-contractive mapping with $\omega \in \Omega$, we have for any $x \in E$ and any $\theta_x \in Tx$,

$$\langle \theta_x - p, j(x - p) \rangle \leq \|x - p\|^2 - \omega(\|x - p\|),$$

and we have

$$\langle (x - p) - (\theta_x - p), j(x - p) \rangle - \omega(\|x - p\|) \geq 0.$$

Moreover, since we have from the definition of γ_x

$$\gamma_x \langle x - p, j(x - p) \rangle = \omega(\|x - p\|) \frac{\langle x - p, j(x - p) \rangle}{\|x - p\|^2 + 1} \leq \omega(\|x - p\|),$$

we have

$$\begin{aligned} &\langle (x - p) - (\theta_x - p) - \gamma_x(x - p), j(x - p) \rangle \\ &= \langle (x - p) - (\theta_x - p), j(x - p) \rangle - \gamma_x \langle (x - p), j(x - p) \rangle \\ &\geq \langle (x - p) - (\theta_x - p), j(x - p) \rangle - \omega(\|x - p\|) \\ &\geq 0. \end{aligned}$$

By Kato's Lemma [5], we obtain

$$\|x - p\| \leq \|x - p + \alpha\{(1 - \gamma_x)(x - p) - (\theta_x - p)\}\|$$

for any $\alpha > 0$. □

Lemma 5. *Let E be a Banach space and let $T : E \to CB(E)$ be uniformly hemi-contractive with $\omega \in \Omega$ satisfying $\omega(t) \leq t^2$ on $[0, \infty)$. Denote by p the unique fixed point of T. For u, v and $y \in E$, let y^* be defined as*

$$y^* = t\theta + (1 - t)y + u,$$

where $\theta \in Tv$ and $t \in [0,1]$. Then we obtain the following inequality for any $\eta \in Ty^*$,

$$\|y^* - p\| \le \frac{1+t(1-\gamma^*)}{1+t}\|y-p\| + \frac{t^2(2-\gamma^*)}{1+t}\|y-\theta\| + \frac{t}{1+t}\|\eta-\theta\| + \frac{1+t(2-\gamma^*)}{1+t}\|u\|,$$

where

$$\gamma^* = \gamma_{y^*} = \frac{\omega(\|y^*-p\|)}{\|y^*-p\|^2+1}.$$

Proof. For any real number γ and any $\eta \in Ty^*$,

$$y = (1+t)y^* + t(1-\gamma)y^* - t\eta + t\eta + t\gamma y^* - 2ty^* - y^* + y.$$

Set $A = (1+t)y^* + t(1-\gamma)y^* - t\eta$, and we have

$$\begin{aligned} y &= A + t\eta + \{t(\gamma-2)-1\}y^* + y \\ &= A + t\eta + \{t(\gamma-2)-1\}\{t\theta + (1-t)y + u\} + y \\ &= A + t\eta + t^2(\gamma-2)(\theta-y) + t(\gamma-2)(y+u) \\ &\quad - t\theta + ty - u \\ &= A + t(\eta-\theta) + t^2(\gamma-2)(\theta-y) + t(\gamma-2)y \\ &\quad + t(\gamma-2)u + ty - u \\ &= A + t^2(\gamma-2)(\theta-y) + t(\eta-\theta) \\ &\quad + t\{(\gamma-2)+1\}y + \{t(\gamma-2)-1\}u \\ &= A + t^2(2-\gamma)(y-\theta) + t(\eta-\theta) \\ &\qquad + t(\gamma-1)y + \{t(\gamma-2)-1\}u. \end{aligned}$$

Since we also have

$$\begin{aligned} p &= (1+t)p - tp + t(1-\gamma)p - t(1-\gamma)p \\ &= (1+t)p + t(1-\gamma)p - tp + t(\gamma-1)p, \end{aligned}$$

we obtain

$$\begin{aligned} y - p &= A - \{(1+t)p + t(1-\gamma)p - tp\} - t(\gamma-1)p \\ &\quad + t^2(2-\gamma)(y-\theta) + t(\eta-\theta) + t(\gamma-1)y + \{t(\gamma-2)-1\}u \\ &= \{(1+t)(y^*-p) + t(1-\gamma)(y^*-p) - t(\eta-p)\} + t^2(2-\gamma)(y-\theta) \\ &\quad + t(\eta-\theta) + t(\gamma-1)(y-p) + \{t(\gamma-2)-1\}u. \end{aligned}$$

Since we have from Lemma 4

$$\begin{aligned}
&\|(1+t)(y^*-p)+t(1-\gamma^*)(y^*-p)-t(\eta-p)\| \\
&= (1+t)\left\|(y^*-p)+\frac{t}{1+t}\{(1-\gamma^*)(y^*-p)-(\eta-p)\}\right\| \\
&\geq (1+t)\left\|y^*-p\right\|,
\end{aligned}$$

thus we have, taking γ^* instead of γ,

$$\begin{aligned}
\|y-p\| \geq\ & \|(1+t)(y^*-p)+t(1-\gamma)(y^*-p)-t(\eta-p)\| \\
& - \left\|t^2(2-\gamma^*)(y-\theta)+t(\eta-\theta)+t(\gamma^*-1)(y-p)+(t(\gamma^*-2)-1)u\right\| \\
& \geq (1+t)\left\|(y^*-p)\right\| \\
& - \left\|t^2(2-\gamma^*)(y-\theta)+t(\eta-\theta)+t(\gamma^*-1)(y-p)+(t(\gamma^*-2)-1)u\right\| \\
& \geq (1+t)\left\|y^*-p\right\| - t^2(2-\gamma^*)\left\|y-\theta\right\| - t\left\|\eta-\theta\right\| \\
& \quad - t(1-\gamma^*)\left\|y-p\right\| - \left|t(2-\gamma^*)-1\right|\left\|u\right\|.
\end{aligned}$$

This means that

$$\begin{aligned}
\|y^*-p\| \leq\ & \frac{1+t(1-\gamma^*)}{1+t}\left\|y-p\right\| + \frac{t^2(2-\gamma^*)}{1+t}\left\|y-\theta\right\| \\
& + \frac{t}{1+t}\left\|\eta-\theta\right\| + \frac{1+t(2-\gamma^*)}{1+t}\left\|u\right\|.
\end{aligned}$$

□

3. Main Results

A set-valued mapping $T : D(T) \subset E \to CB(E)$ is said to be uniformly continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in E$, $\|x-y\| < \delta$ implies $H(Tx, Ty) < \varepsilon$, where $H(\cdot,\cdot)$ is a Hausdorff metric on $CB(E)$, i.e., for any $A, B \in CB(E)$,

$$H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}\left\|x-y\right\|, \sup_{y\in B}\inf_{x\in A}\left\|x-y\right\|\}. \tag{3.1}$$

We shall treat a iterative scheme $\{f_n(T,\cdot,\cdot)\}$ involving a uniformly continuous and uniformly hemi-contractive set-valued mapping T. We shall prove the main theorem by virtue of the previous lemmas.

Theorem 1. *Let E be a Banach space and let $T : E \to CB(E)$ be a uniformly continuous and uniformly hemi-contractive set-valued mapping with a function*

$\omega \in \Omega$ satisfying $\omega(t) \leq t^2$ on $[0, \infty)$. Let p be the unique fixed point of T. Suppose T has a bounded range (i.e., there exists $\tilde{M}$ such that $\sup\{\|\theta\| : \theta \in Tx,\ x \in E\} \leq \tilde{M} < \infty$). For T and a sequence $\{v_n\}$ in E, let an iterative scheme $\{f_n(T, v_n, \cdot)\}_n$ be defined by (1.1). Suppose that a coefficient sequence $\{t_n\}$ in $[0,1]$ and an error term sequence $\{u_n\}$ in E satisfy the following conditions :

$$(h) \sum_{n=0}^{\infty} t_n = \infty, \quad (i) \lim_{n\to\infty} t_n = 0, \quad (j) \sum_{n=0}^{\infty} ||u_n|| < \infty \ .$$

Then the following statements hold.

(I) *If $\{y_n\}$ in E satisfies*

$$\lim_{n\to\infty} ||y_n - v_n|| = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} ||y_{n+1} - f_n(T, v_n, y_n)(\theta_n)|| < \infty$$

for some $\{\theta_n\}$ with $\theta_n \in Tv_n$ for $n \geq 0$ as (1.2), then $y_n \to p$.

(II) *If $\{w_n\}$ in E converges strongly to p, then*

$$\lim_{n\to\infty} ||w_{n+1} - f_n(T, v_n, w_n)(\theta_n)|| = 0$$

for all $\{\theta_n\}$ with $\theta_n \in Tv_n$ for $n \geq 0$.

Proof. (I) Denote $y^*_{n+1} = f_n(T, v_n, y_n)(\theta_n)$ for $\theta_n \in Tv_n$ for any $n \geq 0$, as (1.2), and then, from a property of Hausdorff metric (see [14]), there exists $\eta_n \in Ty^*_{n+1}$ such that

$$\|\eta_n - \theta_n\| \leq 2H(Ty^*_{n+1}, Tv_n). \tag{3.2}$$

Suppose that $\varepsilon_n = ||y^*_{n+1} - y_{n+1}||$ for any $n \geq 0$ and that $M_0 = \sup_{n\geq 0} \|\theta_n - p\| < \infty$. Since we have

$$\begin{aligned} ||y_{n+1} - p|| &\leq \left\|y^*_{n+1} - y_{n+1}\right\| + \left\|y^*_{n+1} - p\right\| \\ &= \varepsilon_n + \|f_n(T, v_n, y_n)(\theta_n) - p\| \\ &= \varepsilon_n + ||t_n\theta_n + (1 - t_n)y_n + u_n - p|| \\ &\leq \varepsilon_n + t_n \|\theta_n - p\| + (1 - t_n) \|y_n - p\| + \|u_n\| \\ &\leq t_n M_0 + (1 - t_n) \|y_n - p\| + (\varepsilon_n + \|u_n\|), \end{aligned}$$

from Lemma 2, the $\{\|y_n - p\|\}$ is bounded. Let $M = \max\{M_0,\ \sup_n \|y_n - p\|\} < \infty$. From Lemma 5, we obtain

$$\begin{aligned} \left\|y^*_{n+1} - p\right\| \leq& \frac{1 + t_n(1 - \gamma_n)}{(1 + t_n)} \|y_n - p\| + \frac{t_n^2(2 - \gamma_n)}{(1 + t_n)} \|y_n - \theta_n\| \\ &+ \frac{t_n}{(1 + t_n)} \|\eta_n - \theta_n\| + \frac{1 + t_n(2 - \gamma_n)}{(1 + t_n)} \|u_n\| , \end{aligned} \tag{3.3}$$

where $\gamma_n = \gamma_{y^*_{n+1}} = \frac{\omega(\|y^*_{n+1}-p\|)}{\|y^*_{n+1}-p\|^2+1} \leq 1$. Noting the following inequality

$$\frac{1+t_n(1-\gamma_n)}{(1+t_n)} \leq 1 - t_n\gamma_n + (t_n)^2,$$

we have from (3. 3)

$$\begin{aligned}\|y_{n+1}-p\| &\leq \|y^*_{n+1}-p\| + \varepsilon_n \\ &\leq (1-t_n\gamma_n)\|y_n-p\| + t_n^2(\|y_n-p\| + \tfrac{2}{1+t_n}\|y_n-\theta_n\|) \\ &\quad + t_n\|\eta_n-\theta_n\| + \{\varepsilon_n + (1+2t_n)\|u_n\|\} \\ &\leq (1-t_n\gamma_n)\|y_n-p\| + t_n^2(5M) \\ &\quad + t_n\|\eta_n-\theta_n\| + (\varepsilon_n + 3\|u_n\|) \\ &\leq (1-t_n\gamma_n)\|y_n-p\| + (\varepsilon_n + 3\|u_n\|) \\ &\quad + t_n(\|\eta_n-\theta_n\| + 5t_nM),\end{aligned}$$

that is,

$$\begin{aligned}\|y_{n+1}-p\| \leq (1-t_n\gamma_n)\|y_n-p\| + (\varepsilon_n + 3\|u_n\|) \\ + t_n(\|\eta_n-\theta_n\| + 5t_nM).\end{aligned} \tag{3. 4}$$

Now, from (3. 4) we shall prove

$$\lim_{n\to\infty}\|y_n-p\| = 0. \tag{3. 5}$$

Since we have from assumptions

$$\begin{aligned}\|y^*_{n+1}-v_n\| &= \|t_n\theta_n + (1-t_n)y_n + u_n - v_n\| \\ &\leq t_n\|\theta_n-y_n\| + \|y_n-v_n\| + \|u_n\| \\ &\leq 2t_nM + \|y_n-v_n\| + \|u_n\| \to 0 \text{ as } n\to\infty,\end{aligned} \tag{3. 6}$$

the inequality (3. 2) and the uniform continuity of T imply

$$\lim_{n\to\infty}\|\eta_n-\theta_n\| \leq \lim_{n\to\infty} 2H(Ty^*_{n+1}, Tv_n) = 0. \tag{3. 7}$$

Set $\gamma = \liminf_{n\to\infty} \gamma_n$, and we have $\gamma > 0$ or $\gamma = 0$. Suppose $\gamma > 0$. For any $\delta \in (0, \frac{\gamma}{2})$ and sufficiently large $n \geq 1$, we have $\gamma - \delta < \gamma_n$. Then the inequality (3. 4) implies

$$\begin{aligned}\|y_{n+1} - p\| \leq \{1 - t_n(\gamma - \delta)\} \|y_n - p\| + (\varepsilon_n + 3\|u_n\|) \\ + t_n (\|\eta_n - \theta_n\| + 5t_n M).\end{aligned}$$

By Lemma 1, we obtain $\lim_{n\to\infty} \|y_n - p\| = 0$. On the other hand, suppose $\gamma = 0$. Since we have for sufficiently large n,

$$\begin{aligned}\left\|y^*_{n+1} - p\right\| &= \|t_n\theta_n - t_n y_n + y_n - p + u_n\| \\ &\leq t_n \|\theta_n - y_n\| + \|y_n - p\| + \|u_n\| \\ &\leq 4M,\end{aligned}$$

we obtain

$$0 = \liminf_{n\to\infty} \gamma_n = \liminf_{n\to\infty} \frac{\omega(\left\|y^*_{n+1} - p\right\|)}{\left\|y^*_{n+1} - p\right\|^2 + 1} \geq \liminf_{n\to\infty} \frac{\omega(\left\|y^*_{n+1} - p\right\|)}{16M^2 + 1} \geq 0,$$

and so we have $\liminf_{n\to\infty} \left\|y^*_{n+1} - p\right\| = 0$. Then we can apply Lemma 3 as follows. Set

$a_n = \|y_n - p\|$, $b_n = \left\|y^*_{n+1} - p\right\|$, $\delta_n = \|\eta_n - \theta_n\| + 5t_n M$, $\rho_n = \varepsilon_n + 3\|u_n\|$ for $n \geq 1$ and $f_1(t) = \omega(t)$, $f_2(t) = t^2 + 1$ for $t \in [0, \infty)$. Then the inequality (3. 4) implies that the conditions (a) and (b) of Lemma 3 are satisfied. From the selection of $\eta_n \in Ty^*_{n+1}$ and the uniform continuity of T, the (3. 4) implies the (c). From the assumptions of $\{\varepsilon_n\}$ and $\{u_n\}$, the (d) is satisfied. From the (3. 6) we have $\liminf_{n\to\infty} \left\|y^*_{n+1} - p\right\| = \liminf_{n\to\infty} b_n = 0$, and

$$\begin{aligned}a_n &= b_n + (a_n - b_n), \\ |a_n - b_n| &\leq \left\|y_n - y^*_{n+1}\right\| \\ &\leq t_n \|\theta_n - p\| + t_n \|y_n - p\| + \|u_n\| \\ &\leq 2t_n M + \|u_n\|,\end{aligned}$$

thus the conditions (e) and (g) are satisfied. Moreover,

$$\begin{aligned}a_{n+1} - a_n &\leq \|y_n - y_{n+1}\| \\ &\leq \varepsilon_n + \left\|y_n - y^*_{n+1}\right\| \\ &\leq \varepsilon_n + 2t_n M + \|u_n\|\end{aligned}$$

implies the condition (f). Therefore we obtain from Lemma 3,

$$\lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} a_n = 0.$$

The (I) is proved completely.

(II) Suppose $w_n \to p$. Then, take any $\{\theta_n\}$ satisfying $\theta_n \in Tv_n$ for $n \geq 1$, and we have

$$||w_{n+1} - f_n(T, v_n, w_n)(\theta_n)|| = \|w_{n+1} - t_n\theta_n - (1-t_n)w_n - u_n\|$$

$$\leq \|w_{n+1} - p\| + t_n\|\theta_n - p\| + (1-t_n)\|w_n - p\| + \|u_n\| \to 0 \quad \text{as } n \to \infty$$

Therefore the statement (II) follows. □

Next we consider generalized Mann and Ishikawa iterative sequences involving the above set-valued mapping T by using the iterative scheme $\{f_n(T, \cdot, \cdot)\}$. A generalized Mann iterative sequence is defined as follows: For any $n \geq 0$,

$$\left\{ \begin{array}{l} x_0 \in E, \\ x_{n+1} \in f_n(T, x_n, x_n) \\ \quad = t_nTx_n + (1-t_n)x_n + u_n. \end{array} \right. \tag{3.8}$$

Similarly, a generalized Ishikawa iterative sequence is defined as follows: For any $n \geq 0$,

$$\left\{ \begin{array}{l} w_0 \in E, \\ w_{n+1} \in f_n(T, w_n^{(1)}, w_n) \\ \quad = t_nTw_n^{(1)} + (1-t_n)w_n + u_n, \end{array} \right. \tag{3.9}$$

where $w_n^{(1)} \in t_n^{(1)}Tw_n + (1-t_n^{(1)})w_n + u_n^{(1)}$ for any $n \geq 0$, and $\{t_n^{(1)}\}$ and $\{u_n^{(1)}\}$ are a coefficient sequence in $[0,1]$ and a error term sequence in E, respectively. By virtue of Theorem 1 we can obtain strong convergence theorems with respect to the generalized Mann and Ishikawa iterative sequences as the following corollaries.

Corollary 1. *Let E be a Banach space, and let $T : E \to CB(E)$ be a uniformly continuous and uniformly hemi-contrative set-valued mapping with a function $\omega \in \Omega$ satisfying $\omega(t) \leq t^2$ on $[0, \infty)$. Suppose T has a bounded range. Let $\{x_n\}$ be generalized Mann iterative sequence involving T defined by (3.8), and assume that $\{t_n\}$ and $\{u_n\}$ satisfy the conditions $(h) - (j)$ in Theorem 1 . Then $\{x_n\}$ converges strongly to a unique fixed point p of T.*

Proof. By the iterative scheme $\{f_n(T, \cdot, \cdot)\}$, the generalized Mann sequence $\{x_n\}$ is represented as follows: For $x_0 \in E$ and any $n \geq 0$,

$$x_{n+1} \in f_n(T, x_n, x_n) = t_nTx_n + (1-t_n)x_n + u_n.$$

Putting $v_n = x_n$ for any $n \geq 0$, Theorem 1 implies strong convergence to p with respect to $\{x_n\}$. $\square$

Corollary 2. *Let E be a Banach space, and let $T : E \to CB(E)$ be a uniformly continuous and uniformly hemi-contrative set-valued mapping with a function $\omega \in \Omega$ satisfying $\omega(t) \leq t^2$ on $[0, \infty)$. Suppose T has a bounded range. Let $\{w_n\}$ be generalized Ishikawa iterative sequence involving T defined by (3.9), and assume that $\{t_n\}$ and $\{u_n\}$ satisfy the conditions $(h) - (j)$ in Theorem 1, and additionary, $\lim_{n\to\infty} t_n^{(1)} = 0$ and $\lim_{n\to\infty} \left\| u_n^{(1)} \right\| = 0$. Then $\{w_n\}$ converges strongly to a unique fixed point p of T.*

Proof. Similarly as Corollary 1, the generalized Ishikawa iterative sequence $\{w_n\}$ is defined as follows: For $x_0 \in E$ and any $n \geq 0$,

$$w_{n+1} \in f_n(T, w_n^{(1)}, w_n), \text{ with } w_n^{(1)} \in t_n^{(1)} T w_n + (1 - t_n^{(1)}) w_n + u_n^{(1)}.$$

Since $\lim_{n\to\infty} \left\| w_n^{(1)} - w_n \right\| = 0$ from the assumptions of T, $\{t_n^{(1)}\}$ and $\{u_n^{(1)}\}$, we can apply Theorem 1 in order to obtain strong convergence to p of $\{w_n\}$. $\square$

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