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EXISTENCE OF HOMOCLINICSOLUTIONS FOR A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

By

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Abstract. Let $N \geq 2$ and $\mathcal{D} \subset \mathbb{R}^{N-1}$ be a bounded domain with smooth boundary. In this paper, we consider the existence of homoclinic solutions for nonlinear elliptic problem

$$\begin{cases} \Delta_u + g(x,u) = 0 & \text{in} \quad \mathbb{R} \times \mathcal{D}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial(\mathbb{R} \times \mathcal{D}), \end{cases}$$

where $\nu(x)$ is the outward pointing normal derivative to ∂D and $g \in C^1((\mathbb{R} \times \overline{\mathcal{D}} \times \mathbb{R}, \mathbb{R})$ has a spacial periodicity.

1. Introduction

Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a cylindrical domain, i.e., $\Omega = \mathbb{R} \times \mathcal{D}$, where $\mathcal{D} \subset \mathbb{R}^{N-1}$ is a bounded open domain with a smooth boundary. In the present paper, we consider the existence of homoclinic solutions of a boundary value problem

(P)
$$\begin{cases} \Delta_u + g(x,u) = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $\nu = \nu(y)$ denotes the outward pointing normal derivative to $\partial(\mathcal{D} \times \mathbb{R})$. For $x \in \Omega$ we set $x = (x_1, y)$, where $x_1 \in \mathbb{R}$ and $y \in \mathcal{D}$. We impose the following conditions on g:

(g1) $g(x,z) \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and is 1-periodic with respect to x_1;

(g2) $G(x,z) = \displaystyle\int_0^z g(x,\tau)d\tau$ is 1-periodic with respect to z.

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In [1], Rabinowitz considered the existence of spacially heteroclinic solutions of problem (P) under the assumptions (g1), (g2) and an additional condition

(g3) $g(x, z)$ is even with respect to $x_1 \in \mathbb{R}$.

In [2] and [4], the existence of the heteroclinic solutions of (P) was established without the evenness condition (g3). Recently, using the results in these papers, the existence of homoclinic solutions of (P) was established in [3]. The purpose of this paper is to investigate the existence of ordered homoclinic solutions of (P) and give sharper characterizations of the solutions. Homoclinic solutions established in [3] are not ordered. We will show that there is a sequence of homoclinic solutions of (P) such that each solution is given as a local minimal of corresponding functional to (P).

2. Preliminaries and Statements of Main Result

Throughout the rest of this paper, we assume that $N \geq 2$, and conditions (g1) and (g2) hold. For $x, y \in \mathbb{R}^N$, we denote by $x \cdot y$ the inner product of x and y. For each bounded open set $U \subset \mathbb{R}^N$, we denote by $\|\cdot\|_{H^1(U)}$ and $\|\cdot\|_{L^2(U)}$ the norm of $H^1(U)$ and $L^2(U)$ defined by $\|u\|^2_{H^1(U)} = \int_U (|u|^2 + |\nabla_u|^2)dx$ and $\|v\|^2_{L^2(U)} = \int_U |v|^2 dx$ for each $u \in H^1(U)$ and $v \in L^2(U)$, respectively. We denoted by $\langle \cdot, \cdot \rangle_U$ the inner product of $H^1(U)$. Put $\Omega_i = [i, i+1] \times \mathcal{D}$ for each $i \in \mathbb{Z}$. For each function $u \in H^1_{loc}(\Omega)$ and $j \in \mathbb{Z}$. we denote by $\tau_j v$ the function denoted by

$$\tau_j v(x_1, y) = v(x_1 - j, y) \qquad \text{for all } (x_1, y) \in \mathbb{R} \times \mathcal{D}.$$

We set

$$L(u)(x) = \frac{1}{2}|\nabla u(x)|^2 - G(x, u) \qquad \text{for } u \in H^1_{loc}(\Omega) \text{ and } x \in \Omega.$$

Put

$$I_i(u) = \int_{\Omega_i} L(u)dx \qquad \text{for } i \in \mathbb{Z} \text{ and } u \in H^1(\Omega_i)$$

and

$$E = \left\{ u \in H^1(\Omega_0) : u \text{ is } 1\text{−periodic in } x_1 \right\}.$$

We put

$$c_0 = \inf_{u \in E} I_0(u) \quad \text{and} \quad M_0 = \{u \in E : I_0(u) = c_0\}.$$

Then the following is known.

Proposition 1. ([2]) *$M_0 \neq \emptyset$ and M_0 is an ordered set, i.e. for each $u, v \in M_0$ with $u \neq v$, $u < v$ on Ω_0 or $u > v$ on Ω_0 holds.*

Here we put

$$a_j(u) = \int_{\Omega_j} L(u)dx - c_0 \qquad \text{for } j \in \mathbb{Z} \text{ and } u \in H^1(\Omega_j),$$

and

$$J_{l,m}(u) = \sum_{j=l}^{m} a_j(u) \quad \text{for } l, m \in \mathbb{Z} \text{ with } l \geq m.$$

We also put

$$\begin{aligned} J(u) &= \liminf_{l \to -\infty} J_{l,0} + \liminf_{m \to \infty} J_{1,m}(u) && \text{for } u \in H^1_{loc}(\Omega), \\ J_{-\infty,m}(u) &= \liminf_{l \to -\infty} J_{l,0} + J_{1,m}(u) && \text{for } u \in H^1_{loc}(\Omega) \text{ and } m \geq 1, \\ J_{m,\infty}(u) &= J_{m,0}(u) + \liminf_{l \to \infty} J_{1,l}(u) && \text{for } u \in H^1_{loc}(\Omega) \text{ and } m \geq 0. \end{aligned}$$

For each $v, w \in M_0$ with $v < w$, we set

$$\begin{aligned} [v, w] &= \left\{ u \in H^1_{loc}(\Omega) : v \leq u \leq w \right\}, \quad [v, w]_m = \{ u|_{\Omega_m} : u \in [v, w] \}, \\ \Gamma_-(z) &= \left\{ u \in [v, w] : J(u) < \infty, \lim_{j \to -\infty} \|u - z\|_{L^2(\Omega_j)} = 0 \right\} \text{ for } z \in \{u, w\}, \\ \Gamma_+(z) &= \left\{ u \in [v, w] : J(u) < \infty, \lim_{j \to \infty} \|u - z\|_{L^2(\Omega_j)} = 0 \right\} \text{ for } z \in \{v, w\}, \end{aligned}$$

and

$$\Gamma(z_1, z_2) = \Gamma_-(z_1) \cap \Gamma_+(z_2) \qquad \text{for } z_1, z_2 \in \{v, w\}.$$

Then we have

Proposition 2. (cf.[3, 4]) *For each $v, w \in M_0$ and $u \in \Gamma(v, w)$, $\lim_{l \to -\infty} J_{l,0}(u)$ and $\lim_{m \to \infty} J_{1,m}(u)$ exist.*

Remark 1. From Proposition 2, it follows that for each $u \in \Gamma_-(v)$,

$$J_{-\infty,m}(u) = \lim_{l \to -\infty} J_{l,0}(u) + J_{1,m}(u) \qquad \text{for } m \geq 1.$$

Similarly, we have for each $u \in \Gamma_+(w)$,

$$J_{m,\infty}(u) = J_{m,0}(u) + \lim_{l \to \infty} J_{1,l}(u) \qquad \text{for } m \leq 0.$$

Let $v, w \in M_0$ and $v < w$. We assume v, w are adjacent in M_0, that is there are no other elements $u_0 \in M_0$ with $v < u_0 < w$. We call $u \in H^1_{loc}(\Omega)$ a heteroclinic solution of (P) in $[v, w]$ if $u \in \Gamma(v, w) \cup \Gamma(w, u)$ and u is a solution of (P). A solution $u \in H^1_{loc}(\Omega)$ of (P) is called a homoclinic solution in $[v, w]$ if $u \in \Gamma(v, v) \cup \Gamma(w, w)$.

We put

$$c(v, w) = \inf_{u \in \Gamma(v,w)} J(u) \quad \text{for } v, w \in M_0$$

and

$$\mathcal{M}(v, w) = \{u \in \Gamma(v, w) : J(u) = c(v, w)\} \quad \text{for } v, w \in M_0.$$

Then we have

Proposition 3. ([1]) *For each $v, w \in M_0$ which are adjacent and $v < w$, $\mathcal{M}(v, w)$ is a nonempty ordered set.*

Let $v_1, v_2 \in \mathcal{M}(v, w)$ and $v_1 < v_2$. If v_1, v_2 are adjacent in $\mathcal{M}(v, w)$, then there are no other elements $v_0 \in \mathcal{M}(v, w)$ with $v_1 < v_0 < v_2$. We will consider the existence of homoclinic solution of (P) under the following conditions:

$(*)$ $v, w \in M_0$ are adjacent with $v < w$.

$(**)$ $v_1, v_2 \in \mathcal{M}(v, w)$ are adjacent with $v_1 < v_2$ and $w_1, w_2 \in \mathcal{M}(w, v)$ are adjacent with $w_1 < w_2$.

$$\inf \left\{ I_0(v) : v \in H^1(\Omega_0) \right\} = c_0. \tag{C}$$

We call $u \in \Gamma(v, v)$ a local minimal of J in $\Gamma(v, v)$ if $J(u + \varphi) \geq J(u)$ for all $\varphi \in H^1(\Omega)$ with $\|\varphi\|_{H^1(\Omega)}$ sufficiently small. We will find a sequence $\{u_n\} \subset \Gamma(v, v)$ of homoclinic solutions of (P) such that each u_n is a local minimal of J in $\Gamma(v, v)$. We can now state our main results:

Theorem 1. *Assume that (g1), (g2), $(*)$, $(**)$ and (C) hold. Then there exist sequences $\{i_n\}, \{j_n\} \subset \mathbb{N}$ and $\{u_n\} \subset \Gamma(v, v)$ of homoclinic solutions of (P) such that*

(1) *each u_n is a local minimal of J in $\Gamma(v, v)$;*
(2) *$u_n \leq u_{n+1}$ for each $n \geq 1$;*
(3) *for each $n \geq 1$, $\tau_{-i_n} v_1 \leq u_n \leq \tau_{-i_n - 1}$ on $(-\infty, p] \times \mathcal{D}$ for some $p \in \mathbb{N}$;*
(4) *for each $n \geq 1$, $\tau_{j_n} w_1 \leq u_n \leq \tau_{j_n + 1} w_2$ on $[q, \infty) \times \mathcal{D}$ for some $q \in \mathbb{N}$ with $q > p$;*
(5) *$\lim_{n \to \infty} J(u_n) = c(v, w) + c(w, v)$.*

Remark 2. The analogous result holds for $\Gamma(w,w)$.

Remark 3. In [3], the existence of homoclinic solutions of (P) was established without assuming the condition (C). Assuming (C), we can get sharper characterizations (2), (3) and (4) for the solutions of (P). The condition (C) is satisfied if the functions satisfy (g3) (cf. [1]).

3. Proof of Theorem 1.

Throughout the rest of this paper, we assume that (g1), (g2), $(*)$, $(**)$ and (C) hold. We put

$$\mathcal{M}_m(v,w) = \{u[m]2 \in C(\Omega_m) : u \in \mathcal{M}(v,w)\} \qquad \text{for } m \in \mathbb{Z}.$$

Then since $\mathcal{M}(v,w)$ is an ordered set, $\mathcal{M}_m(v,w)$ is also an ordered set. Since $v_1, v_2 \in \mathcal{M}(v,w)$ are adjacent, we have that $v_1[m]$ and $v_2[m]$ are adjacent in $\mathcal{M}_m(v,w)$. One can see

$$\begin{aligned}(3.1) \qquad (\tau_n v_1)[m] < (\tau_n v_2)[m] &\le (\tau_{n-1} v_1)[m] \\ &< (\tau_{n-1} v_2)[m] \le (\tau_{n-2} v_1)[m] < (\tau_{n-2} v_2)[m]\end{aligned}$$

for $m, n \in \mathbb{Z}$. Similarly, we have

$$\begin{aligned}(3.2) \qquad (\tau_n w_1)[m] < (\tau_n w_2)[m] &\le (\tau_{n+1} w_1)[m] \\ &< (\tau_{n+1} w_2)[m] \le (\tau_{n+2} w_1)[m] < (\tau_{n+2} w_2)[m]\end{aligned}$$

for $m, n \in \mathbb{Z}$. We put

$$W_0(m) = \left\{u \in L^2(\Omega_0) : (\tau_{-m} v_2)[0] \le u \le (\tau_{-m-1} v_1)[0]\right\} \qquad \text{for each } m \in \mathbb{Z}.$$

Then we find that for each $m \in \mathbb{Z}$,

$$u_1 < u_2 \qquad \text{for all } u_1 \in W_0(m) \text{ and } u_2 \in W_0(m+1).$$

We put

$$\begin{aligned}(3.3) \quad & U_0(m) \\ &= [W_0(m) + \overline{B_{r_m}(0)}] \cap \{u \in [v,w]_0 : (\tau_{-m} v_1)[0] \le u \le (\tau_{-m-1} v_2)[0]\},\end{aligned}$$

where $B_r(0)$ is an open ball in $L^2(\Omega_0)$ centered at 0 with radius $r > 0$ and $\overline{B_r(0)}$ stands for the closure of $B_r(0)$ with respect to the norm of $L^2(\Omega_0)$, and r_m is a positive number. Then $U_0(m)$ is a closed convex set in $H^1(\Omega_0)$. If we choose r_m sufficiently small then

$$U_0(m) \cap U_0(n) = \emptyset \quad \text{for } m, n \in \mathbb{Z} \text{ with } m \neq n. \tag{3.4}$$

For each $m \in \mathbb{Z}$, we denote by $\partial U_0(m)$ the set

$$\partial U_0(m) = \{z \in U_0(m) : d(z, W_0(m)) = r_m\},$$

where $d(z, A) = \inf \left\{ \|z - y\|_{L^2(\Omega_0)} : y \in A \right\}$ for $z \in L^2(\Omega_0)$ and $A \subset L^2(\Omega_0)$.

Lemma 1. *There exists a sequence $\{r_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}^+$ such that the sequence $\{U_0(m)\}_{m\in\mathbb{Z}}$ of closed convex sets in $L^2(\Omega_0)$ defined by (3.3) satisfies the following conditions:*
(i) For each $m \in \mathbb{Z}$,

$$(\tau_{-m}v_1)[0],\ (\tau_{-m-1}v_2)[0] \notin U_0(m).$$

(ii) If $u_1, u_2 \in [v, w]$ are solutions of (P) such that

$$J(u_j) < 2[c(v, w) + c(w, v)] \qquad \text{for } i = 1, 2,$$

and

$$u_1[0] \in U_0(m) \text{ and } u_2[0] \in U_0(m + 1) \quad \text{for some } m \in \mathbb{Z},$$

then

$$\tau_{-m}v_1[0] < u_1[0] < \tau_{-m-1}v_2[0],$$
$$\tau_{-m-1}v_1[0] < u_2[0] < \tau_{-m-2}v_2[0] \quad \text{on } \Omega_0$$

and

$$u_1[0] < u_2[0] \quad \text{on } \Omega_0.$$

Proof. Let $\{r_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}^+$ and let $\{U_0(m)\}$ be the sequence defined by (3.3). It is easy to see that the assertion (i) holds by choosing each r_m sufficiently small. Put

$$S = \{u \in [v, w] : u \text{ is a solution of (P) with } J(u) < 2[c(v, w) + c(w, v)]\}.$$

Then one can see that each $u \in S$ is a classical solution of (P) and there exists $C_0 > 0$ such that

$$\|u\|_{C^1(\Omega_0)} \leq C_0 \qquad \text{for all } u \in S. \tag{3.5}$$

Let $m \in \mathbb{Z}$. Recall that $v_1 < v_2$ on Ω. Then by (3.5), we can choose $\gamma(m) \in (0, \frac{1}{2})$ satisfying the following condition:

(S) If $u \in S$ satisfies (3.6), then $\tau_{-m}v_1[0] < u[0] < \tau_{-m-1}v_2[0]$,

where

$$\begin{aligned}(3.6)\quad &\frac{1}{2}(\tau_{-m}v_1(x_1,y) + \tau_{-m}v_2(x_1,y))\\ &< u(x_1,y)\\ &< \frac{1}{2}(\tau_{-m-1}v_1(x_1,y) + \tau_{-m-1}v_2(x_1,y)) \quad \text{on } [\gamma(m), 1-\gamma(m)] \times \mathcal{D}.\end{aligned}$$

Let $u_1, u_2 \in [v,w]$ be solutions of (P) with $u_1[0] \in U_0(m)$ and $u_2[0] \in U_0(m+1)$. Then since

$$-\Delta(\tau_{-m-1}v_1) = g(x,u_1) - g(x,\tau_{-m-1}v_1)$$

and (g1) hold, we have by standard regularity arguments for elliptic problems that there exists $L > 0$ such that

$$(3.7)\quad |u_1(x_1,y) - \tau_{-m-1}v_1(x_1,y)| \le L\|u_1 - \tau_{-m-1}v_1\|_{L^2(\Omega_0)} \quad \text{for } (x_1,y) \in [\gamma(m), 1-\gamma(m)] \times \mathcal{D}.$$

Similarly, we have

$$(3.8)\quad |u_1(x_1,y) - \tau_m v_2(x_1,y)| \le \|u_1 - \tau_{-m}v_2\|_{L^2(\Omega_0)} \quad \text{for } (x_1,y) \in [\gamma(m), 1-\gamma(m)] \times \mathcal{D}.$$

Then from (3.7) and (3.8), we have by choosing r_m sufficiently small that (3.6) holds. Then by (S), we obtain that $\tau_m v_1[0] < u_1[0] < \tau_{-m-1}v_2[0]$ holds. Similarly we have by choosing r_{m+1} so small that $\tau_{-m-1}v_1[0] < u_2[0] < \tau_{-m-2}v_2[0]$ holds. By a similar argument, we have that $u_1[0] < u2[0]$ holds on Ω_0 by choosing r_m and r_{m+1} sufficiently small. □

In the rest of this paper, we fix $\{U_0(m)\}_{m\in\mathbb{Z}}$ which satisfies the properties (i) and (ii) in Lemma 1. From the definition, we have that

$$(3.9)\quad \sup\Big\{\|u - w\|_{L^2(\Omega_0)} : u \in U_0(m)\Big\} \to 0, \text{ as } m \to \infty.$$

Lemma 2. *Let $u \in [v,w]$. We assume that there exists $\varepsilon_0 > 0$ such that*

$$\liminf_{j\to\infty} \|\tau_{-j}u - v\|_{L^2(\Omega_0)} > \varepsilon_0 \quad \textit{and} \quad \liminf_{j\to\infty} \|\tau_{-j}u - w\|_{L^2(\Omega_0)} > \varepsilon_0.$$

Then there exists a constant $\rho_0 > 0$ such that

$$\lim_{j\to\infty} a_0(\tau_{-j}u) > \rho_0,$$

where ρ_0 is independent of u.

Proof. Let $\{\tau_{-j}u\} \subset [v,w]_0$ satisfy the assumption. Suppose contrary

$$\lim_{j\to\infty} a_0(\tau_{-j}u) = 0.$$

Then we may assume that $\tau_{-j}u \to \overline{u}$ weakly in $H^1(\Omega_0)$ and strongly in $L^2(\Omega_0)$ as $j \to \infty$. Thus we have

$$a_0(\overline{u}) \leq \liminf_{j\to\infty} a_0(\tau_{-j}u).$$

Then by the property of v and w, one can see that $\overline{u} = v$ or $\overline{u} = w$ holds. This is a contradiction. Therefore there exists $\rho_0 > 0$ such that $\lim_{n\to\infty} a_0(\tau_{-j}u) > \rho_0$. □

LEMMA 3. *There exists $m_{v,1} \in \mathbb{N}$ such that for each $u \in \Gamma(v,v)$ with $u[0] \in \cup_{m\geq m_{v,1}} U_0(m)$,*

$$J(u) > c(v,w) + \frac{c(w,v)}{2}. \tag{3.10}$$

Proof. Suppose contrary that there exists $\{u_n\} \subset \Gamma(v,v)$ such that $J(u_n) \leq c(w,v) + \frac{c(v,w)}{2}$ for each $n > 1$ and

$$\lim_{n\to\infty} \|w - u_n\|^2_{L^2(\Omega_0)} = 0.$$

Let $\varphi \in C^\infty(\mathbb{R},[0,1])$ be such that $\varphi(t) = 1$ on $(-\infty,0]$ and $\varphi(t) = 0$ on $[\frac{1}{2},\infty)$. We put

$$\psi_1(u)(x_1,y) = \varphi(x_1)u(x_1,y) + (1-\varphi(x_1))w(x_1,y)$$

and

$$\psi_2(u)(x_1,y) = \varphi(1-x_1)u(x_1,y) + (1-\varphi(1-x_1))w(x_1,y)$$

for $u \in H^1_{loc}(\Omega)$ and $(x_1,y)2 \in \mathbb{R}\times\mathcal{D}$. Then noting that

$$J_{1,\infty}(\psi_1(u)) = J_{-\infty,-1}(\psi_2(u)) = 0 \quad \text{for each } u \in \Gamma(v,v),$$

we have

$$J(\psi_1(u)) + J(\psi_2(u)) = J_{-\infty,-1}(u) + J_{1,\infty}(u) + a_0(\psi_1(u)) + a_0(\psi_2(u))$$

for $u \in \Gamma(v,v)$. We also have

$$\begin{aligned} a_0(\psi_1(u_n)) &= \int_0^1\int_{\mathcal{D}} (\frac{1}{2}|\nabla\psi_1(u_n)|^2 - G(\psi_1(u_n)))dx - c_0 \\ &= \int_0^1\int_{\mathcal{D}} (\frac{1}{2}|(\nabla\varphi(x_1))(u_n - w) + \varphi(x_1)\cdot\nabla u_n + (1-\varphi(x_1))\nabla w|^2 \\ &\quad - G(\psi_1(u_n)))dx - c_0. \end{aligned} \tag{3.11}$$

Similarly we evaluate $a_0(\psi_2(u_n))$. Noting that $u_n \to w$ strongly in $L^2(\Omega_0)$ and weakly in $H^1(\Omega_0)$, we find by (3.11) that

$$\begin{aligned}
&\limsup_{n\to\infty}[a_0(\psi_1(u_n)) + a_0(\psi_2(u_n))] \\
&= \limsup_{n\to\infty}[\int_0^1 \int_{\mathcal{D}} (\frac{1}{2}|\varphi(x_1)\nabla u_n + (1-\varphi(x_1))\nabla w|^2 - G(\psi_1(u_n)))dx - c_0 \\
&\quad + \int_0^1 \int_{\mathcal{D}} (\frac{1}{2}|\varphi(1-x_1)\nabla u_n + (1-\varphi(1-x_1))\nabla w|^2 - G(\psi_2(u_n)))dx - c_0] \\
&= \limsup_{n\to\infty}[\int_0^1 \int_{\mathcal{D}} (\frac{\varphi(x_1)^2}{2}|\nabla u_n|^2 + \frac{1-\varphi(x_1)^2}{2}|\nabla w|^2 - G(\psi_1(u_n))dx - c_0 \\
&\quad + \int_0^1 \int_{\mathcal{D}} (\frac{\varphi(1-x_1)^2}{2}|\nabla u_n|^2 + \frac{1-\varphi(1-x_1)^2}{2}|\nabla w|^2 - G(\psi_2(u_n))dx - c_0] \\
&\le \limsup_{n\to\infty}[\int_0^{1/2} \int_{\mathcal{D}} (\frac{1}{2}|\nabla u_n|^2 - G(\psi_1(u_n))dx + \int_{1/2}^1 \int_{\mathcal{D}} (\frac{1}{2}|\nabla w|^2 - G(w))dx - c_0 \\
&\quad + \int_{1/2}^1 \int_{\mathcal{D}} (\frac{1}{2}|\nabla u_n|^2 - G(\psi_2(u_n))dx + \int_0^{1/2} \int_{\mathcal{D}} (\frac{1}{2}|\nabla w|^2 - G(w))dx - c_0] \\
&= \limsup_{n\to\infty}[\int_0^{1/2} \int_{\mathcal{D}} (\frac{1}{2}|\nabla u_n|^2 - G(u_n))dx + \int_{1/2}^1 \int_{\mathcal{D}} (\frac{1}{2}|\nabla w|^2 - G(w))dx - c_0 \\
&\quad + \int_{1/2}^1 \int_{\mathcal{D}} (\frac{1}{2}|\nabla u_n|^2 - G(u_n)dx + \int_0^{1/2} \int_{\mathcal{D}} (\frac{1}{2}|\nabla w|^2 - G(w))dx - c_0].
\end{aligned}$$

Then noting that

$$\int_0^1 \int_{\mathcal{D}} (\frac{1}{2}|\nabla w|^2 - G(w))dx - c_0 = 0,$$

we have

$$\limsup_{n\to\infty}[a_0(\psi_1(u_n)) + a_0(\psi_2(u_n))] \le \limsup_{n\to\infty} a_0(u_n).$$

Therefore we have

$$\limsup_{n\to\infty} J(u_n) \ge \limsup_{n\to\infty}[J(\psi_1(u_n)) + J(\psi_2(u_n))].$$

Since $\psi_1(u_n) \in \Gamma(v, w)$, we have $J(\psi_1(u_n)) + c(v, w)$. Similarly, we have $\psi_2(u_n) \in \Gamma(w, v)$ and $J(\psi_2(u_n)) \ge c(w, v)$. Therefore we have

$$\limsup_{n\to\infty} J(u_n) \ge c(v, w) + c(w, v).$$

Since $c(w, v)$ is positive by (C), this is a contradiction. This completes the proof. $\square$

Lemma 4. *There exists $\varepsilon_1 > 0$ such that for each $u \in \Gamma_-(v)$ such that $u[0] \in \cup_{m\geq u_{v,1}} U_0(m)$ and $J(u) \leq c(v,w) + \frac{c(w,v)}{4}$,*

$$\inf_{m\geq m_{v,1}} \|v-u\|^2_{L^2(\Omega_m)} \geq \varepsilon_1.$$

Proof. Suppose contrary that there exists $\{u_n\} \subset \Gamma_-(v)$ such that $un[0] \in \cup_{m\geq m_{v,1}} U_0(m)$, $J(u_n) \leq c(v,w) + \frac{c(w,v)}{4}$ for each $n \geq 1$ and

$$\lim_{n\to\infty} \inf_{m\geq m_{v,1}} \|v-u_n\|^2_{L^2(\Omega_m)} = 0.$$

Then there exists a sequence $\{j_n\} \subset \mathbb{N}$ such that $j_n \geq m_{v,1}$ for $n \geq 1$ and

$$\lim_{n\to\infty} \|v-u_n\|^2_{L^2(\Omega_{j_n})} = 0.$$

Let $\varphi \in C^\infty(\mathbb{R},[0,1])$ be the function defined in the proof of Lemma 3. We put

$$\psi_n(u_n) = \varphi(x_1-j_n)u_n(x_1,y) + (1-\varphi(x_1-j_n))v(x_1,y) \quad \text{for } (x_1,y) \in \mathbb{R}\times\mathcal{D}.$$

Then $\psi_n(u_n) \in \Gamma(v,v)$ for $n \geq 1$ and

$$J(\psi_n(u_n)) = J_{-\infty,j_n-1}(u_n) + a_{j_n}(\psi_n(u_n)).$$

Noting that $\tau_{j_n}u_n \to v$ strongly in $L^2(\Omega_0)$ and $\tau_{-j_n}u_n \to v$ weakly in $H^1(\Omega_0)$. Since $a_{j_n}(\psi_n(u_n)) = a_0(\tau_{-j_n}(\psi_n(u_n)))$, by the same argument as in the proof of Lemma 3 we have

$$\limsup_{n\to\infty} a_0(\tau_{-j_n}(\psi_n(u_n))) \leq \limsup_{n\to\infty} a_0(\tau_{-j_n}u_n),$$

thus

$$\limsup_{n\to\infty} a_{j_n}(\psi_n(u_n)) \leq \limsup_{n\to\infty} a_{j_n}(u_n).$$

Therefore we have

$$\limsup_{n\to\infty} J_{-\infty,j_n}(u_n) \geq \limsup_{n\to\infty} J(\psi_n(u_n)).$$

Since $\psi_n(u_n) \in \Gamma(v,v)$ with $u_n[0] \in \cup_{m\geq m_{v,1}} U(m)$, we have by Lemma 3 that $J(\psi_n(u_n)) \geq c(v,w) + c(w,v)/2$. Then

$$\limsup_{n\to\infty} J(u_n) \geq \limsup_{n\to\infty} J_{-\infty,j_n}(u_n) \geq \limsup_{n\to\infty} J(\psi_n(u_n)) \geq c(v,w) + \frac{c(w,v)}{2}.$$

This is a contradiction. □

Lemma 5. *For each $v_1 \in \mathcal{M}(v,w)$ and $u \in \Gamma_-(v)$,*

$$J(\min\{v_1, u\}) \leq J(u).$$

Proof. For each $v_1 \in \mathcal{M}(v,w)$ and $u \in \Gamma_-(v)$, we have

$$J\min\{v_1, u\}) + J\max\{v_1, u\}) = J(v_1) + J(u).$$

Then $\max\{v_1, u\} \in \Gamma(v,w)$, thus by the property of v_1

$$J(\max\{v_1, u\}) \geq J(v_1)$$

Therefore we have

$$J(\min\{v_1, u\}) \leq J(u).$$

□

Here we de.ne a subset $U_n(m) \subset [v,w]_n$ for $m, n \in \mathbb{Z}$. For each $m, n \in \mathbb{Z}$, we put

$$W_n(m) = \left\{u \in L^2(\Omega_n) : (\tau_{-m+n} v_2)[n] \leq u \leq (\tau_{-m-1+n} v_1)[n]\right\}$$

and

$$(3.12) \quad U_n(m) = [W_n(m) + \overline{B_{n,r_m}(0)}] \cap \{u \in [v,w]_n : \tau_{-m+n} v_1)[n] \leq u \leq (\tau_{-m-1+n} v_2)[n]\},$$

where $B_{n,r_m}(0) = \tau_n(B_{r_m}(0))$. Then one can see that for $u \in H^1_{loc}(\Omega)$, $u[0] \in U_0(m)$ if and only if $\tau_n u[n] \in U_n(m)$ for $m, n \in \mathbb{Z}$.

Lemma 6. *For each $n \geq m_{v,1}$, there exist $\delta_{v,1}(n) > 0$ and $m_{v,2}(n) > m_{v,1}$ such that for each $u \in \Gamma_-(v)$ satisfying $J(u) < \infty$ and $u[m_{v,1}] \in \partial U_{m_{v,1}}(n)$,*

$$J_{-\infty,m}(u) \geq c(v,w) + \delta_{v,1}(n) \qquad \textit{for all } m \geq m_{v,2}(n).$$

Proof. Suppose contrary that there exist $n_0 \geq m_{v,1}$ and sequences $\{m_n\} \subset \mathbb{N}$, $\{u_n\} \subset \Gamma_-(v)$ such that $u_n[m_{v,1}] \in \partial U_{m_{v,1}}(n_0)$ for $n \geq 1$, $\lim_{n\to\infty} m_n = \infty$ and

$$\lim_{n\to\infty} J_{-\infty,m_n}(u_n) \leq c(v,w).$$

We need a few steps to prove the assertion.

Step1: We set

$$\Lambda = \left\{u \in [v,w] : u \leq \tau_{-n_0-1+m_{v,1}}, v_2, u[m_{v,1}] \in \partial U_{m_{v,1}}(n_0), J(u) \leq c(v,w)\right\}.$$

Then by the definition of $U_n(m)$, $\Lambda \cap \mathcal{M}(v,w) = \emptyset$. We show Λ is not empty. We put

$$\widetilde{u}_n = \begin{cases} \min\{\tau_{-n_0-1+m_{v,1}}, v_2, u_n\} & \text{on } (-\infty, m_{v,1}+1] \times \mathcal{D} \\ u_n & \text{on } [m_{v,1}+1, \infty) \times \mathcal{D}, \end{cases}$$

$$\phi_n = \begin{cases} \max\{\tau_{-n_0} - 1 + m_{v,1} v_2, u_n\} & \text{on } (-\infty, m_{v,1}+1] \times \mathcal{D} \\ \tau_{-n_0-1+m_{v,1}} v_2 & \text{on } [m_{v,1}+1, \infty) \times \mathcal{D}, \end{cases}$$

for each $n \geq 1$. By the definition of $U_n(m)$, we have that $u_n \leq \tau_{-n_0-1+m_{v,1}} v_2$ on $\Omega_{m_{v,1}}$ for each $n \geq 1$. Then

$$J(\widetilde{u}_n) + J(\phi_n) = J(\tau_{-n_0-1+m_{v,1}} v_2) + J(u_n).$$

Since $\phi_n \in \Gamma(v,w)$, we have $J(\widetilde{u}_n) \leq J(u_n)$ by the argument of Lemma 5. By definition of $\widetilde{u}_n$, we find $J_{-\infty,m}(\widetilde{u}_n) \leq J_{-\infty,m}(u_n)$ for each $n \geq 1$ and $m \geq m_{v,1}$. Since $J_{-\infty,m}(u_n)$ is nondecreasing with respect to m by (C), we find that for each $m \geq m_{v,1}$,

$$\lim_{n\to\infty} J_{-\infty,m}(\widetilde{u}_n) \leq \lim_{n\to\infty} J_{-\infty,m} \leq \lim_{n\to\infty} J_{-\infty,m_n}(u_n) \leq c(v,w).$$

We may assume that $\widetilde{u}_n \to u_0 \in H^1_{loc}(\Omega)$ weakly in $H^1_{loc}(\Omega)$, $\widetilde{u}_n \to u_0$ strongly in $L^2_{loc}(\Omega)$ and pointwise a.e.. Then $u_0 \leq \tau_{-n_0-1+m_{v,1}} v_2$ on $(-\infty, m_{v,1}+1] \times \mathcal{D}$ and $u_0[m_{v,1}] \in \partial U_{m_{v,1}}(n_0)$. For each $m \geq m_{v,1}$ by the weak lower semicontinuity of $J_{l,m}$,

$$J_{l,m}(u_0) \leq \liminf_{n\to\infty} J_{l,m}(\widetilde{u}_n),$$

thus letting $l \to -\infty$ gives

$$J_{-\infty,m}(u_0) \leq \lim_{n\to\infty} J_{-\infty,m}(\widetilde{u}_n) \leq \lim_{n\to\infty} J_{-\infty,m}(u_n) \leq c(v,w)$$

for all $m \geq m_{v,1}$. Then $m \to \infty$ implies

$$J(u_0) \leq \lim_{n\to\infty} J(\widetilde{u}_n) \leq \lim_{n\to\infty} J_{-\infty,m_n}(u_n) \leq c(v,w).$$

Again by the minimization property of $\tau_{-n_0-1+m_{v,1}} v2$, we have that

$$J(\min\{\tau_{-n_0-1+m_{v,1}} v_2, u_0\}) \leq J(u_0).$$

Here we put $\widetilde{u}_0 = \min\{\tau_{-n0-1+m_{v,1}} v_2, u_0\}$. Then $\widetilde{u}_0 \in \Lambda$, i.e. $\Lambda \neq \emptyset$.

Step 2: We put

$$\Gamma = \{u \in \Gamma_-(v) : u \leq \tau_{-n_0-1+m_{v,1}} v_2, \\ \liminf_{m\to\infty} \|u - v\|^2_{L^2(\Omega_m)} \geq \varepsilon_1, J(u) \leq c(v,w) + \varepsilon\}$$

where ε_1 is the constant obtained in Lemma 4, and $\varepsilon \in (0, c(w,v)/4)$. Then by Lemma 4, $\Lambda \subset \Gamma$. Let $\gamma_\Gamma = \inf_{z\in\Gamma} J(z)$. In this step we will see that $\gamma_\Gamma < c(v,w)$. Let $\{\widehat{u}_n\}$ be a sequence in Γ such that $\lim_{n\to\infty} J(\widehat{u}_n) = \gamma_\Gamma$. Then there exists $\{i_n\} \subset \mathbb{N}$ such that

$$\inf_{m\geq 0} \|v - \tau_{i_n}\widehat{u}_n\|^2_{L^2(\Omega_m)} \geq \varepsilon_1 - \frac{1}{n} \qquad \text{for all } n \geq 1$$

We put $\overline{u}_n = \min\{\tau_{-n_0-1+m_{v,1}}v_2, \tau_{-i_n}\widehat{u}_n\}$ for $n \geq 1$. Then $\{\overline{u}_n\} \subset \Gamma$. By Lemma 5 it follows that $J(\overline{u}_n) \leq J(\widehat{u}_n)$ for $n \geq 1$. We may assume that $\overline{u}_n \to \overline{u}_0 \in H^1_{loc}(\Omega)$ weakly in $H^1_{loc}(\Omega)$, strongly in $L^2_{loc}(\Omega)$ and pointwise a.e.. Then we find that $\overline{u}_0 \in \Gamma$ and $J(\overline{u}_0) = \gamma_\Gamma$. Let $t > 0$ sufficiently small, then by Lemma 5, we have

$$J(\min\{\tau_{-n_0-1+m_{v,1}}v_2, \overline{u}_0 + t\psi\}) \leq J(\overline{u}_0 + t\psi).$$

for all $\psi \in C_0^\infty(\Omega)$. Again by the argument in Lemma 5, we also have

$$J(\max\{v, \min\{\tau_{-n_0-1+m_{v,1}}v_2, \overline{u}_0 + t\psi\}\}) \leq J(\min\{\tau_{-n_0-1+m_{v,1}}v_2, \overline{u}_0 + t\psi\})$$

Since $\max\{v, \min\{\tau_{-n_0-1+m_{v,1}}v_2, \overline{u}_0 + t\psi\}\} \in \Gamma$ for all $\psi \in C_0^\infty(\Omega)$,

$$J(\overline{u}_0) \leq J(\max\{v, \min\{\tau_{-n_0-1+m_{v,1}}v_2, \overline{u}_0 + t\psi\}\}) \leq c(v,w) + \varepsilon.$$

Now we find that each $u \in \Gamma$ such that $J(u) = \gamma_\Gamma$ is a solution of (P). We suppose that $\gamma_\Gamma = c(v,w)$. Then noting that $J(\widetilde{u}_0) = c(v,w) = \gamma_\Gamma$, we have that $\widetilde{u}_0$ is a solution of (P). Since $\widetilde{u}_0 \in \Lambda$, we find that $\widetilde{u}_0 \notin \mathcal{M}(v,w)$. Let $m \in \mathbb{N}$ and put $\widetilde{u}_1 = \min\{\tau_{-n_0-1+m_{v,1}}v_2, \tau_{-m}\widetilde{u}_0\} \in \Gamma$. Then one can see that $\widetilde{u}_1 \in \Gamma$. By choosing m sufficiently large, we have $\widetilde{u}_1 \neq \tau_{-n_0-1+m_{v,1}}v_2$ and $\widetilde{u}_1 \neq \tau_{-m}\widetilde{u}_0$. Then we have $J(\max\{\tau_{-n_0-1+m_{v,1}}v_2, \tau_{-m}\widetilde{u}_0\}) > J(\tau_{-n_0-1+m_{v,1}}v_2)$. Therefore $J(\widetilde{u}_1) < J(\widetilde{u}_0) = c(v,w)$. Since $\widetilde{u}_0, \widetilde{u}_1 \in \Gamma$, this is a contradiction. Thus we find that $\gamma_\Gamma < c(v,w)$.

Step 3: Let $\overline{u}_0 \in \Gamma$ such that $J(\overline{u}_0) = \gamma_\Gamma$. Then we show that

$$\liminf_{n\to\infty} \|\overline{u}_0 - w\|^2_{L^2(\Omega_n)} \geq \varepsilon_2 \tag{3.13}$$

holds. Suppose that there exists $\{n_i\} \subset \mathbb{N}$ such that $\lim_{i\to\infty} n_i = \infty$ and

$$\lim_{i\to\infty} \{\overline{u}_0 - w\|^2_{L^2(\Omega_{n_{i-1}} \cup \Omega_{n_i} \cup \Omega_{n_i+1})} = 0.$$

Recall that $\overline{u}_0$ and w are solutions of (P) on Ω. Therefore

$$-\Delta(w - \overline{u}_0) = g(x,w) - g(x,\overline{u}_0).$$

Then by a standard regularity argument for elliptic problem, we find that there exists $M > 0$ such that

$$\|\Delta w - \Delta\overline{u}_0\|_{L^2(\Omega_{n_i})} \leq M\|w - \overline{u}_0\|_{L^2(\Omega_{n_{n-1}} \cup \Omega_{n_i} \cup \Omega_{n_1+1})} \quad \text{for } i \geq 1. \tag{3.14}$$

Let $\varphi \in C^\infty(\mathbb{R}, [0,1])$ be the function defined in the proof of Lemma 3. We put

$$\Psi_i(\overline{u}_0) = \varphi(x_1 - n_i)\overline{u}_0(x_1, y) + (1 - \varphi(x_1 - n_i))w(x_1, y) \quad \text{for } (x_1, y) \in \mathbb{R} \times \mathcal{D}.$$

Then by 3.14), we find that

$$\lim_{i\to\infty} a_{n_i}(\Psi_i(\overline{u}_0)) = \lim_{i\to\infty} a_{n_i}(\overline{u}_0) = 0$$

and then

$$\begin{aligned}\limsup_{i\to\infty} J(\Psi_i(\overline{u}_0)) &= \limsup_{i\to\infty} J_{-\infty,n_i}(\Psi_i(\overline{u}_0)) \\ &= \lim_{i\to\infty} J_{-\infty,n_1-1}(\overline{u}_0) \\ &\le J(\overline{u}_0)\end{aligned}$$

Since $\Psi_i(\overline{u}_0) \in \Gamma(v,w)$, we have $J(\Psi_i(\overline{u}_0)) \ge c(v,w)$. Therefore we find that $J(\overline{u_0}) \ge c(v,w)$. This is a contradiction. Then we have that there exists $\varepsilon_2 > 0$ such that (3.13) holds.

Step 4: Here we put

$$\Gamma_\infty = \Big\{ u \in [v,w] : u \le \tau_{-n_0-1+m_{v,1}} v_2, \liminf_{m\to\infty} \|u - v\|^2_{L^2(\Omega_m)} \ge \varepsilon_1 \\ \liminf_{m\to\infty} \|u - v\|^2_{L^2(\Omega_m)} \ge \varepsilon_2, J(u) \le c(v,w) \Big\}.$$

Then by step 2 and step 3, we have that $\overline{u}_0 \in \Gamma_\infty$ i.e. $\Gamma_\infty \ne \emptyset$. By Lemma 2, if u satisfies $\liminf_{m\to\infty} \|u-v\|^2_{L^2(\Omega_m)} \ge \varepsilon_1$ and $\liminf_{m\to\infty} \|u-w\|^2_{L^2(\Omega_m)} \ge \varepsilon_2$ then u can not satisfy $J(u) \le c(v,w)$. Thus Γ_∞ must be empty. This is a contradiction. Then we obtain that the assertion holds. □

Lemma 7. *For each $n \ge m_{v,1}$ and $\varepsilon > 0$, there exists $m_{v,3}(n,\varepsilon) \in \mathbb{N}$ such that $m_{v,3}(n,\varepsilon) > m_{v,2}(n)$ and*

$$J_{-\infty,m}(u) \ge c(v,w) - \varepsilon$$

for all $m \ge m_{v,3}(n,\varepsilon)$ and $u \in \Gamma_-(v)$ with $u[m_{v,1}] \in U_{m_{v,1}}(n)$.

Proof. Suppose contrary that there exist $n_0 \ge 1$, $\varepsilon_0 > 0$, and sequences $\{u_n\} \subset \Gamma_-(v)$ and $\{m_n\} \subset \mathbb{N}$ such that $u_n[m_{v,1}] \in U_{m_{v,1}}(n_0)$, $lim_{n\to\infty} m_n = \infty$ and

$$J_{-\infty,m_n}(u_n) < c(v,w) - \varepsilon_0.$$

By property of $U_{m_{v,1}}(n_0)$, $u_n \le \tau_{-n_0-1+m_{v,1}} v_2$ on $\Omega_{m_{v,1}}$. We put

$$\overline{u}_n = \begin{cases} \min\{\tau_{-n_0-1+m_{v,1}} v_2, u_n\} & \text{on } (-\infty, m_{v,1}+1] \times \mathcal{D} \\ u_n & \text{on } [m_{v,1}+1, \infty) \times \mathcal{D} \end{cases}$$

for each $n \geq 1$. Then by the argument of Lemma 5,

$$\lim_{n\to\infty} J_{-\infty,m}(\overline{u}_n) \leq \lim_{n\to\infty} J_{-\infty,m}(u_n) \leq \lim_{n\to\infty} J_{-\infty,m_n}(u_n) \leq c(v,w) - \varepsilon_0,$$

for each $m \geq m_{v,1}$. We may assume that $\overline{u}_n \to u \in H^1_{loc}(\Omega)$ weakly in $H^1_{loc}(\Omega)$, strongly in $L^2_{loc}(\Omega)$ and pointwise a.e.. Then we have $u \in \Gamma_-(v)$, $u[m_{v,1}] \in U_{m_{v,1}}(n_0)$ and

$$J(u) < c(v,w) - \varepsilon_0.$$

We have by Lemma 4 $\liminf_{n\to\infty} \|u - v\|_{L^2(\Omega_n)} > 0$. If $\liminf_{n\to\infty} \|u - w\|_{L^2(\Omega_n)} > 0$ holds, then by Lemma 2 we have $J(u) = \infty$. Therefore we find that

$$\liminf_{n\to\infty} \|u - w\|_{L^2(\Omega_n)} = 0$$

holds. That is $u \in \Gamma(v,w)$. This implies that $J(u) \geq c(v,w)$. This is a contradiction. □

Here we put

$$\widetilde{W}_n(m) = \big\{u \in L^2(\Omega_n) : (\tau_{m+n}w_2)[n] \leq u \leq (\tau_{m+1+n}w_1)[n]\big\}$$

and

$$\begin{aligned}(3.15)\quad \widetilde{W}_n(m) = [\widetilde{W}_n(m) + \overline{B_{n,\widetilde{r}_m}(0)} \cap \\ \{u \in [v,w]_n : (\tau_{m+n}w_1)[n] \leq u \leq (\tau_{m+1+n}w_2)[n]\},\end{aligned}$$

for each $m, n \in \mathbb{Z}$.

By analogous arguments as in the proof of Lemma 1, Lemma 3, Lemma 6 and Lemma 7 we have

LEMMA 8. *There exists a sequence $\{\widetilde{r}_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}^+$ such that the sequence $\big\{\widetilde{U}_0(m)\big\}$ of closed convex sets in $L^2(\Omega_0)$ defined by (3.15) satisfies the following conditions:*
(i) For each $m \in \mathbb{Z}$,

$$(3.16)\qquad (\tau_m w_1)[0], (\tau_{m+1}w_2)[0] \notin \widetilde{U}_0(m).$$

(ii) If u_1, u_2 are solutions of (P) such that

$$J(u_i) < 2[c(v,w) + c(w,v)] \qquad \text{for } i = 1,2,$$

and

$$u_1[0] \in \widetilde{U}_0(m) \text{ and } u_2[0]2 \in \widetilde{U}_0(m+1) \quad \text{for some } m \in \mathbb{Z},$$

then

$$\tau_m w_1[0] < u_1[0] < \tau_{m+1} w_2[0],$$
$$\tau_{m+1} w_1[0] < u_2[0] < \tau_{m+2} w_2[0] \quad \text{on } \Omega_0$$

and

$$u_1[0] < u_2[0] \quad \text{on } \Omega_0.$$

Lemma 9. (1) *There exists $m_{w,1} > 0$ such that for each $u \in \Gamma(v, v)$ with $u[0] \in \cup_{m \geq m_{w,1}} \widetilde{U}_0(m)$,*

$$J(u) > c(w, v) + \frac{c(v, w)}{2}. \tag{3.17}$$

(2) *For each $n \geq m_{w,1}$, there exist $\delta_{w,1}(n) > 0$ and $m_{w,2}(n) > m_{w,1}$ such that for each $u \in \Gamma_+(v)$ satisfying $J(u) < \infty$ and $u[-m_{w,1}] \in \partial\widetilde{U}_{-m_{w,1}}(n)$,*

$$J_{-m,\infty}(u) \geq c(w, v) + \delta_{w,1}(n) \qquad \text{for all } m \geq m_{w,2}(n).$$

Lemma 10. *For each $n \geq m_{w,1}$ and $\varepsilon > 0$, there exists $m_{w,3} \in \mathbb{N}$ such that $m_{w,3}(n, \varepsilon) > m_{w,2}(n)$ and*

$$J_{-m,\infty}(u) \geq c(w, v) - \varepsilon$$

for all $m \geq m_{w,3}(n, \varepsilon)$ and $u \in \Gamma_+(v)$ with $u[-m_{w,1}] \in \widetilde{U}_{-m_{w,1}}(n)$.

Proof of Theorem 1. Fix a positive integer $k_1 \geq \max\{m_{v,1}, m_{w,1}\}$. Fix $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{2} \min\{\delta_{v,1}(k_1), \delta_{w,1}(k_1)\},$$

where $\delta_{v,1}$ and $\delta_{w,1}$ are positive numbers obtained in Lemma 6 and Lemma 9. We put $m > \max\{m_{v,3}(k_1, \varepsilon), m_{w,3}(k_1, \varepsilon)\}$, where $m_{v,3}(k_1, \varepsilon)$ and $m_{w,3}(k_1, \varepsilon)$ are positive integers obtained in Lemma 7 and Lemma 10. Let

$$u_0 = \min\{\tau_{-k_1-1+m_{v,1}} v_1, \tau_{k_1+1+2m-m_{w,1}} w_1\}.$$

From the definition of v_1 and w_1, we find that

$$J(u_0) \to c(v, w) + c(w, v), \quad \text{as } m \to \infty. \tag{3.18}$$

Then by choosing $m \geq 1$ sufficiently large, we have that

$$J(u_0) < c_2(k_1) = c(v, w) + c(w, v) + \frac{\varepsilon}{2}. \tag{3.19}$$

We may assume, by choosing m sufficiently large, that

$$u_0[m_{v,1}] = \tau_{-k_1-1+m_1}v_1[m_{v,1}] \text{ and } u_0[2m - m_{w,1}] = \tau_{k_1+1+m_2}w_1[2m - m_{w,1}].$$

Then we have

$$u_0[m_{v,1}] \in U_{m_{v,1}}(k_1) \text{ and } u_0[2m - m_{w,1}] \in \widetilde{U}_{2m-m_{w,1}}(k_1).$$

Here we put $m_1 = m_{v,1}$ and $m_{2,1} = 2m - m_{w,1}$. Then we set

$$\Gamma_1 = \Big\{u \in \Gamma(v, v) : J(u) \le c_2(k_1), u[m_1] \in U_{m_1}(k_1) \text{ and } u[m_{2,1}] \in \widetilde{U}_{m_{2,1}}(k_1).\Big\}$$

Then since $u_0 \in \Gamma_1$, $\Gamma_1 \ne \emptyset$. We put $\gamma = \inf_{z\in\Gamma_1} J(z)$. Let $\{u_n\}$ be a minimizing sequence in Γ_1 such that $\lim_{n\to\infty} J(u_n) = \gamma$. We put

$$\phi_n = \begin{cases} \min\{\tau_{-k_1-1+m_1}v_2, u_n\} & \text{on } (-\infty, m_1 + 1] \times \mathcal{D} \\ u_n & \text{on } [m_1 + 1, \infty) \times \mathcal{D}, \end{cases}$$

for each $n \ge 1$. Since $u_n[m_1] \in U_{m_1}(k_1)$, $u_n \le \tau_{-k_1-1+m_1}v_2$ on Ω_{m_1}. Then the argument of Lemma 5 implies $J(\phi_n) \le J(u_n)$ for each $n \ge 1$. By definition of ϕ_n, we find that $J_{-\infty,m_1}(\phi_n) \le J_{-\infty,m_1}(u_n)$ for each $n \ge 1$. We also put

$$\widetilde{\phi_n} = \begin{cases} u_n & \text{on } (-\infty, m_{2,1}] \times \mathcal{D} \\ \min\{\tau_{k_1+1+m_{2,1}}, w_2, u_n\} & \text{on } [m_{2,1}, \infty) \times \mathcal{D}, \end{cases}$$

for each $n \ge 1$. Then by the same argument as above we have $J_{m_{2,1},\infty}(\widetilde{\phi_n}) \le J_{m_{2,1},infty}(u_n)$. We set

$$\overline{u}_n = \begin{cases} \min\{\tau_{-k_1-1+m_1}, v_2, u_n\} & \text{on } (-\infty, m_1 + 1] \times \mathcal{D} \\ u_n & \text{on } [m_1 + 1, m_{2,1}] \times \mathcal{D} \\ \min\{\tau_{k_1+1+m_{2,1}}, w_2, u_n\} & \text{on } [m_{2,1}, \infty) \times \mathcal{D}. \end{cases}$$

for each $n \ge 1$. Then $\overline{u}_n \in \Gamma_1$. Now we have by above arguments $J(\overline{u}_n) \le J(u_n) \le c_2(k_1)$. We may assume that $\overline{u}_n \to u \in H^1_{loc}(\Omega)$ weakly in $H^1_{loc}(\Omega)$, $\overline{u}_n \to u$ strongly in $L^2_{loc}(\Omega)$ and pointwise a.e.. We find $u \in \Gamma_1$ such that $J(u) = \gamma$. To prove that u is a solution of (P) in Γ_1, it is sufficient to show that $u[m_1] \notin \partial U_{m_1}(k_1)$ and $u[m_{2,1}] \notin \partial\widetilde{U}_{m_{2,1}}(k_1)$. By Lemma 6, we have that if $u[m_1] \in \partial U_{m_1}(k_1)$, then

$$J_{-\infty,m}(u) \ge c(v, w) + \delta_{v,1}(k_1).$$

On the other hand, noting that

$$\tau_{-2m}u[-m_{w,1}] \in \widetilde{U}_{-m_{w,1}}(k_1),$$

we have by Lemma 10 that

$$(3.20)\qquad \begin{aligned} J_{m+1,\infty}(u) &= J_{-m+1,\infty}(\tau_{-2m}u) \\ &\geq c(w,v) - \varepsilon \\ &\geq c(w,v) - \frac{\min\{\delta_{v,1}(k_1), \delta_{w,1}(k_1)\}}{2}. \end{aligned}$$

Then we have that $J(u) \geq c(v,w) + c(w,v) + \delta_{v,1}(k_1)/2$. This is a contradiction. Similarly, we find that $u[m_{2,1}] \notin \partial\widetilde{U}_{m_{2,1}}(k_1)$. Therefore we obtain that $u = u_1 \in \Gamma_1$ is a solution of (P). That is

$$(3.21)\qquad u_1[m_1] \in U_{m_1}(k_1) \text{ and } u_1[m_{2,1}] \in \widetilde{U}_{m_{2,1}}(k_1).$$

Let $u \in \Gamma(v,v)$ be a solution of (P) such that $J(u) < 2(c(v,w) + c(w,v))$,

$$u > \tau_{-k_1-1+m_1}v_2 \quad \text{on } \Omega_{m_1}, \quad u > \tau_{k_1+1+m_{2,1}}w_2 \quad \text{on } \Omega_{m_{2,1}}$$

and

$$u < \tau_{k_1+1+\widehat{m}}w_2 \qquad \text{on} \Omega_{\widehat{m}},$$

for some $\widehat{m} > m_{2,1} + 1$. Then there exists $j \in \mathbb{Z}$ such that $m_1 < j < \widehat{m} - 1$,

$$\|u - v\|_{L^2(\Omega_j)} > 0 \quad \text{and} \quad \|u - w\|_{L^2(\Omega_j)} > 0.$$

Then one can see by an argument as in the proof of Lemma 2 that there exists $\rho > 0$ such that

$$a_j(u) = \int_{\Omega_j} (\frac{1}{2}|\Delta u|^2 - G(x,u))dx - c_0 \geq \rho,$$

where ρ is independent of choice of u and j. We also note that

$$(3.22)\qquad \lim_{n\to\infty} \inf\{J_{-\infty,0}(u) : u \in \Gamma_-(v), u[0] \in U_0(n)\} \geq c(v,w)$$

holds. Let $\varphi \in C^\infty(\mathbb{R}, [0,1])$ be the function defined in the proof of Lemma 3. Then we set, for each $n \geq 1$,

$$\Phi_n(v_1) = \varphi(x_1 - 1)\tau_{-n-1}v_1(x_1, y) + (1 - \varphi(x_1 - 1))w(x_1, y)$$

for $(x_1, y) \in \mathbb{R} \times \mathcal{D}$. Noting that $\Phi_n(v_1)[0] = \tau_{-n-1}v_1[0] \in U_0(n)$, $\Phi_n(v_1) \in \Gamma(v,w) \subset \Gamma_-$. Then

$$J(\Phi_n(v_1)) = J_{-\infty,0}(\tau_{-n-1}v_1) + a_1(\Phi_n(v_1)) \geq c(v,w) \quad \text{for each } n \geq 1.$$

Since $\tau_{-n-1}v_1$ and w are solutions of (P), we have by the argument as in the proof of Step 3 in Lemma 6, there exists $M > 0$ such that

$$\|\Delta\tau_{-n-1}v_1 - \Delta w\|_{L^2(\Omega_1)} \leq M\|\tau_{-n-1}v_1 - w\|_{L^2(\Omega_0\cup\Omega_1\cup\Omega_2)} \quad \text{for } n \geq 1.$$

this implies

$$\lim_{n\to\infty} a_1(\Phi_n(v_1)) = a_1(w) = 0.$$

Thus

$$\lim_{n\to\infty} J_{-\infty,0}(\tau_{-n-1}v_1) \geq c(v,w),$$

therefore (3.22) holds. Similarly, we have

$$\lim_{n\to\infty} \inf \left\{ J_{0,1}(u) : u \in \Gamma_+(v), u[0] \in \widetilde{U}_0(n) \right\} \geq c(w,v). \tag{3.23}$$

Now we fix $k_2 > k_1$ so large that

$$\inf\{J_{-\infty,0}(u) : u \in \Gamma_-(v), u[0] \in U_0(k_2)\} > c(v,w) - \frac{\rho}{4} \tag{3.24}$$

and

$$\inf \left\{ J_{0,\infty}(u) : u \in \Gamma_+(v), u[0] \in \widetilde{U}_0(k_2) \right\} > c(w,v) - \frac{\rho}{4}. \tag{3.25}$$

Here we put

$$\widehat{\varepsilon} < \frac{1}{2}\min\{\delta_{v,1}(k_1), \delta_{w,1}(k_1), \rho\}.$$

Then we find there exists a positive integer $m_{2,2} > m_{2,1}$ and $u_2 \in \Gamma(v,v)$ such that

$$J(u_2) < c(v,w) + c(w,v) + \frac{\widehat{\varepsilon}}{2}$$

and

$$u_2[m_1] \in U_{m_1}(k_2) \text{ and } u_2[m_{2,2}] \in \widetilde{U}_{m_{2,2}}(k_2).$$

We set $c_2(k_2) = c(v,w) + c(w,v) + \frac{\widehat{\varepsilon}}{2}$ and put

$$\Gamma_2 = \left\{ u \in \Gamma(v,v) : J(u) \leq c_2(k_2), u[m_1] \in U_{m_1}(k_2) \text{ and } u[m_{2,2}] \in \widetilde{U}_{m_{2,2}}(k_2) \right\}.$$

The solution u_2 is a local minimal in Γ_2. Then $u_1 < u_2$ on Ω_{m_1}. In fact

$$\tau_{-m_1}u_1[0] \in U_0(k_1) \quad \text{and} \quad \tau_{-m_1}u_2[0] \in U_0(k_2).$$

By Lemma 1, we find that

$$\tau_{-k_1}v_1[0] < \tau_{-m_1}u_1[0] < \tau_{-k_1-1}v_2[0] \quad \text{on } \Omega_0,$$
$$\tau_{-k_2}v_1[0] < \tau_{-m_1}u_2[0] < \tau_{-k_2-1}v_2[0] \quad \text{on}\Omega_0$$

and

$$\tau_{-m_1}u_1[0] < \tau_{-m_1}u_2[0] \text{ on } \Omega_0.$$

Since $u_1 \in \Gamma(v, v)$, we may assume that $m_{2,2}$ is so large that

$$u_1 < \tau_{m_{2,1}+k_1+1}w_2 < \tau_{m_{2,2}+k_2}w_1 < u_2 \quad \text{on } \Omega_{m_{2,1}}.$$

We also have $u_1 < u_2$ on $\Omega_{m_{2,2}}$. In fact, if $u_1(x_1, y) \geq u_2(x_1, y)$ for some $(x_1, y) \in \Omega_{m_{2,2}}$, then $a_j(u_2) \geq \rho$ for some $m_1 < j < m_{2,2} - 1$. Then we obtain by (3.24) and (3.25) that

$$J(u_2) \geq J_{-\infty,m_1}(u_2) + a_j(u_2) + J_{m_{2,2},\infty}(u_2) \geq c(v, w) + c(w, v) + \frac{\rho}{2}.$$

This is a contradiction. Thus we have $u_1 < u_2$ on $\Omega_{m_{2,2}}$.

By the minimality of v_1 and v_2, we will find that

$$\tau_{-k_1+m_1}v_1 \leq u_1 \leq \tau_{-k_1-1+m_1}v_2 \quad \text{on } (-\infty, m_1] \times \mathcal{D} \tag{3.26}$$

and

$$\tau_{-k_2+m_1}v_1 \leq u_2 \leq \tau_{-k_2-1+m_1}v_2 \quad \text{on } (-\infty, m_1] \times \mathcal{D}. \tag{3.27}$$

We show $u_1 \leq \tau_{-k_1-1+m_1}v_2$ holds on $(-\infty, m_1] \times \mathcal{D}$. Since $u_1[m_1] \in U_{m_1}(k_1)$, $u_1 \leq \tau_{-k_1-1+m_1}v_2$ on Ω_{m_1}. We put

$$\overline{u}_1 = \begin{cases} \min\{u_1, \tau_{-k_1-1+m_1}v_2\} & \text{on } (-\infty, m_1] \times \mathcal{D} \\ u_1 & \text{on } [m_1, \infty) \times \mathcal{D}, \end{cases}$$
$$\widetilde{u}_1 = \begin{cases} \max\{u_1, \tau_{-k_1-1+m_1}v_2\} & \text{on } (-\infty, m_1] \times \mathcal{D} \\ \tau_{-k_1-1+m_1}v_2 & \text{on } [m_1, \infty) \times \mathcal{D}, \end{cases}$$

and suppose there exists $(x_1, y) \in (-\infty, m_1] \times \mathcal{D}$ such that $u_1(x_1, y) > \tau_{-k_1-1+m_1}v_2(x_1, y)$. This assumption implies $\overline{u}_1 \neq u_1$ and $\widetilde{u}_1 \neq \tau_{-k_1-1+m_1}v_2$. Then by the argument of Lemma 5, we have $J(u_1) > J(\overline{u}_1)$. Since u_1 is a minimizer in Γ_1, this is contradiction. Therefore we have $u_1 \leq \tau_{-k_1-1+m_1}v_2$ on $(-\infty, m_1] \times \mathcal{D}$. By analogous arguments, we have $\tau_{-k_1+m_1}v_1 \leq u_1$ on $(-\infty, m_1] \times \mathcal{D}$. This implies

$$\tau_{-k_n+m_1}v_1 \leq u_n \leq \tau_{-k_n-1+m_1}v_2 \quad \text{on } (-\infty, m_1] \times \mathcal{D}.$$

We put $-k_n + m_1 = -i_n$ and $m_1 = p$, then (3) of Theorem 1 follows. Similarly by the minimality of w_1 and w_2, we also have

$$\tau_{k_n+m_{2,n}} w_1 \le u_n \le \tau_{k_n+1+m_{2,n}} w_2 \quad \text{on } [m_{2,n}, \infty) \times \mathcal{D}.$$

We put $k_n + m_{2,n} = j_n$ and $m_{2,n} = q$, then (4) of Theorem 1 follows.

We will prove $u_1 \le u_2$ on Ω. Here we put

$$z_1 = \min\{u_1, u_2\} \quad \text{and} \quad z_2 = \max\{u_1, u_2\}.$$

By the argument above, we have $u_1 < u_2$ on Ω_{m_1}, Ω_{m_2}, and $\Omega_{m_{2,2}}$. Then

$$z_1[m_1] = u_1[m_1] \in U_{m_1}(k_1) \text{ and } z_1[m_{2,1}] = u_1[m_2] \in \widetilde{U}_{m_{2,1}}(k_1).$$

Similarly, we have

$$z_2[m_1] \in U_{m_1}(k_2) \text{ and } z_2[m_{2,2}] \in \widetilde{U}_{m_{2,2}}(k_2).$$

Then it follows that

$$J(z_1) \ge J(u_1), J(z_2) \ge J(u_2) \text{ and } J(z_1) + J(z_2) = J(u_1) + J(u_2).$$

This implies that $J(z_1) = J(u_1)$ and then z_1 is a minimizer of Γ_1 i.e., z_1 is a solution of (P). Therefore we find that $u_1 \le u_2$. By repeating the argument above, we have sequences $\{k_n\} \subset \mathbb{N}$ and $\{u_n\} \subset \Gamma(v, v)$ of solutions of (P) such that

$$u_n[m_1] \in U_{m_1}(k_n) \text{ for each } n \ge 1$$

and

$$u_1 \le u_2 \le u_3 \le \cdots .$$

The property (2) follows. This completes the proof. □

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