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AN EXTENSION OF THE MAJORANA UNIQUENESS THEOREM TO HILBERT SPACES

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Abstract. Majorana [3] has considered the question of uniqueness of Cauchy problem for ordinary differential equation in R. The present paper extends these results to a class of differential equations in finite dimensional Hilbert spaces. A uniqueness criterion for generalized differential equations in finite dimensional Hilbert spaces is derived as well.

1. Introduction

Majorana [3] established, with the aid of an auxiliary non-differential equation of the form

$$u = tf(t, u), \tag{1.1}$$

a nonuniqueness result for the following initial value problem

$$x' = f(t, x), \qquad x(0) = 0, \tag{1.2}$$

where x and 0 are reals. And as a consequence a new uniqueness criterion [[3], theorem 2] was deduced.. The advantages of Majorana's result consists in the fact that any of the standard Lipschitz condition types do not apply, i.e., those conditions imposed on the difference $f(t, x) - f(t, y)$. For the sake of simplicity we give here the statement of Majorana's nonuniqeness result.

Theorem 1.1. ([3]) *Let the function $f(t, x)$ be defined in $[0, 1] \times R$, continuous with respect to t, and such that $f(t, 0) = 0$ for every $t \in [0, a]$ $(a < 1)$. Further let (1.2) have two different classical solutions. Then, for every $\varepsilon > 0$, there exists $t \in [0, a]$ such that (1.1) has at least two different roots u with $|u| < \varepsilon$.*

Statement of the problem. Majorana's approach turned out unsuitable to check analogous theorems on the simplest space R^2 as the following example shows

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Example 1.1. Let us consider the inial value problem

$$\widehat{x}' = \widehat{f}(t, \widehat{x}), \quad \widehat{x}(0) = \widehat{0}, \tag{1.3}$$

where

$$\widehat{f}(t, \widehat{x}) = \begin{cases} \left(\frac{2}{\sqrt{\|\widehat{x}\|}}(x_1 + x_2), \frac{2}{\sqrt{\|\widehat{x}\|}}(x_2 - x_1)\right) & \text{if } \widehat{x} \neq \widehat{0}, \\ \widehat{0} & \text{if } \widehat{x} = \widehat{0}. \end{cases}$$

In polar coordinates (1.3) becomes

$$r' = 2\sqrt{r}, \quad \theta' = \frac{-2}{\sqrt{r}}.$$

Thus, besides the trivial solution $\widehat{x} = \widehat{0}$, there is at least another one given by

$$\widehat{x} = \begin{cases} \left(t^2 \cos \ln \frac{1}{t^2}, t^2 \sin \ln \frac{1}{t^2}\right) & \text{if } t \neq 0 \\ \widehat{0} & \text{if } t = 0. \end{cases}$$

We conclude that (1.3) satisfies the assumptions of theorem 1.1[3], however, one can't find more than the trivial root to equations analogous to (1.1), namely,

$$u_1 = \lambda(u_1 + u_2), \quad u_2 = \lambda(u_2 - u_1),$$

where λ is an arbitrary scalar, in particular, $\lambda = \frac{2t}{\sqrt{\|\widehat{u}\|}}$ $\forall t \in [0, 1]$.
Our main concern herein is to prove, by analyzing a scalar equation analogous to (1.1), uniqueness of solutions of each one of the following problems

$$\widehat{x}' = \widehat{f}(t, \widehat{x}), \quad \widehat{x}(0) = \widehat{0}, \tag{1.4}$$

$$\widehat{x}(t) = \widehat{p}(t) + \int_o^t \widehat{f}(s, \widehat{x}(s))ds, \tag{1.5}$$

where $\widehat{f}$ takes values in a real (or complex) Hilbert space H, with $dim(H) < \infty$, $\widehat{x}$ and $\widehat{0}$ are in H. The basic outline of the proof is as in [3]. Throughout the following the notations $<, >$, and $\|.\|$ will be used to denote respectively the inner product and the norm in H. By an abstract function we mean a function taking values in a real (or complex) Hilbert space. By a solution of (1.4) we mean an abstract function $\widehat{\varphi}(t)$, defined, continuous and differentiable on the interval $[0, a], a < 1$, and satisfies (1.4). Peano's existence theorem is not applicable when $dim(H) = \infty$. This is because the continuity of $\widehat{f}(., \widehat{x}(.))$ is not sufficient for local existence of a solution of (1.4) (see, for example, [2,Chap.5]). The reason for this fact is that closed unit ball in an infinite dimensional space is not necessarily relatively compact.

2. Ordinary Differential Equation in Hilbert Space

Theorem 2.1. *Let the abstract function $\widehat{f}(t,\widehat{x})$ be continuous on $[0,1]\times H$ and satisfy, for every $t\in[0,1]$, $\widehat{f}(t,\widehat{0})=\widehat{0}$. Further let (1.4) admit two different solutions defined on $[0,a]$ $(a<1)$. Then, for every $\varepsilon>0$, there exists $t\in[0,a]$ such that the following auxiliary scalar equation*

$$Re < \widehat{u}, \widehat{f}(t,\widehat{u}) >= \frac{1}{t}\|\widehat{u}\|^2 \tag{2.1}$$

has at least two different roots $\widehat{u}$ with $\|\widehat{u}\|<\varepsilon$.

We base the proof on, among other tools, the following simple but useful observation due to J.B. Diaz and R.J. Weinacht [1]. Let $\widehat{\varphi}(t)$ be a solution of (1.4), then $\|\widehat{\varphi}(t)\|$ has a finite derivative on $(0,a]$ given by

$$\frac{d}{dt}\|\widehat{\varphi}(t)\| = \begin{cases} Re\frac{<\widehat{\varphi}'(t),\widehat{\varphi}(t)>}{\|\widehat{\varphi}(t)\|}, & \|\widehat{\varphi}(t)\|\neq 0, \\ 0, & \|\widehat{\varphi}(t)\|=0. \end{cases} \tag{2.2}$$

For completeness we quote Diaz's proof. It is well know that $\|\widehat{\varphi}(t)\|^2$ is differentiable on $[0,a]$, and has the derivative

$$\frac{d}{dt}\|\widehat{\varphi}(t)\|^2 = 2Re < \widehat{\varphi}'(t), \widehat{\varphi}(t) > . \tag{2.3}$$

The differentiability of $\|\widehat{\varphi}(t)\|$ follows from the differentiability of $\|\widehat{\varphi}(t)\|^2$, we thus have two cases to consider. Firstly, if $\widehat{\varphi}(t)\neq\widehat{0}$, then

$$\frac{d}{dt}\|\widehat{\varphi}(t)\| = \frac{d}{dt}(\|\widehat{\varphi}(t)\|^2)^{\frac12} = \frac12\frac{1}{(\|\widehat{\varphi}(t)\|^2)^{\frac12}}\frac{d}{dt}\|\widehat{\varphi}(t)\|^2.$$

Secondly, if $t\in[0,a]$ such that $\widehat{\varphi}(t)=\widehat{0}$, then

$$\begin{aligned}\frac{d}{dt}\|\widehat{\varphi}(t)\| &= \lim_{h\downarrow 0}\frac{\|\widehat{\varphi}(t+h)\|}{h} = \lim_{h\downarrow 0}\frac{<\widehat{\varphi}(t+h),\widehat{\varphi}(t+h)>^{\frac12}}{h} \\ &= \lim_{h\downarrow 0}\frac{|h|}{h}< \frac{\widehat{\varphi}(t+h)}{h}, \frac{\widehat{\varphi}(t+h)}{h} >^{\frac12} \\ &= <\widehat{\varphi}'(t),\widehat{\varphi}'(t)>^{\frac12} = <\widehat{f}(t,\widehat{\varphi}(t)),\widehat{f}(t,\widehat{\varphi}(t))>^{\frac12} \\ &= <\widehat{f}(t,\widehat{0}),\widehat{f}(t,\widehat{0})>^{\frac12} = 0.\end{aligned}$$

Thus (2.2) is established.

Proof of theorem 2.1. It follows, by the assumption $\widehat{f}(t,\widehat{0})=\widehat{0}$, that (1.4) has the zero solution, and we then assume that (1.4) has a nonzero solution $\widehat{\varphi}(t)\neq\widehat{0}$.

Let $\varepsilon > 0$ be given. Since $\widehat{u} = \widehat{0}$ is a root of (2. 1) for every $t \in [0, a]$, it is sufficient to show the existence of $t \in [0, a]$ for which (2. 1) is satisfied by some $\widehat{u} \neq \widehat{0}$ with $\|\widehat{u}\| < \varepsilon$. Let a real function $A(t)$ be defined by setting

$$A(t) = \begin{cases} \frac{\|\widehat{\varphi}(t)\|}{t}, & t \neq 0 \\ 0, & t = 0, \end{cases}$$

$t \in [0, a]$. It follows, by (2. 1) and (2. 2), that $A(t)$ is continuous on $[0, a]$, differentiable on $(0, a)$, and in view of (2. 2), it has the derivative

$$A'(t) = \frac{1}{t^2\|\widehat{\varphi}(t)\|}[Re < t\widehat{f}(t, \widehat{\varphi}(t)), \widehat{\varphi}(t) > -\|\widehat{\varphi}(t)\|^2] \tag{2. 4}$$

for every $t \in (0, a]$.

Fix $t_2 \in (0, a)$ such that $\widehat{\varphi}(t_2) \neq \widehat{0}$ and $\|\widehat{\varphi}(t)\| < \varepsilon$ for every $t \in [0, t_2]$. Denote $t_1 = \sup\{t \in [0, t_2] : A(t) = 0\}$. Clearly $\widehat{\varphi}(t_1) = \widehat{0}$ and $\widehat{\varphi}(t) \neq \widehat{0}$ for every $t \in (t_1, t_2]$. At this point there are just two possibilities:

P1: If there exists $t \in (t_1, t_2]$ such that $A'(t) = 0$, it is clear, by (2. 3), that for such t (2. 1) is satisfied by $\widehat{u} = \widehat{\varphi}(t)$, and hence the proof is accomplished just taking these t and $\widehat{u} = \widehat{\varphi}(t)$.

P2: Otherwise, if $A'(t) \neq 0$ for every $t \in (t_1, t_2]$. According to Darboux property $A'(t)$ has a constant sign in $(t_1, t_2]$. We then put $\widehat{u} = \widehat{\varphi}(t_2)(\neq \widehat{0})$, and define

$$G(t) = Re < t\widehat{f}(t, \widehat{\varphi}(t_2)), \widehat{\varphi}(t_2) > -\|\widehat{\varphi}(t_2)\|^2, \quad \forall t \in [0, a].$$

Now let us suppose that $A'(t) > 0$. $G(0) = -\|\widehat{\varphi}(t_2)\|^2 < 0$. On the other hand, we have $G(t_2) = \|\widehat{\varphi}(t_2)\|t_2^2A'(t_2) > 0$. It follows, by continuity of G, that there exists $t \in (0, t_2]$ such that $G(t) = 0$, and thus (2. 1) holds just in taking this value of t and $\widehat{u} = \widehat{\varphi}(t_2)$.

Let us assume that $A'(t) < 0$, for every $t \in (t_1, t_2]$, By (2. 1),

$$t\frac{d}{dt}\|\widehat{\varphi}(t)\|^2 = 2Re < t\widehat{\varphi}'(t), \widehat{\varphi}(t) >\leq 2\|\widehat{\varphi}(t)\|^2$$

$$t\frac{d}{dt}\|\widehat{\varphi}(t)\|^2 - 2\|\widehat{\varphi}(t)\|^2 \leq 0$$

$$\frac{d}{dt}\frac{1}{t^2}\|\widehat{\varphi}(t)\|^2 \leq 0$$

for $t > t_1$. Moreover,

$$\lim_{t\downarrow t_1}\frac{\|\widehat{\varphi}(t)\|^2}{t^2} = \lim_{t\downarrow t_1} < \frac{\widehat{\varphi}(t)}{t}, \frac{\widehat{\varphi}(t)}{t} >= 0$$

because of continuity of $\widehat{\varphi}$, and $\widehat{\varphi}(t_1) = \widehat{0}$. Thus, the continuous non-negative function $\frac{\|\widehat{\varphi}(t)\|^2}{t^2}$ is non-increasing for $t > t_1$, while its limit, as t approaches t_1 from the right, is zero. Consequently, one must have $\frac{\|\widehat{\varphi}(t)\|^2}{t^2} = 0$ for $t > t_1$, which contradicts the definition $t_1 = \sup\{t \in [0, t_2] : A(t) = 0\}$. Hence the assumption $A'(t) < 0$ is impossible, and the proof will thus be accomplished .

An immediate consequence of theorem 2.1 is the following uniqueness criterion.

Theorem 2.2. *Let the abstract function $\widehat{f}(t, \widehat{x})$ be continuous and satisfy, for every $t \in [0, 1]$, $\widehat{f}(t, \widehat{0}) = \widehat{0}$. Assume further that there exist $\varepsilon > 0$ and $t_0 \in (0, 1]$ such that $\widehat{u} = \widehat{0}$ is the only root of the (2. 1) with $\|\widehat{u}\| < \varepsilon$ for every $t \in [0, t_0]$. Then the (1. 4) admits, on the interval $[0, t_0]$, only the zero solution.*

The crucial point in theorems 2.1 and 2.2 is the assumption that (1. 4) has the zero solution. We follow Majorana's procedure to remove this restriction. If we know a solution $\widehat{\varphi}$ of (1. 4), then, by means of change of variables $\widehat{x} = \widehat{p} + \widehat{\varphi}(t)$,(1. 4) becomes

$$\begin{cases} \widehat{p}' = \widehat{F}(t, \widehat{p}), \\ \widehat{p}(0) = \widehat{0}, \end{cases}$$

where $\widehat{F}(t, \widehat{p}) = \widehat{f}(t, \widehat{p} + \widehat{\varphi}(t)) - \widehat{f}(t, \widehat{\varphi}(t))$.

Remark. In example 1.1 above, the space R^2 endowed with the usual inner product is a Hilbert space, (1. 3) has at least two different solutions, and (2. 1) has the form

$$\frac{2u_1}{\sqrt{\|\widehat{u}\|}}(u_1 + u_2) + \frac{2u_2}{\sqrt{\|\widehat{u}\|}}(u_2 - u_1) = \frac{u_1^2 + u_2^2}{t}.$$

It is easy to see that the trivial root $\widehat{u} = \widehat{0}$ of the above scalar equation corresponds the trivial solution $\widehat{\varphi}(t) = \widehat{0}$. By straightforward calculation one can show that $\widehat{u} = \widehat{\varphi}(2t_2)$ represents the second root of the above scalar equation corresponding the non-trivial solution $\widehat{\varphi}(t)$, where $t_2 \in [0, 1]$ is any for which $\widehat{\varphi}(t_2) \neq \widehat{0}$.

We just have achieved some uniqueness results for ordinary differential equations in Hilbert spaces. Our next main objective is to prove, by analyzing a scalar equation analogous to (2. 1), uniqueness results for the generalized differential equation (1. 5).

3. The generalized differential equations in Hilbert space

THEOREM 3.1. *Suppose that* (C1)–(C3) *below are fulfilled*
(C1) $\widehat{p} : [0,1] \to H$ *is continuous,*
(C2) $\widehat{f} : [0,1] \times H \to H$ *is continuous,*
(C3) *for every* $t \in [0,1]$, $\int_0^t \widehat{f}(s, \widehat{p}(s))ds = \widehat{0}$.

Assume further that (1. 5) has two different solutions defined on $[0,a]$,($a < 1$). Then for every $\varepsilon > 0$ there exists $t \in [0,a]$ such that the following auxiliary scalar equation

$$Re < \widehat{f}(t, \widehat{u} + \widehat{p}(t)), \widehat{u} >= \frac{1}{t}\|\widehat{u}\|^2 \tag{3. 1}$$

has at least two distinct roots $\widehat{u}$ each satisfies $\|\widehat{u}\| < \varepsilon$.

Proof of theorem 3.1. It is obvious, by (C3), that (1. 5) has the solution $\widehat{p}(t)$, we thus may assume that $\widehat{\varphi}(t)$ with $\widehat{\varphi} \neq \widehat{p}$, is a solution of (1. 5). Let $\varepsilon > 0$ be given. Since, for every $t \in [0,1]$, $\widehat{u} = \widehat{0}$ is a root of (3. 1), our task is to show that there exists $t \in [0,1]$ for which (3. 1) is satisfied by some $\widehat{u} \neq \widehat{0}$ with $\|\widehat{u}\| < \varepsilon$. Let $\widehat{F}(t, \widehat{u})$ be the scalar function defined in $[0,1] \times H$ by

$$\widehat{F}(t, \widehat{u}) = \widehat{f}(t, \widehat{u} + \widehat{p}(t)). \tag{3. 2}$$

It follows, by (C2), that $\widehat{F}(t, \widehat{u})$ is continuous on $[0,1] \times H$. By means of the change of variable $\widehat{x}(t) = \widehat{\psi} + \widehat{p}(t)$, (3. 1) is replaced with

$$\widehat{\psi}(t) = \int_0^t \widehat{F}(s, \widehat{\psi}(s))ds, \tag{3. 3}$$

where $\widehat{F}$ is given by (3. 2). Clearly (3. 3) has the zero solution that corresponds to the solution $\widehat{p}(t)$ of (3. 1). Moreover any two different solutions of (1. 5) are mapped into two different solutions of (3. 3). Let $A(t)$ be a real-valued function defined on $[0,a]$, ($a < 1$) by

$$A(t) = \begin{cases} \frac{1}{t}\| \int_0^t \widehat{F}(s, \widehat{\psi}(s))ds\| & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

It is obvious, by (C2) and (3. 2), that $A(t)$ is differentiable on $(0,a]$. By (3. 2) and (3. 3) we obtain

$$A'(t) = \frac{1}{t^2\|\widehat{\psi}(t)\|}[Re < t\widehat{F}(t, \widehat{\psi}(t)), \widehat{\psi}(t) > -\|\widehat{\psi}(t)\|^2]$$

for every $t \in (0,a]$.
From now on we follow step by step arguments similar to that in the proof of theorem 2.1.

As a consequence of theorem 3.1 we deduce the following uniqueness criterion for (1. 5).

THEOREM 3.2. *Suppose that* (D1)–(D3) *below are fulfilled*
(D1) $\widehat{p} : [0,1] \to H$ *is continuous,*
(D2) $\widehat{f} : [0,1] \times H \to H$ *is continuous,*
(D3) *there exist* $\varepsilon > 0$ *and* t_0 *such that for every* $t \in [0, t_0]$, (3. 1) *has a unique root* $\widehat{u} \neq \widehat{0}$ *with* $\|\widehat{u}\| < \varepsilon$. *Then* $\widehat{p}(t)$ *is the only solution of* (1. 5) *on the interval* $[0, t_0]$.

Remark. There are two observations worth making here, firstly, the truly interesting thing in the present paragraph is that theorem 3.1 generalizes theorem 2.1. Secondly, although the norm in Banach space could not in general be defined in terms of inner product, the inner product should be replaced with duality mapping to establish similar results in Banach space, which we hope to consider elsewhere.

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