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DISTINGUISHING CHROMATIC NUMBERS OF PLANAR GRAPHS

By

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Abstract. A graph G is said to be *d-distinguishing colorable* if there is a d-coloring of G such that any automorphism of G except the identity map does not preserve colors. We shall prove that every 3-connected planar graph is 6-distinguishing colorable and every maximal planar graph is 5-distinguishing colorable except $K_{2,2,2}$ and $C_6+\overline{K}_2$, establishing a general theorem on the distinguishing colorability of graphs faithfully embedded on closed surfaces.

Introduction

Let G be a graph and $c : V(G) \to \{1, 2, \ldots, d\}$ a labeling of vertices with d numbers (or "colors"), which is not assumed to be a proper coloring now. Define $\mathrm{Aut}(G, c)$ as the set of automorphisms $\sigma : G \to G$ such that $c(\sigma(v)) = c(v)$ for all vertices $v \in V(G)$. Then $\mathrm{Aut}(G, c)$ forms a subgroup in the automorphism group $\mathrm{Aut}(G)$ of G. If $\mathrm{Aut}(G, c)$ consists only of the identity, c is said to be *d-distinguishing*.

A graph G is said to be *d-distinguishable* if G admits a d-distinguishing labeling. The *distinguishing number* of G is defined as the minimum number d such that G is d-distinguishable and is denoted by $D(G)$. There have been many papers written on the distinguishing number of graphs, say [1, 2], and also we can find some recent papers [4, 6, 8] on this topic with topological aspect. In particular, Fukuda, Negami and Tucker have discussed the distinguishing number of planar graphs, as follows.

Theorem 1. (Fukuda, Negami and Tucker [4]) *Every 3-connected planar graph is 2-distinguishable, except K_4, $K_{2,2,2}$, W_4, W_5, $C_3+\overline{K_2}$, $C_5+\overline{K_2}$ and Q_3.*

Here W_n and Q_3 denote the wheel with rim of length n and the 3-cube, respectively. The others are given by the standard notations. For example, $C_n+\overline{K_2}$ presents the double pyramid with n-gonal base, or which is often called a *double*

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wheel.

On the other hand, Collins and Trenk [3] have defined a variety of the distinguishing chromatic number concerning vertex coloring of graphs. A graph G is said to be *d-distinguishing colorable* if G has a d-distinguishing coloring, which should be a proper coloring, and the *distinguishing chromatic number* $\chi_D(G)$ is defined as the minimum number d such that G is d-distinguishing colorable. It is obvious that $D(G) \leq \chi_D(G)$. For example, it is not difficult to see that $D(K_{n,n,n}) = n + 1$ ($n \geq 2$) and $\chi_D(K_{n,n,n}) = 3n$.

In this paper, we shall carry out topological arguments similar to that in [6], using the notion of "faithfulness of embedding", and establish a general theorem on the distinguishing chromatic number of graphs embedded on closed surface. We shall prove the following theorem on planar graphs as one of its corollaries:

Theorem 2. *Every 3-connected planar graph is 6-distinguishing colorable.*

A *maximal planar graph* is a simple graph G such that G can be embedded on the plane and that $G + e$ cannot be for any new edge e joining two nonadjacent vertices in G. It is easy to see that any maximal planar graph with at least four vertices can be embedded on the sphere as a *triangulation*, that is, so that each face is bounded by a cycle of length 3 and that it is 3-connected. Thus, it follows from Theorem 2 that such a maximal planar graph is 6-distinguishing colorable. However, we can prove a stronger theorem on maximal planar graphs:

Theorem 3. *Every maximal planar graph is 5-distinguishing colorable unless it is isomrphic to $K_{2,2,2}$ or $C_6 + \overline{K}_2$.*

We shall show a more essential fact that $\chi_D(G) \leq \chi(G) + 2$ for a 3-connected planar graph G with some exceptions as an immediate consequence of our general arguments in Section 1. As we know, "Four Color Theorem" states that every planar graph is 4-colorable. This and the above inequality imply our main theorem, Theorem 2. On the other hand, we use not only Four Color Theorem but also the planarity of graphs explicitly to prove Theorem 3.

1. Faithfully embedded polyhedral graphs

Let G be a graph embedded on a closed surface F^2. If any automorphism σ of G extends to an auto-homeomorphism $h : F^2 \to F^2$ with $h|_G = \sigma$, then G is said to be *faithfully embedded* on F^2. As the following lemma suggests, the assumption of being faithfully embedded works to extend local arguments around a vertex to a global one over the whole surface:

Lemma 4. *Let G be a graph embedded on a closed surface F^2 and v a vertex of degree at least* 3*. If an automorphism σ of G extends to an auto-homeomorphism over F^2 and fixes each of three vertices u, v and w which form a corner uvw of a face A, then σ must be the identity map over G.*

Proof. Let Ω be the set of faces that σ fixes. Then $\Omega \neq \emptyset$ since it contains A at least; if $\deg v = 2$, then σ might swap the two faces meeting along the path uvw. If a face B does not belong to Ω, we can rechoose it so that B meets another face $B' \in \Omega$ along an edge xy since G is connected. However, σ must fix B, too since there are only two faces sharing xy, which are B and B' and since σ fixes B'. This is a contradiction. Therefore, there is no face not belonging to Ω. ■

Here we shall modify the distinguishing chromatic number for more general use, as follows. Consider a pair (G, Γ) of a graph G and a subgroup Γ in $\mathrm{Aut}(G)$. Let Γ_c denote the subgroup consisting of automorphisms $\sigma \in \Gamma$ that preserve colors given by a coloring $c : V(G) \to \{1, 2, \dots, d\}$ of G. If $\Gamma_c = \{\mathrm{id}_G\}$, then c is called a *d-distinguishing coloring* of (G, Γ). The pair (G, Γ) is *d-distinguishing colorable* if it admits a d-distinguishing coloring and the *distinguising chromatic number* of the pair is defined as the minimum number d such that (G, Γ) is d-distinguishing colorable and is denoted by $\chi_D(G, \Gamma)$. It is clear that $\chi_D(G, \Gamma) \le \chi_D(G)$ for any subgroup Γ in $\mathrm{Aut}(G)$ and that $\chi(G) = \chi_D(G, \{\mathrm{id}_G\})$, where $\chi(G)$ stands for the *chromatic number* of G in the usual sense.

A 3-connected graph G is said to be *polyhedral* on a closed surface F^2 if each face is bounded by a cycle and if the boundary cycles of two faces intersect in at most one vertex or one edge. A pair (G, Γ) is said to be *faithfully embedded* on F^2 if any automorphism $\sigma \in \Gamma$ extends to an auto-homeomorphism $h : F^2 \to F^2$ with $h|_G = \sigma$.

Theorem 5. *Let G be a polyhedral graph G on a closed surface F^2 and Γ a subgroup in* $\mathrm{Aut}(G)$*. If (G, Γ) is faithfully embedded on F^2, then $\chi_D(G, \Gamma) \le$* $\max\{6, \chi(G) + 2\}$.

Proof. Put $k = \chi(G)$ for convenience and consider a k-coloring $c : V(G) \to \{1, 2, \dots, k\}$ of G. First suppose that G has two faces of different sizes sharing an edge uv, say A and B. Define a $(k+2)$-coloring $c_1 : V(G) \to \{1, 2, \dots, k, k+1, k+2\}$ by $c_1(u) = k+1$, $c_1(v) = k+2$ and $c_1(x) = c(x)$ for any other vertex x. It is clear that any automorphism $\sigma \in \Gamma$ fixes each of u and v if σ preserves colors of vertices given by c_1 since they are unique vertices colored with $k+1$ and $k+2$, respectively. Such an automorphism σ fixes each of the two faces A and B since they are only faces incident to uv and are of different sizes. This implies that σ is the identity map over G by Lemma 4, and hence c_1 is a $(k+2)$-distinguishing coloring of (G, Γ).

Now we may assume that all faces have the same size. Let v be any vertex of G with neighbors $u_0, u_1, \ldots, u_{d-1}$ lying around v in this cyclic order and A the face having the corner u_0vu_1. Define a $(k+2)$-coloring $c_2 : V(G) \to \{1, 2, \ldots, k, k+1, k+2\}$ by $c_2(u_0) = k+1$, $c_2(u_1) = k+2$ and $c_2(x) = c(x)$ for any other vertex x. Take any automorphism $\sigma \in \Gamma$ which preserves this new coloring c_2. Then σ fixes u_0 and u_1 since they are unique vertices colored with $k+1$ and $k+2$, respectively. If σ fixes v, then it fixes the corner u_0vu_1 and hence σ becomes the identity map over G, by Lemma 4. This implies that c_2 is a $(k+2)$-distinguishing coloring of (G, Γ).

Assume that $\sigma(v) \neq v$. Then σ maps the corner u_0vu_1 to another courner of a face B, whose boundary cycle contains both u_0 and u_1. Since G is polyhedral, there are two possibilities; (i) $A = B$ and it is a quadrilateral, or (ii) $A \neq B$ and they are triangular faces sharing the edge u_0u_1. It follows from the first argument in this proof that the faces of G are all quadrilateral or all triangular, corresponding to (i) or (ii).

In Case (i), G is a quadrangulation on F^2 and has a k-coloring c such that each diagonal pair of vertices in any face get a common color. This implies that only two colors appear along the boundary of each face. It is clear that the neighboring face has the same two colors and hence c must be a 2-coloring with colors 1 and 2. Choose any two faces A and B sharing one edge uv and let uvw_1w_2 and $uvw_1'w_2'$ be their boundary cycles. We may assume that $c(u) = c(w_1) = c(w_1') = 1$ and $c(v) = c(w_2) = c(w_2') = 2$. Define a $(k+2)$-coloring $c_3 : V(G) \to \{1, 2, 3, 4\}$ with $k = 2$ by $c_3(u) = c_3(w_1) = k+1$, $c_3(v) = c_3(w_2') = k+2$ and $c_3(x) = c(x)$ for any other vertex x. Then A is a unique face incident to two vertices colored with $k+1$, so is B for color $k+2$ and uv is a unique edge shared by A and B since G is polyhedral. Thus, any automorphism $\sigma \in \Gamma$ preserving colors given by c_3 fixes uv and also each of A and B. This forces σ to be the identity map over G by Lemma 4, and hence c_3 is a $(k+2)$-distinguishing coloring of (G, Γ).

In Case (ii), G is a trinagulation on F^2 and has a k-coloring c such that any two triangular faces uvw and $uv'w$ sharing one edge uw get three common colors, that is, $c(v) = c(v')$. Since G is connected, it follows that c is a 3-coloring with colors 1, 2 and 3 and that the degree of each vertex in G is an even number. Take a face with boundary uvw and define a 6-coloring $c_4 : V(G) \to \{1, 2, 3, 4, 5, 6\}$ by $c_4(u) = 4$, $c_4(v) = 5$, $c_4(w) = 6$ and $c_4(x) = c(x)$ for any other vertex x. Then any automorphism $\sigma \in \Gamma$ fixes each of u, v and w since there are no other vertices with colors 4, 5 and 6, and hence it fixes the face with boundary uvw. This implies that σ is the identity map over G by Lemma 4 and that c_4 is a 6-distinguishing coloring of (G, Γ). Thus, $\chi_D(G, \Gamma) \leq 6$.

Therefore, we have constructed a $(k+2)$-distinguishing coloring of (G, Γ) in all cases but the last and hence $\chi_D(G,\Gamma) \le \chi(G)+2$. Unifying this and the last case, we obtain the theorem. ■

Note that $\chi_D(G,\Gamma) \le \chi(G)+2$ unless G is a 3-colorable triangulation on a closed surface. Although this might hold for the exceptional case, Theorem 5 works well enough to prove Theorem 2 and corollaries below.

A graph G with a fixed embedding on a closed surface F^2 is often called a *map* on the surface. We denote such a map here by $M(G)$. A *map-automorphism* of G or an automorphism of $M(G)$ is defined as an automorphism σ of G which carries each face to a face, and hence σ extends to an auto-hoemomorphism over F^2. We denote the set of map-automorphisms of G by $\mathrm{Aut}(M(G))$ and call it the *automorphism group* of $M(G)$. Clearly, $\mathrm{Aut}(M(G))$ forms a subgroup of $\mathrm{Aut}(G)$.

We can defined the *distinguishing chromatic number* $\chi_D(M(G))$ of a map $M(G)$ in the same way as for an abstract graph, restricting automorphisms to map-automorphisms. That is, we have $\chi_D(M(G)) = \chi_D(G, \mathrm{Aut}(M(G)))$. Besides, $(G, \mathrm{Aut}(M(G)))$ is faithfully embedded on the surface. Therefore, the following corollary is an immediate consequence of Theorem 5. A map $M(G)$ is said to be *polyhedral* if its underlying graph G is polyhedral.

Corollary 6. *Let $M(G)$ be a polyhedral map on a closed surface F^2 with underlying graph G. Then $\chi_D(M(G)) \le \max\{6, \chi(G)+2\}$.* ■

"Map Color Theorem" [7] gives us an exact upper bound for the chromatic number of graphs embedded on a given closed surface. Using this and the above corollary, we can establish the following corollary immediately; it suffices to see the right hand of the inequality in the corollay is greater than or equal to 6 for any closed surface.

Corollary 7. *For any polyhedral map M on a closed surface F^2, we have:*

$$\chi_D(M) \le \left\lfloor \frac{7+\sqrt{49-24\,\varepsilon(F^2)}}{2} \right\rfloor + 2$$

where $\varepsilon(F^2)$ stands for the Euler characteristic of F^2. ■

2. Graphs on the sphere

It is well-known that every 3-connected planar graph is uniquely embedded on the sphere, which follows from the uniqueness of its dual, proved by Whitney [9]. As is pointed out in [5], the uniqueness of duals implies the faithfulness of embedding. That is, every 3-connected planar graph can be faithfully embedded on the sphere. Furthermore, it is easy to see that a 3-connected planar graph is polyhedral. Thus, we can apply Theorem 5 to prove our main theorem.

Proof of Theorem 2. Let G be a 3-connected planar graph embedded on the sphere, which is polyhedral and is faithfully embedded. Put $\Gamma = \mathrm{Aut}(G)$. Then (G, Γ) is faithfully embedded on the sphere. By Theorem 5, we have $\chi_D(G) = \chi_D(G, \Gamma) \le \max\{6, \chi(G) + 2\}$. By Four Color Theorem, $\chi(G) \le 4$ and hence $\chi_D(G) \le 6$. ■

Here, we shall determie the distinguishing chromatic number of the exceptions in Theorem 3. Both of them can be regarded as double wheels; $K_{2,2,2} \cong C_4 + \overline{K}_2$ in particular.

LEMMA 8. *We have the following formulas:*

(i) $\chi_D(C_n) = 3$ *if* $n \ne 4, 6$, $\quad \chi_D(C_4) = \chi_D(C_6) = 4$

(ii) $\chi_D(C_n + \overline{K}_2) = 5$ *if* $n \ne 4, 6$, $\quad \chi_D(C_4 + \overline{K}_2) = \chi_D(C_6 + \overline{K}_2) = 6$

Proof. (i) Let $C_n = u_0 u_1 \cdots u_{n-1}$ be the cycle of length n. First we shall show that $\chi_D(C_n) \ge 3$. If n is odd, then clearly $3 = \chi(C_n) \le \chi_D(C_n)$. If n is even, then there is a unique 2-coloring of C_n with colors 1 and 2, up to exchanging colors. Since a reflexion preserves the 2-coloring, C_n is not 2-distinguishing colorable and hence we have $\chi_D(C_3) \ge 3$ in this case, too.

Now we try to construct a 3-distinguishing coloring of C_n. Define a 3-coloring $c : V(C_n) \to \{1, 2, 3\}$ by $c(u_0) = c(u_3) = 3$, $c(u_i) = 1$ for odd numbers $i \ne 3$ and $c(u_j) = 2$ for even numbers $j \ne 0$. If $n = 3$, then clearly c is a 3-distinguishing coloring since the vertices of C_n get all different colors. However, it does not work if $n = 4$; u_0 and u_3 are adjacent to each other, but they get the same color 3. It is easy to see that any 3-coloring of C_4 assigns the same color to an antipodal pair of veritces, say u_0 and u_2. Then the reflexion fixing u_1 and u_3 preserves the colors. This implies that C_4 is not 3-distinguishing colorable and hence $\chi_D(C_4) = 4$.

Suppose that $n = 5$ or ≥ 7 and take any automorphism $\sigma \in \mathrm{Aut}(G, c)$. It is clear that $\sigma(\{u_0, u_3\}) = \{u_0, u_3\}$ since they are the only vertices colored with 3. They divide C_n into two segments of length 3 and $n - 3$. Since $n = 5$ or ≥ 7 now, these segments have different lengths and hence σ cannot exchange them. Furthermore, the coloring $3, 1, 2, 3$ along $u_0 u_1 u_2 u_3$ forces σ to fixe the segments pointwise and to be the identity map over C_n. Thus, c is a 3-distinguishing coloring of C_n and it follows that $\chi_D(C_n) = 3$ for $n = 5$ or ≥ 7.

The remaining case is when $n = 6$. We shall write here $c = (c_0, c_1, c_2, c_3, c_4, c_5)$

to express that $c(u_i) = c_i$ for $i = 0, 1, 2, 3, 4, 5$. Consider any 3-coloring c of C_6. Suppose that there is a pair of vertices at distance 3 with the same color. We may assume that $c = (3, 1, 2, 3, *, *)$ with $* = 1$ or 2. The rotation in 180° preserves $(3, 1, 2, 3, 1, 2)$ while the reflexion fixing u_0 and u_3 preseves $(3, 1, 2, 3, 2, 1)$. If there is no such pair, then we may assume that $c = (3, 1, 2, 1, 2, 1)$. The reflexion fixing u_0 and u_3 preserves this. Therefore, C_6 admits no 3-distinguishing coloring. It is easy to see that the 4-coloring $c' = (3, 1, 2, 3, 4, 1)$ is 4-distinguishing and hence $\chi_D(C_6) = 4$.

(ii) Put $G = C_n + \overline{K}_2$ for convenience. It suffices to show that $\chi_D(G) = \chi_D(C_n) + 2$. Let c be any d-distinguishing coloring of G and $\{x, y\}$ the independent pair of vertices corresponding to $\overline{K}_2$. Since both x and y are adjacent to all vertices lying along C_n, their colors $c(x)$ and $c(y)$ are different from the colors of vertice on C_n. Furthermore, we have $c(x) \neq c(y)$ since there is an automorphism of G that exchanges x and y, fixing each vertex on C_n. Let $\bar{c}$ be the $(d-2)$-coloring of C_n obtained as the restriction of c to C_n. It is clear that any automorphism $\bar{\sigma} \in \mathrm{Aut}(C_n, \bar{c})$ extends to an automorphism $\sigma \in \mathrm{Aut}(G, c)$ with $\sigma(x) = x$ and $\sigma(y) = y$. This implies that $\bar{c}$ is a $(d-2)$-distinguishing coloring of C_n and it follows that $\chi_D(G) \geq \chi_D(C_n) + 2$.

Conversely, take any $(d-2)$-distinguising coloring $\bar{c}$ of C_n with colors $1, \ldots, d-2$. Define a d-coloring c of G by $c(x) = d-1$, $c(y) = d$ and $c(z) = \bar{c}(z)$ for each vertex z on C_n. If c is not d-distinguishing, then there is an automorhism $\sigma \in \mathrm{Aut}(G, c)$ which is not the identity map over G. Since x and y are unique vertices colored with $d-1$ and d respectively, σ fixes each of x and y and hence it must move some vertices on C_n. That is, $\sigma|_{C_n} \in \mathrm{Aut}(C_n, \bar{c})$ is not the identity map over C_n. However, this is contrary to our assumption of $\bar{c}$ being $(d-1)$-distinguising. Therefore, c is a d-distinguishing coloring of G. This implies that $\chi_D(G) \leq \chi_D(C_n) + 2$. ■

Theorem 3 cannot be obtained as an easy corollay of Theorem 5. We need more detailed arguments with the planarity of graphs.

Proof of Theorem 3. Let G be a maximal planar graph on the sphere. Then G has a vertex of degree 3, 4 or 5, as well-known and also G is 4-colorable by Four Color Theorem. Let $c' : V(G) \to \{1, 2, 3, 4\}$ be its 4-coloring. We shall modify c' to be a 5-distinguishing coloring $c : V(G) \to \{1, 2, 3, 4, 5\}$, as follows.

Case 1. *G has a vertex v of degree at least 7.* Let $C = u_0 u_1 \cdots u_{k-1}$ be the link of v, with the indices taken modulo k. Without loss of generality, we may assume that $c(v) = 4$ and $u_0, u_1, \ldots, u_{k-1}$ are colored by 1, 2 and 3. For $i = 0, 1, \ldots, k-1$, we define a 5-coloring $c_i : V(G) \to \{1, 2, 3, 4, 5\}$ by $c_i(u_i) = c_i(u_{i+3}) = 5$ and

$c_i(x) = c'(x)$ for any other vertex $x \in V(G) - \{u_i, u_{i+3}\}$ if u_i is not adjacent to u_{i+3}; otherwise, c_i should be "undefined". If c_i is undefined, then c_{i+1} and c_{i+2} can be defined since the edge $u_i u_{i+3}$ prevents u_{i+1} and u_{i+2} from being adjacent to u_{i+4} and u_{i+5}, respectively. Thus, we may assume that c_0 can be defined at least.

Take any automorphism $\sigma_0 \in \mathrm{Aut}(G, c_0)$. If $\sigma_0(v) = v$, then σ_0 sends the cycle C onto itself, fixing $\{u_0, u_3\}$ setwise since the only vertices colored with 5 in c_0 are u_0 and u_3. The set $\{u_0, u_3\}$ divides the link of v into two segments, $u_1 u_2$ and $u_4 u_5 \cdots u_{k-1}$; their lengths are different since $k \geq 7$. This implies that σ_0 fixes each of these two segments and besides $\sigma_0(u_1) = u_1$ and $\sigma_0(u_2) = u_2$ since they have different colors. Therefore, σ_0 fixes these segments pointwise and hence does the whole of the link of v totally and extends to the identity map by Lemma 4. Thus, G is 5-distinguishing colorable in this case. Otherwise, there is another vertex $v_0 = \sigma_0(v)$ that is colored by 4 in c' and is adjacent to both u_0 and u_3. Now we found a path $u_0 v_0 u_3$ of length 2 outside the star neighborhood of v.

Carry out the same arguments for c_i $(i = 0, 1, \dots k-1)$ as for c_0 in the above. If c_i can be defined, then either we can conclude that G is 5-distinguishing colorable or can find a path $u_i v_i u_{i+3}$ outside the star neighborhood of v with $v_i = \sigma_i(v)$ for some automorphism $\sigma_i \in \mathrm{Aut}(G, c_i)$; v_i does not lie in the link of v since $c'(v) = c_i(v) = 4$. If the first case does not happen for all i's, then each pair $\{u_i, u_{i+3}\}$ is joined by a path $u_i v_i u_{i+3}$ or an edge $u_i u_{i+3}$, which corresponds to the case when c_i is undefined. It is easy to that the planarity excludes the latter and forces $v_0, v_1, \dots, v_{k-1}$ to be one vertex. That is, the subgraph H induced by $\{v, u_0, u_1, \dots, u_{k-1}, v_0\}$ in G is isomorphic to $C_k + \overline{K}_2$ and we have $\deg_G v = \deg_G v_0 = \deg_H v_0 = k$. This implies that there is no vertex inside the region bounded by $v_0 u_i u_{i+1}$ and hence $G = H \cong C_k + \overline{K}_2$. Thus, G is 5-distinguishing colorable in this case by Lemma 8.

Case 2. *G has a vertex v of degree* 3. Define a 5-coloring $c : V(G) \to \{1, 2, 3, 4, 5\}$ by $c(v) = 5$ and $c(x) = c'(x)$ for any other vertex $x \in V(G) - \{v\}$. Then any automorphism $\sigma \in \mathrm{Aut}(G, c)$ must fix v and hence it leaves the neighbors of v invariant. Since the three neighbors have three distinct colors in c, σ fixes them, too and becomes the indentity map of G by Lemma 4. Thus, G is 5-distinguishing colorable.

Case 3. *G has a vertex v of degree* 5. Let u_1, u_2, u_3, u_4 and u_5 be the five neighbors of v lying around v in this order. That is, $u_1 u_2 u_3 u_4 u_5$ forms the link of v. Without loss of generality, we may assume that $c'(v) = 4$, $c'(u_1) = c'(u_3) = 1$, $c'(u_2) = c'(u_4) = 2$ and $c'(u_5) = 3$. Define a 5-coloring $c : V(G) \to \{1, 2, 3, 4, 5\}$ by $c(v) = 5$ and $c(x) = c'(x)$ for any other vertex $x \in V(G) - \{v\}$. It is clear that any automorphism $\sigma \in \mathrm{Aut}(G, c)$ fixes v and each of its neighbors and hence it becomes the identify map of G by Lemma 4. Thus, c is 5-distinguishing coloring and hence G is 5-distinguishing colorable.

CASE 4. *Each vertex in G has dergee* 4 *or* 6. This is the final case; assume that none of Cases 1 to 3 happen. Since all vertices have even degree, G is 3-colorable and c' may be assumed to be a 3-coloring with colors 1, 2 and 3.

First, suppose that there are two adjacent vertices v and u of degree 6. Let $C_v = u_0u_1u_2u_3u_4u_5$ be the link around v with $u = u_0$ and $C_u = v_0v_1v_2v_3v_4v_5$ the link around u with $v = v_0$, $u_1 = v_5$ and $u_5 = v_1$. Then we may assume that $c'(u) = c'(u_2) = c'(u_4) = 1$, $c'(v) = c'(v_2) = c'(v_4) = 2$ and $c'(u_1) = c'(u_3) = c'(v_1) = c'(v_3) = 3$. By the planarity, at least one of the pairs $\{u_2, v_2\}$ and $\{u_4, v_4\}$ is not adjacent. Up to symmetry, we assume that u_2 is not adjacent to v_2 and define a 5-coloring $c : V(G) \to \{1, 2, 3, 4, 5\}$ by $c(v) = c(v_2) = 4$, $c(u) = c(u_2) = 5$ and $c(x) = c'(x)$ for the other vertices x. Then the subgraph induced by vertices with colors 4 and 5 forms a path u_2vuv_2 of length 3 and clearly any automorphism $\sigma \in \mathrm{Aut}(G, c)$ fixes this path pointwise. It follows that σ is the identity map over G by Lemma 4. Thus, c is a 5-distinguishing coloring and G is 5-distinguishing colorable in this case.

The remaining case is when the set of vertices of degree 6 is independent. If there is a vertex of degree 6, then it must be adjacent to six vertices of degree 4. It is easy to see that G is isomorphic to $C_6 + \overline{K}_2$, in this case. Otherwise, all vertices of G has degree 4 and G must be isomorphic to $K_{2,2,2}$ These two graphs are listed as the exceptions in the theorem. ■

We have carried out only local arguments in the previous proofs. It might be possible to prove the following by more global arguments about colorings on the plane:

CONJECTURE 1. Every 3-connected planar graph is 5-distinguishing colorable with a finite number of exceptions.

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