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ON CONGRUENT EMBEDDINGS OF A TETRAHEDRON INTO A CIRCULAR CYLINDER

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Abstract. We prove that for any pair of congruent regular tetrahedra embedded in a fixed infinite circular cylinder, one tetrahedron can be superposed to the other tetrahedron by a rigid motion within the cylinder. Thus, all congruent embeddings of a regular tetrahedron into a circular cylinder are equivalent modulo rigid motions within the cylinder. We also present an elementary proof of the fact that a regular tetrahedron of unit edge can be inscribed in a circular cylinder of radius r if and only if $1/2 \le r \le 3\sqrt{2}/8$.

1. Introduction

Many authors considered problems related to embedding or inscribing simplices in circular cylinders, mostly to study the outer j-radii of simplices or to compute cylinders through the vertices of a simplex. See e.g., Brandenberg, et al. [1, 2], Devillers, et al. [3], Schömer, et al. [7], and the references in these articles. In this note, we classify the embeddings of a regular tetrahedron in circular cylinders. (It will be also interesting to consider the classifications of embeddings of other regular polyhedra in circular cylinders.)

For $r > 0$, an infinite circular cylinder $C(r)$ of radius r is defined to be the closed r-neighborhood of the z-axis in $\mathbb{R}^3$. A regular tetrahedron with unit edge is simply called a *unit tetrahedron*. A unit tetrahedron T contained in $C(r)$ is also called an *embedding* of a unit tetrahedron in $C(r)$. If all vertices of T in $C(r)$ lie on the boundary $\partial C(r)$ of $C(r)$, then T is said to be *inscribed* in $C(r)$.

It is known that $C(r)$ can contain a unit tetrahedron if and only if $r \ge 1/2$ (e.g., Pukhov [6], Brandenberg, et al. [1]) and a unit tetrahedron can be inscribed in $C(r)$ if and only if $1/2 \le r \le 3\sqrt{2}/8$ (e.g., Devillers, et al. [3]). We present an elementary proof of these facts (Corollary 2).

Let T_1, T_2 be a pair of unit tetrahedra in $C(r)$. If we can superpose T_1 on T_2 by moving T_1 within $C(r)$ as a rigid body, then T_1 is said to be *equivalent in $C(r)$ to T_2*. Let $\nu_\circ(r)$ denote the maximum number of mutually non-equivalent

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unit tetrahedra in $C(r)$. We present an elementary proof of the following:

Theorem 1.

$$\nu_\circ(r) = \begin{cases} 0 & \text{for } r < 1/2 \\ 1 & \text{for } r \geq 1/2. \end{cases}$$

This result is simple but not trivial. If circular cylinders are replaced by equilateral triangular prisms $\Delta(t) \times \mathbb{R}$, the result is not so simple, where $\Delta(t)$ denotes the equilateral triangle of edge length t. Let $\nu_\Delta(t)$ denote the maximum number of mutually non-equivalent embeddings of a unit tetrahedron in $\Delta(t) \times \mathbb{R}$, modulo rigid motions within the prism. Then, as proved in [5], the result becomes:

$$\nu_\Delta(t) = \begin{cases} 0 & \text{for} \quad t < t_0 := \frac{1+\sqrt{2}}{\sqrt{6}} \\ 6 & \text{for} \quad t_0 \leq t < t_1 := \frac{\sqrt{3}+3\sqrt{2}}{6} \\ 18 & \text{for} \quad t_1 \leq t < 1 \\ 1 & \text{for} \quad 1 \leq t. \end{cases}$$

2. Tetrahedra inscribed in a cylinder

Note that $\{C(r) \mid r > 0\}$ is a *nested* family, that is, for every $0 < s < r$, we always have $C(s) \subset C(r)$. If a unit tetrahedron contained in $C(r)$ has a vertex lying in the interior of $C(r)$, then the vertex is called an *interior vertex* in $C(r)$. A unit tetrahedron inscribed in $C(r)$ has no interior vertex in $C(r)$.

Lemma 1. *If a unit tetrahedron in $C(r)$ has an interior vertex, then it is equivalent in $C(r)$ to a unit tetrahedron contained in the interior* Int $C(r)$ *of $C(r)$, and hence equivalent in $C(r)$ to a unit tetrahedron contained in a $C(s)$, for some $s < r$.*

Proof. Note that (1) if a triangle has three vertices on the boundary of $C(r)$, then a rotation (through arbitrary small angle) around an edge sends one vertex into Int $C(r)$, and (2) if a line segment L has both endpoints on the boundary of $C(r)$, then a rotation of L (through arbitrary small angle) around the midpoint of L in the vertical plane containing L sends the both endpoints into Int $C(r)$. Applying these (1)(2), the lemma is obtained. $\square$

By applying a standard technique on convergence, the next corollary is obtained.

Corollary 1. *Every unit tetrahedron T contained in $C(r)$ is equivalent in $C(r)$ to a unit tetrahedron inscribed in some $C(s)$, $s \le r$.* $\square$

For every point $P = (a, b, c) \in \mathbb{R}^3$, its z-coordinate c is denoted by $z(P)$, and the orthogonal projection of P on the xy-plane (that is, the point $(a, b, 0)$) is denoted by $\bar{P}$. For a unit tetrahedron $T \subset \mathbb{R}^3$, let $\bar{T}$ denote the orthogonal projection of T into the xy-plane.

The next is the key lemma in this paper.

Lemma 2. *Let $T = ABCD \subset C(r)$ be an inscribed unit tetrahedron such that*

$$z(A) \le z(B) \le z(C) \le z(D). \tag{1}$$

If $\bar{A} = \bar{D}$ then $\bar{T}$ is an isosceles triangle with base $\bar{B}\bar{C}$, otherwise, $\bar{A}\bar{D} \parallel \bar{B}\bar{C}$ (and hence $\bar{T}$ is an isosceles trapezoid).

Proof. Let M be the midpoint of AD. Let H be the plane that perpendicularly bisects the line segment AD. Then $B, C, M \in H$. If $\bar{A} = \bar{D}$, then H is a horizontal plane, and since $MB = MC$, we have $\bar{A}\bar{B} = \bar{A}\bar{C}$. Thus, $\bar{T}$ is an isosceles triangle with base $\bar{B}\bar{C}$.

Suppose $\bar{A} \neq \bar{D}$. Then H is an inclined plane. We may suppose that AD is parallel to the xz-plane and $M = (0, -h, 0)$ for some $h > 0$. Then the section of $\partial C(r)$ by H is an ellipse Ω with center at $(0, 0, 0)$ whose major axis lies on the xz-plane and whose minor axis lies on the y-axis, see Figure 1 left. The two vertices B, C of T lie on this ellipse. Since $1 = AB^2 = AM^2 + MB^2 = 1/4 + MB^2$, we have $MB = \sqrt{3/4}$, and similarly $MC = \sqrt{3/4}$. Hence the two vertices B, C also lie on a circle Γ on H with center M, radius $\sqrt{3/4}$, see Figure 1 right. Thus B, C are intersection points of the ellipse Ω and the circle Γ. If Ω and Γ intersect only in two points, then it is obvious that BC is parallel to the xz-plane, and hence we have $\bar{B}\bar{C} \parallel \bar{A}\bar{D}$.

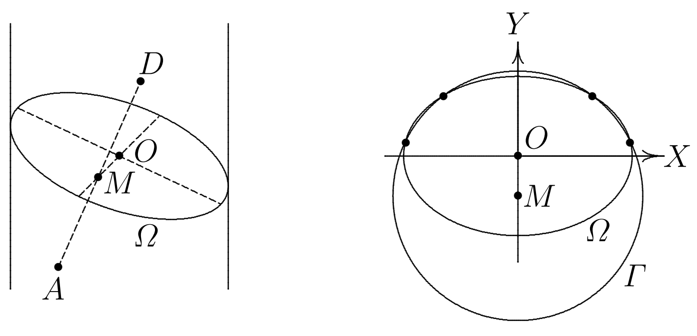

Figure 1 The ellipse Ω and the circle Γ

Suppose that $\bar{B}\bar{C}$ and $\bar{A}\bar{D}$ are not parallel (that is, BC is not parallel to the xz-plane). Then Ω and Γ must intersect in four points. Let us introduce rectilinear coordinate system (X, Y) on the inclined plane H as follows: the Y-axis is the same as the y-axis of $\mathbb{R}^3$, and the X-axis is the line of the major axis of Ω, pointing either direction, see Figure 1 right. Since BC is not parallel to the X-axis, the midpoint N of BC does not lie on the Y-axis. Now, Ω and Γ are represented by the equations:

$$\frac{X^2}{a^2} + \frac{Y^2}{r^2} = 1 \tag{2}$$

$$X^2 + (Y + h)^2 = 3/4 \tag{3}$$

where $a > 0$ is the half length of the major axis of Ω, and r (the radius of the cylinder) is the half length of the minor axis of Ω. Let θ be the angle between AD and the xy-plane. Since the minimum width of T (i.e., the minimum distance of a pair of parallel planes between which T can lie) is $1/\sqrt{2}$ (see, e.g., [8],[4]), and since T lies between the planes $z = z(A)$ and $z = z(D)$, it follows that $\theta \geq \pi/4$. Then, since H perpendicularly bisects AD, the angle $\pi/2 - \theta$ between the X-axis and the x-axis is at most $\pi/4$. Therefore, $r/a = \cos(\pi/2 - \theta) \geq \cos(\pi/4) = 1/\sqrt{2}$. Thus $a^2/r^2 \leq 2$.

By eliminating X from (2) and (3), we have

$$(a^2 - r^2)Y^2 - 2r^2hY - r^2(h^2 + a^2 - 3/4)r^2 = 0. \tag{4}$$

Since Ω and Γ intersect in four points, the equation (4) must have two solutions, say, α, β. Since B, C have different Y-coordinates (otherwise, we have $\bar{B}\bar{C} \parallel \bar{A}\bar{D}$), we may suppose the B has Y-coordinate α, and C has Y-coordinate β. Then the midpoint N of BC has Y-coordinate $(\alpha + \beta)/2$. Since

$$\alpha + \beta = \frac{2r^2h}{a^2 - r^2} = \frac{2h}{a^2/r^2 - 1} \geq 2h,$$

we have $(\alpha+\beta)/2 \geq h$. Thus the Y-coordinate (the y-coordinate) of N is at least h. Then $\bar{N}$ (which is the midpoint of $\bar{B}\bar{C}$) has the y-coordinate at least h. Since $\bar{N}$ does not lie on the y-axis (for otherwise, $\bar{B}\bar{C} \parallel \bar{A}\bar{D}$), we have $O\bar{N} > h$. This implies $O\bar{N} > O\bar{M}$. However, in the two chords $\bar{B}\bar{C}$, $\bar{A}\bar{D}$ of the circle in the xy-plane with center O and radius r, we have $(\bar{B}\bar{C})^2 = 1 - (z(C) - z(D))^2$, $(\bar{A}\bar{D})^2 = 1 - (z(D) - z(A))^2$. Hence (1) implies that $\bar{B}\bar{C} \geq \bar{A}\bar{D}$, which implies that $O\bar{N} \leq O\bar{M}$, a contradiction. $\square$

3. Radius of a cylinder with an inscribed tetrahedron

Lemma 3. *Let $T = ABCD$ be a unit tetrahedron inscribed in $C(r)$ and suppose that $z(A) \le z(B) \le z(C) \le z(D)$. Let θ be the angle between the line AD and the horizontal plane. Then $\pi/4 \le \theta \le \pi/2$ and*

$$r^2 = f(\theta) := \frac{9}{32} - \frac{1}{32}(\sin 2\theta)^2. \tag{5}$$

Proof. Let M be the midpoint of AD and N be the midpoint of BC. Then $\bar{M}$ is the midpoint of $\bar{A}\bar{D}$ and $\bar{N}$ is the midpoint of $\bar{B}\bar{C}$. Since $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are vertices of an isosceles trapezoid (isosceles triangle if $\bar{A} = \bar{D}$) with base $\bar{B}\bar{C}$ by Lemma 2, we have $\bar{M}\bar{N} \perp \bar{B}\bar{C}$. Hence MN is perpendicular to the vertical plane containing BC. Therefore, MN is horizontal. Hence, if the angle between AD and the horizontal plane is θ, then the angle between BC and the horizontal plane is $\pi/2 - \theta$. Since $\theta \ge \pi/2 - \theta$ by the assumption $z(A) \le z(B) \le z(C) \le z(D)$, we have $\pi/4 \le \theta \le \pi/2$.

To show (5), first consider the case $\theta = \pi/2$. In this case, AD is vertical, $\bar{A} = \bar{D}$, and isosceles triangle $\bar{A}\bar{B}\bar{C}$ is horizontally inscribed in $C(r)$. Since $\bar{B}\bar{C} = 1$, and the height of the isosceles triangle $\bar{A}\bar{B}\bar{C}$ is equal to $\bar{M}\bar{N} = MN = 1/\sqrt{2}$, its circumradius is equal to $3/\sqrt{32}$, and hence $r^2 = 9/32 = f(\pi/2)$.

Now, suppose $\pi/4 \le \theta < \pi/2$. We may suppose that M, N lies on the y-axis on $\mathbb{R}^3$. Since $MN = 1/\sqrt{2}$, we may further put

$$M = (0, -t, 0), \quad N = (0, \frac{1}{\sqrt{2}} - t, 0). \tag{6}$$

Then, $r^2 = OM^2 + M\bar{A}^2 = ON^2 + N\bar{B}^2$, where $O = (0,0,0)$, the origin. Since $M\bar{A}^2 = \frac{1}{4}\cos^2\theta$ and $N\bar{B}^2 = \frac{1}{4}\sin^2\theta$, we have

$$t^2 + \frac{1}{4}\cos^2\theta = \left(\frac{1}{\sqrt{2}} - t\right)^2 + \frac{1}{4}\sin^2\theta,$$

from which we have

$$t = \frac{1}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}\sin^2\theta. \tag{7}$$

Therefore

$$\begin{aligned} r^2 &= \left(\frac{1}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}\sin^2\theta\right)^2 + \frac{1}{4}\cos^2\theta \\ &= \frac{9}{32} - \frac{1}{32}(\sin 2\theta)^2. \end{aligned}$$

□

Corollary 2. *A unit tetrahedron can be inscribed in $C(r)$ if and only if $\frac{1}{2} \leq r \leq \frac{3\sqrt{2}}{8}$.*

Proof. If a unit tetrahedron can be inscribed in $C(r)$, then since $f(\theta)$ of (5) is strictly monotone increasing in $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$, we have $f(\frac{\pi}{4}) \leq r^2 \leq f(\frac{\pi}{2})$, and $\frac{1}{2} \leq r \leq \frac{3\sqrt{2}}{8}$.

Conversely, if $\frac{1}{2} \leq r \leq \frac{3\sqrt{2}}{8}$, then we can determine $\theta\,(\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2})$ by (5). Determine t by (7), and take M, N as in (6). Then we can take a line in the plane $y = -t$ that passes through M and makes angle θ with the horizontal line in this plane. Let A, D be the points on this line at distance $1/2$ from M. Similarly, we can take B, C in the plane $y = 1/\sqrt{2} - t$. Then $ABCD$ is a unit tetrahedron inscribed in $C(r)$. □

Corollary 3. *If T is a unit tetrahedron contained in $C(1/2)$, then T is inscribed in $C(1/2)$, and $\bar{T}$ is a square.* □

4. Proof of Theorem 1

Proof. Let $T_0 \subset C(r)$ be a unit tetrahedron. By Corollary 1 and Lemma 2, T_0 is equivalent in $C(r)$ to a unit tetrahedron T inscribed in $C(s)$, $s \leq r$, whose projection $\bar{T}$ is an isosceles trapezoid (or an isosceles triangle). Using the same notations as in Lemma 3, if $\theta > \pi/4$, then, since $f(\theta)$ is monotone increasing in $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$, we can rotate T around MN with decreasing the circumradius of $\bar{T}$ till θ becomes $\pi/4$. Hence, we can move T_0 within $C(r)$ so that it becomes inscribed in $C(1/2)$.

Let T_1, T_2 be a pair of unit tetrahedra inscribed in $C(1/2)$. Then $\bar{T}_1, \bar{T}_2$ are squares by Corollary 3, and in both T_1, T_2, their angle θ must be equal to $\pi/4$. Hence, by a suitable rotation of T_1 around the z-axis and a suitable translation within $C(r)$, we can superpose T_1 to T_2. Thus, every pair of unit tetrahedra in $C(r)$ are equivalent in $C(r)$. □

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