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LOGARITHMIC-TYPE FUNCTIONS OF THE DIFFERENTIAL OPERATOR

By

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Abstract. The logarithmic function and its related functions of the differential operator d/dx are defined by extending the framework of Mikusiński's operational calculus. The operation of $\varphi(d/dx)$ on a function $F(x)$, which vanishes for $x < 0$, is expressed as a convolution of a distribution $\Omega(x)$ and $F'(x)$. For various examples of $\varphi(d/dx)$, the explicit expressions for $\Omega(x)$ are found and their properties are investigated. Extension to the several-variable case is also considered.

1. Introduction

Recently, in order to find models of deformed canoninal commutation relations, Asada[4] has studied non-integer powers of differentiation (fractional calculus) and the logarithmic differentiation. His analysis is mainly based on the Borel transformation [3], but he has considered the logarithmic differentiation also in some other ways.

Let x be a real variable; we write the differential operator d/dx as D. Let $F(x)$ be an arbitrary C^1-class function whose support is included in $\mathbb{R}_+ \equiv \{x|\ x \geqq 0\}$. Then, according to Asada, we have

$$(\log D)F(x) = -\gamma F(x) - \int_0^x dy\, \log(x-y)F'(y), \tag{1.1}$$

where γ is Euler's constant. If $F(x)$ has a finite jump at $x = 0$, (1.1) still remains valid by understanding that $F'(x)$ contains a $\delta(x)$ (Dirac measure) term.

We can derive (1.1) from the well-known formula for a non-integer order "derivative" (see next section)

$$D^{-\alpha}F(x) = \frac{1}{\Gamma(\alpha)} \int_0^x dy\, (x-y)^{\alpha-1}F(y) \tag{1.2}$$

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with $\Re\alpha > 0$. We differentiate (1.2) with respect to $-\alpha$ and take the limit of $\alpha \to 0$. When differentiated, two terms arise according to the Leibniz rule, and both are divergent as $\alpha \to 0$. The divergent parts of both terms cancel, and the remainder yields the right-hand side of (1.1).

As shown by Asada [4], it is not difficult to derive the following formulae from (1.1):

$$(\log D)x^t\theta(x) = \Big(-\log x - \gamma + \sum_{n=1}^{\infty} \frac{t}{n(n+t)}\Big) x^t\theta(x), \tag{1.3}$$

$$(\log D)(\log x)^m\theta(x) = \Big(-(\log x)^{m+1} - \gamma(\log x)^m - \sum_{k=0}^{m-1} \frac{(-1)^{m-k} m!\, \zeta(m-k+1)}{k!} (\log x)^k \Big)\theta(x), \tag{1.4}$$

where $\zeta(s)$ denotes Riemann's zeta function and $\theta(x)$ stands for the Heaviside step function, that is, $\theta(x) = 1$ for $x \geqq 0$, $\theta(x) = 0$ for $x < 0$. We have explicitly written $\theta(x)$ for clarity.

Now, as is well known, Mikusiński [6] made mathematical justification of Heaviside's operational calculus on the basis of the convolution of functions whose support is included in $\mathbb{R}_+$. He succeeded in defining various functions of D, but $\log D$ cannot be defined as an operator in the sense of his theory.

In the present paper, we extend the concept of the operator of Mikusiński's operational calculus so as to include Asada's formula (1.1). In this extended framework, various logarithmic-type functions of D can be defined. We investigate their properies and calculate explicit formulae for simple cases.

The present paper is organized as follows. In Sec.2, we propose a definition of logarithmic-type functions of D in terms of convolution. In Sec.3, we calculate the defining function of $(\log D)^m$, m being a positive integer. In Sec.4, its generating function is discussed. In Sec.5, $(\log D)^\beta$ for β complex and $\log(\log D)$ are considered. In Sec.6, we show that it is possible to discuss logarithmic-type functions of D in the Mikusiński framework if we consider a commutator. Extension to the several-variable case is discussed in the final section.

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2. Extention of Mikusiński's operator

Mikusiński's operational calculus [6] is based on the commutative algebra $\mathcal{C}$; it is the totality of continuous functions defined in $\mathbb{R}_+$ and its product is defined by the convolution on $\mathbb{R}_+$. According to Titchmarsh's theorem, $\mathcal{C}$ has no zero divisor. Hence $\mathcal{C}$ can be extended to a field, $\mathcal{Q}$, of quotients. An operator is defined as an element of $\mathcal{Q}$. It can be shown that all usual operations in the functional analysis are transcribed into the corresponding ones in $\mathcal{Q}$.

A "function of the differential operator", denoted by $\varphi(D)$,[*1] is given by

$$\varphi(D)F(x) = \int_0^x dy\, \Phi(x-y)F(y), \tag{2. 1}$$

where $\Phi(x)$ and $F(x)$ are elements of $\mathcal{Q}$. Mikusiński calculated the explicit expressions for $\Phi(x)$ corresponding to various functions $\varphi(D)$, such as rational functions of D, functions expandable into a power series of D^{-1}, non-integer powers of rational functions of D, various kinds of exponential functions of D, etc. However, it is not possible to construct $\Phi(x)$ corresponding to $\log D$. In order to include $\log D$ and its related functions, it is necessary to deform (2.1) slightly.

In Mikusiński's operational calculus, it is essential that $\Phi(x)$ and $F(x)$ are treated in a symmetric way. But since our purpose is to define $\varphi(D)$ by $\Phi(x)$, we need not adhere to the symmetric treatment. By giving up the symmetry between $\Phi(x)$ and $F(x)$, we can give a definition of the function of D in a more flexible way.

We propose to define $\varphi(D)$ by the formula

$$\varphi(D)F(x) \equiv \int_{-0}^x dy\, \Omega(x-y)F'(y). \tag{2. 2}$$

Here, $F(x)$ is a function of $\mathcal{C}^1$-class in $\mathbb{R}_+$; since its support is included in $\mathbb{R}_+$, we may write $F(x) = F(x)\theta(x)$. The symbol -0 means to take the limit 0 from the $x<0$ side; hence $F(-0) = 0$ always. If $F(+0)$, i.e., the $x \to 0$ limit of $F(x)$ from the $x > 0$ side is nonvanishing and equal to a finite value, we understand that $F'(x)$ contains $F(+0)\delta(x)$. As for $\Omega(x)$, whose support is, of course, included in $\mathbb{R}_+$, we suppose, for a moment, that it is a function of $\mathcal{C}^1$-class for $x > 0$.

By integrating (2.2) by parts, we obtain

$$\varphi(D)F(x) = \Big[\Omega(x-y)F(y)\Big]_{-0}^x + \int_{-0}^x dy\, \Omega'(x-y)F(y). \tag{2. 3}$$

[*1] Mikusiński used a symbol s instead of D.

If the first term of the right-hand side vanishes, then (2.3) reduces to (2.1) by setting $\Omega' = \Phi$. Since $F(-0) = 0$, the first term vanishes if $\Omega(+0) = 0$. If $\Omega(+0) \neq 0$ is a definite finite value, we redefine $\Omega(x) - \Omega(+0)$ as $\Omega(x)$ so that the situation reduces to the case of $\Omega(+0) = 0$. On the other hand, if $\Omega(+0)$ is not a definite finite value, (2.2) no longer coincides with (2.1), but becomes its *finite part*.

If we set $F(x) = \theta(x)$ in (2.2), we have

$$\varphi(D)\theta(x) = \Omega(x). \tag{2.4}$$

In particular, if $\varphi(x) = 1$, then $\Omega(x) = \theta(x)$; we thus see that (2.2) is a representation more natural than (2.1).

Now, as is known in the theory of distributions [7], a convolution of two distributions is well defined if at least one of them has a compact support. Hence, in (2.2), we can regard $\Omega(x)$ as a distribution. On the other hand, according to the structure theorem of the distribution, any distribution can be represented as a finite order derivative of a continuous function. Hence $\Phi(x)$ in the Mikusiński theory can be identified with a distribution.

The most typical example in which $\Omega(+0)$ does not exist is the case of $\varphi(D) = \log D$. Identifying D^{-1} with Mikusiński's integration operator l, we consider

$$D^{-\alpha}F(x) = \int_{-0}^{x} dy \, \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)} F(y). \tag{2.5}$$

Since $\lim_{x\to 0} x^\alpha/\Gamma(\alpha+1) = 0$ for $\Re\alpha > 0$, (2.3) implies

$$D^{-\alpha}F(x) = \int_{-0}^{x+0} dy \, Y_{\alpha+1}(x-y)F'(y). \tag{2.6}$$

Here we have employed Schwartz's pseudofunction [7] defined by

$$\begin{aligned} Y_\lambda(x) &= \mathrm{Pf.}\frac{x^{\lambda-1}}{\Gamma(\lambda)}\theta(x) \quad \text{for } \lambda \neq 0, -1, -2, \cdots \\ &= \delta^{(n)}(x) \qquad \text{for } \lambda = -n = 0, -1, -2, \cdots , \end{aligned} \tag{2.7}$$

where Pf. means a finite part. As a distribution, $Y_\lambda(x)$ can be analytically continued to the whole complex plane and give an entire function of λ. Hence (2.6) is meaningful for any value of α. Especially, for $-\alpha = n = 0, 1, 2, \ldots$, the right-hand side of (2.6) correctly reproduces the nth order derivative of $F(x)$; thus the identification of D^{-1} with the integration operator l is reasonable.

Differentiating both sides of (2.6) with respect to $-\alpha$ and setting $\alpha = 0$, we can define $\log D$. Then, without encountering divergent terms, we obtain

$$(\log D)F(x) = \int_{-0}^{x} dy \, [-\log(x-y) - \gamma]F'(y), \tag{2.8}$$

which is nothing but (1.1). In particular,

$$(\log D)\theta(x) = (-\log x - \gamma)\theta(x). \tag{2.9}$$

Furthermore, for $\alpha > 0$, we have

$$(\log(D+\alpha))\theta(x) = \Big(\log\alpha + \int_{\alpha x}^{\infty} dt\, \frac{e^{-t}}{t}\Big)\theta(x), \tag{2.10}$$

because $[(\log(D+\alpha) - \log D]\theta(x)$ is given by

$$(\log(1+\alpha D^{-1}))\theta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\alpha^n D^{-n}\theta(x) = \int_0^x dy\, \frac{1-e^{-\alpha y}}{y}\theta(x). \tag{2.11}$$

Asada caluclated (2.8) for $F(x) = e^{\alpha x}\theta(x)$[*2] ; comparing his formula with (2.10), we find that the following interesting relation is seen to hold:

$$(\log(D+\alpha))\theta(x) = e^{-\alpha x}(\log D)e^{\alpha x}\theta(x). \tag{2.12}$$

3. Positive-integer power of $\log D$

In this section, we consider the case of $\varphi(D) = (\log D)^m$, where m is a positive integer. Of course, it is defined by differentiating (2.6) m times with respect to $-\alpha$ and then setting $\alpha = 0$. But, because analytic continuation preserves any analytic relation, we can calculate it successively by the recurrence formula $(\log D)^{m+1}\theta(x) = (\log D)[(\log D)^m\theta(x)]$.

First, we set $L(x) \equiv \log x + \gamma$, and rewrite (1.4) in terms of $L(x)$ in order to simplify our calculation. It is easy to show

$$\begin{aligned}(\log D)&L^m(x)\theta(x)\\ &= \Big(-L^{m+1}(x) - \sum_{k=0}^{m-1} \frac{(-1)^{m-k}m!\,\zeta(m-k+1)}{k!}L^k(x)\Big)\theta(x)\end{aligned} \tag{3.1}$$

by means of mathematical induction. Then, by using (3.1), we calculate

[*2] Private communication.

$(\log D)^m\theta(x)$ successively:

$$(\log D)^1\theta(x) = -L(x)\theta(x), \tag{3.2}$$

$$(\log D)^2\theta(x) = \Big(L^2(x) - \zeta(2)\Big)\theta(x), \tag{3.3}$$

$$(\log D)^3\theta(x) = \Big(-L^3(x) + 3\zeta(2)L(x) - 2\zeta(3)\Big)\theta(x), \tag{3.4}$$

$$(\log D)^4\theta(x) = \Big(L^4(x) - 6\zeta(2)L^2(x) + 8\zeta(3)L(x) - 6\zeta(4) + 3(\zeta(2))^2\Big)\theta(x), \tag{3.5}$$

$$(\log D)^5\theta(x) = \Big(-L^5(x) + 10\zeta(2)L^3(x) - 20\zeta(3)L^2(x) + 30\zeta(4)L(x) - 24\zeta(5) - 15(\zeta(2))^2L(x) + 20\zeta(2)\zeta(3)\Big)\theta(x). \tag{3.6}$$

For general m, we find

$$(\log D)^m\theta(x) = \Big(\sum_{l=0}^{[m/2]} \frac{1}{l!} \sum_{k_1 \geqq 2, \cdots, k_l \geqq 2} (-1)^{m-|k|-l} \frac{m! \prod_{j=1}^{l} \zeta(k_j)}{\prod_{j=1}^{l} k_j \cdot (m-|k|)!} L^{m-|k|}(x)\Big)\theta(x), \tag{3.7}$$

where $|k| \equiv \sum_{j=1}^{l} k_j$. The upper limit of the sums over k_j's is $|k| \leqq m$, but since this is automatically guaranteed by the existence of $(m-|k|)!$ in the denominator, we have omitted to write so explicitly.

Proof of (3.7). We employ mathematical induction with respect of m. Its validity for $m = 1$ is self-evident. Hence assuming the validity of (3.7), we calculate $(\log D)[(\log D)^m\theta(x)]$. It is sufficient to consider each part characterized by the number l of zeta function factors. Because the right-hand side of (3.1) is linear with respect to zeta functions, the l part of $(\log D)[(\log D)^m\theta(x)]$ consists of two parts: the part arising from the l part of $(\log D)^m\theta(x)$ and the first term of (3.1) and the part arising from the $l-1$ part of $(\log D)^m\theta(x)$ and the remainder of (3.1). The former is simply $-L(x)$ times (3.7). The latter is given by

$$\frac{1}{(l-1)!} \sum_{k_1 \geqq 2, \cdots, k_{l-1} \geqq 2} (-1)^{m-|k|'-l+1} \frac{m! \prod_{j=1}^{l-1} \zeta(k_j)}{\prod_{j=1}^{l-1} k_j \cdot (m-|k|')!} \cdot \sum_{p=2}^{m-|k|'+1} (-1)^p \frac{(m-|k|')!\, \zeta(p)}{(m-|k|'-p+1)!} L^{m-|k|'-p+1}(x)\theta(x), \tag{3.8}$$

where $|k|' \equiv \sum_{j=1}^{l-1} k_j = |k| - k_l$. We set $p = k_l$ and symmetrize the expression of (3.8) with respect to $k_1, \cdots, k_{l-1}$ and k_l after cancellation of $(m-|k|')!$. That is, we make the following rewriting:

$$\sum_{k_1,\cdots,k_{l-1}} \sum_{k_l} \frac{S(k_1,\cdots,k_l)}{\prod_{j=1}^{l-1} k_j} = \frac{1}{l} \sum_{k_1,\cdots,k_l} \frac{|k| S(k_1,\cdots,k_l)}{\prod_{j=1}^{l} k_j}, \tag{3.9}$$

where $S(k_1, \cdots, k_l)$ is a totally symmetric quantity. By reducing to a common denominator, we see that the sum of both parts coincides with the l term of (3.7) for $(\log D)^{m+1}\theta(x)$. □

It is instructive to confirm that (3.7) is consistent with the following fundamental property of the logarithmic function:

$$(\log D + a)^m F(x) = (\log e^a D)^m F(x), \tag{3.10}$$

where a is a constant.

Direct check of (3.10). According to the definition (2.2), we write

$$(\log D)^m F(x) \equiv \int_{-0}^{x} dy\, \Omega_m(x-y) F'(y), \tag{3.11}$$

where $\Omega_m(x)$ is given by (3.7). The left-hand side of (3.10) is

$$(\log D + a)^m F(x) = \int_{-0}^{x} dy \sum_{p=0}^{m} \frac{m!}{p!(m-p)!} a^p \Omega_{m-p}(x-y) F'(y) \tag{3.12}$$

On the other hand, the right-hand side of (3.10) is

$$(\log D')^m F(e^a x') = \int_{-0}^{x'} dy'\, \Omega_m(x'-y') e^a F'(e^a y'), \tag{3.13}$$

where we have set $x \equiv e^a x'$, so that $e^a D \equiv D'$. By the transformation $y' = e^{-a} y$, (3.13) is rewritten as

$$(\log e^a D)^m F(x) = \int_{-0}^{x} dy\, \Omega_m(e^{-a}(x-y)) F'(y). \tag{3.14}$$

Hence, to verify (3.10), it is sufficient to prove

$$\Omega_m(e^{-a}x) = \sum_{p=0}^{m} \frac{m!}{p!(m-p)!} a^p \Omega_{m-p}(x). \tag{3.15}$$

The left-hand side of (3.15) is equal to the right-hand side of (3.11) with the replacement of $L(x)$ by $L(x) - a$ in (3.7). After expanding $(L(x) - a)^{m-|k|}$, we arrange the resultant expression in powers of a^p. Then, by making the simple rewriting

$$\frac{m!}{(m-|k|)!}\cdot\frac{(m-|k|)!}{p!(m-|k|-p)!}=\frac{m!}{p!(m-p)!}\cdot\frac{(m-p)!}{(m-p-|k|)!}, \tag{3.16}$$

we find that it coincides with the right-hand side of (3.15). □

4. Generating function

By definition, $Y_{-\alpha+1}(x)$ (see (2.6) and (2.7)) should be the generating function of (3.7). In this section, we confirm that the power series constructed by (3.7) indeed reproduces $Y_{-\alpha+1}(x)$:

$$J(x;\alpha)\equiv\sum_{m=0}^{\infty}\frac{\alpha^m}{m!}(\log D)^m\theta(x)=e^{\alpha\log D}\theta(x)=D^{\alpha}\theta(x)=Y_{-\alpha+1}(x). \tag{4.1}$$

Proof of (4.1). Substituting (3.7) into $J(x;\alpha)$, we have

$$J(x;\alpha) = \sum_{m=0}^{\infty}\alpha^m\sum_{l=0}^{[m/2]}\frac{1}{l!}\sum_{k_1\geqq 2,\cdots,k_l\geqq 2}(-1)^{m-|k|-l}\frac{\prod_{j=1}^{l}\zeta(k_j)}{\prod_{j=1}^{l}k_j\cdot(m-|k|)!}L^{m-|k|}(x)\theta(x). \tag{4.2}$$

Setting $m-|k|=n$, we change the order of summations; then

$$J(x;\alpha)=\sum_{n=0}^{\infty}\frac{(-\alpha L(x))^n}{n!}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\sum_{k_1\geqq 2,\cdots,k_l\geqq 2}\frac{\prod_{j=1}^{l}\zeta(k_j)}{\prod_{j=1}^{l}k_j}\alpha^{|k|}\theta(x). \tag{4.3}$$

Since $|k|=\sum_{j=1}^{l}k_j$, (4.3) becomes

$$\begin{aligned}J(x;\alpha)&=\sum_{n=0}^{\infty}\frac{(-\alpha L(x))^n}{n!}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\Big(\sum_{k\geqq 2}\frac{\zeta(k)}{k}\alpha^k\Big)^l\theta(x)\\&=\exp\Big(-\alpha L(x)-\sum_{k\geqq 2}\frac{\zeta(k)}{k}\alpha^k\Big)\theta(x).\end{aligned} \tag{4.4}$$

Hence noting $L(x)=\log x+\gamma$ and using the formula

$$\log\Gamma(-\alpha+1)=\gamma\alpha+\sum_{k\geqq 2}\frac{\zeta(k)}{k}\alpha^k, \tag{4.5}$$

we find that $J(x;\alpha)$ is equal to $[\Gamma(-\alpha+1)]^{-1}x^{-\alpha}\theta(x)$. □

Digression Looking at (3.3)-(3.6), we become aware of the fact that the total sum of the coefficients in each formula for $m \geqq 2$ is always zero. One cannot believe that this fact is merely accidental. That is, it is natural to conjecture that the expression that is obtained by formally setting $\zeta(k)$ and $L(x)$ equal to 1 in (3.7) for $m \geqq 2$ is always equal to 0. That is, we should have an identity

$$\sum_{l=0}^{[m/2]} \frac{1}{l!} \sum_{k_1 \geqq 2, \cdots, k_l \geqq 2} (-1)^{m-|k|-l} \frac{m!}{\prod_{j=1}^{l} k_j \cdot (m-|k|)!} = 0 \qquad \text{for } m \geqq 2. \tag{4.6}$$

This is indeed true. Direct proof is supposed to be very difficult, but we can prove (4.6) very simply if we employ its generating function.

Proof of (4.6). As is seen from (4.4), the generating function of the left-hand side of (4.6) is written as

$$\exp\Big(-\alpha - \sum_{k \geqq 2} \frac{\alpha^k}{k} \Big) = e^{\log(1-\alpha)} = 1 - \alpha. \tag{4.7}$$

Thus it contains no terms nonlinear with respect to α. Therefore, (4.6) holds. □

5. Complex power of $\log D$

First, we consider $(\log D)^{-m}\theta(x)$, m being a positive integer. It is obtained by integrating $Y_{\alpha+1}(x)$ with respect to α m times and then setting $\alpha = 0$. Therefore, the following "ν function" [5] becomes important:

$$\nu(x) \equiv \int_0^\infty dt \, \frac{x^t}{\Gamma(t+1)}. \tag{5.1}$$

Now, rewriting (1.3) as

$$\begin{aligned}(\log D)x^t\theta(x) &= \Big[-x^t(\log x + \gamma) + x^t \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} - \frac{1}{n+t+1} \Big) \Big] \theta(x) \\ &= \Big(-x^t \log x + x^t \frac{\Gamma'(t+1)}{\Gamma(t+1)} \Big) \theta(x),\end{aligned} \tag{5.2}$$

we find

$$(\log D) \frac{x^t}{\Gamma(t+1)} \theta(x) = -\frac{\partial}{\partial t} \frac{x^t}{\Gamma(t+1)} \theta(x). \tag{5.3}$$

By using (5.3) repeatedly, we obtain a beautiful result

$$(\log D)^m Y_{t+1}(x) = \Big(-\frac{\partial}{\partial t} \Big)^m Y_{t+1}(x). \tag{5.4}$$

Reversing (5.4), we should have

$$\left(-\frac{\partial}{\partial t}\right)^{-m} Y_{t+1}(x) = (\log D)^{-m} Y_{t+1}(x). \tag{5.5}$$

Here, $(-\partial/\partial t)^{-1}$ means the integration over t from t to $+\infty$ because $\Gamma(t) \sim t^t e^{-t}$ as $t \to +\infty$. Setting $t = 0$ in (5.5), therefore, we have

$$\begin{aligned}(\log D)^{-m}\theta(x) &= \int_0^\infty dt_1 \int_{t_1}^\infty dt_2 \cdots \int_{t_{m-1}}^\infty dt_m Y_{t_m+1}(x) \\ &= \frac{1}{(m-1)!}\int_0^\infty dt\, t^{m-1} Y_{t+1}(x) \\ &= \frac{1}{(m-1)!}(xD)^{m-1}\nu(x)\theta(x). \end{aligned} \tag{5.6}$$

As confirmed in (3.10), we know $\log D - \alpha = \log(e^{-\alpha}D) = \log(d/d(e^\alpha x))$. Hence, (5.6) implies

$$(\log D - \alpha)^{-m}\theta(x) = \frac{1}{(m-1)!}(xD)^{m-1}\nu(e^\alpha x)\theta(x). \tag{5.7}$$

We can extend the above result for $(\log D)^{-m}$ to $(\log D)^\beta$, β being a complex number. By noting the second line of (5.6), it is natural to define it by

$$(\log D)^\beta \theta(x) = \int_0^\infty dt\; Y_{-\beta}(t) Y_{t+1}(x). \tag{5.8}$$

Indeed, for $\beta = m$, we easily see from (2.7) and (5.4) that the right-hand side of (5.8) reduces to $(\log D)^m\theta(x)$, that is, the left-hand side of (5.8) for $\beta = m$.

From (5.8) together with[*3]

$$D^\alpha Y_{t+1}(x) = Y_{-\alpha+t+1}(x), \tag{5.9}$$

we obtain

$$D^\alpha (\log D)^\beta \theta(x) = \int_0^\infty dt\, Y_{-\beta}(t) Y_{-\alpha+t+1}(x). \tag{5.10}$$

Finally, differentiating (5.8) with respect to β and then taking the limit of $\beta \to 0$, we obtain the formula for $\log\log D$:

$$(\log\log D)\theta(x) = -\gamma\theta(x) + \int_0^\infty dt\; \log t \frac{\partial}{\partial t} Y_{t+1}(x). \tag{5.11}$$

This calculation is similar to the derivation of (1.1) from (1.2).

[*3] (5.9) is the generating-function version of (5.4).

6. Commutator representation

As emphasized in Section 2, in order to define the logarithmic-type functions of D, it is necessary to extend the framework of Mikusiński's operational calculus. However, if we consider only the commutator between a function of D and a function of x,[*4] we can work in the Mikusiński framework.

Let $f(x)$ be a C^1-class function. From (2.1), we obtain

$$[\varphi(D),\ f(x)]F(x) = -\int_{-0}^{x} dy\, \Phi(x-y)(f(x)-f(y))F(y). \tag{6.1}$$

If we integrate (6.1) by parts as in (2.3), the first term vanishes owing to the presence of the factor $f(x) - f(y)$, as long as the singularity of $\Omega(x-y)$ at $x = y$ is weaker than that of $(x-y)^{-1}$. This means that in (6.1) our finite-part definition coincides with that in the Mikusiński framework.

For $f(x) = x$, (6.1) becomes

$$[\varphi(D),\ x]F(x) = -\int_{-0}^{x} dy\, \Phi(x-y)(x-y)F(y). \tag{6.2}$$

We set

$$\Psi(x) \equiv D[\varphi(D),\ x]\theta(x) = D\varphi'(D)\theta(x). \tag{6.3}$$

Then, (6.2) with $F(x) = \theta(x)$ implies

$$\Psi(x) = -x\Phi(x). \tag{6.4}$$

From (6.1) and (6.4), we obtain

$$[\varphi(D),\ f(x)]F(x) = \int_{-0}^{x} dy\, \Psi(x-y)\frac{f(x)-f(y)}{x-y}F(y). \tag{6.5}$$

In particular, for $\varphi(D) = \log D$, we have $\Psi(x) = \theta(x)$; hence

$$[\log D,\ f(x)]F(x) = \int_{-0}^{x} dy\, \frac{f(x)-f(y)}{x-y}F(y). \tag{6.6}$$

It is interesting to compare (6.5) with the following formula concerning a commutator of "operators" [1].

Let A and B be two elements of a non-commutative algebra, such that A does not commute with $[A,\ B]$; for example, they are two generators of a free tensor algebra. For an analytic function $f(z)$, we can *formally* write

$$[f(A),\ B] = \frac{f(A_{\mathrm{L}}) - f(A_{\mathrm{R}})}{A_{\mathrm{L}} - A_{\mathrm{R}}}\,[A,\ B], \tag{6.7}$$

[*4] The idea is similar to the renormalization in quantum field theory.

where $A_{\rm L}$ and $A_{\rm R}$ denote the A lying in the left of $[A,\ B]$ and the A lying in the right of $[A,\ B]$, respectively, without regard to their positions written actually [2].

The basis of considering (6.7) is the following fact. From the definition of the commutator, we directly see that

$$[A^n,\ B] = \sum_{j=0}^{n-1} A_{\rm L}^j A_{\rm R}^{n-1-j}[A,\ B] = \frac{A_{\rm L}^n - A_{\rm R}^n}{A_{\rm L} - A_{\rm R}}\,[A,\ B]. \tag{6.8}$$

Hence, if $f(z)$ is a polynomial, (6.7) is valid. Therefore, it is natural to expect that the above statement is justifiable in a certain sense by expanding $f(A)$ into a formal power series. We point out that our present consideration provides a mathematical justification of (6.7) in the Mikusiński framework.

Setting $A = x$ and $B = \varphi(D)$ in (6.7) and using (6.2) with (6.4), we can make the following formal calculation:

$$\begin{aligned}[\varphi(D),\ f(x)]F(x) &= \frac{f(x_{\rm L}) - f(x_{\rm R})}{x_{\rm L} - x_{\rm R}}[\varphi(D),\ x]F(x) \\ &= \frac{f(x_{\rm L}) - f(x_{\rm R})}{x_{\rm L} - x_{\rm R}} \int_{-0}^{x} dy\, \Psi(x-y)F(y) \\ &= \int_{-0}^{x} dy\, \frac{f(x) - f(y)}{x - y}\Psi(x-y)F(y). \end{aligned} \tag{6.9}$$

Thus, (6.5) can be regarded as an integral representation of (6.7).

7. Several-variable case

In this section, we consider the case of a function of n variables $x_1, \cdots, x_n$. Our discussion is restricted only to a function $\varphi(|D|)$ of $|D| \equiv \sum_{j=1}^n D_j$, where $D_j \equiv \partial/\partial x_j$ [*5]. The operand function is denoted by $F(\boldsymbol{x}) \equiv F(x_1, \cdots, x_n)$. Since its support belongs to $\mathbb{R}_+^n$, we can write

$$F(\boldsymbol{x}) = F(\boldsymbol{x}) \prod_{j=1}^{n} \theta(x_j) = F(\boldsymbol{x})\theta(\xi), \tag{7.1}$$

where

$$\xi \equiv \min\{x_1, \cdots, x_n\}. \tag{7.2}$$

[*5] Extension to a function of $\sum_{j=1}^n \lambda_j D_j$ with $\lambda_j > 0$ is straightforward.

It is convenient to make the following transformation of the variables:

$$\overline{x} = \frac{\sum_{j=1}^{n} x_j}{n},$$
$$u_j = x_j - x_{j+1} \;\; (j = 1, \cdots, n-1); \tag{7.3}$$

conversely,

$$x_k = \overline{x} + \sum_{j=k}^{n-1} u_j - \frac{1}{n}\sum_{j=1}^{n-1} j u_j. \tag{7.4}$$

From (7.3), we obtain

$$D_k = \frac{1}{n}\frac{\partial}{\partial \overline{x}} - \frac{\partial}{\partial u_{k-1}} + \frac{\partial}{\partial u_k} \quad \left(\frac{\partial}{\partial u_0} \equiv \frac{\partial}{\partial u_n} \equiv 0\right), \tag{7.5}$$

so that

$$|D| = \sum_{k=1}^{n} D_k = \frac{\partial}{\partial \overline{x}}. \tag{7.6}$$

Thus, in this coordinate system, the problem essentially reduces to that in the one-variable case.

From (2.2), therefore, we obtain

$$\varphi(|D|)F(\boldsymbol{x}) = \int_{-0}^{\overline{x}} dy\; \Omega(\overline{x} - y)\frac{\partial}{\partial y}F(y - \overline{x} + \boldsymbol{x}), \tag{7.7}$$

where $F(y - \overline{x} + \boldsymbol{x})$ means $F(y - \overline{x} + x_1, \cdots, y - \overline{x} + x_n)$. In writing (7.7), we have made use of (7.4), that is,

$$y + \sum_{j=k}^{n-1} u_j - \frac{1}{n}\sum_{j=1}^{n-1} j u_j = y - \overline{x} + x_k. \tag{7.8}$$

As in the one-variable case, if we can adjust to have $\Omega(+0) = 0$, we can write

$$\varphi(|D|)F(\boldsymbol{x}) = \int_{-0}^{\overline{x}} dy\; \Phi(\overline{x} - y)F(y - \overline{x} + \boldsymbol{x}), \tag{7.9}$$

where $\Omega' = \Phi$.

Now, we consider the case $\varphi(|D|) = |D|^{-\alpha}$. From (7.7) and (2.6), we obtain

$$|D|^{-\alpha}F(\boldsymbol{x}) = \int_{-0}^{\overline{x}} dy\; Y_{\alpha+1}(\overline{x} - y)\frac{\partial}{\partial y}F(y - \overline{x} + \boldsymbol{x}). \tag{7.10}$$

If $F(\boldsymbol{x})$ is a function of $\overline{x}$ only apart from $\prod_j \theta(x_j) = \theta(\xi)$, (7.10) becomes

$$|D|^{-\alpha} F(\overline{x})\theta(\xi) = \int_{-0}^{\overline{x}} dy\, Y_{\alpha+1}(\overline{x}-y) \frac{\partial}{\partial y}[F(y)\theta(y-\overline{x}+\xi)]. \tag{7.11}$$

In particular,

$$|D|^{-\alpha}\theta(\xi) = Y_{\alpha+1}(\xi). \tag{7.12}$$

Differentiating (7.11) with respect to $-\alpha$ and setting $\alpha = 0$, we have

$$(\log|D|)F(\overline{x})\theta(\xi) = \int_{-0}^{\overline{x}} dy\, [-\log(\overline{x}-y)-\gamma] \frac{\partial}{\partial y}[F(y)\theta(y-\overline{x}+\xi)]. \tag{7.13}$$

In particular,

$$(\log|D|)\theta(\xi) = (-\log\xi - \gamma)\theta(\xi). \tag{7.14}$$

As for the commutator representation, from (7.7) and the definition of a commutator, we obtain

$$\begin{aligned}
&[\varphi(|D|),\ f(\boldsymbol{x})]F(\boldsymbol{x}) \\
&= -\int_{-0}^{\overline{x}} dy\, \Omega(\overline{x}-y)\Big[f(x)\frac{\partial}{\partial y}F(y-\overline{x}+\boldsymbol{x}) - \frac{\partial}{\partial y}\Big(f(y-\overline{x}+\boldsymbol{x})F(y-\overline{x}+\boldsymbol{x})\Big)\Big] \\
&= -\Big[\Omega(\overline{x}-y)\Big(f(\boldsymbol{x})-f(y-\overline{x}+\boldsymbol{x})\Big)F(y-\overline{x}+\boldsymbol{x})\Big]_{-0}^{\overline{x}} \\
&\qquad + \int_{-0}^{\overline{x}} dy\, \frac{\partial}{\partial y}\Omega(\overline{x}-y)\cdot\Big(f(\boldsymbol{x})-f(y-\overline{x}+\boldsymbol{x})\Big)F(y-\overline{x}+\boldsymbol{x}) \\
&= -\int_{-0}^{\overline{x}} dy\, \Phi(\overline{x}-y)\Big(f(\boldsymbol{x})-f(y-\overline{x}+\boldsymbol{x})\Big)F(y-\overline{x}+\boldsymbol{x}),
\end{aligned} \tag{7.15}$$

where we have assumed

$$\lim_{\varepsilon\to 0} \Omega(\varepsilon)[f(\boldsymbol{x})-f(\boldsymbol{x}-\varepsilon)] = 0. \tag{7.16}$$

Writing $\Psi(v) \equiv -v\Phi(v)$, we obtain the formula

$$\begin{aligned}
[\varphi(|D|),\ f(\boldsymbol{x})]F(\boldsymbol{x}) &= \int_{-0}^{\overline{x}} dy\, \Psi(\overline{x}-y)\frac{f(\boldsymbol{x})-f(y-\overline{x}+\boldsymbol{x})}{\overline{x}-y}F(y-\overline{x}+\boldsymbol{x}) \\
&= \int_{0}^{\overline{x}+0} dv\, \Psi(v)\frac{f(\boldsymbol{x})-f(\boldsymbol{x}-v)}{v}F(\boldsymbol{x}-v).
\end{aligned} \tag{7.17}$$

Here, $\Psi(v)$ is expressed as follows. Setting $f(\boldsymbol{x}) = x_j$ in (7.17), we find

$$\varphi'(|D|)F(\boldsymbol{x}) = [\varphi(|D|),\ x_j]F(\boldsymbol{x}) = \int_{0}^{\overline{x}+0} dv\, \Psi(v)F(\boldsymbol{x}-v). \tag{7.18}$$

Furthermore, for $F(\boldsymbol{x}) = \theta(\xi)$, it reduces to

$$\varphi'(|D|)\theta(\xi) = \int_0^{\xi} dv\, \Psi(v). \tag{7.19}$$

Since this is a function of ξ only as in the one-variable case, we find

$$\Psi(v) = (d/dv)\varphi'(d/dv)\theta(v). \tag{7.20}$$

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