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FRACTIONAL CALCULUS AND INFINITE ORDER DIFFERENTIAL OPERATORS

By

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Abstract. Leibniz rules of fractional order and logarithm of differentiations are presented. They provide infinite order differential operator expressions of fractional order and logarithm of differentiations. Including higher order cases, commutation relations involving fractional order and logarithm of differentiations are also studied. Special values of Riemann's zeta function appear in higher order commutation relations involving logarithm of differentiation.

1. Introduction

Since $I^n f(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)1} f(t)dt$ satisifies $\frac{d^n}{dx^n} I^n f(x) = f(x)$, the integral

$$I^a f(x) = \frac{1}{\Gamma(a)} \int_0^x (x-t)^{a-1} f(t)dt, \quad a > 0,$$

can be regarded as the a-th order indefinite integral. Simplified form of Abel's integral equation $\int_0^x \frac{y(t)}{\sqrt{x-t}} dt = F(x)$, $F(x)$ is given, is the first example of such integral transformation.

Since $I^a I^b = I^{a+b}$ and $\lim_{a\to 0} I^a = \mathrm{I}(= I^0)$, the identity map, $\{I^a | a \geq 0\}$ is a 1-parameter semigroup. Its generating operator A is given by

$$Af(x) = (\log x + \gamma) f(x) + \int_0^x \log\left(1 - \frac{t}{x}\right) f'(t)dt,$$

where γ is the Euler constant. We define logarithm of differentiation $\log\left(\frac{d}{dx}\right)$ by $-A$. It seems this operator did not take attentions of researchers.

The following are two definitions of fractional order differentiations;

$$\frac{d^{n-a} f(x)}{dx^{n-a}} = \frac{d^n}{dx^n} I^a f(x), \quad \frac{d^{n-a} f(x)}{dx^{n-a}} = I^a \left(\frac{d^n f(x)}{dx^n}\right).$$

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The first is called Riemann-Liouville's fractional derivative, and the second is called Caputo's fractional derivative. They are different. But although not widely noticed (cf.[1]), if we consider the domain and range of fractional order differentiation are the space of Mikusinski's operators ([10]), they coincide.

Fractional order differentiation and related calculus are called fractional calculus. It is convenient to the study of functions having singularity of the form x^{-a}. Mainly by this reason, fractional calculus is used in applied mathematics ([8], [9], [12]). On the other hand, the domain and range of fractional calculus remain unclear. This may be the reason why most of pure mathematicians are not interested in fractional calculus.

In this paper, assuming g is a Gevrey class function of index $\alpha < 1$, that is f is smooth and $|f^{(n)}(x)| \le M_x(n!)^\alpha$, the following Leibniz rules are derived.

$$\frac{d^a}{dx^a}(fg) = \frac{d^a f}{dx^a} g + \sum_{n=1}^{\infty} \frac{a(a-1)\cdots(a-n+1)}{n!} \frac{d^{a-n} f}{dx^{a-n}} \frac{d^n g}{dx^n},$$

$$\log\left(\frac{d}{dx}\right)(fg) = \left(\log\left(\frac{d}{dx}\right) f\right) g + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} I^n f \frac{d^n g}{dx^n}.$$

Here, $\frac{d^{a-n}}{dx^{a-n}}$ means I^{n-a}. If a is not a positive integer, these Leibniz rules are not symmetric in f and g. As for logarithm of differentiation, Nakanishi dicovered a symmetric Leibniz rule ([11]). But by using these asymmetries, if f is a Gevrey class function of index $\alpha < 1$, we obtain

$$\frac{d^a f(x)}{dx^a} = \frac{x^{-a}}{\Gamma(1-a)} \left(f(x) + \sum_{n=1}^{\infty} \frac{(-1)^n a x^n}{(n-a)n!} \frac{d^n f(x)}{dx^n} \right),$$

$$\log\left(\frac{d}{dx}\right) f(x) = -(\log x + \gamma) f(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n \cdot n!} \frac{d^n f(x)}{dx^n}.$$

These are the first main results in this paper. If f is an entire function, and $|f^{(n)}(x)| = O(r^n), n \to \infty$, $r < 1$, we also have

$$\log\left(\frac{d}{dx}\right) f(\log x) = \big(-(X+\gamma) + \mathfrak{d}_X\big) f(X)|_{X=\log x},$$

$$\mathfrak{d}_X = \sum_{k=1}^{\infty} (-1)^{k-1} \zeta(k+1) \frac{d^k}{dX^k}, \quad \zeta(k+1) = \sum_{n=1}^{\infty} n^{-(k+1)},$$

$$-\gamma + \mathfrak{d}_X = \left(\frac{d}{dt} \log(\Gamma(1+t))\right)\Big|_{t=\frac{d}{dX}}.$$

These expressions may relate fractional calculus and noncommutative field theory (NCFT), because the maximal order of differentiation in NCFT is infinite ([5], cf.[7]).

By Leibniz rules, we also have

$$\left[\frac{d^a}{dx^a}, x\right] = a\frac{d^{a-1}}{dx^{a-1}}, \quad \left[\log\left(\frac{d}{dx}\right), x\right] = I^1.$$

Suggested by these realtions, we use

$$\begin{aligned} \mathrm{H}_a &= \left\{ \sum_{n=1}^{\infty} c_n x^{an-1} \,\middle|\, \sum_{n=1}^{\infty} |c_n|^2 < \infty \right\}, \quad 0 < a < 1, \\ \mathrm{H}_{\log} &= \left\{ \sum_{n=0}^{\infty} c_n (\log x)^n \,\middle|\, \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\} \end{aligned}$$

as the Hilbert spaces having $\frac{d^a}{dx^a}$, $0 < a < 1$, and $\mathcal{R} = \log\left(\frac{d}{dx}\right) + \gamma + \log x$ as deformed annihilation operators. We take x^a and $\log x$ as the corresponding deformed creation operators. The Lie algebras $\mathfrak{g}_a$ and $\mathfrak{g}_{\log}$, generated by $\frac{d^a}{dx^a}$ and x^a, and by $\log\left(\frac{d}{dx}\right)$ and $\log x$ respectively, are projective limits of nilpotent Lie algebras. These suggest there might exist some relations between fractional calculus and nilpotent analysis (cf.[6]). We set

$$\mathrm{C}_a = \left[\frac{d^a}{dx^a}, x^a\right], \quad \mathrm{B}_{\log} = \left[\log\left(\frac{d}{dx}\right), \log x\right].$$

Then C_a is a p-Schatten class diagonal form operator and $\mathrm{B}_{\log} = \zeta(2)\mathrm{I} + N_{\log}$. Here $p > 1/(1-a)$ and $N_{\log}$ is a generalized nilpotent operator (cf.[14]). In general, $\overbrace{[\tfrac{d^a}{dx^a}, [\cdots [\tfrac{d^a}{dx^a},}^{p} \overbrace{[x^a, [\cdots, [}^{q} x^a, \mathrm{C}_a], \cdots]$ is an m-Schatten class operator if $m > 1/((p+1)(1-a)+q)$ and

$$[X_1, [\cdots, [X_m, \mathrm{B}_{\log}] \cdots] = (m+1)!\zeta(m+2)\mathrm{I} + N_{\log,m}.$$

Here X_i is either of $\log\left(\frac{d}{dx}\right)$ or $-\log x$ and $N_{\log,m}$ is a generalized nilpotent operator. Therefore, fractional order and logarithm of differentation provide deformations of canonical commutation relation (cf.[13]). These are the second main results in this paper.

This paper is organized as follows: §2 reviews fractional calculus. Alternative definition of fractional calculus by using extended Borel transfromation is also sketched (cf.[2]). §3 derives Leibniz rules and expresses fractional order and logarithm of differentiation as infinite order differential operators. Infinite order differential operator expressions of fractional calculus allows to consider fractional order and logarithm of differentations of functions defined on $\mathbb{R}$ and to investigate variable change of fractional calculus. These are explained in §4. §5 and §6 deal with higher order commutation relations involving fractional calculus. Higher order commutation relations involving logarithm of differentiation are derived in §7 and §8.

Acknowledgement. In [2], we have extended Borel transfromation and applied it to the study of fractional calculus. Fractional calculus was applied to infinite dimensional analysis in [3]. $\mathfrak{g}_a$ and $\mathfrak{g}_{\log}$ were defined in [4]. But higher order commutation relations were not considered in [4].

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2. Review on fractional calculus

Let a be a positive number. We use

$$I^a f(x) = \frac{1}{\Gamma(a)} \int_0^x (x-t)^{a-1} f(t)dt, \tag{1}$$

as the definition of a-th order indefinite integal. Riemann-Liouville's and Caputo's fractional derivatives are defined by

$$\frac{d^{n-a}}{dx^{n-a}} f(x) = \frac{d^n}{dx^n} I^a f(x), \quad \frac{d^{n-a}}{dx^{n-a}} f(x) = I^a \left(\frac{d^n f(x)}{dx^n} \right),$$

respectively ([1]). They are different and $\frac{d^b}{dx^b}\left(\frac{d^c}{dx^c}\right)$ are not be equal to $\frac{d^{b+c}}{dx^{b+c}}$, in general.

These ambuigities are resolved if we use f_+; $f_+(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$ instead of f, because we have

$$I^a f(x) = \frac{1}{\Gamma(a)} x_+^{a-1} * f_+(x), \quad u * v(x) = \int_{-\infty}^{\infty} u(x-t)v(t)dt.$$

In this case, we have $\frac{d^a}{dx^a}\frac{d^b}{dx^b} = \frac{d^{a+b}}{dx^{a+b}}$. Hence we can denote $\frac{d^a}{dx^a} = I^{-a}$ and $I^a = \frac{d^{-a}}{dx^{-a}}$. On the other hand, the constant function 1 is replaced by the Heaviside function Y and we can not consider $\frac{d^a 1}{dx^a}$. Since

$$\frac{df_+(x)}{dx} = \frac{df(x)}{dx} + f(0)\delta,$$

δ is the Dirac function, we need distribution in this case (cf.[10]).

$\{I^a | a \geq 0\}$, $I^0 = \mathrm{I}$, the identity map, is a semigroup. Its generating operator $Af(x) = \lim_{a\to 0} \frac{dI^a f(x)}{da}$ is given by

$$\begin{aligned} Af(x) &= \gamma f(x) + \int_0^x \log(x-t)\frac{df_+(t)}{dt}dt \\ &= (\log x + \gamma)f(x) + \int_0^x \log\left(1 - \frac{t}{x}\right)\frac{df(t)}{dt}dt. \end{aligned}$$

Definition 1. We define logarithm of differentiation $\log\left(\frac{d}{dx}\right)$ by

$$\log\left(\frac{d}{dx}\right)f(x) = -\left((\log x + \gamma)f(x) + \int_0^x \log\left(1 - \frac{t}{x}\right)\frac{df(t)}{dt}dt\right). \tag{2}$$

The following examples are used later.

$$\frac{d^a}{dx^a}x^c = \frac{\Gamma(c+1)}{\Gamma(c-a+1)}x^{c-a}, \quad \frac{d^a}{dx^a}1 = \frac{1}{\Gamma(1-a)}x^{-a}. \tag{3}$$

$$\log\left(\frac{d}{dx}\right)x^c = -\left(\log x + \gamma - \sum_{n=1}^{\infty}\frac{c}{n(n+c)}\right)x^c. \tag{4}$$

Here, none of c, $-a$ and $c-a$ are negative integers. We also use

$$\begin{aligned} &\log\left(\frac{d}{dx}\right)(\log x)^n \\ &= -(\log x + \gamma)(\log x)^n + \sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}n!\zeta(n-k+1)}{k!}(\log x)^k. \end{aligned} \tag{5}$$

By (5), we obtain

Proposition 1. *Let* $\mathfrak{d}_x$ *be*

$$\mathfrak{d}_x = \sum_{k=1}^{\infty}(-1)^{k-1}\zeta(k+1)\frac{d^k}{dx^k}.$$

Then we have

$$\log\left(\frac{d}{dx}\right)(\log x)^n = \big(-(X+\gamma) + \mathfrak{d}_X\big)X^n|_{X=\log x}. \tag{6}$$

By (6), if $f(x) = \sum_n c_n x^n$ satisfies $|f^{(n)}(x)| = O(r^n), n \to \infty,\ r < 1$ on (positive) real axis, then

$$\log\left(\frac{d}{dx}\right)f(\log x) = \big(-(X+\gamma) + \mathfrak{d}_X\big)f(X)|_{X=\log x}.$$

There are several alternative definitions of fractional calculus. Among them, we sketch the definition by Borel transformation, which provides a simple proof of the formula $e^{a\log(\frac{d}{dx})} = \frac{d^a}{dx^a}$.

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$, then its Borel transform $\mathcal{B}[f(\zeta)](z)$ is defined by

$$\mathcal{B}[f](z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n = \frac{1}{2\pi i} \oint e^{\frac{z}{\zeta}} \frac{f(\zeta)}{\zeta} d\zeta.$$

Borel transformation is linear and satisfies

$$\frac{d}{dz}\mathcal{B}[f(\zeta)](z) = \mathcal{B}[\zeta^{-1}f(\zeta)](z), \quad \mathcal{B}[fg](z) = \mathcal{B}[f]\sharp\mathcal{B}[g],$$

where $u\sharp v$ is $\frac{d}{dx}\int_0^x u(x-t)v(t)dt$. Originally, Borel transforms of $\log x$ and x^a, $a \notin \mathbb{Z}$, are not defined. But since inverse Borel transformation $\mathcal{B}^{-1}$ is given by $\mathcal{B}^{-1}[f] = \int_0^\infty e^{-x} f(xt)dt$, we define Borel transforms of $\log x$ and x^a by

$$\mathcal{B}[\log\zeta](z) = \log z + \gamma, \quad \mathcal{B}[\zeta^a] = \frac{z^a}{\Gamma(a+1)}.$$

We have

$$\lim_{\epsilon\to 0}\mathcal{B}[(\zeta+\epsilon)^a](z) = \frac{z^a}{\Gamma(a+1)}, \quad \lim_{\epsilon\to 0}\mathcal{B}[\log(\zeta+\epsilon)] = \log z + \gamma,$$

only on $\{z|\Re z > 0\}$. Hence extended Borel transformation is defined only for functions defined on $\{z|\Re z > 0\}$, or at least functions defined on $\mathbb{C}\setminus\{x \le 0\}$. We note that since $\mathcal{B}^{-1}[\delta](z) = z^{-1}$, δ is the Dirac function, we define $\mathcal{B}[\zeta^{-1}] = \delta$ in the extended Borel transformation.

It is shown

$$e^{\sharp t\log x} = \frac{e^{-\gamma t}}{\Gamma(1+t)} x^t, \quad e^{\sharp f} = \sum_{n=0}^{\infty} \frac{f^{\sharp n}}{n!},$$

where $f^{\sharp n} = \overbrace{f\sharp\cdots\sharp f}^{n}$ ([2], (5) follows from the proof of this formula). Hence we can define

$$\frac{d^a}{dz^a}\mathcal{B}[f(\zeta)](z) = \mathcal{B}[\zeta^{-a}f(\zeta)](z), \quad \log(\frac{d}{dz})\mathcal{B}[f(\zeta)](z) = -\mathcal{B}[\log\zeta f(\zeta)](z).$$

By definitions, we have $e^{a\log(\frac{d}{dz})} = \frac{d^a}{dz^a}$ on the one hand, and

$$-\mathcal{B}[\log\zeta f(\zeta)](x) = -(\gamma u(x) + \int_0^x \log(x-t)\frac{du(t)}{dt}dt), \quad u = \mathcal{B}[f],$$

on the other hand. Since extended Borel transformation is defined for functions on positive real axis, this shows definition of logarithm of differentiation by using Borel transformation coincides with our previous definition.

3. Leibniz rules and infinite order differential operator expressions

Since $I^1 f = If = \int_0^x f(t)dt$, if g is sufficiently regular, *e.g.*, if g is a Gevrey class function of index $\alpha < 1$, that is, g is smooth and $|g^{(n)}(x)| \leq M_x(n!)^\alpha$, we have

$$I^1(fg) = (I^1 f)g - (I^2 f)g' + \cdots + (-1)^{n-1}(I^n f)g^{(n-1)} + \cdots .$$

Because we have $|I^n f(x)| \leq \frac{C_x|x|^n}{n!}$, for some constant $C_x > 0$.

Replacing f by $f_a(t) = \frac{(x-t)^{a-1}}{\Gamma(a)} f(t)$ in this equality, and using

$$I^n f_a(x) = \frac{1}{(n-1)!\Gamma(a)} \int_0^x (x-t)^{n+a-2} f(t)dt = \frac{\Gamma(n+a)}{(n-1)!\Gamma(a)} I^{n+a-1} f(x),$$

we obtain

$$\begin{aligned} I^a(fg) = (I^a f)g - a(I^{a+1})f)g' + \cdots + \\ +(-1)^{n-1}\frac{\Gamma(n+a-1)}{(n-1)!\Gamma(a)}(I^{a+n-1}f)g^{(n-1)} + \cdots . \end{aligned} \tag{7}$$

if g is a Gevrey class function of index $\alpha < 1$. Then by the definition of Riemann-Liouville's fractional derivative, we have

Proposition 2. *If g is a Gevrey class function of index $\alpha < 1$, then*

$$\begin{aligned} \frac{d^a}{dx^a}\big(f(x)g(x)\big) = \frac{d^a f(x)}{dx^a} g(x) + a\frac{d^{a-1}f(x)}{dx^{a-1}}\frac{dg(x)}{dx} + \cdots + \\ + \frac{a(a-1)\cdots(a-n+1)}{n!}\frac{d^{a-n}f(x)}{dx^{a-n}}\frac{d^n g(x)}{dx^n} + \cdots . \end{aligned} \tag{8}$$

Here $\frac{d^{a-n}f(x)}{dx^{a-n}}$ means $I^{n-a}f(x)$.

Since $\frac{d}{dt}(\frac{d^t f}{dx^t}) = \log(\frac{d}{dx})(\frac{d^t f}{dx^t})$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{d^t}{dx^t}(fg)\right)\\
&= \frac{d}{dt}\left(\frac{d^t f}{dx^t}g + t\frac{d^{t-1}f}{dx^{t-1}}g' + \cdots + \frac{t\cdots(t-n+1)}{n!}\frac{d^{t-n}f}{dx^{t-n}}g^{(n)} + \cdots\right)\\
&= \log\left(\frac{d}{dx}\right)\left(\frac{d^t f}{dx^t}\right)g + \frac{d^{t-1}f}{dx^{t-1}}g' + t\frac{d}{dt}\left(\frac{d^{t-1}f}{dx^{t-1}}\right)g' + \cdots +\\
&\quad + \frac{(t-1)\cdots(t-n+1)}{n!}\frac{d^{t-n}f}{dx^{t-n}}g^{(n)} +\\
&\quad + t\frac{d}{dt}\left(\frac{(t-1)\cdots(t-n+1)}{n!}\frac{d^{t-n}f}{dx^{t-n}}g^{(n)}\right) + \cdots,
\end{aligned}$$

by (8). Hence we obtain

Proposition 3. *If g is a Gevrey class function of index $\alpha < 1$, then*

$$\begin{aligned}
&\log\left(\frac{d}{dx}\right)\big(f(x)g(x)\big)\\
&= \left(\log\left(\frac{d}{dx}\right)f(x)\right)g(x) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\big(I^n f(x)\big)\frac{d^n g(x)}{dx^n}. \qquad (9)
\end{aligned}$$

If $g = c$, a constant function, then we have

$$\frac{d^a}{dx^a}(f\cdot c) = \left(\frac{d^a f}{dx^a}\right)c, \quad \log\left(\frac{d}{dx}\right)(f\cdot c) = \left(\log\left(\frac{d}{dx}\right)f\right)c$$

by (8) and (9). On the other hand, if a is not a positive integer, we have

$$\frac{d^{a-n}}{dx^{a-n}}1 = \frac{1}{\Gamma(n+1-a)}x^{n-a} \neq 0,$$

for all positive integers n, by (3). We also have $\log(\frac{d}{dx})1 = -(\log x + \gamma) \neq 0$ by (4). Hence using $f(x) = 1\cdot f(x)$ and

$$\frac{a(a-1)\cdot(a-n+1)}{n!\Gamma(n+1-a)} = \frac{(-1)^{n-1}a}{n!(n-a)\Gamma(1-a)},$$

we obtain by (8) and (9)

Theorem 1. *If f is a Gevrey class function of index $\alpha < 1$, then*

$$\frac{d^a f(x)}{dx^a} = \frac{x^{-a}}{\Gamma(1-a)}\left(f(x) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}ax^n}{(n-a)n!}\frac{df^n(x)}{dx^n}\right), \qquad (10)$$

$$\log\left(\frac{d}{dx}\right)f(x) = -(\log x + \gamma)f(x) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n\cdot n!}\frac{d^n f(x)}{dx^n}. \qquad (11)$$

Note. (10) does not have meanings if a is a positive integer. But since $\frac{1}{\Gamma(1-a)} = \frac{\sin(\pi a)\Gamma(a)}{\pi}$, we have

$$\begin{aligned}
&\frac{d^{m-\epsilon} f(x)}{dx^{m-\epsilon}} \\
&= \frac{\sin(\pi\epsilon)\Gamma(m-\epsilon)x^{\epsilon}}{\pi}\left(x^{-m} f(x) + \sum_{n\neq m}^{\infty} \frac{(m-\epsilon)x^{n-m}}{(n-m+\epsilon)n!}\frac{d^m f(x)}{dx^m}\right) + \\
&\quad + \frac{\sin(\pi\epsilon)\Gamma(m-\epsilon)x^{\epsilon}}{\pi\epsilon}\frac{(m-\epsilon)}{m!}\frac{d^m f(x)}{dx^m}.
\end{aligned}$$

Hence in the sense of pointwise convergence, we obtain

$$\lim_{a\to m} \frac{x^{-a}}{\Gamma(1-a)}(f(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a x^n}{(n-a)n!}\frac{d^n f(x)}{dx^n}) = \frac{d^m f(x)}{dx^m}.$$

4. Fractional calculus of functions defined on domains other than $\mathbb{R}_+$

Originally, fractional calculus is defined for functions defined on $\mathbb{R}_+ = \{x | x \geq 0\}$. But (10) and (11) allow to investigate fractional order and logarithm of differentiations for functions defined on a domain other than $\mathbb{R}_+$.

If f is a Gevrey class function of index $\alpha < 1$ defined on $\mathrm{D} \subset \mathbb{C}$; D is simply connected and $0 \notin \mathrm{D}$, we can define its fractional order and logarithm of differentiations by (10) and (11). They also allow to consider fractional order and logarithm of differentiations of functions defined on covering spaces of such domains. In §5 and §7, we use the cases $\mathrm{D} = \{z | |z| = 1, z \neq -1\}$ and its covering space $\{e^{i\theta} | -\pi/a < \theta < \pi/a\}$, $0 < a < 1$.

To define fractional order differentiation and logarithm of differentiation of functions defined on $\mathbb{R}$, we define functions $x_{\pm}^a$ and $\log_{\pm} x$ on $\mathbb{R}$ by

$$x_{\pm}^a = \begin{cases} x^a, & x \geq 0, \\ e^{\pm a\pi i}|x|^a, & x < 0, \end{cases} \qquad \log_{\pm} x = \begin{cases} \log x, & x \geq 0, \\ \log x \pm \pi i, & x < 0. \end{cases}$$

Definition 2. Let f be a Gevrey class function of index $\alpha < 1$ defined on $\mathbb{R}$. Then we set

$$\frac{d^a}{dx^a}_{\pm} f(x) = \frac{x_{\pm}^{-a}}{\Gamma(1-a)}\left(f(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a x^n}{(n-a)n!}\frac{d^n f(x)}{dx^n}\right), \tag{12}$$

$$\log_{\pm}\left(\frac{d}{dx}\right) f(x) = -(\log_{\pm} x + \gamma) f(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n \cdot n!}\frac{d^n f(x)}{dx^n}. \tag{13}$$

On the other hand, the operators

$$x^a \frac{d^a}{dx^a} = \frac{1}{\Gamma(1-a)}\left(1 + \sum_{n-1}^{\infty} \frac{(-1)^{n-1} a x^n}{(n-a)n!} \frac{d^n}{dx^n}\right),$$

$$\log\left(\frac{d}{dx}\right) + \log x = -\gamma + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n \cdot n!} \frac{d^n}{dx^n}$$

are defined for Gevrey class functions of index $\alpha < 1$ on $\mathbb{R}$. $\{x^t \frac{d^t}{dx^t} | t \geq 0\}$ is not a semigroup. But we have

$$\frac{d}{dt}\left(x^t \frac{d^t}{dx^t}\right)|_{t=0} = \log\left(\frac{d}{dx}\right) + \log x.$$

We can investigate variable change $x \to x(t)$ of fractional order or logarithm of differentiation by using (10) and (11), if the change $x \to x(t)$ preserves orientation. If $x \to x(t)$ reverses orientation, we need to use (12) and (13). For example, if $x = ct$, we have

$$\frac{d^a}{dt^a}_{\pm} = c^a_{\pm} \frac{d^a}{dx^a}_{\pm}, \quad \log_{\pm}(\frac{d}{dt}) = \log_{\pm} c + \log_{\pm}(\frac{d}{dx}).$$

5. Deformed Hardy space H_a

By (8), we have $[\frac{d^a}{dx^a}, x] = a\frac{d^{a-1}}{dx^{a-1}}$.Therefore Hilbert spaces such as $L^2(\mathbb{R}_+)$ are not appropriate to treat commutation relations involving $\frac{d^a}{dx^a}$, if a is not an integer. We propose

$$\mathrm{H}_a = \left\{\sum_{n=1}^{\infty} c_n x^{an-1} | \sum_{n=1}^{\infty} |c_n|^2 < \infty\right\} \tag{14}$$

as the Hilbert space to treat commutation relations involving $\frac{d^a}{dx^a}$. In (14), a is arbitrary. But we assume $0 < a < 1$ in the rest.

$\mathrm{H}_1 = \mathrm{H}$ is the Hardy space. Hence we may regard H_a to be a deformed Hardy space. We define a Hilbert space isometry $\rho_a : \mathrm{H}_a \to \mathrm{H}$ by

$$\rho_a(x^{an-1}) = x^{n-1}.$$

We regard $f(x) \in \mathrm{H}_a$ to be a function on $\{e^{i\theta} | -\pi/a < \theta < \pi/a\}$. Then the inner product (f, g) of $f, g \in \mathrm{H}_a$ is given by

$$(f, g) = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta.$$

The function x^a does not belong to H_a. But the multiplication operator x^a : $f(x) \to x^a f(x)$ is defined. Since $(x^a x^{an-1}, x^{am-1}) = (x^{an-1}, x^{a(m-1)-1})$, $m \geq 2$, and

$$(x^a x^{an-1}, x^{a-1}) = \frac{a}{2\pi}\int_{-\pi/a}^{\pi/a} e^{ani\theta} d\theta = 0,$$

we have

$$x^{a\dagger} = x^{-a}; x^{-a}x^{an-1} = x^{a(n-1)-1},\ n \geq 2, \quad x^{-a}x^{a-1} = 0.$$

Hence as an operator on H_a, we have

$$\frac{d^a}{dx^a}x^{an-1} = \frac{\Gamma(an)}{\Gamma(a(n-1))}x^{a(n-1)-1},\ n \geq 2, \quad \frac{d^a}{dx^a}x^{a-1} = 0. \tag{15}$$

$\frac{d^{an}}{dx^{an}}$ also maps H_a into H_a. Owing to the second equality of (15), we do not have $(\frac{d^a}{dx^a})^n = \frac{d^{an}}{dx^{an}}$ in general.

Note. Precisely saying, $\frac{d^{an}}{dx^{an}}$ is not defined on H_a. We can introduce Sobolev norm and Sobolev space W_a^{an} by using these derivations. But we do not use this aspect in the rest.

We define diagonal form operators $A_{a,\pm}$ by

$$A_{a,+}x^{an-1} = \frac{\Gamma(a(n+1))}{\Gamma(an)}x^{an-1},$$
$$A_{a,-}x^{an-1} = \frac{\Gamma(an)}{\Gamma(a(n-1))}x^{an-1},\ n \geq 2, \quad A_{a,-}x^{a-1} = 0.$$

By definitions, we have

$$\frac{d^a}{dx^a} = x^{-a}A_{a,-} = A_{a,+}x^{-a}, \quad (\frac{d^a}{dx^a})^\dagger = A_{a,-}x^a = x^a A_{a,+}. \tag{16}$$

Lemma 1. *Let* C_a *be* $[\frac{d^a}{dx^a}, x^a]$. *Then we have*

$$\mathrm{C_a} = A_{a,+} - A_{a,-}. \tag{17}$$

Proof. By (16), we have $\mathrm{C}_a = A_{a,+}x^{-a}x^a - x^a x^{-a}A_{a,-} = A_{a,+} - A_{a,-}$. Hence we obtain Lemma.

Lemma 2. C_a *is a p-Schatten class operator if* $p > 1/(1-a)$ *and* $\lim_{a\to 1}\rho_a \mathrm{C}_a \rho_a^{-1} = \mathrm{I}$ *by the strong topology of operators.*

Proof. C_a is a diagonal form operator where the diagonal element $c_{a,n}$ is

$$c_{a,n} = \frac{\Gamma(a(n+1))}{\Gamma(an)} - \frac{\Gamma(an)}{\Gamma(a(n-1))},\ n \geq 2,$$

by (17). Since $\Gamma(an) = \sqrt{2\pi}e^{-an}(an)^{an-1/2}\big(1+O(\frac{1}{n})\big)$ by Stirling's formula, we have

$$\frac{\Gamma(a(n+1))}{\Gamma(an)} = e^{-a}(a(n+1))^a\left(1+\frac{1}{an}\right)^{an-1/2}\left(1+O\left(\frac{1}{n}\right)\right).$$

Hence we get

$$c_{a,n} = \big(a(n+1)\big)^a\left(1+O\left(\frac{1}{n}\right)\right) - (an)^a\left(1+O\left(\frac{1}{n}\right)\right) = an^{a-1}\left(1+O\left(\frac{1}{n}\right)\right).$$

Therefore $\lim_{a\to1} c_{a,n} = 1$ on the one hand, and $\sum_{n=1}^{\infty}|c_{a,n}|^p < \infty$ if $p > 1/(1-a)$ on the other hand. Hence we have Lemma.

6. Higher order commutation relations involving fractional order differentiation

In this section, we consider $\frac{d^a}{dx^a}$ and x^a to be operators on H_a.

As opertors on H_a, neither $\frac{d^a}{dx^a}$ nor x^a commute with C_a. Hence there are non-trivial higher order commutation relations. As for x^a, we have

$$[x^a, \mathrm{C}_a]x^{an-1} = (c_{a,n} - c_{a,n+1})x^{a(n+1)-1}.$$

Since $c_{a,n} = an^{a-1}(1+O(1/n))$, we have

$$c_{a,n} - c_{a,n+1} = -a^2n^{a-2}\left(1+O\left(\frac{1}{n}\right)\right).$$

Repeating this, we obtain

$$\begin{aligned}&\overbrace{[x^a,[\cdots,[}^{m}x^a,\mathrm{C}_a]\cdots]x^{an-1}\\ &= (-1)^m a^{m+1}n^{a-(m+1)}\left(1+O\left(\frac{1}{n}\right)\right)x^{a(n+m)-1}.\end{aligned} \tag{18}$$

As for $\frac{d^a}{dx^a}$, we have

$$\left[\frac{d^a}{dx^a}, \mathrm{C}_a\right]x^{an-1} = (c_{a,n} - c_{a,n+1})\frac{\Gamma(na)}{\Gamma((n-1)a)}x^{a(n-1)-1},\ n \geq 2.$$

Hence we have

$$\left[\frac{d^a}{dx^a}, \mathrm{C}_a\right] x^{an-1} = a^2 e^{-a} n^{2(a-1)} \left(1 + O\left(\frac{1}{n}\right)\right) x^{a(n-1)-1}, \ n \geq 2.$$

Repeating this, we obtain

$$\begin{aligned}&\overbrace{\left[\frac{d^a}{dx^a}, [\cdots, \left[\frac{d^a}{dx^a}\right.\right.}^{m}, \mathrm{C}_a\Big] \cdots \Big] x^{an-1} \\ &= a^{m+1} e^{-ma} n^{(m+1)(a-1)} \big(1 + O(\frac{1}{n})\big) x^{a(n-m)-1}, \ n > m. \end{aligned} \tag{19}$$

By (18) and (19), we have

$$\begin{aligned}&\overbrace{\left[\frac{d^a}{dx^a}, \left[\cdots, \left[\right.\right.\right.}^{p} x^a, \overbrace{\left[\cdots, \left[\right.\right.}^{q} x^a, \mathrm{C}_a\Big] \cdots \Big] x^{an-1} \\ &= (-1)^q a^{p+q} e^{-pa} n^{(p+1)(a-1)-q} \left(1 + O\left(\frac{1}{n}\right)\right) x^{a(n-p+q)-1}, \end{aligned} \tag{20}$$

where we assume $n > p$. Hence we obtain

Proposition 4. *Let $\mathfrak{I}_m$ be the m-Schatten ideal. Then*

$$\overbrace{\left[\frac{d^a}{dx^a}, \left[\cdots, \left[\right.\right.\right.}^{p} x^a, \overbrace{\left[\cdots, \left[\right.\right.}^{q} x^a, \mathrm{C}_a\Big] \cdots \Big] \begin{cases} \in \mathfrak{I}_m, \ m > \frac{1}{(p+1)(1-a)+q}, \\ \notin \mathfrak{I}_m, \ m \leq \frac{1}{(p+1)(1-a)+q}. \end{cases} \tag{21}$$

Definition 3. We denote the Lie algebra generated by $\frac{d^a}{dx^a}$ and x^a by $\mathfrak{g}_a$.

Note 1. We do not consider topology of $\mathfrak{g}_a$. Hence $Y \in \mathfrak{g}_a$ takes the following form.

$$Y = \sum_{j=1}^{m} c_j [X_{j_1}, [\cdots, [X_{j,n_j-1}, X_{j,n_j}] \cdots].$$

Here $X_{j,k}$ is either $\frac{d^a}{dx^a}$ or x^a.

Note 2. I does not belong to $\mathfrak{g}_a$. We denote $\mathbb{C}\mathrm{I} \oplus \mathfrak{g}_a$ *by* $\tilde{\mathfrak{g}}_a$.

$\frac{d^a}{dx^a}$ is unbounded. But other elements of $\mathfrak{g}_a$ are bounded by Proposition 4. Therefore $\mathfrak{i}_{a,p} = \mathfrak{g}_a \cap \mathfrak{I}_p$ is an ideal of $\mathfrak{g}_a$. By Proposition 4, we also have

$$\bigcap_{p>0} \mathfrak{i}_{a,p} = \{0\}. \tag{22}$$

Since $\mathfrak{i}_{a,p} \subset \mathfrak{i}_{a,q}$, $p < q$, there is a homomorphism

$$j_q^p : \mathfrak{g}_a/\mathfrak{i}_{a,q} \to \mathfrak{g}_a/\mathfrak{i}_{a,p},\ p < q.$$

By definition, we have $j_q^p \circ j_r^q = j_r^p$, if $p < q < r$. Hence if $\{p_n\}$ is a series such that $p_1 > p_2 > \cdots$, $\lim_{n\to\infty} p_n = 0$, then we have a projective system $\{\mathfrak{g}_a/\mathfrak{i}_{a,p_n}; j_{p_n}^{p_{n+1}}\}$.

NOTE. $\mathfrak{g}_a/\mathfrak{i}_p$ looks like a truncation. But (21) shows that it is not a truncation.

By definition, each $\mathfrak{g}_a/\mathfrak{i}_{a,p_n}$ is a nilpotent Lie algebra and by (22), we have

$$\varprojlim \{\mathfrak{g}_a/\mathfrak{i}_{a,p_n}; j_{p_n}^{p_{n+1}}\} \cong \mathfrak{g}_a. \tag{23}$$

Hence we obtain

THEOREM 2. $\mathfrak{g}_a$ *is a projective limit of nilpotent Lie algebras.*

7. The spaces $\mathrm{H}_{\log}$ and $\mathrm{F}_{\log}$

By (9), we have $[\log(\frac{d}{dx}), x] = I^1$, the indefinite integral operator. Hence spaces such as $L^2(\mathbb{R}_+)$ are not appropriate to the study of commutation relations involving logarithm of differentiation. We propose

$$\mathrm{H}_{\log} = \left\{ \sum_{n=0}^{\infty} c_n (\log x)^n \Big| \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\} \tag{24}$$

as the Hilbert space to treat commutation relations involving logarithm of differentiation.

Let $f(w) = \sum_{n=0}^{\infty} c_n w^n$, $w = \log x$, be an element of $\mathrm{H}_{\log}$. Then considering $f(w)$ to be a function on $S^1 \setminus \{-1\} = \{x \in \mathbb{C} \,|\, |x| = 1, x \neq -1\}$, $f(w)$ becomes a power series $f(i\theta) = \sum_{n=0}^{\infty} c_n (i\theta)^n$, $-\pi < \theta < \pi$. Therefore we may regard $\mathrm{H}_{\log}$ to be a function space on $(-\pi, \pi)$. As a Hilbert space, we can identify $\mathrm{H}_{\log}$ and $W^{1/2}(-\pi, \pi)$, the Sobolev 1/2-space over $(-\pi, \pi)$. $\{(\log x)^n | n = 0, 1, \ldots\}$ is a complete basis of $\mathrm{H}_{\log}$. It is not an orthogonal system. We may use Legendre polynomials of $\log x$ as a complete orthogonal basis of $\mathrm{H}_{\log}$. But in the rest, we do not use these arguments.

Let $\log x$ be the multiplication operator $\log x : f(\log x) \to \log x f(\log x)$. Then by (5), we have

$$
\begin{aligned}
&[\log(\frac{d}{dx}), \log x](\log x)^n \\
&= -(\log x)^{n+1}(\log x + \gamma) + \sum_{k=0}^{n} \frac{(-1)^{n-k}(n+1)!\zeta(n+2-k)}{k!}(\log x)^k + \\
&\quad +(\log x)^{n+1}(\log x + \gamma) - \sum_{k=0}^{n-1} \frac{(-1)^{n-k+1}n!\zeta(n+1-k)}{k!}(\log x)^{k+1} \\
&= \sum_{k=1}^{n}(-1)^{n-k}\Big(\frac{(n+1)!}{k!} - \frac{n!}{(k-1)!}\Big)\zeta(n+2-k)(\log x)^k + \\
&\quad +(-1)^n(n+1)!\zeta(n+2) \\
&= \sum_{k=0}^{n}(-1)^{n-k}(n+1-k)\frac{n!}{k!}\zeta(n+2-k)(\log x)^k \\
&= \zeta(2)(\log x)^n - 2n\zeta(3)(\log x)^{n-1} + \cdots .
\end{aligned}
$$

Hence we obtain

Lemma 3. *We have*

$$\left[\log\left(\frac{d}{dx}\right), \log x\right] = \zeta(2)\mathrm{I} + N_{\log}. \tag{25}$$

Here $N_{\log}$ satisfies $N_{\log}^n(\log x)^m = 0$, $n > m$.

We can regard $\log x$ as a deformed creation operator. But we can not regard $\log(\frac{d}{dx})$ as a deformed annihilation operator, because it does not annihilate 1. We use

$$\mathcal{R} = \log(\frac{d}{dx}) + \log x + \gamma,$$

as a deformed annihilation operator instead of $\log(\frac{d}{dx})$. Then, we have

$$[\mathcal{R}, \log x] = \left[\log\left(\frac{d}{dx}\right), \log x\right] = \zeta(2)\mathrm{I} + N_{\log}.$$

In the rest, we denote $[\mathcal{R}, \log x]$ by $\mathrm{B}_{\log,1}$. We also use $\mathrm{B}_{\log}$ instead of $\mathrm{B}_{\log,1}$.

$\mathcal{R}$ and $N_{\log}$ are unbounded operators on $\mathrm{H}_{\log}$. As a convenient representation space of $\mathcal{R}$ (and $\log(\frac{d}{dx})$), we use

$$\mathrm{F}_{\log} = \left\{ f(\log x) | f(x) = \sum_{n=0}^{\infty} a_n x^n,\ \left|\frac{d^n f(x)}{dx^n}\right| = O(r^n), n \to \infty, r < 1 \right\}.$$

We also set $\mathrm{F} = \{f(x) = \sum_{n=0}^{\infty} a_n x^n || \frac{d^n f(x)}{dx^n}| = O(r^n), n \to \infty, r < 1\}$. $\mathrm{F}_{\log}$ is a dense subspace of $\mathrm{H}_{\log}$. But we do not consider topologies of F and $\mathrm{F}_{\log}$, although we consider convergence of a series $\{f_n(x)\}$ or $\{f_n(\log x)\}$, where $f_n(x) \in \mathrm{F}$.

We define a vector space isomorphism $\kappa : \mathrm{F}_{\log} \cong \mathrm{F}$ by $\kappa(f(\log x)) = f(x)$. By definition, we have $\kappa \circ \log x \circ \kappa^{-1} = x$, where $\log x$ and x are regarded as multiplication operators on $\mathrm{F}_{\log}$ and on F. By (6), we also have

$$\kappa \circ \log\left(\frac{d}{dx}\right) \circ \kappa^{-1} = -(x+\gamma) + \mathfrak{d}_x. \tag{26}$$

Definition 4. We denote the Lie algebra generated by $\log(\frac{d}{dx})$ and $\log x$ *by* $\mathfrak{g}_{\log}$.

Here, $\log(\frac{d}{dx})$ and $\log x$ are considered to be operators on $\mathrm{F}_{\log}$. The Lie algebra generated by $\mathcal{R}$ and $\log x$ is denoted by $\mathfrak{g}_{\mathcal{R}}$. Similarly to $\mathfrak{g}_a$, we do not consider topologies of $\mathfrak{g}_{\log}$ and $\mathfrak{g}_{\mathcal{R}}$.

$\mathfrak{g}_{\log}$ and $\mathfrak{g}_{\mathcal{R}}$ are different. But as for $\tilde{\mathfrak{g}}_{\log} = \mathbb{C}\mathrm{I} \oplus \mathfrak{g}_{\log}$, we have

$$\tilde{\mathfrak{g}}_{\log} = \mathbb{C}\mathrm{I} \oplus \mathfrak{g}_{\log} = \mathbb{C}\mathrm{I} \oplus \mathfrak{g}_{\mathcal{R}}. \tag{27}$$

By (26), we have

Lemma 4. *$\kappa\mathfrak{g}_{\log}\kappa^{-1}$ is generated by $-(x+\gamma)+\mathfrak{d}_x$ and x. Similarly, $\kappa\mathfrak{g}_{\mathcal{R}}\kappa^{-1}$ is generated by $\mathfrak{d}_x$ and x.*

We denote $\kappa \circ \mathrm{B}_{\log} \circ \kappa^{-1} = \mathrm{B}_{\mathfrak{d}_x} = \mathrm{B}_{\mathfrak{d}_x,1}$. Then we have

$$\mathrm{B}_{\mathfrak{d}_x} = [\mathfrak{d}_x, x] = \zeta(2)\mathrm{I} + \sum_{k=1}^{\infty} (-1)^k (k+1)\zeta(k+2)\frac{d^k}{dx^k}.$$

Hence we obtain

$$N_{\log}\big(f(\log x)\big) = \sum_{k=1}^{\infty} (-1)^k (k+1)\zeta(k+2)\frac{d^k f(X)}{dX^k}|_{X=\log x}.$$

8. Higher order commutation relations involving logarithm of differentiation

In this Section, we consider $\log(\frac{d}{dx})$ and $\log x$ acting on $\mathrm{F}_{\log}$, and $\mathfrak{d}_x$ and x acting on F.

Since $[x, \frac{d^k}{dx^k}] = k\frac{d^{k-1}}{dx^{k-1}}$, we have

$$[x, \mathrm{B}_{\mathfrak{d}_x,1}] = -2\zeta(3)\mathrm{I} + \sum_{k=1}^{\infty} (-1)^{k+1}(k+1)(k+2)\zeta(k+3)\frac{d^k}{dx^k}.$$

Definition 5. We define $\mathrm{B}_{\mathfrak{d}_x,m}$ inductively by

$$\mathrm{B}_{\mathfrak{d}_x,m} = [x, \mathrm{B}_{\mathfrak{d}_x,m-1}], \quad m \geq 2. \tag{28}$$

$\kappa^{-1} \circ \mathrm{B}_{\mathfrak{d}_x,m} \circ \kappa$ is denoted by $\mathrm{B}_{\log,m}$.

Directly, $\mathrm{B}_{\log,m}$ is defined by

$$[\log x, \mathrm{B}_{\log,m-1}] = \mathrm{B}_{\log,m}, \ m \geq 2.$$

By definition, we have

$$\begin{aligned} \mathrm{B}_{\mathfrak{d}_x,m} &= (-1)^m (m+1)!\zeta(m+2)\mathrm{I} + \\ &\quad + \sum_{k=1}^{\infty} (-1)^{k+m} \frac{(k+m)!}{k!} \zeta(k+m+1) \frac{d^k}{dx^k}, \qquad (29) \\ &= (-1)^m (m+1)!\zeta(m+2)\mathrm{I} + N_{\mathfrak{d}_x,m}, \qquad (30) \end{aligned}$$

where $N_{\mathfrak{d}_x,m}$ is a generalized nilpotent operator. That is, we have

$$N^k_{\mathfrak{d}_x,m}(x^l) = 0, \ k > l.$$

Each $\mathrm{B}_{\mathfrak{d}_x,m}$, $m \geq 1$, is a constant coefficient linear differential operator of degree infinite. Since the coefficient of k-th degree term of $\mathrm{B}_{\mathfrak{d}_x,m}$ is evaluated by k^m, we have

$$[\mathrm{B}_{\mathfrak{d}_x,p}, \mathrm{B}_{\mathfrak{d}_x,q}] = 0, \quad p \geq 1, \ q \geq 1, \tag{31}$$

as operators on F.

By (31) and (26), we have

$$[\log(\frac{d}{dx}), \mathrm{B}_{\log,m}] = -[\log x, \mathrm{B}_{\log,m}]. \tag{32}$$

Hence we obtain

Theorem 3. *Let $p+q$ be equal to m. Then we have*

$$\begin{aligned} &\overbrace{\left[\log\left(\frac{d}{dx}\right), \left[\cdots, \left[\log\left(\frac{d}{dx}\right),\right.\right.\right.}^{p} \overbrace{\left[\log x, \cdots, \left[\log x, \mathrm{B}_{\log,m}\right]\cdots\right]}^{q} \\ &= (-1)^q (m+1)!\zeta(m+2)\mathrm{I} + (-1)^p N_{\log,m}. \qquad (33) \end{aligned}$$

Here, $N_{\log,m} = \kappa^{-1} \circ N_{\mathfrak{d}_x,m} \circ \kappa$ is a generalized nilpotent operator.

By (29), explicit form of $N_{\log,m}$ is given by

$$N_{\log,m}\big(f(\log x)\big) = \sum_{k=1}^{\infty}(-1)^{k+m}\frac{(k+m)!}{k!}\zeta(k+m+1)\frac{d^k f(X)}{dX^k}|_{X=\log x}.$$

By (32), we have $[\log(\frac{d}{dx}), \mathfrak{g}_{\log}] = [\log x, \mathfrak{g}_{\log}]$ and so on. We set

$$\mathfrak{i}_{\log,1} = [\log x, \mathfrak{g}_{\log}], \quad \mathfrak{i}_{\log,m} = [\log x, \mathfrak{i}_{\log,m-1}], \; m \geq 2. \tag{34}$$

$\mathfrak{i}_{\log,m}$ is an Abelian ideal of $\mathfrak{g}_{\log}$ and $\mathfrak{i}_{\log,n}$, $n < m$. As a vector space, $\mathfrak{i}_{\log,m}$ is spanned by $\{\mathrm{B}_{\log,k} : k \geq m\}$. $\mathfrak{g}_{\log}/\mathfrak{i}_{\log,1}$ is an Abelian Lie algebra and $\mathfrak{g}_{\log}/\mathfrak{i}_{\log,2}$ is isomorphic to Heisenberg Lie algebra. Hence we may consider $\mathfrak{g}_{\log}$ to be an Abelian extension of Heisenberg Lie algebra. We denote

$$k_q^p : \mathfrak{g}_{\log}/\mathfrak{i}_{\log,q} \to \mathfrak{g}_{\log}/\mathfrak{i}_{\log,p}, \; q > p.$$

Then, since $\cap_{m\geq 1}\mathfrak{i}_{\log,m} = \{0\}$, we have

$$\mathfrak{g}_{\log} \cong \varprojlim\{\mathfrak{g}_{\log}/\mathfrak{i}_{\log,m}; k_{m+1}^m\}.$$

Since each $\mathfrak{g}_{\log}/\mathfrak{i}_{\log,m}$ is a nilpotent Lie algebra, we obtain

Theorem 4. $\mathfrak{g}_{\log}$ *is a projective limit of nilpotent Lie algebras.*

By (26), we have $[\mathcal{R}, \mathrm{B}_{\log,m}] = 0$ and

$$[\log x, \mathfrak{g}_{\mathcal{R}}] = \mathfrak{i}_{\log,1}.$$

Hence $\mathfrak{i}_{\log,m}$, $m \geq 1$, is contained in $\mathfrak{g}_{\mathcal{R}}$, and $\mathfrak{g}_{\mathcal{R}}$ is also a projective limit of nilpotent Lie algebras. Similarly to $\mathfrak{g}_{\log}$, $\mathfrak{g}_{\mathcal{R}}$ can be regarded as an Abelian extension of Heisenberg Lie algebra.

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