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B. Y. CHEN'S INEQUALITY AND ITS APPLICATIONS TO SLANT IMMERSIONS INTO LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLDS

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Abstract. In this paper we establish B.Y. Chen's inequality for a submanifold of a locally conformal almost cosymplectic manifold of pointwise constant ϕ-sectional curvature, tangent to the structure vector field of the ambient space. Some applications to slant submanifolds are also discussed.

1. Introduction

In the study of submanifold theory, one of the basic interests is to find relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Let M be an n-dimensional Riemannian manifold. For each point $p \in M$, put

$$(\inf K)(p) = \inf\{K(\pi) : \ \textit{plane sections } \pi \subset T_pM\}$$

where $K(\pi)$ denotes the sectional curvature of M associated with π. Let

$$\delta_M(p) = \tau(p) - \inf K(p), \tag{1.1}$$

where τ is scalar curvature of M. Then δ_M is a Riemannian invariant introduced by Chen [4], [6].

For an n-dimensional submanifold M in a real-space form $\overline{R}^m(c)$, Chen established the following basic inequality involving the intrinsic invariant δ_M and the squared mean curvature of the immersion

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)} \| H \|^2 + \frac{1}{2}(n+1)(n-2)c, \tag{1.2}$$

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The above inequality is also true for anti-invariant submanifolds in complex space forms $\overline{M}(4c)$, [7]. In [1], A. Carriazo established a contact version of Chen's inequality for submanifolds of a Sasakian space form. In [19], authors established similar inequality for submanifolds of a Kenmotsu manifold. As we know, there is an interesting class of almost contact metric manifolds, which are locally conformal almost cosymplectic manifolds. This class of manifolds includes Kenmotsu manifolds. In [18] authors established a basic inequality for submanifolds in a locally conformal almost cosymplectic manifolds and also dicussed its some applications. The purpose of the present paper is to study slant submanifolds tangent to the structure vector field in a locally conformal almost cosymplectic manifold of pointwise constant ϕ-sectional curvature.

2. Preliminaries

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η a one form and g the Riemannian metric on $\overline{M}$, such that

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi),$$

for any vector fields X, Y on $\overline{M}$.

The almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $\overline{M} \times \mathbb{R}$ defined by $J(X, \lambda\frac{d}{dt}) = (\phi X - \lambda\xi, \eta(X)\frac{d}{dt})$ is integrable, where X is tangent to $\overline{M}$, t the coordinate on $\mathbb{R}$ and λ a smooth function on $\overline{M} \times \mathbb{R}$. The manifold $\overline{M}$ is said to be normal if the almost complex structure J is integrable which is equivalent to vanishing of the torsion tensor $[\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. Let Φ denote the fundamental 2-form of $\overline{M}$ defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X, Y tangent to $\overline{M}$. If the fundamental 2-form Φ and 1-form η are closed, then $\overline{M}$ is said to be almost cosymplectic manifold. A normal almost cosymplectic manifold is cosymplectic [10]. $\overline{M}$ is called a locally conformal almost cosymplectic manifold [21] if there exists a 1-form ω such that

$$d\Phi = 2\omega \wedge \Phi, \ d\eta = \omega \wedge \eta \text{ and } d\omega = 0.$$

A necessary and sufficient condition for a structure to be normal locally conformal almost cosymplectic is [16]

$$(\overline{\nabla}_X \phi)Y = f(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.1}$$

where $\overline{\nabla}$ is the Levi-Civita connection of the Riemannian metric g and $\omega = f\eta$. From (2.1) it follows that

$$\overline{\nabla}_X \xi = f(X - \eta(X)\xi). \tag{2.2}$$

A plane section σ in $T_p\overline{M}$ of an almost contact metric manifold $\overline{M}$ is called a ϕ-section if it is spanned by X and ϕX, where X is an unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called ϕ-sectional curvature. $\overline{M}$ is of pointwise constant ϕ-sectional curvature if at each point $p \in \overline{M}$, the sectional curvature $\overline{K}(\sigma)$ does not depend on the choice of the ϕ-section σ of $T_p\overline{M}$ and in this case for $p \in \overline{M}$ and for any ϕ-section σ of $T_p\overline{M}$, the function c defined by $c(p) = \overline{K}(\sigma)$ is called the ϕ-sectional curvature of $\overline{M}$. A locally conformal almost cosymplectic manifold $\overline{M}$ of dimension ≥ 5 is of pointwise constant ϕ-sectional curvature if and only if its curvature tensor $\overline{R}$ is of the form [21]

$$\begin{aligned}(2.3)\quad \overline{R}(X,Y,Z,W) &= \frac{c-3f^2}{4}\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ &+ \frac{c+f^2}{4}\{g(X,\phi W)g(Y,\phi Z) - g(X,\phi Z)g(Y,\phi W) \\ &- 2g(X,\phi Y)g(Z,\phi W)\} - (\frac{c+f^2}{4} + f')\{g(X,W)\eta(Y)\eta(Z) \\ &- g(X,Z)\eta(Y)\eta(W) + g(Y,Z)\eta(X)\eta(W) - g(Y,W)\eta(X)\eta(Z)\}\end{aligned}$$

where f is the function such that $\omega = f\eta$, $f' = \xi f$; and c is the pointwise constant ϕ-sectional curvature of $\overline{M}$.

Let M be an $(n+1)$-dimensional submanifold of a $(2m+1)$-dimensional normal locally conformal almost cosymplectic manifold $\overline{M}(c)$ of pointwise constant ϕ-sectional curvature c. The Gauss and Weingarten formulae are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\overline{\nabla}$, ∇ and $\nabla^{\perp}$ are Riemannian, induced Riemannian and induced normal connections in $\overline{M}$, M and the normal bundle $T^{\perp}M$ of M respectively and h is the second fundamental form related to the shape operator A by

$$g(h(X,Y),N) = g(A_N X, Y).$$

From equation (2.2), we have

$$h(X,\xi)=0. \tag{2.4}$$

The equation of Gauss is given by

$$\begin{aligned}\overline{R}(X,Y,Z,W) &= R(X,Y,Z,W)\\ &\quad + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)),\end{aligned} \tag{2.5}$$

for any vector fields X, Y, Z and W tangent to M, where we denote as usual

$$R(X,Y,Z,W) = -g(R(X,Y)Z,W).$$

Let $p \in M$ and $\{e_1, e_2, \ldots, e_{n+1}\}$ an orthonormal basis of the tangent space T_pM. The mean curvature vector $H(p)$ at $p \in M$ is

$$H(p) = \frac{1}{n+1}\sum_{i=1}^{n+1} h(e_i,e_i). \tag{2.6}$$

The submanifold M is totally geodesic in $\overline{M}$ if $h = 0$ and minimal if $H = 0$. We set

$$h^r_{ij} = g(h(e_i,e_j),e_r) \text{ and } \|h\|^2 = \sum_{i,j=1}^{n+1} g(h(e_i,e_j),h(e_i,e_j)). \tag{2.7}$$

For any $X \in TM$, we put $\phi X = TX + NX$, where TX (resp. NX) is the tangential (resp. normal) component of ϕX.

We assume that the structure vector field ξ is tangential to M. Then the tangent bundle TM can be decomposed as $TM = D \oplus \langle\xi\rangle$, where D is orthogonal distribution to $\langle\xi\rangle$ in TM.

Given a local orthonormal frame $\{e_1, \ldots, e_n\}$ of D, we can define the squared norms of T and N by

$$\|T\|^2 = \sum_{i,j=1}^{n} g^2(e_i,Te_j),\ \|N\|^2 = \sum_{i=1}^{n} |Ne_i|^2, \text{ respectively.} \tag{2.8}$$

It is easy to see that both $\|T\|^2$ and $\|N\|^2$ are independent of the choice of the above orthonormal frame. The submanifold M is said to be invariant if N is identically zero i.e. $\phi X \in TM$ for any $X \in TM$. On the other hand, M is said to be anti-invariant if T is identically zero, that is, $\phi X \in T^\perp M$ for any $X \in TM$.

For each non-zero $X \in T_pM$, such that X is not proportional to ξ_p, let $\theta(X)$ be the angle between ϕX and T_pM. Then M is said to be slant [14] if the angle $\theta(X)$ is a constant, which is independent of the choice of $p \in M$ and $X \in T_pM - \langle \xi_p \rangle$. The angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

For a θ-slant submanifold M of an almost contact metric manifold $\overline{M}$, we have [14]

$$g(TX, TY) = \cos^2\theta(g(X, Y) - \eta(X)\eta(Y)) \tag{2.9}$$

$$g(NX, NY) = sin^2\theta(g(X, Y) - \eta(X)\eta(Y)) \tag{2.10}$$

for any $X, Y \in TM$. We also have

$$\sum_{j=1}^{n} g^2(e_i, \phi e_j) = \cos^2\theta, \tag{2.11}$$

for any $i = 1, 2, \ldots, n$ where $\{e_1, e_2, \ldots, e_n, \xi\}$ is a local orthonormal frame of TM.

For an orthonormal basis $\{e_1, e_2, \ldots, e_{n+1}\}$ of $T_pM,\ p \in M$, the scalar curvature τ is defined by

$$\tau = \sum_{i<j} K(e_i \wedge e_j), \tag{2.12}$$

where $K(e_i \wedge e_j)$ denotes the scalar curvature of M associated with the plane section spanned by e_i, e_j.

In particular for $e_{n+1} = \xi_p$, we have

$$2\tau = \sum_{i \neq j}^{n} K(e_i \wedge e_j) + 2\sum_{i=1}^{n} K(e_i \wedge \xi). \tag{2.13}$$

3. B.Y. Chen's inequality

Let M be an $(n+1)$-dimensional submanifold of $\overline{M}(c)$, tangent to the structure vector field ξ and $\pi \subset D_p$ a plane section at $p \in M$, orthogonal to ξ_p. Then

$$\alpha(\pi) = g^2(e_1, \phi e_2), \tag{3.1}$$

is a real number in $[0,1]$, independent of the choice of the orthonormal basis $\{e_1,e_2\}$ of π. Let τ and $K(\pi)$ be the scalar curvature and the sectional curvature of M associated with π respectively.

We recall following Lemma [6]:

LEMMA 3.1. *Let $a_1,\ldots,a_k,c$ be $k+1(k\geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^{k} a_i\right)^2 = (k-1)\left(\sum_{i=1}^{k} a_i^2 + c\right).$$

Then, $2a_1a_2 \geq c$ with equality holding if and only if

$$a_1+a_2=a_3=\cdots=a_k.$$

Now, we have following theorem:

THEOREM 3.2. *Let M be an $(n+1)$-dimensional $(n\geq 2)$ submanifold isometrically immersed in a $(2m+1)$-dimensional normal locally conformal almost cosymplectic manifold $\overline{M}(c)$ of pointwise constant ϕ-sectional curvature c, tangent to the structure vector field ξ. Then for each point $p\in M$ and any plane section $\pi\subset D_p$, we have*

$$\begin{aligned}(3.2)\qquad \tau-K(\pi) \leq{}& \frac{(n+1)^2(n-1)}{2n}\,\|H\|^2-\frac{1}{2}(n+2)(n-1)f^2-nf' \\ &+\left(\frac{c+f^2}{8}\right)\left(3\,\|T\|^2-6\alpha(\pi)+(n+1)(n-2)\right)-\|N\|^2.\end{aligned}$$

The equality in (3.2) holds at $p\in M$ if and only if there exists an orthonormal basis $\{e_1,\ldots,e_{n+1}\}$ of T_pM and an orthonormal basis $\{e_{n+1},\ldots,e_{2m+1}\}$ of $T_p^{\perp}M$ such that

(a) $e_{n+1}=\xi_p$,
(b) *π is spanned by e_1,e_2 and*
(c) *the shape operators $A_r=A_{e_r},\ r=n+2,\ldots,2m+1$,*

take the following forms:

$$(3.3)\qquad A_{n+2}=\begin{pmatrix}\lambda & 0 & 0 & \mu_1^{n+2}\\ 0 & \mu & 0 & ..\\ 0 & 0 & (\lambda+\mu)I_{n-2} & \mu_n^{n+2}\\ \mu_1^{n+2} & .. & \mu_n^{n+2} & 0\end{pmatrix}$$

$$(3.4)\qquad A_r=\begin{pmatrix}h_{11}^r & h_{12}^r & 0 & \mu_1^r\\ h_{12}^r & -h_{11}^r & 0 & ..\\ 0 & 0 & (\lambda+\mu)I_{n-2} & \mu_n^r\\ \mu_1^r & .. & \mu_n^r & 0\end{pmatrix},\quad r=n+3,\ldots,2m+1,$$

where $\mu_i^r = g(\phi e_i, e_r)$ *for* $i = 1, \ldots, n; r = n+2, \ldots, 2m+1$.

Proof. From (2.3), (2.5), (2.8) and (2.13), we have following relation between the scalar curvature and mean curvature of M:

$$2\tau = (n+1)^2 \|H\|^2 - \|h\|^2 + n(n+1)\left(\frac{c-3f^2}{4}\right) - 2n\left(\frac{c+f^2}{4} + f'\right) + 3\frac{c+f^2}{4}\|T\|^2, \tag{3.5}$$

where $\|h\|^2$ denotes the norm of the second fundamental form h.

Let us put

$$\epsilon = 2\tau - \frac{(n+1)^2(n-1)}{n}\|H\|^2 - (n+1)(n-2)\left(\frac{c-3f^2}{4}\right) - \frac{3(c+f^2)}{4}\|T\|^2 + 2n\left(\frac{c+f^2}{4} + f'\right). \tag{3.6}$$

Then, from (3.5) and (3.6), we get

$$(n+1)^2\|H\|^2 = n\|h\|^2 + n\left(\epsilon - \frac{2(c-3f^2)}{4}\right). \tag{3.7}$$

Let $p \in M$ and $\pi \subset D_p$ be a plane section. We choose an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ of T_pM and an orthonormal basis $\{e_{n+2}, \ldots, e_{2m+1}\}$ of $T_p^\perp M$ such that $e_{n+1} = \xi_p$, $\pi = span\{e_1, e_2\}$ and the mean curvature vector $H(p)$ is parallel to e_{n+2}.

Hence, from (3.7), we get

$$\left(\sum_{i=1}^{n+1} h_{ii}^{n+2}\right)^2 = n\left\{\sum_{i=1}^{n+1}(h_{ii}^{n+2})^2 + \sum_{i\neq j}(h_{ij}^{n+2})^2 + \sum_{r=n+3}^{2m+1}\sum_{i,j=1}^{n+1}(h_{ij}^r)^2 + \epsilon - \frac{2(c-3f^2)}{4}\right\}. \tag{3.8}$$

Now, applying Lemma (3.1), we have

$$2h_{11}^{n+2}h_{22}^{n+2} \geq \sum_{i\neq j}(h_{ij}^{n+2})^2 + \sum_{r=n+3}^{2m+1}\sum_{i,j=1}^{n+1}(h_{ij}^r)^2 + \epsilon - \frac{2(c-3f^2)}{4}. \tag{3.9}$$

From (2.3),(2.5), we have

$$K(\pi) = \frac{c-3f^2}{4} + \frac{3}{4}(c+f^2)\alpha(\pi) + h_{11}^{n+2}h_{22}^{n+2} - (h_{12}^{n+2})^2 + \sum_{r=n+3}^{2m+1}(h_{11}^r h_{22}^r - (h_{12}^r)^2). \tag{3.10}$$

Then, using (3.9),(3.10), we have

$$K(\pi) \geq \sum_{r=n+2}^{2m+1}\sum_{j=3}^{n}\{(h^r_{1j})^2+(h^r_{2j})^2\}+\frac{1}{2}\sum_{i\neq j>2}^{n}(h^{n+2}_{ij})^2 + \frac{1}{2}\sum_{r=n+3}^{2m+1}\sum_{i,j=3}^{n}(h^r_{ij})^2+\frac{1}{2}\sum_{r=n+3}^{2m+1}(h^r_{11}+h^r_{22})^2 + \frac{\epsilon}{2}+\frac{3}{4}(c+f^2)\alpha(\pi)+\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h^r_{in+1})^2. \tag{3. 11}$$

Now, from (2.4) and (2.8), we get

$$\sum_{r=n+2}^{2m+1}\sum_{i=1}^{n}(h^r_{in+1})^2 = \|N\|^2 . \tag{3. 12}$$

In view of (3.6), (3.11) and (3.12), we obtain (3.2).

If the equality in (3.2) holds, then the inequalities in (3.9) and (3.11) become equalities. So using Lemma (3.1) and (2.4), we have

$$h^{n+2}_{1j}=0,\ h^{n+2}_{2j}=0,\quad h^{n+2}_{ij}=0,\ 2<i\neq j<n;$$

$$h^r_{1j}=h^r_{2j}=h^r_{ij}=0,\ r=n+3,\dots,2m+1,\ i,j=3,\dots,n,$$

$$h^{n+3}_{11}+h^{n+3}_{22}=\cdots=h^{2m+1}_{11}+h^{2m+1}_{22}=0;$$

$$h^{n+2}_{11}+h^{n+2}_{22}=h^{n+2}_{33}=\cdots=h^{2m+1}_{n+1n+1}=0.$$

Hence, if we also choose e_1, e_2 such that $h^{n+2}_{12}=0$, then we obtain (3.3) and (3.4). The converse can be proved by straightforward computation.

Now, for each point $p \in M$, we define

$$(\inf_D K)(p)=\inf\{K(\pi):\ \text{plane section } \pi\subset D_p\}.$$

Thus, $\inf_D K$ is a well-defined function on M.

Let us define δ^D_M as the difference between the scalar curvature and $\inf_D K$, that is,

$$\delta^D_M(p)=\tau(p)-\inf_D K(p). \tag{3. 13}$$

Then, from (1.1) and (3.13) it follows that

$$\delta^D_M \leq \delta_M. \tag{3. 14}$$

On the other hand, if M is anti-invariant immersion, then $\|T\|^2 = 0$, $\|N\|^2 = n$ and $\alpha(\pi) = 0$, for any plane section π orthogonal to ξ. Hence from (3.2), we have

Corollary 3.3. *Let M be an $(n+1)$-dimensional anti-invariant submanifold isometrically immersed in a $(2m+1)$-dimensional normal locally conformal almost cosymplectic manifold $\overline{M}(c)$ of pointwise constant ϕ-sectional curvature c. Then we have*

$$\delta_M^D \le \frac{(n+1)^2(n-1)}{2n}\|H\|^2 - \frac{1}{2}(n+2)(n-1)f^2 - nf' - n + \frac{c+f^2}{8}(n+1)(n-2). \tag{3.15}$$

4. Applications to slant immersions

Let M be an $(n+1)$-dimensional $(n \ge 2)$, θ-slant submanifold of an almost contact metric manifold. Then using (2.8), (2.10) and (2.11), we get

$$\|T\|^2 = n\cos^2\theta \text{ and } \|N\|^2 = n\sin^2\theta. \tag{4.1}$$

Now, from (3.2) and (4.1), we have

Theorem 4.1. *Let M be an $(n+1)$-dimensional $(n \ge 2)$, θ-slant submanifold isometrically immersed in a $(2m+1)$-dimensional normal locally conformal almost cosymplectic manifold $\overline{M}(c)$ of pointwise constant ϕ-sectional curvature c. Then, for each point $p \in M$ and any plane section $\pi \subset D_p$, we have*

$$\tau - K(\pi) \le \frac{(n+1)^2(n-1)}{2n}\|H\|^2 - \frac{1}{2}(n+2)(n+1)f^2 - nf' - n\sin^2\theta + \frac{c+f^2}{8}(3n\cos^2\theta - 6\alpha(\pi) + (n+1)(n-2)). \tag{4.2}$$

Corollary 4.2. *Let M be a 3-dimensional θ-slant submanifold of $\overline{M}(c)$, then*

$$\delta_M^D \le \frac{9}{4}\|H\|^2 - 2(f^2 + f' + \sin^2\theta), \tag{4.3}$$

with equality holding if $f = 1$ and M is minimal invariant submanifold.

Proof. For $n = 2$, we have

$$\delta_M^D = \tau - K(D) \quad \text{and} \quad \alpha(\pi) = \cos^2\theta. \tag{4.4}$$

So (4.3) follows from (4.2) and (4.4).

On the other hand, we have

$$\tau = K(e_1 \wedge e_2) + K(e_1 \wedge \xi) + K(e_2 \wedge \xi) \quad \text{and}$$
$$K(D) = K(e_1 \wedge e_2).$$

So, $\tau - K(D) = K(e_1 \wedge \xi) + K(e_2 \wedge \xi) = -(f^2 + 2f')$.

Therefore, equality in (4.3) holds if $f = 1$ and M is minimal invariant submanifold.

5. Chen's inequality for semi-slant submanifolds

At first, we recall:

Definition 5.1. ([13]) A differentiable distribution D on M is called a slant distribution if for each $x \in M$ and each non-zero vector $X \in D_x$, the angle $\theta_D(X)$ between ϕX and the vector subspace D_x is constant, which is independent of the choice of $x \in M$ and $X \in D_x$. In this case, the constant angle θ_D is called the slant angle of the distribution D.

Definition 5.2. ([13]) A submanifold M tangent to ξ is said to be a semi-slant submanifold of $\overline{M}$ if there exist two orthogonal distributions D_1 and D_2 on M such that

(i) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2 \oplus \{\xi\}$,
(ii) the distribution D_1 is an invariant distribution, that is $\phi(D_1) = D_1$,
(iii) the distribution D_2 is slant with angle $\theta \neq 0$.

In this section, we establish Chen's inequality for proper semi-slant submanifolds in a locally conformal almost cosymplectic manifold of pointwise constant ϕ-sectional curvature. We consider plane sections σ orthogonal to ξ and invariant by P and denote $\dim D_1 = 2d_1$ and $\dim D_2 = 2d_2$.

Theorem 5.1. *Let M be an n-dimensional proper semi-slant submanifold in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\overline{M}(c)$ of* *pointwise constant ϕ-sectional curvature c. Then*

$$(5.1)\qquad K(\pi) \geq \tau - \frac{n-2}{2}\left\{\frac{n^2}{n-1}\,\|H\|^2 - \left(\frac{c-3f^2}{4}\right)(n+1)\right\} + \frac{c+f^2}{4}\{3d_2\cos^2\theta + 3(d_1-1) - (n-1)\} - (n-1)f',$$

for any plane section π invariant by P and tangent to D_1, and

$$(5.2)\qquad K(\pi) \geq \tau - \frac{n-2}{2}\left\{\frac{n^2}{n-1}\,\|H\|^2 - \left(\frac{c-3f^2}{4}\right)(n+1)\right\} + \frac{c+f^2}{4}\{3(d_2-1)\cos^2\theta + 3d_1 - (n-1)\} - (n-1)f',$$

for any plane section π invariant by P and tangent to D_2.

The equality case of inequalities (5.1) *and* (5.2) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_n = \xi\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}, e_{2m+1}\}$ of $T_p^{\perp}M$ such that the shape operators of M in $\overline{M}(c)$ at p have the following forms:*

$$(5.3)\qquad A_{n+1} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & \mu I_{n-2} & 0 \end{pmatrix},\ a+b=\mu,$$

$$(5.4)\qquad A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & 0 \\ h_{12}^r & -h_{11}^r & 0 & 0 \\ 0 & 0 & 0_{n-2} & 0 \end{pmatrix},\ r \in \{n+2, \ldots, 2m+1\}.$$

Proof. Let $p \in M$. We choose an orthonormal basis $\{e_1, e_2, \ldots, e_n = \xi\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}, e_{2m+1}\}$ of $T_p^{\perp}M$.

For $X = Z = e_i$ and $Y = W = e_j$, $\forall i, j \in \{1, 2, \ldots, n\}$, the equation (2.3) implies that that

$$(5.5)\qquad \sum_{i,j} \overline{R}(e_i, e_j, e_i, e_j) = \frac{c-3f^2}{4}(n-n^2) - \frac{3}{4}(c+f^2)\sum_{i,j=1}^{n} g^2(\phi e_i, e_j) + 2(n-1)\left(\frac{c+f^2}{4} + f'\right).$$

Let M be a proper semi-slant submanifold of $\overline{M}(c)$ and $\dim M = n = 2d_1 +$

$2d_2 + 1$. We consider an adapted orthonormal frame

$$\begin{aligned}(5.6)\qquad & e_1, e_2 = \phi e_1, \ldots, e_{2d_1-1}, e_{2d} = \phi e_{2d_1-1},\\ & e_{2d_1+1}, e_{2d_1+2} = \frac{1}{\cos\theta} P e_{2d_1+1}, \ldots, e_{2d_1+2d_2-1},\\ & e_{2d_1+2d_2} = \frac{1}{\cos\theta} P e_{2d_1+2d_2-1}, e_{2d_1+2d_2+1} = \xi.\end{aligned}$$

From (5.6), we have

$$\begin{aligned}(5.7)\qquad g^2(\phi e_i, e_{i+1}) &= 1, \text{ for } i \in \{1, \ldots, 2d_1 - 1\}\\ &= \cos^2\theta, \text{ for } i \in \{2d_1 + 1, \ldots, 2d_1 + 2d_2 - 1\}.\end{aligned}$$

Then, we get

$$(5.8)\qquad \sum_{i,j=1}^{n} g^2(\phi e_i, e_j) = 2(d_1 + d_2 \cos^2\theta).$$

From (5.5), we have

$$\begin{aligned}(5.9)\qquad \sum_{i,j} \overline{R}(e_i, e_j, e_i, e_j) = {} & \frac{c - 3f^2}{4}(n - n^2) - \frac{3}{2}(c + f^2)(d_1 + d_2\cos^2\theta)\\ & + 2\Big(\frac{c + f^2}{4} + f'\Big)(n - 1).\end{aligned}$$

The equation (5.9) implies that

$$\begin{aligned}(5.10)\qquad 2\tau = {} & n^2 \,\|H\|^2 - \|h\|^2 - \Big(\frac{c - 3f^2}{4}\Big) n(n-1)\\ & - \frac{3}{2}(c + f^2)(d_1 + d_2\cos^2\theta) + 2\Big(\frac{c + f^2}{4} + f'\Big)(n - 1).\end{aligned}$$

We set

$$\begin{aligned}(5.11)\qquad \rho = {} & 2\tau - \frac{n^2}{n-1}(n-2)\,\|H\|^2 + \Big(\frac{c - 3f^2}{4}\Big) n(n-1)\\ & + \frac{3}{2}(c + f^2)(d_1 + d_2\cos^2\theta) - 2(n-1)\Big(\frac{c + f^2}{4} + f'\Big).\end{aligned}$$

Then we obtain

$$(5.12)\qquad n^2\,\|H\|^2 = (n-1)(\rho + \|h\|^2).$$

Let $p \in M$, $\pi \subset T_pM$, $\dim\ \pi = 2$, π orthogonal to ξ and invariant by P. Now, we consider two cases:

(i) The plane section π is tangent to D_1. We may assume that $\pi = span\{e_1, e_2\}$. We choose $e_{n+1} = \frac{H}{\|H\|}$.

By using (5.12), we have

(5.13)
$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left\{\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 + \rho\right\}.$$

From Lemma (3.1) and (5.13), we have

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 + \rho. \tag{5.14}$$

From the Gauss equation, for $X = Z = e_1$ and $Y = W = e_2$, we have

$$\begin{aligned} K(\pi) &= -\frac{c-3f^2}{4} - \frac{3}{4}(c+f^2) + \sum_{r=n+1}^{2m+1}(h_{11}^r h_{22}^r - (h_{12}^r)^2) \\ &\geq -\left(\frac{c-3f^2}{4}\right) - \frac{3}{4}(c+f^2) + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 \\ &\quad + \frac{\rho}{2} + \sum_{r=n+2}^{2m+1} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m+1}(h_{12}^r)^2 \\ &= -\frac{c-3f^2}{4} - 3\left(\frac{c+f^2}{4}\right) + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j>2}(h_{ij}^r)^2 \\ &\quad + \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r + h_{22}^r)^2 + \sum_{j>2}\left((h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2 + \frac{\rho}{2}\right). \end{aligned} \tag{5.15}$$

From (5.15), it follows that

$$K(\pi) \geq -\left(\frac{c-3f^2}{4}\right) - 3\left(\frac{c+f^2}{4}\right) + \frac{\rho}{2}. \tag{5.16}$$

In view of (5.11) and (5.16), we obtain (5.1).

(i) Similarly, if π is tangent to D_2, we obtain (5.2).

The equality in (5.1) holds if and only if (5.14), (5.15), (5.16) and Lemma 3.1

become equalities. In this case, we have

$$
\begin{aligned}
&h_{ij}^{n+1} = 0,\ \forall\ i \neq j,\ i,j > 2\\
&h_{ij}^{r} = 0,\ i \neq j,\ i,j > 2,\ r = n+1,\ldots,2m+1\\
&h_{11}^{r} + h_{22}^{r} = 0,\ \forall\ r = n+2,\ldots,2m+1\\
&h_{1j}^{n+1} = h_{2j}^{n+1} = 0,\ j > 2\\
&h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}.
\end{aligned}
$$

Furthermore, if we choose e_1 and e_2 so that $h_{12}^{n+1} = 0$ and denote $a = h_{11}^{r},\ b = h_{22}^{r}$ and $\mu = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}$, then the shape operators take the desired forms.

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