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DIAGONAL SLIDES AND ROTATIONS IN QUADRANGULATIONS ON THE SPHERE

By

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Abstract. We shall show that any two quadrangulations on the sphere with n vertices can be transformed into each other by at most $6n-32$ diagonal slides and rotations if $n \geq 6$.

1. Introduction

A *quadrangulation* G on a closed surface F^2 is a map of a simple graph (with no loops and no multiple edges) embedded on F^2 such that each face is quadrilateral. Suppose that a quadrangulation G has a hexagonal region $v_1v_2v_3v_4v_5v_6$ with unique diagonal v_1v_4 and no inner vertices. The *diagonal slide* is an operation replacing the diagonal v_1v_4 with v_2v_5, or with v_3v_6 (see Figure 1). If a diagonal slide yields multiple edges or loops, then we don't apply it. This operation clearly transforms a quadrangulation to another quadrangulation on the same surface.

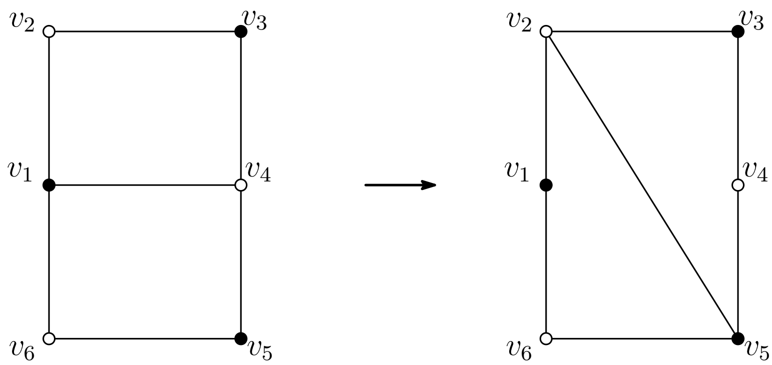

Figure 1 The diagonal slide.

Let f be a 2-cell region of F^2, bounded by a cycle of length 4 in G, which

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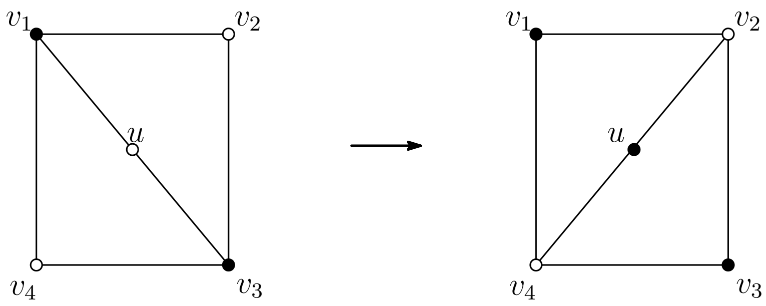

Figure 2 The diagonal rotation.

contains only a single vertex u of degree 2. Let v_1, v_2, v_3 and v_4 be four vertices of G lying on the boundary of f in this order and assume that u is adjacent to v_1 and v_3. The *diagonal rotation* is an operation replacing the edges uv_1 and uv_3 with uv_2 and uv_4, respectively. The diagonal rotation also transforms a quadrangulation into a quadrangulation on the same surface (see Figure 2). Two quadrangulations G and G' on a closed surface F^2 are said to be *equivalent* to each other if they can be transformed into each other by a finite sequence of diagonal slides and diagonal rotations, up to homeomorphism.

The following theorem has been proved by the first author of the paper [3], and related topics are in [4, 5].

Theorem 1. *For any closed surface F^2, there exists a positive integer $M(F^2)$ such that any two bipartite quadrangulations G_1 and G_2 with $|V(G_1)| = |V(G_2)| \geq M(F^2)$ are equivalent to each other under diagonal slides and diagonal rotations, up to homeomorphism.*

In this paper, we deal with quadrangulations only on the sphere S^2. It is known that every quadrangulation on S^2 is bipartite, and $M(S^2) = 4$ in the above theorem. In this paper, focusing on the number of diagonal slides and rotations needed to transform given two quadrangulations, we shall prove the following.

Theorem 2. *Any two quadrangulations G_1 and G_2 with $|V(G_1)| = |V(G_2)| = n$ on the sphere can be transformed into each other, up to homeomorphism, by at most $6n - 32$ diagonal slides and rotations if $n \geq 6$.*

This research for quadrangulations has been motivated by the earlier works for *diagonal flips* in triangulations; e.g., see [1, 2, 6]. (A *triangulation* on a closed surface F^2 is a graph on F^2 such that each face is triangular, and a *diagonal flip* of an edge e is to replace e in the quadrilateral formed by the two triangular faces sharing e by another diagonal.)

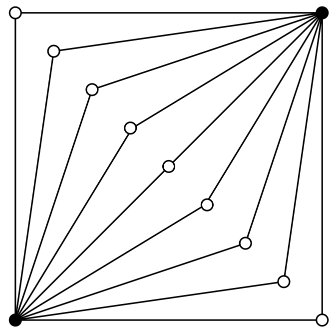

Figure 3 The standard form of quadrangulations on the sphere.

2. Proof of the theorem

The vertex set $V(G)$ of a bipartite graph G can be uniquely decomposed into two independent sets, called the *partite sets* of G. We denote these by $V_B(G)$ and $V_W(G)$ respectively and always consider a fixed 2-vertex coloring which assigns black to each vertex in $V_B(G)$ and white to one in $V_W(G)$.

If a quadrangulation G is bipartite, then $\{v_1, v_3, v_5\}$ and $\{v_2, v_4, v_6\}$ in Figure 1 are contained in $V_B(G)$ and $V_W(G)$ separately before and after the deformation. So a diagonal slide preserves the bipartiteness of quadrangulations and does not change the partite sets. On the other hand, a diagonal rotation also preserves the bipartiteness but change the partite sets, that is, it changes the color of u in Figure 2. Thus, both a diagonal slide and a diagonal rotation are needed to transform two bipartite quadrangulations with the same number of vertices but different size of partite sets.

Since both of diagonal slide and diagonal rotation preserve the bipartiteness of quadrangulations, a bipartite quadrangulation and a non-bipartite one on a same surface can never be equivalent to each other even if they have the same and sufficiently large number of vertices. The following lemma is just an exercise.

Lemma 3. *Every quadrangulation on the sphere is bipartite.*

Let Γ_n denote a complete bipartite graph $K_{2,n+2}$ embedded on the sphere as a quadrangulation (see Figure 3). Note that Γ_n has $n+2$ vertices of degree 2 and two vertices of degree $n+2$. Thus, Γ_0 represents the unique minimum quadrangulation on the sphere with 4 vertices. We call it a *standard form* of quadrangulations on the sphere.

Let G be a quadrangulation on the sphere. Let u and v be two vertices of G lying on the boundary 4-cycle of a face of G but are not adjacent on the 4-

cycle. We call $\{u, v\}$ a *diagonal pair* of vertices. To prove Theorem 2, we define a constant $d_G(u, v)$ for a diagonal pair $\{u, v\}$ by:

$$d_G(u, v) = 2 \deg u + \deg v.$$

LEMMA 4. *Let G be a quadrangulation on the sphere with n vertices and let b and b' be a diagonal pair. Then G can be transformed into Γ_{n-4}, up to homeomorphism, by $3n - 6 - (2 \deg b + \deg b')$ diagonal slides and diagonal rotations.*

Proof. Let $bwb'w'$ be a face of G where $b, b' \in V_B(G)$ and $w, w' \in V_W(G)$. We first suppose that $\deg w \geq 3$ and let $b, b', b_1, b_2, \ldots, b_l$ be the neighbors of w lying in this order. If b_1 has degree 2, we apply a diagonal rotation around b_1. If b_1 has degree at least 3, then let w_1, w_2 be two distinct white vertices adjacent to b_1 such that $wb'w_1b_1$ and $wb_1w_2b_2$ form two faces of G sharing the edge wb_1. Under these conditions, if w_2 is not adjacent to b', then we apply a diagonal slide of wb_1 to join b' and w_2 and we can increase $d_G(b, b')$ by 1. If w_2 is adjacent to b', then we can replace wb' with bw_1 since $bw_1 \notin E(G)$ by the planarity. Note that this operation decreases the degree of b' but increases that of b. Therefore in both cases, applying exactly one operation, we can increase $d_G(b, b')$ by at least 1.

Next, suppose that $\deg w = 2$ and let ω_1 be the white vertex such that $bwb'\omega_1$ bounds a face of G. If $\omega_1 = w'$, then G consists of only four vertices $\{b, b', w, w'\}$ and is isomorphic to Γ_0. If $\omega_1 \neq w'$, then we carry out the same deformation inside the quadrangle $bw'b'\omega_1$ not including w as we did for the $bw'b'w$. (Now, if $\deg \omega_1 = 2$, let ω_2 be the vertex such that $b\omega_1b'\omega_2$ forms a face of G.)

Eventually, this algorithm stops when $\omega_{n-3} = w'$ and we get a sequence of vertices $w = \omega_0, \omega_1, \omega_2 \ldots, \omega_{n-3} = w'$ such that each of them has degree 2 and each $b\omega_kb'\omega_{k+1}$ bounds a face of the quadrangulation where the subscripts are taken modulo $n - 2$. This final form is clearly isomorphic to Γ_{n-4}, in which both b and b' have degree $n - 2$, and $d_{\Gamma_{n-4}}(b, b') = 3n - 6$. Since one diagonal slide or diagonal rotation corresponds to the increment 1 of $d_G(b, b')$ through the above deformations, the total number of those diagonal slides and rotations in this algorithm does not exceed:

$$d_{\Gamma_{n-4}}(b, b') - d_G(b, b') = 3n - 6 - (2 \deg b + \deg b').$$

Thus, the lemma follows. ■

Now we shall prove Theorem 2.

Proof of Theorem 2. First, we shall estimate the number of diagonal slides and rotations which transform a given quadrangulation G into the standard form

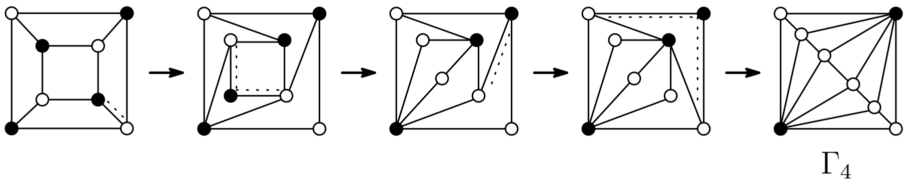

Figure 4 Cube transformed into Γ_4.

Γ_{n-4}. We would like to choose a diagonal pair $\{b, b'\}$ in G so as to maximize $d_G(b, b') = 2 \deg b + \deg b'$.

If $n \geq 9$, there is a vertex of degree at least 4. This follows from the formula for quadrangulations

$$\sum_{i \geq 2} (4 - i) V_i = 8$$

where V_i stands for the number of vertices of degree i. Choose such a vertex as b. Since a quadrangulation has no vertex of degree 1, we have $d_G(b, b') \geq 2 \times 4 + 2 = 10$ and we need at most $3n - 16$ diagonal slides and rotations to obtain Γ_{n-4} from G.

Now, we consider the case when the number of vertices n is less than 9. Assume that $n = 8$, G either has a vertex of degree 4 or is isomorphic to a cube with 8 vertices by the above formula again. Even if the latter happens, the number of operations, which transform G into Γ_4, is at most 4 (see Figure 4).

Next, we use the complete list of quadrangulations on the sphere with $4 \leq n \leq 7$ in Figure 5. When $n = 7$, it is easy to check that the numbers of operations transforming those graphs into Γ_3 are at most 2. Furthermore, if $n = 6$, a single diagonal rotation is sufficient for the theorem.

At last, consider any two quadrangulations G_1 and G_2 with n vertices on the sphere. Since each of them can be transformed into Γ_{n-4}, G_1 and G_2 can be transformed into each other via Γ_{n-4} by twice many operations as shown above. For any integer $n \geq 6$, $6n - 32$ diagonal slides and rotations are sufficient and hence the theorem holds. (Note that if $n = 4, 5$, no diagonal slide and diagonal rotation is needed since each number of vertices admits the unique quadrangulation.) ■

3. Lower bounds

In this section, we shall estimate a lower bound for the number of diagonal slides and rotations which transform a given quadrangulation into another and show

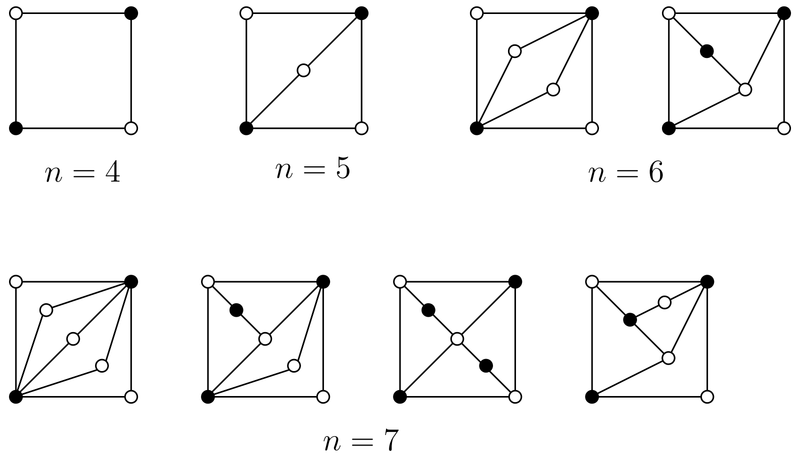

Figure 5 Quadrangulations on the sphere with $n \leq 7$.

that the linear order of the bound in Theorem 1 is best possible with respect to the number of vertices n of quadrangulations.

Let G and G' be two quadrangulations on the sphere with $V(G) = \{v_1, \ldots v_n\}$ and $V(G') = \{v'_1, \ldots v'_n\}$ and suppose that

$$\deg v_1 \leq \cdots \leq \deg v_n; \quad \deg v'_1 \leq \cdots \leq \deg v'_n.$$

Then we define the *degree difference* $D(G, G')$ by:

$$D(G, G') = \sum_{i=1}^{n} |\deg v_i - \deg v'_i|.$$

THEOREM 5. *Let G and G' be two quadrangulations on the sphere. Any sequence of diagonal slides and rotations which transforms G into G' contains at least $\frac{1}{4}D(G, G')$ those deformations.*

Proof. Let D_σ denote the number of diagonal slides and rotations in the sequence and suppose that each vertex v_i of G corresponds to a vertex $v'_{\sigma(i)}$ of G' through the sequence. We need at least $|\deg v_i - \deg v'_{\sigma(i)}|$ diagonal deformations to adjust the degree of v_i while each of them changes the degree of four vertices simultaneously. Thus,

$$D_\sigma \geq \frac{1}{4} \sum_{i=1}^{n} |\deg v_i - \deg v'_{\sigma(i)}|.$$

Considering the permutation σ over $\{1, \ldots, n\}$, we have that

$$D_\sigma \geq \frac{1}{4} \min_{\sigma} \sum_{i=1}^{n} |\deg v_i - \deg v'_{\sigma(i)}|.$$

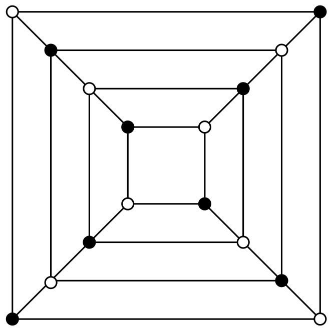

Figure 6 Quadrangulation with vertices of degree 3 and 4.

It is not so difficult to show that $\frac{1}{4}D(G, G')$ gives the right hand of this inequality. Let $d_i = \deg v_i$ and $d'_i = \deg v'_i$ and assume that $d_1 \leq \cdots \leq d_n$ and $d'_1 \leq \cdots \leq d'_n$. We can easily show the following inequality by a routine.

$$(|d_i - d'_k| + |d_j - d'_h|) - (|d_i - d'_h| + |d_j - d'_k|) \leq 0 \quad (i < j; k < h)$$

For example, when $d'_k \leq d_i \leq d'_h \leq d_j$, then

$$(|d_i - d'_k| + |d_j - d'_h|) - (|d_i - d'_h| + |d_j - d'_k|) = 2(d_i - d'_h) \leq 0.$$

This implies that

$$\sum_{i=1}^{n} |d_i - d'_i| \leq \sum_{i=1}^{n} |d_i - d'_{\sigma(i)}|.$$

Thus, we got a conclusion. ■

The standard form Γ_{n-4} has the degree sequence $(2, \ldots, 2, n-2, n-2)$ which contains $n-2$ 2's. On the other hand, we can construct a quadrangulation G_n of sufficiently large size $n = 8 + 4m$ with degree sequence $(3, \ldots, 3, 4, \ldots, 4)$ including eight 3's and $n-8$ 4's (see the quadrangulation given in Figure 6). Then, we have:

$$D(\Gamma_{n-4}, G_n) = 8 \cdot (3-2) + (n-10) \cdot (4-2) + 2 \cdot (n-2-4) = 4n - 40.$$

Thus, we need at least $n - 10$ diagonal slides and rotations to transform G_n into Γ_{n-4}, by Theorem 5. This example implies that the order of the bound in Theorem 2 is the best, as mentioned above.

References

[1] R. Brunet, A. Nakamoto, S. Negami, Diagonal flips of triangulations on closed surface preserving specified properties, *J. Combin Theory Ser. B.* **68** (1996), 295–309.

[2] H. Komuro, A. Nakamoto, S. Negami, Diagonal flips in triangulations on closed surfaces with minimum degree at least 4, *J. Combin. Theory Ser. B* **76** (1999) 68–92.

[3] A. Nakamoto, Diagonal transformations in quadrangulations of surfaces, *J. Graph Theory* **21** (1996), 289–299.

[4] A. Nakamoto, Diagonal transformations and cycle parities of quadrangulations on surfaces, *J. Combin. Theory Ser.* B **67** (1996) 202–211.

[5] A. Nakamoto, Quadrangulations on closed surfaces, *Interdiscip. Inform. Sci.* **7** (2001), 77–98.

[6] S. Negami, Diagonal flips of triangulations on surfaces, a survey. *Yokohama Math. J.* **47** (1999), 1–40.

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