Yokohama Mathematical
Journal Vol. 55, 2009

THE DISTINGUISHING NUMBERS OF 4-REGULAR QUADRANGULATIONS ON THE KLEIN BOTTLE

By

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(Received September 21, 2008)

Abstract. A graph G is said to be *d-distinguishable* if there is an assignment of d labels to vertices such that no automorphism of G other than the identity map preserves the labels of vertices. We shall prove that 4-regular quadrangulations on the Klein bottle are 2-distinguishable with few exceptions, after reviewing their classification.

Introduction

The "distinguishing number" in this paper is a combinatorial invariant defined for an abstract graph, concerning the symmetry of graphs as follows. Let G be a graph and $c : V(G) \to \{1, 2, \ldots, d\}$ an assignment of labels to the vertices of G. Such a labeling c is called a *d-distinguishing labeling* of G if no automorphism of G other than the identity map preserves the labels given by c. In other words, a d-distinguishing labeling blocks the symmetry of G up to automorphism. The *distinguishing number* of G is defined as the minimum number d such that G is d-distinguishable and is denoted by $D(G)$.

We can find some arguments on the distinguishing number of graphs in [1, 2, 7]. Furthermore, Negami [6] has established a general theorem on the distinguishing number of graphs embedded on closed surfaces, using some technique in topological graph theory. In his theory, "the faithfulness of embedding", defined later, plays an important role. Acutually, he has proved that polyhedral graphs faithfully embedded on a closed surface are 2-distinguishable with finitely many exceptions. In particular, he has developed a theory to analyze the distinguishing number of triangulations on closed surfaces, applying re-embedding theory for them established in [4]. So one might expect a theory on that of quadrangulations on closed surfaces.

As one of such attempts, Fukuda and Negami have determined the distinguishing number of 4-regular quadrangulations on the torus; most of them are

2000 Mathematics Subject Classification: 05C10, 05C25, 05C78
Key words and phrases: distinguishing, quadrangulations, Klein bottle

2-distinguishable and there are two infinite series of those quadrangulations on the torus that are not 2-distinguishable. In this paper, we shall focus on the 4-regular quadrangulations on the Klein bottle in turn and prove the following theorem:

Theorem 1. *Every 4-regular quadrangulatoin on the Klein bottle is 2-distinguishable unless it is isomorphic to one of* $Q_l(4,2)$ *and* $Q_m(2,r)$ *with* $r \geq 3$.

In general, a graph G embedded on a closed surface F^2 is said to be r-*representative* if any simple closed curve on F^2 intersects G in at least r points. The assumption of being 3-representative excludes all exceptions in the above theorem and hence we have:

Corollary 2. *Every 3-representative 4-regular quadrangulation on the Klein bottle is 2-distinguishable.* ■

There have been classified the 4-regular quadrangulations on the Klein bottle with their standard forms in [5]. The notations $Q_l(4,2)$ and $Q_m(2,r)$ appear in the classification and will be described with general cases in Section 1. Furthermore, we shall discuss the faithfulness of embedding of 4-regular quadrangulations on the Klein bottle in Section 2 and determine their distinguishing numbers in Section 3 to prove the above main theorem.

1. Classification and standard forms

A *quadrangulation* on a closed surface is a simple graph embedded on the surface so that each face is bounded by a cycle of length 4. If a quadrangulation on the Klein bottle is regular, then it is necessarily 4-regular by Euler's formula. According to the classification of 4-regular quadrangulations on the Klein bottle in [5], there are three types of those, describled below.

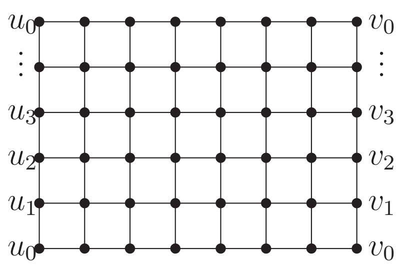

Figure 1 A planar grid used for grid and ladder types

Prepare the cylinder $C_p \times P_{r+1}$, which is obtained from the planar grid $P_{p+1} \times$

P_{r+1} depicted in Figure 1 by identifying the two horizontal sides of length r. We regard this cylinder as one embedded on an annulus in the natural way. Let $u_0u_1 \cdots u_{p-1}$ and $v_0v_1 \cdots v_{p-1}$ be the two cycles lying along the boundary of the annulus, which correspond to the two vertical sides of the grid. Their indices are given modulo p so that u_i and v_i are joined by a horizontal path in the grid for $i = 0, 1, \dots, p-1$. Identify these cycles to obtain the Klein bottle.

This identification can be exhibited by an isomorphism τ between the two cycles, which flips the direction and we have two possibilities; there is an index $i \in \{0, 1, \dots, p-1\}$ with $\tau(u_i) = v_i$ or not. If it happens, we may assume that $i = 0$ after re-labeling, and hence we have $\tau(u_i) = v_{-i}$. This is the same identification as to obtain the Klein bottle from a rectangle usually. That is, identify the two horizontal sides in parallel and the two vertical sides in anti-parallel. Then we obtain a 4-regular quadrangulation on the Klein bottle. We call this type a *grid type* and denote it by $Q_g(p, r)$. If p is odd, then we always get this type.

In the other case, the parameter p must be an even number, say $p = 2s$, and we may assume that $\tau(u_i) = v_{p-1-i}$. We have $\tau(u_0u_{p-1}) = v_{p-1}v_0$ and $\tau(u_{s-1}u_s) = v_sv_{s-1}$ in particular. Each of the two pairs of edges $\{u_0u_{p-1}, v_0v_{p-1}\}$ and $\{u_{s-1}u_s, v_{s-1}v_s\}$ are joined by a "ladder" placed horizontally in the grid and such a ladder forms what is called a *Möbius ladder* on the Klein bottle. If we remove the two Möbius bands corresponding to these Möbius ladders from the Klein bottle, then we obtain a cylinder $C_{2r} \times P_s$ although C_{2r} runs horizontally and P_s goes vertically in the grid. We call this type a *ladder type* and denote it by $Q_l(2r, s)$. Cutting the Klein bottle along each of cycles corresponding to C_{2r} results in two Möbius bands. Such a cycle or a simple closed curve is called an *equator* on the Klein bottle. Thus, $Q_l(2r, s)$ contains s equators lying in parallel.

One of common properties of the gird type $Q_g(p, q)$ and the ladder type $Q_l(2r, s)$ is that they have a *geodesic 2-factor* given as a union of cycles of length p placed along "meridians" on the Klein bottle; a *meridian* is a simple closed curve on the Klein bottle such that cutting open the Klein bottle along it results in an annulus. The difference between them is the existence of a "longitude". A *longitude* is a simple closed curve on the Klein bottle whose tubular neighborhood is homeomorphic to a Möbius band and cutting open the surface along it results in a Möbius band. We can take two disjoint logitudes, as simple closed curves, on the Klein bottle. There is only one cycle of length r in $Q_g(p, r)$ which is a longitude on the Klein bottle if p is odd while there are two if p is even. On the other hand, there is no cycle in $Q_l(2r, s)$ which runs along a longitude.

It may be convenient in some cases to regard them as the same type, say *handle types*, with notation $Q_h(p, r, \varepsilon)$. Here $\varepsilon \in \{0, 1\}$ stands to distinguish the original two types. Prepare the cylinder $C_p \times P_{r+1}$ with two ends $u_0u_1 \cdots u_{p-1}$

and $v_0v_1 \cdots v_{p-1}$ as well as in the previous. Identify these ends by an isomorphism τ between them with $\tau(u_i) = v_{p-(i+\varepsilon)}$ for $i = 0, 1, \ldots, p-1$ to obtain $Q_h(p, r, \varepsilon)$. Then we have $Q_g(p, r)$ as $Q_h(p, r, 0)$ while $Q_h(p, r, 1)$ is isomorphic to $Q_l(2r, s)$ if p is an even number $2s$. If p is odd, then both $Q_h(p, r, 0)$ and $Q_h(p, r, 1)$ are isomorphic to $Q_g(p, r)$.

There is one more type of a 4-regular quadrangulation on the Klein bottle. Now prepare the rectangular region given as $\Omega = \{(x, y) \in \boldsymbol{R}^2 : 0 \le x \le r,\ 0 \le y \le 2p\}$ on the xy-plane $\boldsymbol{R}^2$ and take the points (x, y) in Ω with integral coodinates as vertices only if $x - y \equiv 0 \pmod 2$. Add edges so that each vertex (x, y) is adjacent to $(x \pm 1, y \pm 1)$. We denote the resulting graph on the plane by $M_{p,r}$; this is depicted as in Figure 2. The two corners at the right side of $M_{p,r}$ are occupied by two vertices if r is even, while no vertex there if r is odd, as in the figure.

First identify the pair of horizontal sides of Ω to get a cylinder and next the two ends of this cylinder to obtain the Klein bottle so that it contains a 4-regular quadrangulation. This is called a *mesh type* and we denote it by $Q_m(p, r)$. One might be anxious about the ambiguity in the second identificaion, but we obtain a unique quadrangulation on the Klein bottle, up to isomorphism, in fact.

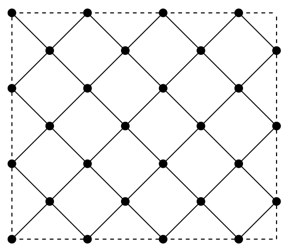

Figure 2 A planar grid used for mesh types

To see this, give labels $u_0, u_1, \ldots u_{p-1}$ to the vertices on the left side and $v_0, v_1, \ldots, v_{p-1}$ to the right side upward, as well as in case of the handle types. Each pair of u_i and v_i lie in the same horizontal level if r is even while their y-coodinates differ by 1 if r is odd. An isomorphism τ between the cycles $u_0u_1 \cdots u_{p-1}$ and $v_0v_1 \cdots v_{p-1}$ indicates the identification of two ends of the annulus. If p is odd, we always find an index i so that $\tau(u_i) = v_i$ and hence we obtain a unique result of the identification, up to isomorphism.

On the other hand, if p is even, there are two cases for the identification by τ; either $\tau(u_i) = v_i$ for two i's, or there are not such i's. In the second case, we have $\tau(u_iu_{i+1}) = v_{i+1}v_i$ for two i's modulo p. Cut open the Klein bottle along a meridian which contains the second column of vertices (x, y) with $x = 1$. It is easy to see that the identification to recover the Klein bottle for this cutting becomes the first type. Therefore, the two types of identification result in the same 4-regular quadrangulation on the Klein bottle, up to isomorphism.

Theorem 3. (Nakamoto and Negami [5]) *Every 4-regular quadrangulation on the Klein bottle is isomorphic to one of the grid types $Q_g(p, r)$, the ladder types $Q_l(2r, s)$ and the mesh types $Q_m(p, r)$ with suitable parameters p and r.*

There are some restrictions on these parameters to keep the simpleness of graphs. For the grid types, we have $p \geq 3$ and $r \geq 3$; if $r = 2$, then the longitude contained in $Q_g(p, r)$ forms a cycle of length 2. For the ladder types $Q_l(2r, s)$, we have $r \geq 2$ and $s \geq 2$; if $s = 1$, then rungs of two Möbius ladders form multiple edges. For the mesh types $Q_m(p, r)$, we have $p \geq 2$ and $r \geq 3$. For each type, the two parameters are designed so that their product is equal to the number of vertices.

There are several ways to derive another 4-regular quadrangulation from a given one. One of those is to take the dual. For example, $Q_h(p, r, 0)$ and $Q_h(p, r, 1)$ can be embedded together on the Klein bottle so that they are dual to each other after moving slightly one of them. In such an embedding, the longitude of $Q_h(p, r, 0)$ corresponding to the horizontal sides of the grid runs within the Möbius ladder of $Q_h(p, r, 1)$, crossing each of its rungs at a point. On the other hand, it is easy to see that the dual of $Q_m(p, q)$ is isomorphic to itself. We may express these situations with the following formulas:

$$Q_h(p, r, 0)^* = Q_h(p, r, 1), \quad Q_h(p, r, 1)^* = Q_h(p, r, 0), \quad Q_m(p, q)^* = Q_m(p, q)$$

Another way to make a 4-regular quadrangulation is to take “the radial graph”, as follows. Let G be a graph 2-cell embedded on a closed surface in general. Put a vertex x in each face A of G, which can be ragarded as a vertex in the dual G^* of G and add new edges from x to all vertices lying along the boundary cycle of A. The resulting graph with all edges of G removed is called the *radial graph* of G and is denoted by $R(G)$. Obviously, the same graph can be obtained from G^* in the same way. Thus, we have $R(G) = R(G^*)$.

Also, it is clear that if G is a 4-regular quadrangulation, then so is $R(G)$. Thus, $R(Q_h(p, q, \varepsilon))$ and $R(Q_m(p, r))$ can be expressed by the standard forms with suitable parameters. The former should be a mesh type since it contains neither a meridian nor a Möbius ladder. The latter should be a grid type since it contains a longitude. It is easy to see the following formulas:

$$R(Q_h(p, r, 0)) = R(Q_h(p, r, 1)) = Q_m(p, 2r), \quad R(Q_m(p, r)) = Q_g(2p, 2r)$$

The dual of the radial graph $R(G)$ of G is called the *medial graph* with notation $M(G)$ and it also is a 4-regular quadrangulation on the Klein bottle if so is G. However, $M(Q_h(p, r, \varepsilon)) = Q_m(p, 2r)^*$ is isomorphic to $Q_m(p, 2r)$ and $M(Q_m(p, r)) = Q_g(2p, 2r)^*$ is isomorphic to $Q_h(2p, 2r, 1) = Q_l(4r, p)$. Therefore, the standard forms given in [5] are closed under taking duals, radial and medial graphs, as above.

2. Faithfulness of embeddings

A graph G embedded on a closed surface F^2 is said to be *faithfully embedded* on F^2 if any automorphism $\sigma : G \to G$ extends to an auto-homeomorphism $h : F^2 \to F^2$ with $h|_G = \sigma$. The faithfulness of embeddings is very important to analyze the distinguishing number of graphs since the symmetry of a graph can be regarded as that over the surface containing it. Here we shall discuss how to recognize the faithfulness of embedding of 4-regular quadrangulations on the Klein bottle.

Let G be a graph embedded on a closed surface F^2 so that each face is bounded by a cycle. A cycle is said to be *facial* or *non-facial* if it bounds a face or not. Let $f : G \to F^2$ be another embedding of G to F^2, which is an injective continuous map between two topological spaces G and F^2. A face of G or its boundary cycle C is said to be a *panel* or to be *paneled* for f if $f(C)$ is facial in $f(G)$. The graph G is said to be *full-paneled* for f if the faces of G are all panelled for f. In this case, the embedding $f : G \to F^2$ extends to an auto-homeomorphism $h : F^2 \to F^2$ with $h|_G = f$.

An automorphism $\sigma : G \to G$ can be regarded as an embedding map $\sigma : G \to F^2$ of G to the surface F^2 where G is embedded. Thus, G is faithfully embedded on F^2 if and only if G is full-paneled for all automorphisms of G. On the other hand, if $f(G) \neq G$, then f cannot be regarded as an automorphism of G and it is another embedding of G to F^2. If G is full-paneled for such an embedding f, then we may consider that G and $f(G)$ are the same embeddeding up to auto-homeomorphism over the surface. This motivates us to define the following notions.

Two embeddings f_1 and $f_2 : G \to F^2$ of a graph G on a closed surface F^2 are said to be *equivalent* up to homeomorphism if there is an auto-homeomorphism $h : F^2 \to F^2$ over F^2 with $hf_1 = f_2$. In particular, if G admits only one equivalence class of embeddings on F^2, then G is said to be *uniquely embedded* on F^2, up to homeomorphism.

When G is embedded on F^2, the identity map id_G over G can be regarded as an embedding map of G on F^2, which is often called the *inclusion map* of G, and any embedding $f : G \to F^2$ with $f(G) = G$ induces an automorphism of G. Thus, if G is uniquely embedded on F^2, then G is faithfully embedded on F^2 since f extends to an auto-homeomorphism. However, a faithfully embedded graph does not need to be uniquely embedded.

On the other hand, if G is not faithfully embedded on F^2, then there is an automorphism σ of G which cannot extend to any auto-homeomorphism over F^2 and this σ can be regarded as an embedding of G not equivalent to the inclusion map id_G. Thus, G is not uniquely embedded on F^2, up to homeomorphism, in this case. However, it happens that all embeddings of G on F^2 looks the same if we neglect the labels of vertices and one might want to say "uniquely embedded" in such a case, too. The following definition works as he expects.

Two embeddings f_1 and $f_2 : G \to F^2$ of a graph G on a closed surface F^2 are said to be *congruent* if there exist an auto-homeomorphism $h : F^2 \to F^2$ over F^2 and an automorphism σ of G such that $hf_1 = f_2\sigma$. A graph G is uniquely embedded on F^2 *up to congruence* if all embeddings of G on F^2 are congruent to one another. In particular, all automorphisms of G are congruent to each other if we regard them as embeddings of G on F^2.

By definitons above, G is faithfully embedded on F^2 if and only if G is full-paneled for all automorphisms of G. So we would like to find many panels in the 4-regular quadrangulations on the Klein bottle to conclude that most of them are faithfully embedded. The following lemma will help us to do it. However, we do not need to take account of Condition (ii) in most cases as we shall see in our proof of Lemma 6. An embedding $f : G \to F^2$ is said to be *quadrangular* if $f(G)$ is a quadrangulation on F^2.

Lemma 4. *Let G be a 4-regular quadrangulation on the Klein bottle K^2 and v a vertex of G with four neighbors u_0, u_1, u_2 and u_3 lying around v in this cyclic order. Let $C_i = vu_iw_iu_{i+1}$ be the boundary cycles of the four faces of G which are incident to v for $i \equiv 0, 1, 2, 3 \pmod 4$. Suppose that:*

(i) *C_0 and C_1 are paneled for an quadrangular embedding $f : G \to K^2$, and*
(ii) *there is no vertex w, other than w_2, such that vu_2wu_3 forms a cycle of length 4.*

Then C_2 is paneled for f.

Proof. Since C_0 and C_1 are paneled, the rotation around v in $f(G)$ contains the segment $u_0u_1u_2$ and hence this must be completed to be $u_0u_1u_2u_3$. Thus, u_2vu_3 becomes a corner of a face in $f(G)$. By the second conditionm $vu_2w_2u_3$ is the unique candidate for the boundary cycle of the face. Thus, $vu_2w_2u_3$ is paneled for f. ■

First, we shall discuss the faithfulness of embedding of 4-regular quadrangulations of grid and ladder types on the Klein bottle, dealing with them together as handle types.

Lemma 5. *A 4-regular quadrangulation $Q_h(p, r, \varepsilon)$ of handle type on the Klein bottle contains a non-facial cycle of length 4 if and only if it is isomorphc to one of $Q_h(4, r, \varepsilon)$, $Q_h(p, 2, 1)$, $Q_h(p, 3, 1)$ and $Q_h(p, 4, 0)$ with suitable parameters (p, r, ε).*

Proof. Let $G = Q_h(p, r, \varepsilon)$ be a 4-regular quadrangulation of handle type on the Klein bottle. The integral grid on the xy-plane consisting of vertical and horizontal lines covers naturally G, according to the definition of handle types. Thus, each vertical line covers a meridian in G on the Klein bottle and the vertices corresponding to a common vertex of G occur along it in period p. Two vertical lines at distance r cover a common meridian in G. Each of vertices in this planar grid has coordinates (x, y) with integers x and y. We denote by $u_{(x,y)}$ the vertex of G corresponding to the vertex (x, y) in the planar grid.

Suppose that G contains a non-facial cycle C of length 4. Let v be one of the four vertices lying on C and assume that $v = u_{(0,0)}$ for convenience. Then C can be lifted to a path of length 4 starting from $v_0 = (0, 0)$. Let $v_1 = (s, t)$ be the other end of this path with $v = u_{(s,t)}$. We may assume that $s \geq 0$ and $t \geq 0$, up to symmetry and there are the eight cases on (s, t) depicted in Figure 3, where v_0 and v_1 are encircled in each and the sequences of short arrows give examples for the lift of C. However, some of them should be excluded immediately because of the simpleness of G; (ii) implies $p = 2$, (iii) and (iv) $r = 1$, (vi) $(r, \varepsilon) = (2, 0)$, and $u_{(0,1)}u_{(1,1)}u_{(2,1)}$ would be a cycle of length 2 in (v). Discuss the remaining cases (i), (vii) and (viii) in details below.

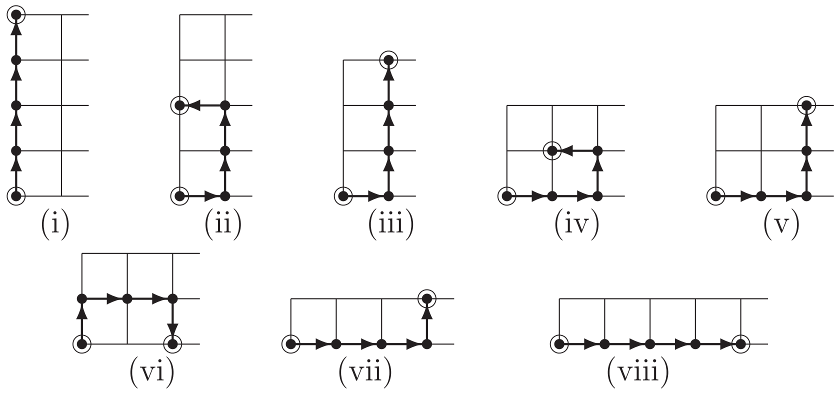

Figure 3 Non-facial cycles of length 4 in handle types

Case (i) implies that $p = 4$ and G is isomorphic to $Q_h(4, r, \varepsilon)$ for any $r \geq 3$ and any $\varepsilon \in \{0, 1\}$. On the other hand, we have $r = 3$ and 4 in Cases (vii) and (viii), respectively. It is easy to see that G is isomorphic to $Q_h(p, 3, 1)$ in Case (vii). However, we should be more careful in Case (viii); the candidates for G are not only $Q_h(p, 4, 0)$ but also $Q_h(p, 2, 1)$. The cycle C of length 4 forms a longitude in the former and an equator in the latter. ■

The above lemma suggests that it is very rare for handle types not to be faithfully embedded on the Klein bottle and so is it actually as shown in the following lemma:

Lemma 6. *A 4-regular quadrangulation $Q_h(p,q,r)$ of handle type on the Klein bottle is full-paneled for all of its quadrangular embeddings unless it is isomorphic to either $Q_h(4,2,1)$ or $Q_h(4,3,1)$.*

Proof. Let G be a 4-regular quadrangulation on the Klein bottle K^2 and suppose that G is not full-paneled for a quadrangular embedding $f : G \to K^2$. Then the inverse map $f^{-1}(G) : f(G) \to K^2$ of f also can be regarded as a quadrangular embedding of $f(G)$ to the Klein bottle and $f(G)$ is not full-paneled for f^{-1}, too. That is, there exist some non-facials cycles C in G such that $f(C)$'s are facial in $f(G)$. By Lemma 5, G is isomorphic to one of $Q_h(4,r,\varepsilon)$, $Q_h(p,2,1)$, $Q_h(p,3,1)$ and $Q_h(p,4,0)$.

Before we discuss each case, we shall investigate when Condition (ii) in Lemma 4 holds since we use it frequently. Suppose that the condition does not hold for $G = Q_h(p,r,\varepsilon)$. That is, there is a vertex w of G such that vu_2wu_3 forms a non-facial cycle of length 4 and we find one of the two situations, up to symmetry, depicted in Figure 4; two adjacent panels lie vertically or horizontally. Each of the faces containing a circle inside in the figure is a panel for f. For convenience, we use the same notations for vertiecs on the girds as in the previous proof, with $v = u_{(0,0)}$.

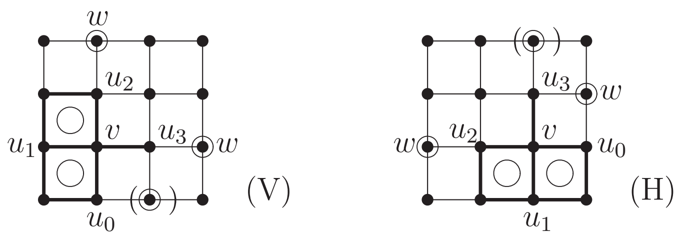

Figure 4 Around two consecutive panels in handle types

If Case (V) happens, then we have $r = 2$. However, $u_2 = u_{(0,1)}$ and $u_{(2,1)}$ must be identical in G and there would be multiple edges between u_2 and $u_{(1,1)}$, a contradiction. If Case (H) happens, then we have $r = 2$ or 3 and the former case implies the same contradiction as in Case (V). Therefore, Condition (ii) in Lemma 4 holds with (i) unless G is isomorphic to $Q_h(p,3,1)$. More precisely speaking, it does not hold only when u_2vu_3 forms a corner of a face within a Möbius ladder with three rungs.

Case 1. $G \cong Q_h(p,2,1)$: In this case, G is isomorphic to the ladder type $Q_l(4,s)$ with $p = 2s$. First suppose that $p \geq 6$. Then there are precisely s non-facial cycles C of length 4 in G and each of them forms an equator on the Klein bottle. The number of faces of G is equal to $4s$ and $f(G)$ has the same number of faces as G. Let n be the number of panels in G for f. Then the boundary cycles of $4s-n$ faces in $f(G)$ are the images of non-facial cycles in G by f. Thus, we have $4s-n \leq s$ and hence G has at least $3s$ panels for f. In this case, there are two adjacent panels clearly. That is, Condisiton (i) in Lemma 4 holds and also (ii) does by the above argument. Using the lemma repeatedly, we can recognize many panels around these two and extend it to the whole of G. Therefore, G is full-paneled for f if $p \geq 6$.

On the other hand, it is easy to see that $Q_h(4,2,1) \cong Q_l(4,2)$ is isomorphic to $K_4 \times K_2$ and it admits a re-embedding which sends a facial cycle of length 4 in the Möbius ladder to its rim. Thus, $Q_h(4,2,1)$ should be excluded as one of exceptions of the lemma.

Case 2. $G \cong Q_h(p,3,1)$: Suppose that $p \geq 5$. Then only Case (vii) in Figure 3 happens and any non-facial cycle of length 4 in G lies within one of the Möbius ladders with three rungs. Such a cycle contains only one rung and each rung is covered by precisely two such cycles. Thus, there are six non-facial cycles of length 4 within each Möbius ladder.

Suppose that four of those non-facial cycles of length 4 in one Möbius ladder are mapped to four cycles bounding faces in $f(G)$. Since the Möbius ladder has only three rungs, two of the four non-facial cycles contain a common rung and cover all edges lying along the rim once together. If the other two non-facial cycles cover the other two rungs, then we find an edge covered by three of those cycles, which is contrary to that precisely two faces are incident to each edge in $f(G)$. Otherwise, that is, if they cover a common rung, then the first two and they cover all edges along the rim doubly. In this case, any cycle of length 4 covering the third rung, facial or not, cannot be mapped to a facial cycle in $f(G)$. For, if it could, then an edge on the rim would be covered by three cycles which are mapped to facial cycles in $f(G)$, a contradiction.

Therefore, at most three of non-facial cycles of length 4 contained in one Möbius ladder are mapped to facial cycles in $f(G)$ and there are at most six in total since G has one or two Möbius ladders, depending on the parity of p. By the same argument as in the previous case, it follows that G has at least $3p-6$ panels since G has precisely $3p$ faces.

Since $p \geq 5$, we have $3p/2 < 3p-6$. This implies that there are two adjacent panels lying in a column of p faces in the grid. That is, we find an adjacent pair of panels as depicted in Case (V) in Figure 4; neglect the existence of w. Then we can find other panels in order around these two panels, confirming the two conditions (i) and (ii) in Lemma 4. Finally, we conclude that all faces except at most six faces within the two Möbius ladders are paneled for f. Furthermore, it is easy to see that the rotation around each vertex lying along the rims of the Möbius ladders is preserved by f and that the remaining faces also are paneled for f. Thus, G is full-paneled for f if $p \geq 5$.

If $p = 3$ or 4, then G is isomorphic to either $Q_h(3,3,1)$ or $Q_h(4,3,1)$. The formar is isomorphic to $Q_h(3,3,0) \cong Q_g(3,3)$ and the latter is $Q_l(6,2)$. The same argument as above works for $Q_h(3,3,1)$ since it contains only one Möbius ladder and has only three non-facial cycles of length 4. Thus, $Q_h(3,3,1)$ has at least $9-3=6$ panels for f and there are two adjacent ones among them since $9/2 < 6$. Thus, $Q_h(3,3,1)$ is full-paneled and is not an exception of the lemma.

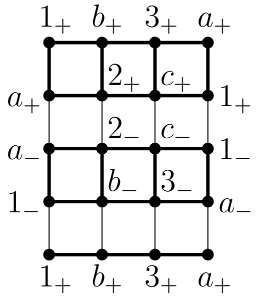

Figure 5 $Q_l(6,2)$

On the other hand, $Q_h(4,3,1) \cong Q_l(6,2)$ is an actual exception. This is isomorphic to $K_{3,3} \times K_2$ and contains two Möbius ladders, each of which is isomorphic to $K_{3,3}$. Each of them is drawn by thick lines in Figure 5 and has six vertices $1_\pm$, $2_\pm$, $3_\pm$, $a_\pm$, $b_\pm$ and $c_\pm$, labled so that two vertices with the same symbol and different signs are joined by an edge. In this case, there is an automorphism σ of $Q_h(4,3,1)$ which exchanges 1_+ and 2_+, and 1_- and 2_-, fixing the other vertices and it does not extend to any auto-homeomorphism over the Klein bottle. For example, the facial cycle $b_+2_+c_+3_+$ is mapped to $b_+1_+c_+3_+$ by σ and the latter is not faicial. Since σ can be regarded as an embedding of $Q_h(4,3,1)$ to the Klein bottle, $b_+2_+c_+3_+$ is not paneled for σ. Therefore, $Q_h(4,3,1)$ is not full-paneled.

Case 3. $G \cong Q_h(p,4,0)$: In this case, G contains at most two non-facial cycles of length 4, each of which is a longitude, if $p \neq 4$ and four meridians are non-facial cycles of length 4 in addition if $p = 4$. Thus, there are at most six non-facial cycles of length 4 in either case and we have $4p/2 < 4p-6$ if $p \geq 4$. Since $4p$ is equal to the number of faces in G and since G has at least $4p-6$ panels, there are two adjacent panels for f in G. Using Lemma 4 repeatedly, we can conclude that all facial cycles are paneled for f and hence G is full-paneled for f if $p \geq 4$.

If $p = 3$, then G is isomorphic to $Q_h(3,4,0)$ and it contains only one non-facial cycle of length 4, which is a longitude. Thus, G has at most one facial cycle which is not paneled for f. However, it is clear that a facial cycle surrounded by eight panels are paneled for f, too. Therefore, Thus, $Q_h(3,4,0)$ is not an exception of the lemma.

Case 4. $G \cong Q_h(4,r,\varepsilon)$: This is the remaining case. Suppose that the previous cases do not happen. Then G contains precisely r non-facial cycles of length 4, each of which forms a meridian on the Klein bottle, while the number of faces in G is equal to $4r$. This implies that there are at least $3r$ panels for f and there are two adjacent panels for f. By the same argument as above, we conclude that G is full-paneled. There is no exception in this case. ■

Now we shall discuss the faithfulness of embedding of the mesh types. Although there are infinitely many exceptions in this case, it is easier than the previous case.

Lemma 7. *A 4-regular quadrangulation $Q_m(p,r)$ of mesh type on the Klein bottle contains a non-facial cycle of length 4 if and only if either $p=2$ or $r=4$.*

Proof. Let G be a 4-regular quadrangulation $Q_m(p,r)$ of mesh type on the Klein bottle. The planar grid consisting of lines with slope ± 1, like Figure 2, covers G. Suppose that G contains a non-facial cycle C of length 4. Then C is lifted to a path of length 4 in the planar grid. We may assume that one of two ends of the path is located at $(0,0)$. Let (x,y) be the other end. Since they are joined by a path of length 4, we have $x \leq 4$ and $y \leq 4$ and both x and y are even numbers. Thus, we have $(x,y) \in \{0,2,4\} \times \{0,2,4\} - \{(0,0)\}$. However, if $x=2$, then we would have $r=2$, but G would not be simple in this case. The remaining case is when $x=0$ or 4, each of which corresponds to $p=2$ or $r=4$, respectively. Conversely, it is clear that G contains non-facial cycles of length 4 actually in these cases. ■

Lemma 8. *Every 4-regular quadrangulation $Q_m(p,r)$ of mesh type on the Klein bottle is full-paneled for all of its quadrangular embeddings if and only if $p \geq 3$ and $r \neq 4$.*

Proof. By Lemma 7, if $p \neq 2$ and $r \neq 4$, then G contains no non-facial cycle of length 4 and hence all facial cycles are paneled for any quadrangular embedding. Thus, the sufficiency is clear. To show the necessity, it suffices to show that $Q_m(2,r)$ and $Q_m(p,4)$ are not full-paneled for some quadrangular embedding.

It is easy to see that $Q_m(2,r)$ is isomorphic to the graph H_r constructed as follows. Prepare r pairs of vertices $\{x_i^+, x_i^-\}$ for $i \equiv 0,1,\dots,r-1 \pmod r$, and join $x_i^\pm$ and $x_{i+1}^\pm$ for every pair of signs $\pm$. By the isomorphism between $Q_m(2,r)$ and H_r, we may identify their vertices so that $x_i^- = u_{(i,0)}$ and $x_i^+ = u_{(i,2)}$ if i is even while $x_i^- = u_{(i,1)}$ and $x_i^+ = u_{(i,3)}$ if i is odd. It is clear that there is an automorphism σ_i which exchanges two vertices in $\{x_i^+, x_i^-\}$, fixing the other vertices. For example, σ_0 carries a facial cycle $u_{(0,2)}u_{(1,1)}u_{(2,2)}u_{(1,3)}$ to a non-facial cycle $u_{(0,0)}u_{(1,1)}u_{(2,2)}u_{(1,3)}$ and hence $Q_m(2,r)$ is not full-paneled for σ_0, which can be regarded as a quadrangular embedding.

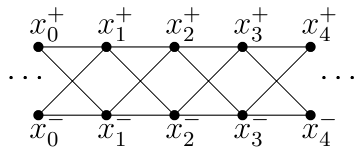

Figure 6 The isomorphism type of $Q_m(2, r)$

On the other hand, $Q_m(p, 4)$ also contains two distinct vertices w_1 and w_2 which can be exchanged by an automorphism σ fixing the other vertices; there are two pairs of those in fact. Since any identification for $Q_m(p, r)$ results in a unique quadrangulation up to isomorphism, we may assume that $u_{(0,0)} = u_{(4,2)}$ and $u_{(0,2)} = u_{(4,0)}$ in $Q_m(p, 4)$. Then $u_{(1,1)}$ and $u_{(3,1)}$ have four common neighbors $u_{(0,0)}$, $u_{(0,2)}$, $u_{(2,0)}$ and $u_{(2,2)}$, and we can choose these two as w_1 and w_2. Similarly to the previous, σ carries a facial cycle $u_{(1,1)}u_{(0,2)}u_{(1,3)}u_{(2,2)}$ to a non-facial cycle $u_{(3,1)}u_{(0,2)}u_{(1,3)}u_{(2,2)}$, and hence $Q_m(p, 4)$ is not full-paneled for a quadrangular embedding σ. Therefore, if $Q_m(p, r)$ is full-paneled for all quadrangular embeddings, then we have $p \neq 2$ and $r \neq 4$. ■

The following theorem is our goal in this section and will play an important role to decide the distinguishing number of 4-regular quadrangulations on the Klein bottle in the next section:

Theorem 9. *Every 4-regular quadrangulation on the Klein bottle has a unique quadrangular embedding on the Klein bottle, up to congruence. It is faithfully embedded on the Klein bottle, except $Q_l(4, 2)$, $Q_l(6, 2)$, $Q_m(2, r)$ with $r \geq 3$ and $Q_m(p, 4)$ with $p \geq 3$.*

Proof. Let G be any 4-regular quadrangulation on the Klein bottle, isomorphic to none of $Q_l(4, 2) \cong Q_h(4, 2, 1)$, $Q_l(6, 2) \cong Q_h(4, 3, 1)$, $Q_m(2, r)$ and $Q_m(p, 4)$. By Lemmas 6 and 8, G is full-paneled for all of its quadrangular embeddings and hence every quadrangular embedding of G extends to an auto-homeomorphism over the Klein bottle. This implies that any quadrangular embedding of G is equivalent to the inclusion map of G, up to homeomorphism and that G is faithfully embedded on the Klein bottle since any automorphism of G can be regarded as a quadrangular embedding.

On the other hand, the exceptions of the faithfulness are not isomorphic to one another as graphs if they have different types and different parameters. For example, if $Q_l(2r, 2)$ were isomorphic to $Q_m(2, r')$, then we would have $r' = 2r$, comparing the number of their vertices. It is not difficult to see that $Q_l(4, 2) \ncong Q_m(2, 4)$ and that $Q_l(6, 2) \ncong Q_m(2, 6)$. If $Q_l(2r, 2)$ were isomorphic to $Q_m(p, 4)$, then we would have $r = p$, but $Q_l(4, 2) \ncong Q_m(2, 4)$ and $Q_l(6, 2) \ncong Q_m(3, 4)$. These imply that each of the exceptions cannot be re-embedded on the Klein bottle as another exception. Therefore, each of them has a unique congruence class of quadrangular embeddings on the Klein bottle. They are not exceptions for the uniqueness up to congruence

We have already shown that each of the exceptions admits an automorphism which does not extend to any auto-homeomorphism over the Klein bottle, in the previous arguments. Thus, they are actually exceptions for the faithfulness and are those for the uniqueness up to homeomorphism, too. ■

Notice that this theorem concerns only quadrangular embeddings and says nothing about the existence of non-quadrangular embeddings. If a 4-regular quadrangulation G on the Klein bottle admits a non-quadrangular embedding on the Klein bottle, then it should have triangular faces since Euler's formula forces the average of faces sizes to be 4 for any embedding of G on the Klein bottle. Thus, G must contain a cycle of length 3, which is non-facial of course. Clearly, the candidates for such a G are very few. We shall leave the detailed arguments on the uniqueness of embedding for studies in future.

In the classification in [5], there has been given only the standard form of 4-regular quadrangulations on the Klein bottle, but the relation among those standard forms has never been discussed. Now we can refer to it after establishing the above theorem. A couple of parameters present the same 4-regular quadrangulations on the torus, as is shown in [3], while the type and parameters correspond bijectively to the isomorphism type of those on the Klein bottle.

Corollary 10. *Two* 4*-regular quadrangulations on the Klein bottle are isomorphic to each other as abstract graphs if and only if they have the same type, grid, ladder or mesh, with the same parameters.*

Proof. First neglect the exceptions for the faithfulness given in Theorem 9. Then each of the 4-regular quadrangulations on the Klein bottle has a unique quadrangular embedding on the Klein bottle, up to heomeorphism, which is nothing but its inclusion map. This implies that any isomorphism between two of those preserves any topological property of them on the Klein bottle. For example, The handle types contain meridians as their cycles while the mesh types do not. Furthermore, any grid type contains a longitude as its cycle while any ladder type does not. These properties distinguish the three types from one another. The parameters determine the length of meridians, longitude and so one and discriminate the isomorphism types of the same types. Also we can conclude the same fact for the exceptions neglected individually. ■

3. Distinguishability

Now we shall show that most of 4-regular quadrangulations on the Klein bottle are 2-distinguishable. To do it, it is convenient to rephrase the property of being 2-distinguishable as follows. A graph G is 2-distinguishable if and only if there is a subset S in $V(G)$ such that the only automorphism σ of G with $\sigma(S) = S$ is the identity map over G; specifying such a set S is nothing but specifying vertices which get label "1" in a 2-distinguishing labeling with labels "1" and "2".

First, we shall determine the distinguishing number of the exceptional ladder types that are not faithfully embedded on the Klein bottle:

LEMMA 11. *$D(Q_l(4, 2)) = 3$ and $D(Q_l(6, 2)) = 2$.*

Proof. First, we shall determine the distinguishing number of $G = Q_l(4, 2)$. This is isomorphic to $K_4 \times K_2$. Let $V_\pm = \{1_\pm, 2_\pm, 3_\pm, 4_\pm\}$ be the vertex sets of two K_4's so that i_+ and i_- are joined by an edge corresponding to the second factor K_2 in the product.

Let $c : V(G) \to \{1, 2\}$ be a 2-distinguishing labeling of G and suppose that there are two vertices in one of two K_4's which get the same label, say $c(1_+) = c(2_+) = 1$. If $c(1_-) = c(2_-)$, then the automorphism of G which exchanges 1_+ and 2_+, and 1_- and 2_-, fixing the other vertices preserves the labels of vertices given by c. This is contrary to c being 2-distinguishing and hence we have $c(1_-) \neq c(2_-)$. If $c(3_+) = 1$ in addition, then we would have $c(3_-) \neq c(1_-)$ and $\neq c(2_-)$, but this is impossible since there are only two labels. By similar arguments, we conclude, up to symmetry, that:

$$c(1_+) = c(2_+) = c(1_-) = c(3_-) = 1, \quad c(3_+) = c(4_+) = c(2_-) = c(4_-) = 2$$

However, the automorphism σ defined below preserves the labels:

$$\sigma(1_\pm) = 1_\mp, \quad \sigma(2_\pm) = 3_\mp, \quad \sigma(3_\pm) = 2_\mp, \quad \sigma(4_\pm) = 4_\mp$$

This is contrary to the assumption on c, too. Therefore, there is no 2-distinguishing labeling of G and G is not 2-distinguishable.

Now consider the following labeling $c : V(G) \to \{1, 2, 3\}$:

$$c(1_\pm) = c(2_+) = c(3_-) = c(4_-) = 1, \quad c(2_-) = c(3_+) = 2, \quad c(4_+) = 3$$

Let σ be any automorphism of G preserving this labeling. It is clear that any automorphism of $G \cong K_4 \times K_2$ fixes each of $V_\pm$ setwise or exchanges them. Since V_+ contains "3" but V_- does not, the former holds for σ and hence we have

$\sigma(\{1_+, 2_+\}) = \{1_+, 2_+\}$, $c(3_+) = 3_+$ and $c(4_+) = 4_+$. The last two imply directly that σ fixes 3_- and 4_- and it follows that σ fixes 1_- and 2_- since $c(1_-) \neq c(2_-)$. This implies that σ fixes 1_+ and 2_+ and hence σ is the identity map over G. Therefore, G is 3-distinguishable and we have $D(G) = 3$.

Now put $G = Q_l(6, 2)$. This is isomorphic to $K_{3,3} \times K_2$ and contains two disjoint $K_{3,3}$'s, say K_+ and K_-, as shown in Figure 7. Let $X_\pm = \{1_\pm, 2_\pm, 3_\pm\}$ and $Y_\pm = \{a_\pm, b_\pm, c_\pm\}$ be the partite sets of $K_\pm$ so that each pair of vertices with the same symbol and different signs are joined by an edge in G. In the figure, $1_\pm$, $2_\pm$ and $3_\pm$ lie at the top of $K_\pm$ and $a_\pm$, $b_\pm$ and $c_\pm$ at the bottom of $K_\pm$ in order. It is clear that G is a bipartite graph with partite sets $X_+ \cup Y_-$ and $Y_+ \cup X_-$; the vertices in the former are black and those in the latter are white. Define a labeling $c : V(G) \to \{1, 2\}$ as shown in the figure:

$$c(1_+) = c(2_+) = c(c_+) = c(1_-) = c(a_-) = 1$$
$$c(3_+) = c(a_+) = c(b_+) = c(2_-) = c(3_-) = c(b_-) = a(c_-) = 2$$

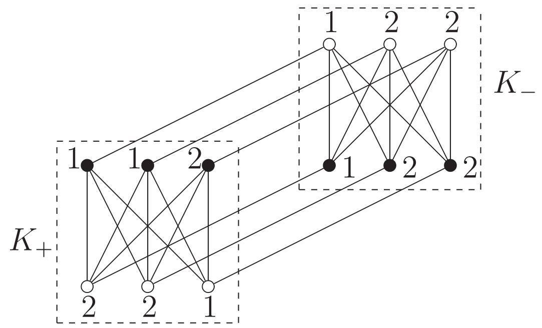

Figure 7 A 2-distinguishing labeling of $K_{3,3} \times K_2$

Let σ be any automorphism of G preserving this labeling. Since the number of "1"'s in K_+ is different from that in K_-, σ cannot exchange K_+ and K_-, and hence we have $\sigma(K_+) = K_+$ and $\sigma(K_-) = K_-$. Similarly, we have $\sigma(X_+) = X_+$ and $\sigma(Y_+) = Y_+$ since the number of "1"'s in X_+ is different from that in Y_+. This imlies that $\sigma(3_+) = 3_+$ and $\sigma(c_+) = c_+$. If σ exchanged 1_+ and 2_+, then it would exchange 1_- and 2_-, but this is forbidden since they have different labels. Thus, we have $\sigma(1_+) = 1_+$ and $\sigma(2_+) = 2_+$, and similary $\sigma(a_+) = a_+$ and $\sigma(b_+) = b_+$. That is, $\sigma|_{K_+}$ must be the identity map and this forces $\sigma|_{K_-}$ to be the identity map, too. Therefore, the whole of σ also is the identity map over G and we conclude that σ is a 2-distinguishing labeling of G and that G is 2-distinguishable. Since $\mathrm{Aut}(G)$ is not trivial, we have $D(G) = 2$. ■

The following lemma suggests the reason why the faithfulness of embedding works well to analyze the distinguishing number of graphs embedded on closed surfaces; local arguments extend to the global one.

Lemma 12. *Let G be a connected graph faithfully embedded on a closed surface F^2. If an automorphism of G fixes a corner of a face, then it is the identity map over G.*

Proof. Let A be a face with a corner uvw and σ an automorphism of G fixing the corner, that is, $\sigma(u) = u$, $\sigma(v) = v$ and $\sigma(w) = w$. Since G is faithfully embedded on the surface, σ extends to an auto-homeomorphism h over F^2 and h sends each face to a face of G. In particular, we have $h(A) = A$ since σ fixes the corner of A. This implies that σ fixes each of the vertices along the boundary cycle of A. Furthermore, σ fixes a coner of each face adjacent to A since h preserves the rotation around each vertex on the surface. Repeating this argument, we can conclude that σ fixes the vertices lying along the boundary cycles of every face since G is connected. Thus, σ is the identity map over G. ■

The following two theorems and Lemma 11 determine the distinguishing number of grid types and ladder types completely. The basic idea to prove the theorems is to make a coner fixed by an automorphism, setting S suitably.

Theorem 13. *Every 4-regular quadrangulation $Q_g(p, r)$ of grid type on the Klein bottle is 2-distinguishable.*

Proof. Let G be a grid type $Q_g(p, r)$ with $p \geq 3$ and $r \geq 3$. Then G contains at least one longitude of length r. Let $w_0w_1w_2w_3$ be the boundary cycle of a face A containing one edge w_0w_1 on one of the longitudes.

Put $S = \{w_0, w_1, w_3\}$. Then the subgraph H induced by S is isomorphic to a path of length 2 and forms a corner of A. Take any automorphism σ of G with $\sigma(S) = S$. This extends to an auto-homeomorphism h over the Klein bottle since G is faithfully embedded by Theorem 9. Since $\deg_H w_0 = 2$ and $\deg_H w_1 = \deg_H w_3 = 1$, σ fixes w_0 and $\sigma(\{w_1, w_3\}) = \{w_1, w_3\}$. If σ swapped w_1 and w_3, then h would carry the longitude containing w_0w_1 to a cycle containing w_0w_3, which is a meridian of length p. Since this is impossible however, σ must fix each of w_1 and w_3 and hence it fixes the corner $w_3w_0w_1$ of the face A. By Lemma 12, σ becomes the identity map over G. This implies that G is 2-distinguishable. ■

Theorem 14. *A 4-regular quadrangulation $Q_l(2r, s)$ of ladder type on the Klein bottle is 2-distinguishable unless it is isomorphic to $Q_l(4, 2)$.*

Proof. Every ladder type $G = Q_l(2r, s)$ contains two Möbius ladders as its subgraphs and each of their rims forms an equator on the Klein bottle. Take one of edges lying on the rims, say uv, and let $uvwx$ be the face incident to the edge uv and lying outside the Möbius ladder. Then vw is contained in a meridian of length $2s$. Set $S = \{u, v, w\}$ and let σ be any automorphism of G with $\sigma(S) = S$.

By Theorem 9, if $(2r, s) \neq (4, 2)$ and $(6, 2)$, then G is faithfully embedded on the Klein bottle and hence σ extends to an auto-homeomorphism h over the surface. It is clear that $\sigma(v) = v$ since the only vertex of degree 2 in the subgraph induced by S is v. Thus, there are two possibilities; σ fixes each of u and w, or exchanges them. However, h would carry the equator containing vu to the meridian containing vw in the latter case, but this is impossible. Therefore, σ fixes the corner uvw and must be the identity map over G by Lemma 12. This implies that G is 2-distinguishable.

By Lemma 11, $Q_l(6, 2)$ also is 2-distinguishable and hence is not an exception of the theorem. Thus, we have only one exception, which is $Q_l(4, 2)$. ■

We need a slightly big task to analyze the distinguishing number of mesh types since there are infinitely many exceptions for the faithfulness of their embeddings. We shall determine the distinguishing number of the exceptions before establishing the theorem for mesh types.

Lemma 15.

$$D(Q_m(2,r)) = \begin{cases} 3 & (r = 3,\ r \geq 5) \\ 5 & (r = 4) \end{cases}$$

Proof. First we shall show that $D(Q_m(2, r)) \geq 3$. As is shown in our proof of Lemma 8, $Q_m(2, p)$ is isomorphic to H_r, which consists of r pairs $\{x_i^+, x_i^-\}$ for $i \equiv 0, 1, \cdots, r-1$ and it has an automorphism σ_i which exchanges x_i^+ and x_i^-, fixing the other vertices. Suppose that there is a 2-distinguishing labeling $c : V(H_r) \to \{1, 2\}$ of H_r. If $c(x_i^+) = c(x_i^-)$ for some i, then σ_i preserves this labeling, contrary to c being 2-distinguishing. Thus, $c(x_i^+) \neq c(x_i^-)$ for all i and we may assume that $c(x_i^+) = 1$ and $c(x_i^-) = 2$, up to symmetry. However, there is an automorphis which shifts $x_i^{\pm}$ to $x_{i+1}^{\pm}$, preserving the labels in this case. This is contrary to our assumption, again. Therefore, there is no 2-distinguishing labeling of H_r and hence we have $Q_m(2, r) \geq 3$.

Now suppose that $r \geq 5$ and define a labeling $c : V(H_r) \to \{1, 2, 3\}$ by:

$$\begin{aligned} &c(x_0^+) = 1, \quad c(x_j^+) = 2 \ \ (j = 2, \ldots, r-1) \\ &c(x_i^-) = 3 \ \ (i = 0, 1, \ldots, r-2), \quad c(x_{r-1}^-) = 1 \end{aligned}$$

It is clear that any automorphism σ of H_r preserving this labeling fixes the two cycles $x_0^+ x_1^+ \cdots x_{r-1}^+$ and $x_0^- x_1^- \cdots x_{r-1}^-$ setwise, say C^+ and C^-, respectively. In particular, σ fixes x_0^+ and x_{r-1}^- since each of them is a unique vertex on each cycle with label "1". Thus, σ flips each of $C^\pm$ or fixes it pointwise. If the former happens for C^+, then x_{r-1}^+ is mapped to x_1^+ by σ. However, this is impossible since x_{r-1}^+ lies on a cycle of length 4 which contains two "1"'s, but x_1^+ does not. Thus, σ must fix C^+ pointwise and also C^- similarly, and hence it is the identity map over H_r. This implies that c is a 3-distinguishing labeling and that H_r is 3-distinguishable. Therefore, $D(Q_m(2,r)) = D(H_r) = 3$ for $r \geq 5$.

On the other hand, it is easy to see that $Q_m(2,3) \cong K_{2,2,2}$ and $Q_m(2,4) \cong K_{4,4}$. Also we have $D(K_{2,2,2}) = 3$ and $D(K_{4,4}) = 5$. These complete the formula in the lemma. ■

Lemma 16. *The mesh type $Q_m(p,4)$ is 2-distinguishable if $p \geq 3$.*

Proof. Consider the meshed rectangles as given in Figure 8 to construct the mesh type $Q_m(p,4)$ and use the notation $u_{(x,y)}$ to denote the vertex in $Q_m(p,4)$ corresponding to a point (x,y) on the xy-plane; the left bottom corner has coodinates $(0,0)$ in particular. Since any identification to obtain the Klein bottle results in a unique graph up to isomorphism, we may assume that $u_{(0,i)} = u_{(4,2p-i)}$ for $i = 0, 2, \ldots, 2p$.

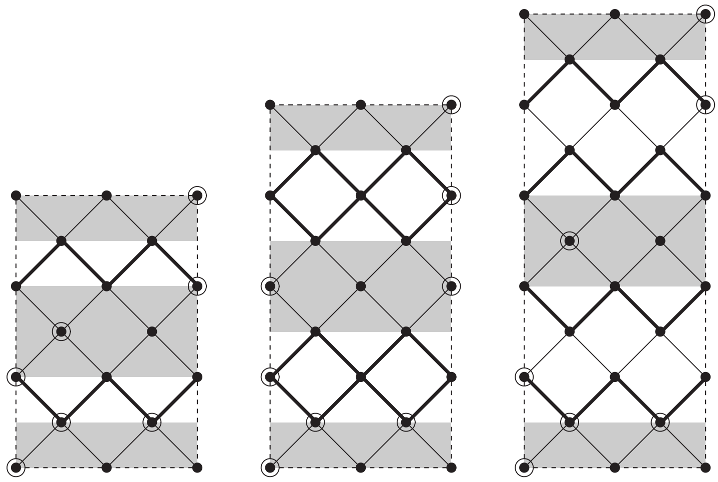

Figure 8 The mesh type $Q_m(p,4)$

As we have seen in our proof of Lemma 8, $Q_m(p,4)$ has two pairs of those vertices, say X_1 and X_2, that have four common neighbors. They are $X_1 = \{u_{(0,0)}, u_{(2,0)}\}$ and $X_2 = \{u_{(1,p)}, u_{(3,p)}\}$ if p is odd and $X_2 = \{u_{(0,p)}, u_{(2,p)}\}$ if p is even. The shaded regions in each meshed rectangle in Figure 8 form two Möbius bands containing X_1 and X_2 in cases of $p = 3$, 4 and 5. The boundary of each of the Möbius bands contains the four neighbors of two vertices in X_i. Then we can find a unique cycle of length 8 such that the four neighbors and others lie along it alternately. Let C_i be the cycle containing the four neighbors of X_i for $i = 1, 2$; they are drawn by thick lines in the figure.

$$\begin{aligned} C_1 &= u_{(0,2)}u_{(1,1)}u_{(2,2)}u_{(3,1)}u_{(4,2)}u_{(1,2p-1)}u_{(2,2p-2)}u_{(3,2p-1)} \\ C_2 &= u_{(0,p-2)}u_{(1,p-1)}u_{(2,p-2)}u_{(3,p-1)}u_{(4,p-2)}u_{(1,p+1)}u_{(2,p+2)}u_{(3,p+1)} \ (p\text{: even}) \\ &= u_{(0,p-1)}u_{(1,p-2)}u_{(2,p-1)}u_{(3,p-2)}u_{(4,p-1)}u_{(1,p+2)}u_{(2,p+1)}u_{(3,p+2)} \ (p\text{: odd}) \end{aligned}$$

The uniqueness of these cycles implies that $\sigma(C_i) = C_j$ and $\sigma(X_i) = X_j$ with $i, j \in \{1, 2\}$, possibly equal, for any automorphism σ of $Q_m(p, 4)$. Also, there is an automorphism σ_i which exchanges the two vertices in X_i, fixing the other vertices. This implies that any distinguishing labeling assigns two distinct labels to these two vertices. Put $S = \{u_{(0,0)}, u_{(0,p)}, u_{(0,2)}, u_{(1,1)}, u_{(3,1)}\}$ if p is even and replace $u_{(0,p)}$ in S with $u_{(1,p)}$ if p is odd; the vertices belonging to S are encircled in the figure. Define a labeling $c : V(Q_m(p, 4)) \to \{1, 2\}$ by $c(v) = 1$ for $v \in S$ and $c(v) = 2$ for $v \notin S$. We shall show that c is a 2-distinguishing labeling to conclude that $Q_m(p, 4)$ is 2-distinguishable. Let σ be any automorphism σ of $Q_m(p, 4)$ which preserves the labels given by c.

If $p = 3$, then we have $C_1 = C_2$ and $Q_m(3, 4)$ consists of this cycle C_1 and the four vertices in $X_1 \cup X_2$. First, the occurences of "1" along C_1 forces σ to fix each of the vertices lying on C_1. The four neighbors of X_1 get two "1"'s and two "2"'s while those of X_2 get one "1" and three "2"'s. This implies that σ cannot exchange X_1 and X_2, and hence it fixes them. Therefore, σ must be the identity map over $Q_m(3, 4)$ and we conclude that c is 2-distinguishing.

Suppose that $p \geq 4$. Then C_1 and C_2 are distinct cycles and C_1 has three vertices with label "1". Since the number of "1"'s occurring along C_2 is 1 or 0, σ cannot exchange C_1 and C_2 and hence we have $\sigma(C_i) = C_i$ and $\sigma(X_i) = X_i$ for $i = 1, 2$. In particular, σ fixes each of the vertices lying on C_1 as well as in the previous case.

Here we shall use the following logic; if σ fixes each of $u_{(x-1,y)}$, $u_{(x,y-1)}$ and $u_{(x+1,y)}$, then σ fixes $u_{(x,y+1)}$, too. However, this is incorrect if there is a vertex w, other than $u_{(x,y+1)}$, adjacent to both $u_{(x-1,y)}$ and $u_{(x+1,y)}$ since σ might exchange $u_{(x,y+1)}$ and w. For example, if $y = p - 1$, then this unexpected case happens actually. Conversely, we can carry out the above logic repeatedly whlie $2 \leq y \leq p - 2$ and conclude that σ fixes the vertices in the lower white regoin. Similarly, σ fixes those in the upper white region and hence it fixes each of the vertices lying on C_2. Also σ fixes each of the two vertices in X_i since they get different labels. Now we have seen that σ fixes all vertices, that is, it is the identity map over $Q_m(p, 4)$. Therefore, c is a 2-distinguishing labeling and hence $Q_m(p, 4)$ is 2-distinguishable. ■

Now all we need to prove the following theorem has been prepared:

THEOREM 17. *A 4-regular quadrangulation $Q_m(p,r)$ of mesh type on the Klein bottle is 2-distinguishable if and only if $p \geq 3$.*

Proof. The necessity is clear by Lemma 15. To prove the sufficiency, we assume that $p \geq 3$ and we may assume that $Q_m(p,r)$ is faithfully embedded on the Klein bottle since we have already discussed the case when it is not, in Lemmas 15 and 16. That is, we have $r = 3$ or $r \geq 5$.

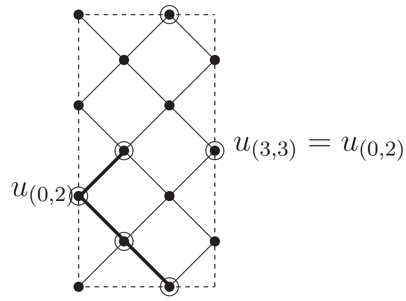

Figure 9 A 2-distinguishing labeling of $Q_m(p,3)$

Put $S = \{u_{(0,2)}, u_{(1,1)}, u_{(1,3)}, u_{(2,0)}\}$ and let H be the subgraph in $Q_m(p,r)$ induced by S. It is clear that H is a path of length 3 and $u_{(1,3)}u_{(0,2)}u_{(1,1)}$ form a corner of a face if $r \geq 5$. On the other hand, if $r = 3$, then we need to take account of the identification of the left and right sides of the meshed rectangle. In this case, we may assum that $u_{(0,2)}$ is identified with $u_{(3,3)}$ in $Q_m(p,3)$ after changing identification along the vertical sides; recall that this does not change the isomorphism type of $Q_m(p,3)$. Thus, if $p \geq 3$, then H forms a path of length 3 bending at a corner of a face like "Γ", as well as in case of $r \geq 5$.

Now take any automorphism σ of $Q_m(p,r)$ with $\sigma(S) = S$. Then σ extends to an auto-homeomorphism h over the Klein bottle since $Q_m(p,r)$ is faithfully embedded. Since h preserves any corner of a face, it is clear that σ fixes the corner $u_{(1,3)}u_{(0,2)}u_{(1,1)}$ and hence it must be the identity map over $Q_m(p,r)$ by Lemma 12. Therefore, $Q_m(p,r)$ is 2-distinguishable if $p \geq 3$ and if $r = 3$ or $r \geq 5$, and so is it in case of $r = 4$ by Lemma 16. ■

We do not need to prove Theorem 1 any longer because it just states Theorems 13, 14 and 17 together. It should be noticed that the infinite series of exceptions $Q_m(2,r)$ in our theorem can be embedded on the torus as 4-regular quadrangulations and hence they also appear as exceptions in the theorem in [3]. On the other hand, the other exception $Q_l(4,2) \cong K_4 \times K_2$ in our thoerem cannot be a quadrangulation on the torus although it can be embedded there.

Corollary 18. *The distinguishing number of* 4*-regular quadrangulations on the Klein bottle takes only three values* 2*,* 3 *and* 5*.*

Proof. It is clear that the automorphism group of any 4-regular quadrangulation on the Klein bottle is non-trivial and hence $D(G) \geq 2$. We find only 2, 3 and 5 in the theorems, looking through this section. ■

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